Programming Research Group

COMPLETE POSITIVITY WITHOUT POSITIVITY AND WITHOUT COMPACTNESS

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Abstract

Given any †-symmetric monoidal category $\mathbf{C}$ we construct a new category $\mathbf{Mix}(\mathbf{C})$, which, in the case that $\mathbf{C}$ is a †-compact category, is isomorphic to Selinger's $\mathbf{CPM}(\mathbf{C})$ [Sel]. Hence, if $\mathbf{C}$ is the category $\mathbf{FdHilb}$ of finite dimensional Hilbert spaces and linear maps we exactly obtain completely positive maps as morphisms. This means that *mixedness* of states and operations, within the categorical quantum axiomatics developed in [AC1, AC2, Sel, CPv, CPq], is a concept which exists independently of the quantum and classical structure. Moreover, since our construction does not require †-compactness, it can be applied to categories which have infinite dimensional Hilbert spaces as objects. Finally, in general $\mathbf{Mix}(\mathbf{C})$ is not a †-category, so does not admit a notion of positivity. This means that, in the abstract, the notion of 'complete positivity' can exist independently of a notion of 'positivity', which points at a very unfortunately terminology.

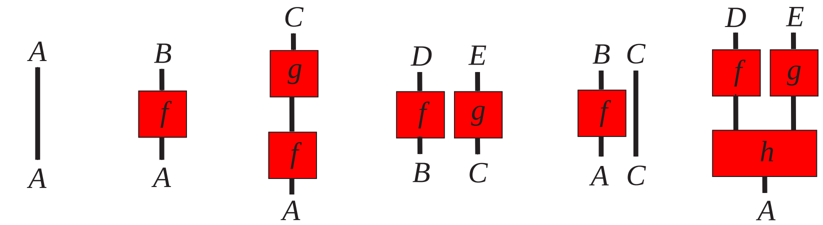
1 Definition of $\mathbf{Mix}(\mathbf{C})$

A *†-symmetric monoidal category* [Sel] $(\mathbf{C}, \otimes, \dagger)$ is a symmetric monoidal category $(\mathbf{C}, \otimes)$ with an involutive identity-on-objects contravariant functor $(-)^\dagger$, which preserves the tensor, and relative to which all the natural isomorphisms of the symmetric monoidal structure are unitary, that is, their adjoint coincides with their inverse.

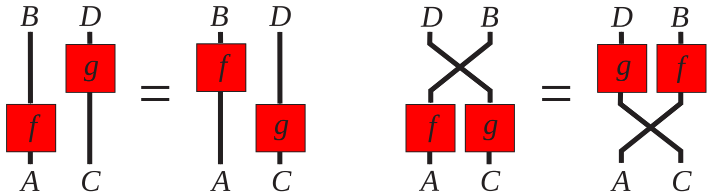
We will rely on the graphical calculus for †-symmetric monoidal categories of [Sel]. It extends the graphical calculus for symmetric monoidal categories of [JS]. For example

$$1_A \qquad f \qquad g\circ f \qquad f\otimes g \qquad f\otimes 1_C \qquad (f\otimes g)\circ h$$

for $f : A \to B$, $g : B \to C$ and $h : E \to A \otimes B$ depict as:



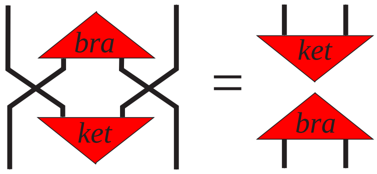
These boxes are subject to the intuitively obvious graphical rules:



which capture commutation of the diagrams:

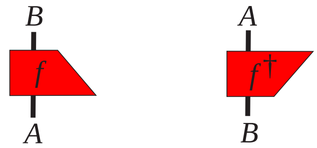
$$\begin{array}{ccc} A\otimes C & \xrightarrow{1_A\otimes g} & A\otimes D \\ {\scriptstyle f\otimes 1_C}\downarrow & & \downarrow{\scriptstyle f\otimes 1_D} \\ B\otimes C & \xrightarrow{1_B\otimes g} & B\otimes D \end{array} \qquad\qquad \begin{array}{ccc} A\otimes C & \xrightarrow{\sigma_{A,C}} & C\otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ B\otimes D & \xrightarrow{\sigma_{B,D}} & D\otimes B \end{array}$$

More sophisticated ones such as:



(1)

follow by coherence. The adjoint depicts as reversal i.e. $f : A \to B$ and $f^\dagger : B \to A$ depict as:

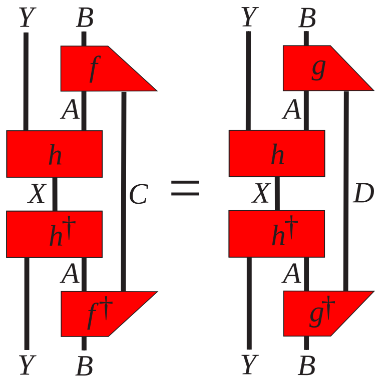


where asymetric boxes are required to make this reversal visible.

For $f : A \otimes C \to B$ and $h : X \to Y \otimes A$ let

$$\Xi(f, h) := (1_Y \otimes f) \circ ((h \circ h^\dagger) \otimes 1_C) \circ (1_Y \otimes f)^\dagger : Y \otimes B \to Y \otimes B\,.$$

For $f : A \otimes C \to B$ and $g : A \otimes D \to B$ we set $f \simeq g$ iff for all $X, Y \in |\mathbf{C}|$ and for all $h : X \to Y \otimes A$ we have that $\Xi(f, h) = \Xi(g, h)$ i.e.



Clearly $\simeq$ defines an equivalence relation on $\bigcup_{C\in|\mathbf{C}|} \mathrm{Hom}(A \otimes C, B)$.

Lemma 1.1. *If* $f \simeq f' : A \otimes D^{(\prime)} \to B$ *and* $g \simeq g' : B \otimes E^{(\prime)} \to C$ *then*

$$g \circ (f \otimes 1_E) \simeq g' \circ (f' \otimes 1_{E'}) : A \otimes (D^{(\prime)} \otimes E^{(\prime)}) \to C\,,$$

and if $f \simeq f' : A \otimes E^{(\prime)} \to B$ *and* $g \simeq g' : C \otimes F^{(\prime)} \to D$ *then,*

$$(f \otimes g) \circ (1_A \otimes \sigma_{C,E} \otimes 1_F) \simeq (f' \otimes g') \circ (1_A \otimes \sigma_{C,E'} \otimes 1_{F'}) : (A \otimes C) \otimes (E^{(\prime)} \otimes F^{(\prime)}) \to B \otimes D\,,$$

where for notational convenience we took associativity to be strict and used brackets merely to group the input types A *and* C *and variable types* $D^{(\prime)}$, $E^{(\prime)}$ *and* $F^{(\prime)}$.

Proof: Straightforward verification. □

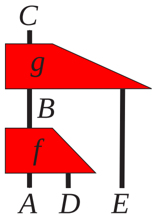
Definition 1.2. Given any †-symmetric monoidal category $\mathbf{C}$ define a new category $\mathbf{Mix}(\mathbf{C})$ as follows: **i.** it has the same objects as $\mathbf{C}$; **ii.** its hom-sets are

$$\mathbf{Mix}(\mathbf{C})(A, B) := \left\{ [f] \;\middle|\; f \in \bigcup_{C \in |\mathbf{C}|} \mathrm{Hom}(A \otimes C, B) \right\}$$

where $[f]$ denotes the equivalence class containing f for the equivalence relation $\simeq$; **iii.** its identities are inherited from $\mathbf{C}$; **iv.** for $[f] : A \to B$ and $[g] : B \to C$ the composite $[g] \circ [f] : A \to C$ is the equivalence class in $\bigcup_{F \in |\in C|} \mathrm{Hom}(A \otimes F, C)$ for $\simeq$ which contains

$$g \circ (f \otimes 1_E) \circ \alpha_{A,D,E} : A \otimes (D \otimes E) \to B\,,$$

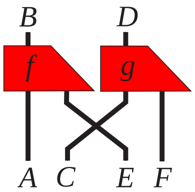
i.e., diagrammatically



v. for $[f] : A \to B$ and $[g] : C \to D$ the tensor $[g] \otimes [f] : A \to C$ is the equivalence class in $\bigcup_{G \in |\in C|} \mathrm{Hom}((A \otimes C) \otimes G, B \otimes D)$ for $\simeq$ which contains

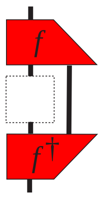
$$(f \otimes g) \circ \alpha^{\dagger}_{A,C,E,F} \circ (1_A \otimes \sigma_{C,E} \otimes 1_F) \circ \alpha_{A,C,E,F} : (A \otimes C) \otimes (E \otimes F) \to B \otimes D\,,$$

i.e., diagrammatically

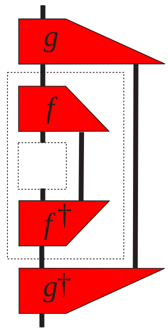


By Lemma 1.1 this composition and this tensor are both well-defined and the reader can easily verify that we indeed obtain a symmetric monoidal category.

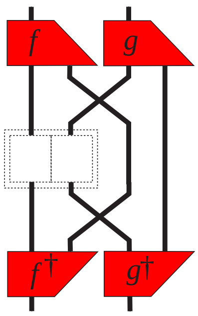
One can conveniently depict the morphisms of $\mathbf{Mix}(\mathbf{C})$ as containing "holes":



and composition boils down to inserting a morphism in the hole of another one:



while the tensor involves combining two holes into one:



2 Mix(†-CC) $\simeq$ CPM(†-CC)

For a comprehensive account on †-compact categories we refer the reader to the existing literature e.g. [AC1, AC2, Sel, CPv]. Here we limit ourselves to recalling the following:

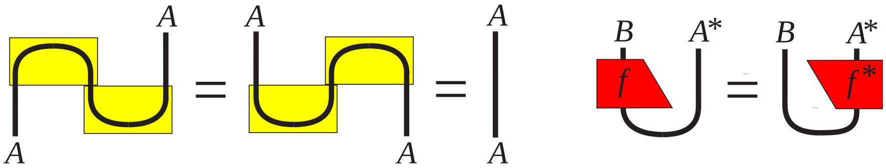
- For each object A there exists an object A^* and a morphism $\eta_A = \mathrm{I} \to A \otimes A^*$ such that $A^{**} = A$ and

$$\lambda_A^\dagger \circ (\eta_A^\dagger \otimes 1_A) \circ (1_A \otimes \eta_{A^*}) \circ \rho_A = \rho_A^\dagger \circ (1_A \otimes \eta_{A^*}^\dagger) \circ (\eta_A \otimes 1_A) \circ \lambda_A = 1_A \,. \tag{2}$$

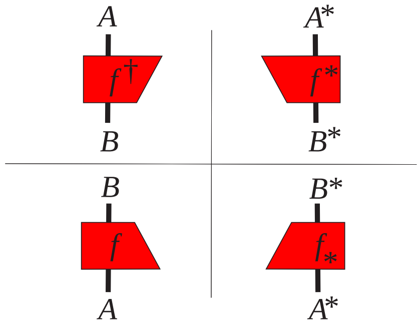
- For each morphism $f : A \to B$ there exist morphisms $f_* : A^* \to B^*$ and $f^* : B^* \to A^*$ which are such that $(f_*)^* = (f^*)_* = f^\dagger$, $f_{**} = f^{**} = f$ and

$$(f \otimes 1_{A^*}) \circ \eta_A = (1_B \otimes f^*) \circ \eta_B \,. \tag{3}$$

Graphically equations eq.(2) and eq.(3) become:



where the yellow boxes depict η and $\eta^\dagger$. The table

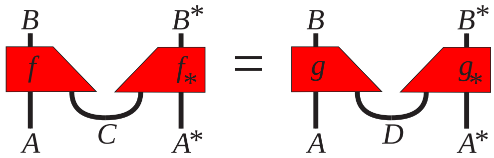


shows how in [Sel] all the from $f : A \to B$ derived morphisms are depicted.

For $f : A \otimes C \to B$ let

$$\Gamma(f) := (f \otimes f_*) \circ (1_A \otimes \eta_C \otimes 1_{A^*}) \circ (\rho_A \otimes 1_{A^*}) : A \otimes A^* \to B \otimes B^* \,.$$

For $f : A \otimes C \to B$ and $g : A \otimes D \to B$ we set $f \mathrel{\dot{\simeq}} g$ iff $\Gamma(f) = \Gamma(g)$ i.e.



Clearly $\dot{\simeq}$ defines an equivalence relation on $\bigcup_{C \in |\mathbf{C}|} \mathrm{Hom}(A \otimes C, B)$.

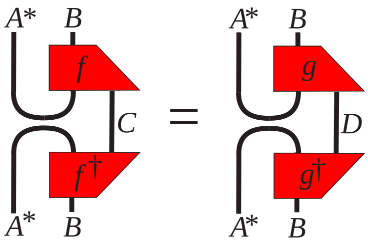
Proposition 2.1. *If in Definition 1.2 we replace* $\simeq$ *by* $\dot{\simeq}$ *we obtain Selinger's* $\mathbf{CPM}(\mathbf{C})$.

Proof: Follows straightforwardly from the definition given in [Sel]. □

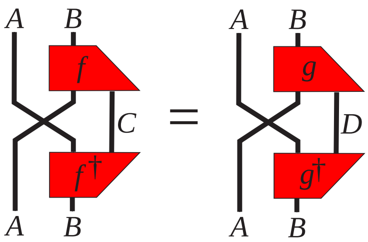
Lemma 2.2. *If* $\mathbf{C}$ *is* $\dagger$*-compact then for* $f : A \otimes C \to B$ *and* $g : A \otimes D \to B$ *TFAE:*

(1) $f \simeq g$

(2)

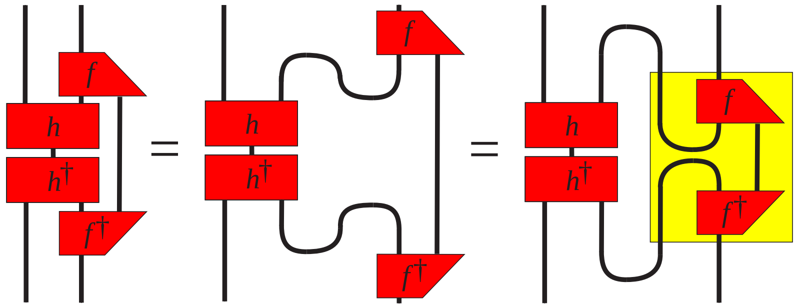


(3)

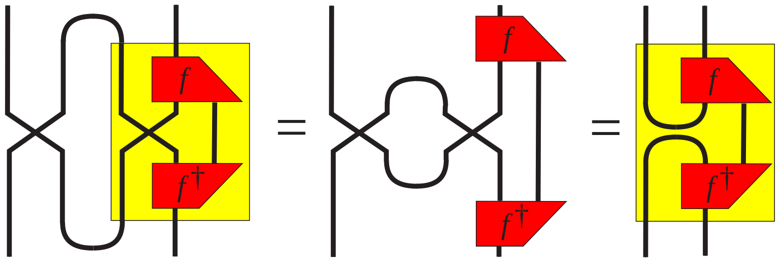


(4) $f \mathrel{\dot{\simeq}} g$

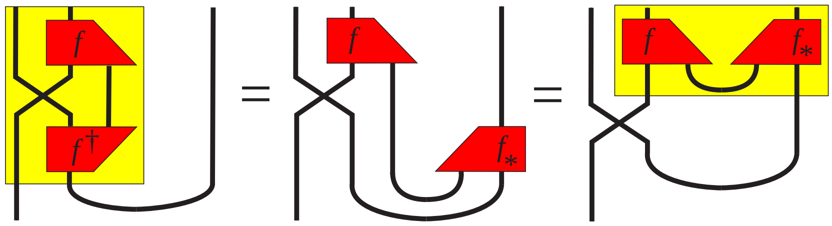
Proof: Setting $h := \eta_{A^*}$ in Definition 1.2 realises **(1)**⇒**(2)**. By eq.(2) we have



so **(2)** implies $\Xi(f,h) = \Xi(g,h)$ and hence $f \simeq g$ i.e. **(2)**⇒**(1)**. By eq.(1) we also have



from which it easily follows that **(3)**⇔**(2)**. The converse **(2)**⇔**(3)** to this is established similarly. Finally, applying eq.(3) to $f^\dagger$ we obtain:



from which we can conclude that also **(3)**⇔**(4)**. □

Theorem 2.3. *If* $\mathbf{C}$ *is* †*-compact then* $\mathbf{Mix}(\mathbf{C})$ *and* $\mathbf{CPM}(\mathbf{C})$ *are isomorphic.*

Proof: Follows by Proposition 2.1 and Lemma 2.2. □

Corollary 2.4. *The morphisms of* $\mathbf{Mix}(\mathbf{FdHilb})$ *are the completely positive maps.*

3 The role of positivity

Concretely, i.e. in the standard quantum mechanics literature, a map is completely positive if it is a morphism in $\mathbf{CPM}(\mathbf{FdHilb})$. More abstractly, as shown in [Sel], for $\mathbf{C}$ any †-compact category, the morphisms of $\mathbf{CPM}(\mathbf{C})$ are completely positive maps. Theorem 2.3 and Corollary 2.4 show that we can generalise this even further to the morphisms of $\mathbf{Mix}(\mathbf{C})$ for any †-symmetric monoidal category $\mathbf{C}$. Conceptually, the 'hidden object' $C \in |\mathbf{C}|$ in Definition 1.2 represents the variables relative to which we mix (e.g. lack of knowledge are an ancillary external system).

Now recall that an endomorphism $f : A \to A$ is *positive* whenever it can be written as $f = g^\dagger \circ g$ for some $g : A \to B$. The reader can verify that $\mathbf{Mix}(\mathbf{C})$ does not canonically inherit the †-structure from $\mathbf{C}$. Since $\mathbf{Mix}(\mathbf{C})$ in general does not come with a †-structure the canonical abstract generalisation of the notion of positivity does not apply to this category. This means that, in the abstract, the notion of 'complete positivity' can exist independently of a notion of 'positivity', which points at a very unfortunately terminology.

4 Remarks

We speculate on the following:

- Rather than requiring symmetry most probably it suffices to have a braid-structure, as it is also the case for the traced monoidal categories in [JSV].
- If the †-symmetric monoidal category $\mathbf{C}$ also comes with a trace-structure in the sense of [JSV] then it seems that $\mathbf{Mix}(\mathbf{C})$ does come with a †-structure.

5 Acknowledgements

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