Categories, Proofs and Processes
Lecture 4: Natural Transformations

Samson Abramsky & Nikos Tzevelekos

Natural transformations

> *"Categories were only introduced to allow functors to be defined; functors were only introduced to allow natural transformations to be defined."*

Just as categories have morphisms between them, namely functors, so functors have morphisms between them too —*natural transformations*.

Natural Transformations Defined

Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. A natural transformation $t : F \longrightarrow G$ is a family of $\mathcal{D}$-morphisms

$$\{ \, t_A : FA \longrightarrow GA \, \}_{A \in Ob(\mathcal{C})}$$

indexed by objects A of $\mathcal{C}$, such that, for all $f : A \longrightarrow B$, the following diagram commutes.

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "t_A"'] & FB \arrow[d, "t_B"] \\
GA \arrow[r, "Gf"'] & GB
\end{tikzcd}$$

If each t_A is an isomorphism, we say that t is a *natural isomorphism*:

$$t : F \xrightarrow{\cong} G$$

Examples

- Let Id be the identity functor on **Set**, and Δ the diagonal functor taking each set X to (X, X) and each function f to (f, f). Then there is a natural transformation

  $$\delta : \mathsf{Id} \longrightarrow \Delta$$

  $$\delta_X : X \longrightarrow X \times X := x \mapsto (x, x).$$

  (This is in fact the *only* natural transformation between these functors.)

- Natural transformations on lists:

  $$\mathsf{reverse}_X : \mathsf{List}(X) \longrightarrow \mathsf{List}(X) := [x_1, \ldots, x_n] \mapsto [x_n, \ldots, x_1]$$

  $$\mathsf{unit}_X : X \longrightarrow \mathsf{List}(X) := x \mapsto [x]$$

  $$\begin{aligned} \mathsf{flatten}_X &: \mathsf{List}(\mathsf{List}(X)) \longrightarrow \mathsf{List}(X) \\ &:= [[x^1_1, \ldots, x^1_{n_1}], \ldots, [x^k_1, \ldots, x^k_{n_k}]] \mapsto [x^1_1, \ldots \ldots, x^k_{n_k}] \end{aligned}$$

Examples ctd.

- Projections and terminal arrows in a category $\mathcal{C}$ with finite products:

$$\pi_{1(A,B)} : A \times B \longrightarrow A$$
$$\tau_A : A \longrightarrow \mathbf{1}$$

  Also, in such a $\mathcal{C}$, diagonals:

$$\delta_A : A \longrightarrow A \times A$$
$$\delta_A : A \longrightarrow A \times A := \langle \mathsf{id}_A, \mathsf{id}_A \rangle$$

- Consider the following functor,

$$U \times U : \textbf{Mon} \longrightarrow \textbf{Set} := (M, \cdot, 1) \mapsto M \times M,\ f \mapsto f \times f$$

  Then, the monoid operation yields a natural transformation $t : U \times U \longrightarrow U$, defined by:

$$t_{(M,\cdot,1)} : M \times M \longrightarrow M := (m, m') \mapsto m \cdot m'$$

Natural isomorphisms for products

If a category $\mathcal{C}$ has binary products and a terminal object, then there are natural isomorphisms

$$a_{A,B,C} : A \times (B \times C) \xrightarrow{\cong} (A \times B) \times C$$

$$l_A : \mathbf{1} \times A \xrightarrow{\cong} A \qquad r_A : A \times \mathbf{1} \xrightarrow{\cong} A$$

$$s_{A,B} : A \times B \xrightarrow{\cong} B \times A$$

For example:

$$a_{A,B,C} = \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle.$$

Since natural isomorphisms are a *self-dual* notion, the same holds if a category has binary coproducts and an initial object.

Natural transformations between Hom-functors

Let $f : A \longrightarrow B$ in $\mathcal{C}$. Then this induces a natural transformation

$$t : \mathcal{C}(B, \_) \longrightarrow \mathcal{C}(A, \_)$$
$$t_C := (g : B \to C) \mapsto (g \circ f : A \to C)$$

Naturality is

$$\begin{tikzcd} C \arrow[d, "h"'] \\ D \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(B,C) \arrow[r, "t_C"] \arrow[d, "\mathcal{C}(B,h)"'] & \mathcal{C}(A,C) \arrow[d, "\mathcal{C}(A,h)"] \\ \mathcal{C}(B,D) \arrow[r, "t_D"'] & \mathcal{C}(A,D) \end{tikzcd}$$

Starting from $g : B \to C$, we compute:

$$\mathcal{C}(A, h)(t_C(g)) = h \circ (g \circ f) = (h \circ g) \circ f = t_D(\mathcal{C}(B, h)(g)).$$

We write t as $\mathcal{C}(f, \_)$.

The Yoneda Lemma

There is a remarkable result, the *Yoneda Lemma*, which says that *every* natural transformation between Hom-functors comes from a (unique) arrow in $\mathcal{C}$ in this way:

Lemma Let A, B be objects in $\mathcal{C}$. For each natural transformation $t : \mathcal{C}(A, \_) \longrightarrow \mathcal{C}(B, \_)$, there is a unique arrow $f : B \longrightarrow A$ such that

$$t = \mathcal{C}(f, \_)$$

Proof: Take any such A, B and t and let

$$f : B \longrightarrow A := t_A(\mathsf{id}_A)$$

We want to show that $t = \mathcal{C}(f, \_)$. For any object C and any arrow $g : A \longrightarrow C$, naturality of t means that

$$\begin{tikzcd} \mathcal{C}(A,A) \arrow[r, "\mathcal{C}(A,g)"] \arrow[d, "t_A"'] & \mathcal{C}(A,C) \arrow[d, "t_C"] \\ \mathcal{C}(B,A) \arrow[r, "\mathcal{C}(B,g)"'] & \mathcal{C}(B,C) \end{tikzcd}$$

commutes. Starting from id_A we have that

$$t_C(\mathcal{C}(A, g)(\mathsf{id}_A)) = \mathcal{C}(B, g)(t_A(\mathsf{id}_A)) \;\; \textit{i.e.} \;\; t_C(g) = g \circ f$$

Hence, noting that and $\mathcal{C}(f, C)(g) = g \circ f$, we have $t = \mathcal{C}(f, \_)$.
For uniqueness, we have that, for any $f, f' : B \longrightarrow A$, if $\mathcal{C}(f, \_) = \mathcal{C}(f', \_)$ then

$$f = \mathsf{id}_A \circ f = \mathcal{C}(f, A)(\mathsf{id}_A) = \mathcal{C}(f', A)(\mathsf{id}_A) = \mathsf{id}_A \circ f' = f'.$$

■

Vertical composition and Functor Categories

Suppose we have functors $F, G, H : \mathcal{C} \longrightarrow \mathcal{D}$, and natural transformations

$$t : F \longrightarrow G, \qquad u : G \longrightarrow H.$$

Then we can compose these natural transformations and obtain $u \circ t : F \longrightarrow H$,

$$(u \circ t)_A := F(A) \xrightarrow{t_A} G(A) \xrightarrow{u_A} H(A).$$

This composition is associative, and has as identity the natural transformation

$$I_F : F \longrightarrow F := \{ \, (I_F)_A := \mathsf{id}_A : F(A) \longrightarrow F(A) \, \}_A.$$

In this way can define the *functor category* $\mathsf{Func}(\mathcal{C}, \mathcal{D})$:

- Objects: functors $F : \mathcal{C} \longrightarrow \mathcal{D}$.
- Arrows: natural transformations $t : F \longrightarrow G$.

Thus we see that in the category **Cat** of categories and functors, each hom-set **Cat**$(\mathcal{C}, \mathcal{D})$ *itself has the structure of a category*.
In fact, **Cat** is the basic example of a "2-category".

Examples of Functor Categories

- $\mathsf{Func}(G, \textbf{Set})$: G-actions on sets and *equivariant functions*: $f : X \to Y$ such that $f(g \cdot x) = g \cdot f(x)$.
- $\mathsf{Func}(\bullet \rightrightarrows \bullet, \textbf{Set})$: Graphs and graph homomorphisms.
- If $F, G : P \longrightarrow Q$ are monotone maps between posets, then $t : F \longrightarrow G$ means that

$$\forall x \in P.\, Fx \leq Gx.$$

Note that a natural isomorphism is exactly an isomorphism in the functor category.

Alternative definition of equivalence

Another way of defining equivalence of categories, $\mathcal{C} \simeq \mathcal{D}$:

There are functors $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms

$$G \circ F \cong \mathsf{Id}_{\mathcal{C}}, \qquad F \circ G \cong \mathsf{Id}_{\mathcal{D}}.$$

This definition is equivalent (assuming the Axiom of Choice) to the one given previously.