Construction of Minimum Euclidean Distance MIMO Precoders and Their Lattice Classifications

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***Abstract*—This correspondence deals with the construction of minimum Euclidean distance precoders for multiple-input multiple-output (MIMO) systems with up to four transmit antennas. By making use of a state-of-the-art technique for optimization over the unitary group, we can numerically optimize the MIMO precoders. The correspondence then proceeds by identifying the obtained precoders as well-known lattices (square Z_2, Schläfli D_4, D_6, Gosset E_8). With three transmit antennas, the results are slightly different compared with other numbers of transmit antennas since the obtained precoder is not an instance of the densest 6-dimensional lattice. The overall conclusions of the correspondence are that the found precoders for MIMO transmission are highly structured and that, even with small constellations, lattice theory can be used for the design of MIMO precoders.**

***Index Terms*—Lattice theory, MIMO, minimum Euclidean distance, precoding.**

I. Introduction

The problem of constructing efficient linear precoders for multiple-input multiple-output (MIMO) channels has been extensively studied for many different scenarios. Under the assumption of MMSE detection and with perfect channel state information at both the transmitter and receiver, optimal precoder design is treated in depth in [1]. This correspondence differs from [1] by assuming maximum-likelihood (ML) detection at the receiver side. Under ML detection, the ultimate limit of a MIMO system is determined by the mutual information between the input and the output. In [2], it was shown that the capacity of a MIMO system is achieved by a Gaussian constellation, together with water-filling over the eigenmodes of the channel. However, for practical input constellations, such as QAM, optimal mutual information precoder design is difficult since no closed form solutions for the differential entropy of the channel output exist. A suboptimal technique is to convert the MIMO vector channel into a set of parallell channels, where the optimal power allocation in this case is found through the mercury/waterfilling (MWF) principle [3]. For practical input constellations though, simple power allocation over the eigenmodes is no longer optimal. This was demonstrated in [4], where a nondiagonal precoder was found through a numerical technique, that gave higher information rate than MWF. However, the technique in [4] involves significant complexity and is not guaranteed to converge to the global optimal solution.

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At high signal-to-noise-ratios (SNRs), the optimal mutual information precoder converges to the optimal minimum distance precoder [5]. Under ML detection and uncoded transmission, the predominant parameter to be optimized at high SNR is the minimum square Euclidean distance[1] of the received signal constellation. Unfortunately, the minimum distance precoder optimization problem is known to be NP-hard [5]. The first paper dealing with optimization of the minimum distance within a MIMO context is [6]; the approach is to maximize the smallest eigenvalue of the effective channel (channel times precoder). In [7], a suboptimal precoder design based on linear programming is proposed, that gives large minimum distances for any MIMO constellation. With two transmit antennas and BPSK/QPSK constellations, optimal minimum distance precoders are given in [8]; the outcome is that only two different precoder structures are optimal for all channels. That work has later been extended to 16-QAM [9], where eight precoder structures were found. For three or more transmit antennas, only suboptimal results exist so far [10], [11].

The main contributions of this correspondence can be summarized as follows.

- We construct codebooks containing matrix structures from which the precoders are selected. We consider MIMO systems with up to 4 transmit antennas and QPSK inputs. With three transmit antennas, our results improve upon [11], which confirms that [11] is not optimal. For four transmit antennas, a codebook of 77 precoder structures is found. Thirty-one of these are optimal for 99.9% of 4×4 complex Gaussian distributed channels. The fact that all codebooks are finite makes the results practical.
- We investigate the efficiency of an iterative optimization procedure to optimize the precoder for MIMO transmissions. The iterative optimization uses the fact that the power allocation of the precoder can be found via linear programming. The rotation matrix of the precoder is obtained by making use of an advanced technique for optimization over the unitary group [12], [13]. Our finding is that the iterative optimization procedure is efficient as it converges to what we believe is a global optimum.
- We discover that all full rank precoders that we have found are instances of well known dense packing lattices. For example, with four transmit antennas the obtained precoder is the E_8 Gosset lattice, expressed through a specific basis of the lattice. When the channel matrix changes, the precoder is still the Gosset lattice, but expressed in another basis. This is important since it implies that the precoder optimization can be simplified since one may specify the lattice at start and then only optimize the basis. For channels with one or more very weak eigenmodes, the precoder is not full rank. The precoder is an instance of a well-known lattice in some cases, but in other cases, it is not a lattice. In the latter case, its structure still contains well-known sublattices.

A. System Model and Problem Formulation

We consider linearly precoded MIMO channels in additive white Gaussian noise. Both the transmitter and the receiver are assumed to have perfect knowledge of the communication channel. The received signal vector is

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{F}\boldsymbol{x} + \boldsymbol{w} \tag{1}$$

where $\boldsymbol{H}$ is an $M \times N$ complex-valued matrix that represents the channel. The matrix $\boldsymbol{F}$ is an $N \times N$ precoding matrix which satisfies the power constraint

$$\mathrm{Tr}(\boldsymbol{F}\boldsymbol{F}^{\dagger}) = 1 \tag{2}$$

[1]This is subsequently referred to as minimum distance only.

where "Tr" denotes the trace operation and "†" means Hermitian transpose. In this correspondence, we only consider $2 \leq N \leq 4$. The case $N = 1$ is trivial since it implies that $\boldsymbol{F}$ is a scalar that equals unity. The data vector $\boldsymbol{x}$ comprises N independent unit energy QPSK symbols unless stated otherwise. The noise vector $\boldsymbol{w}$ comprises independent zero-mean circulary symmetric complex Gaussian noise samples, each one with the variance N_0.

Let $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^\dagger \boldsymbol{x}}$ denote the 2-norm of a column vector. Under ML detection and at high SNR, the error probability is approximately given by $\mathrm{P_e} \approx \mathcal{Q}\left(\sqrt{\frac{\Delta^2_{\min}}{2N_0}}\right)$, where $\mathcal{Q}$ is the complementary error function and

$$\Delta^2_{\min} \triangleq \min_{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_1 \neq \boldsymbol{x}_2} \|\boldsymbol{H}\boldsymbol{F}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\|^2. \tag{3}$$

The quantity $\Delta^2_{\min}$ is referred to as the minimum distance. Expanding (3) yields

$$\Delta^2_{\min} = \min_{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_1 \neq \boldsymbol{x}_2} (\boldsymbol{x}_1 - \boldsymbol{x}_2)^\dagger \boldsymbol{F}^\dagger \boldsymbol{H}^\dagger \boldsymbol{H}\boldsymbol{F}(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \min_{\boldsymbol{e} \neq \boldsymbol{0}} \boldsymbol{e}^\dagger \boldsymbol{G} \boldsymbol{e}^\dagger \tag{4}$$

where $\boldsymbol{G} \triangleq \boldsymbol{F}^\dagger \boldsymbol{H}^\dagger \boldsymbol{H}\boldsymbol{F}$ is the $N \times N$ *Gram matrix* and $\boldsymbol{e} \triangleq \boldsymbol{x}_1 - \boldsymbol{x}_2$. Observe that if $M < N$, then $\boldsymbol{G}$ is of reduced rank. However, the case $M < N$ does not need separate attention since $\boldsymbol{G}$ can also be of reduced rank if the channel $\boldsymbol{H}$ has vanishing eigenvalues.

The objective of the first part of the correspondence is to solve the optimization $\max_{\boldsymbol{F}} \Delta^2_{\min}$ under the constraint (2) by means of numerical methods. Then the correspondence proceeds by lattice identification of the obtained precoders. Let $\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\dagger$ denote the singular value decomposition of the channel matrix $\boldsymbol{H}$. Further, let $\boldsymbol{G} = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^\dagger$ be the eigenvalue decomposition of $\boldsymbol{G}$. It can be shown [7] that the $\boldsymbol{F}$ with least energy that satisfies $\boldsymbol{G} = \boldsymbol{F}^\dagger \boldsymbol{H}^\dagger \boldsymbol{H}\boldsymbol{F}$ is

$$\boldsymbol{F} = \boldsymbol{V}\begin{bmatrix}\boldsymbol{\Sigma}^+ & \boldsymbol{0}_{N,\max(0,N-M)}\end{bmatrix}\sqrt{\boldsymbol{D}}\boldsymbol{Q}^\dagger \tag{5}$$

where $\boldsymbol{0}_{a,b}$ denotes the $a \times b$ all-zero matrix and $\boldsymbol{\Sigma}^+$ is the pseudo-inverse of $\boldsymbol{\Sigma}$, i.e., the strictly positive elements in $\boldsymbol{\Sigma}$ are inverted. For convenience, we define $\hat{\boldsymbol{\Sigma}} \triangleq [\boldsymbol{\Sigma}^+ \boldsymbol{0}_{N,\max(0,N-M)}]$, which is essentially the pseudo-inverse of $\boldsymbol{\Sigma}$ stacked with a $N - M$ all-zero columns if $M < N$. Let $\hat{\Sigma}_m$ and D_m denote the diagonal elements of $\hat{\boldsymbol{\Sigma}}$ and $\boldsymbol{D}$, respectively.

Altogether, the optimization problem can be expressed over $\boldsymbol{Q}$ and $\boldsymbol{D}$ as

$$\max_{\boldsymbol{Q},\boldsymbol{D}} \min_{\boldsymbol{e}} \boldsymbol{e}^\dagger \underbrace{\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^\dagger}_{\boldsymbol{G}} \boldsymbol{e}$$
$$\text{subject to}$$
$$\sum_{m=1}^{N} \hat{\Sigma}^2_m D_m = 1 \text{ and } D_m \geq 0, \text{ for } 1 \leq m \leq M. \tag{6}$$

The solution to the precoder optimization problem (6) naturally depends on the particular realization of $\boldsymbol{\Sigma}$. Observe that scaling $\boldsymbol{\Sigma}$ with its largest value only scales the solution to (6), without affecting the structure of the solution. In this correspondence, we will solve the precoder optimization for all $N \times N$ diagonal matrices $\boldsymbol{\Sigma}$ with diagonal elements

$$\Sigma_m = \begin{cases} 1, & m = 1 \\ n\Delta, & 2 \leq m \leq N,\ n \in \{0, 1, 2, \ldots, \frac{1}{\Delta}\} \end{cases}.$$

For $N = 3$, we use $\Delta = \frac{1}{100}$, thus, we solve the optimization for $101^2 = 10201$ different $\boldsymbol{\Sigma}$. For $N = 4$, the step size Δ is 1/20, which yields $21^3 = 9261$ different $\boldsymbol{\Sigma}$. Note that the case $M < N$ is included into this optimization since it corresponds to $\Sigma_m = 0$ for $m = M + 1, \ldots, N$.

Solving the optimization (6) for each $\boldsymbol{\Sigma}$ is a challenging task as it is not an instance of convex optimization. In the following, we propose an advanced iterative optimization approach for solving this problem.

II. Iterative Precoder Optimization

Our optimization procedure alternates between optimization of $\boldsymbol{D}$ and $\boldsymbol{Q}$. The steps are the following: 1) Given a unitary matrix $\boldsymbol{Q}$, optimize (6) over $\boldsymbol{D}$; 2) with the obtained $\boldsymbol{D}$, optimize (6) over $\boldsymbol{Q}$; and 3) iterate the first two steps, until the increase in the value of the objective function $\Delta^2_{\min}$ becomes negligible.

Next, each of these optimization steps will be explained in detail.

A. Optimization over $\boldsymbol{D}$

For a fixed $\boldsymbol{Q}$, the objective function of (6) can be written as

$$\boldsymbol{D}_{\text{opt}} = \arg\max_{\boldsymbol{D}} \min_{\boldsymbol{e}} \boldsymbol{e}^\dagger \boldsymbol{G} \boldsymbol{e} = \arg\max_{\boldsymbol{D}} \min_{\tilde{\boldsymbol{e}}} \tilde{\boldsymbol{e}}^\dagger \boldsymbol{D} \tilde{\boldsymbol{e}} \tag{7}$$

where $\tilde{\boldsymbol{e}} \triangleq \boldsymbol{Q}^\dagger \boldsymbol{e}$, and thus the optimization (6) reduces to

$$\max_{\boldsymbol{D}} \min_{\tilde{\boldsymbol{e}}} \tilde{\boldsymbol{e}}^\dagger \boldsymbol{D} \tilde{\boldsymbol{e}}$$
$$\text{subject to}$$
$$\sum_{m=1}^{N} \hat{\Sigma}^2_m D_m = 1 \text{ and } D_m > 0, \text{ for } 1 \leq m \leq M. \tag{8}$$

An equivalent reformulation of this is to minimize the constraint $\mathrm{Tr}(\boldsymbol{F}\boldsymbol{F}^\dagger)$ subject to the objective function $\Delta^2_{\min}$ being larger than an arbitrary constant. In other words,

$$\min_{\boldsymbol{D}, D_m > 0 \text{ for } 1 \leq m \leq M} \sum_{m=1}^{N} \hat{\Sigma}^2_m D_m$$
$$\text{subject to}$$
$$\tilde{\boldsymbol{e}}^\dagger \boldsymbol{D} \tilde{\boldsymbol{e}} \geq 1, \quad \forall \tilde{\boldsymbol{e}}. \tag{9}$$

Hence, the optimization is a linear problem over $\boldsymbol{D}$ and can be trivially solved by means of standard techniques, such as the simplex algorithm [14].

B. Optimizing for the Unitary Matrix $\mathbf{Q}$

For a fixed $\boldsymbol{D}$, the constraint in (6) is fulfilled and thus only the objective function in (6) has to be optimized over $\boldsymbol{Q}$. The problem of optimizing objective functions under a unitary matrix constraint has been extensively treated in [12] and [13]. In our case, the unitary optimization problem is

$$\mathbf{Q}_{\text{opt}} = \arg\max_{\mathbf{Q}} \min_{\mathbf{e}} \mathbf{e}^\dagger \mathbf{Q}\mathbf{D}\mathbf{Q}^\dagger \mathbf{e}, \text{ subject to } \mathbf{Q}^\dagger\mathbf{Q} = \mathbf{I}_N. \tag{10}$$

For a given N and symbol constellation, there is a finite number of error vectors $\mathbf{e} \in \{\mathbf{e}_1, \ldots, \mathbf{e}_L\}$.

The optimization problem (10) is equivalent to maximizing the minimum of a set of L continuously-differentiable objective functions $\mathcal{J}_\ell(\mathbf{Q})$ over the Lie group of unitary matrices $U(N)$:

$$\mathbf{Q}_{\text{opt}} = \arg\max_{\mathbf{Q}} \min_{\ell} \{\mathcal{J}_\ell(\mathbf{Q})\}_{\ell=1}^{L}, \quad \mathbf{Q} \in U(N) \tag{11}$$

where each of the objective functions is defined as $\mathcal{J}_\ell(\mathbf{Q}) \triangleq \mathbf{e}_\ell^\dagger \mathbf{Q}\mathbf{D}\mathbf{Q}^\dagger \mathbf{e}_\ell$, $\ell = 1, \ldots, L$. Since the number of error vectors L is considerably large (order of thousands), the complexity of the optimization problem needs to be reduced. Without loss of generality, we assume that the set of vectors $\{\mathbf{e}_\ell\}$ are sorted in an ascending order, according to the values of the objective function $\mathcal{J}_m$ they produce, i.e., $\mathcal{J}_1 \leq \mathcal{J}_2 \leq \ldots \leq \mathcal{J}_L$. Maximizing the minimum

value of the inner objective function $\mathcal{J}_\ell$, is equivalent to maximizing $\mathcal{J}_1 = \min_\ell \{\mathcal{J}_\ell(\mathbf{Q})\}_{\ell=1}^L$ w.r.t $\mathbf{Q} \in U(N)$. The optimization w.r.t. the unitary matrix $\mathbf{Q}$ is done by using the Riemannian Steepest Ascent (SA) algorithm on the unitary group given in [12, Table I].[2] After each iteration of the SA algorithm, the vectors $\{\mathbf{e}_\ell\}$ are again sorted in ascending order, and the new obtained objective function $\mathcal{J}_1(\mathbf{Q})$ is maximized. The Euclidean gradient of $\mathcal{J}_1(\mathbf{Q})$ at a point $\mathbf{Q}_k \in U(N)$ is given by $\mathbf{\Gamma}_{1|k} = \mathbf{e}_1\mathbf{e}_1^\dagger\mathbf{Q}_k\mathbf{D}$, and represents the steepest ascent direction on the Euclidean space. The Riemannian gradient is a skew-Hermitian matrix $\mathbf{S}_{1|k} = \mathbf{\Gamma}_{1|k}\mathbf{Q}_k^\dagger - \mathbf{Q}_k\mathbf{\Gamma}_{1|k}^\dagger$, and represent the steepest ascent direction on the constrained parameter space $U(N)$ at $\mathbf{Q}_k$, translated to a group identity element. A rotational update is performed, such that the unitary matrix constraint is maintained at every iteration: $\mathbf{Q}_{k+1} = \exp(+\alpha\mu\mathbf{S}_{1|k})\mathbf{Q}_k$, where $\exp(\cdot)$ is the standard matrix exponential,[3] and μ is a step size. Note that the update is multiplicative since a product of unitary matrices is unitary.

The complexity of the unitary optimization itself is of order $\mathcal{O}(N^3)$. However, the complexity of sorting the error vectors $\{\mathbf{e}_\ell\}$ in ascending order is $\mathcal{O}(LN^2)$. Since $N \ll L$, the complexity of the entire optimization is therefore dominated by the sorting operation. It is possible to reduce the number of error vectors $\{\mathbf{e}_\ell\}$, but this has not been done since with a standard work-station, the entire optimization is a matter of fractions of a second.

An initial, accurate size μ is selected by using the polynomial-based line search method given in [13, Table 1]. The scaling factor $\alpha \in (0, 1)$ prevents the objective function $\mathcal{J}_1(\mathbf{Q})$ to increase too quickly. Too much increase in $\mathcal{J}_1(\mathbf{Q})$ may produce a decrease in the other objective functions $\{\mathcal{J}_\ell(\mathbf{Q})\}_{\ell=2}^L$, even below the initial value of $\mathcal{J}_1$. This is because their corresponding gradients $\mathbf{S}_{m|k}$ do not necessarily point in directions that increase the minimum value of the objective functions. In that case, instead of increasing the minimum value of $\{\mathcal{J}_\ell(\mathbf{Q})\}_{\ell=1}^L$, the value decreases. Therefore, small steps $\alpha\mu$ are preferred.[4]

III. Optimization Results

The above described optimization procedure produces a set of Gram matrices $\boldsymbol{G} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\dagger$, from which the precoder is obtained by (5). Due to this bijection between $\boldsymbol{G}$ and $\boldsymbol{F}$, we will sometimes refer to this codebook as a "precoder codebook," and call the elements in the codebook as precoders.

The convergence properties of the iterative optimization method described in Section II depend heavily on the step size $\alpha\mu$. Simulation results show that $\alpha \approx 10^{-2}$ produces the best codebook. A larger α gives more rapid convergence, but then the optimization often converges to a local optimum. We have found that the starting point does not have a significant effect for the $\alpha\mu$ that is used. For each $\mathbf{\Sigma}$ in the grid, we have chosen as starting point the precoder for the previously considered $\mathbf{\Sigma}$, which is close in Euclidean distance to the current $\mathbf{\Sigma}$, but also, 10 randomly chosen starting points. Then we take as output the best of the 11 solutions, but most often they are all the same. As an overall conclusion of the iterative optimization, we find the method to be efficient, but we anyway reckon that it should be carried out offline since several hundreds of iterations are needed, where the main complexity arises from sorting the error vectors for each iteration. If suboptimal solutions are tolerated, using a larger α results in much faster convergence, so that the optimization can be carried out online.

It may appear problematic to even carry out the optimization offline since the size of the precoder codebook may be prohibitive. However, it turns out that for $\mathbf{\Sigma}$s that are "close," the corresponding Gram matrices $\boldsymbol{G} = \boldsymbol{F}^\dagger\boldsymbol{H}^\dagger\boldsymbol{H}\boldsymbol{F}$ are scaled versions of each other. Thus, the size of the codebook becomes small, and the need of online optimization is circumvented. The same is not true for the mutual information precoders from [4]. In that case, no precoder codebook can be tabulated since each channel $\mathbf{\Sigma}$ has a unique Gram matrix. This is a strong motivation to consider minimum distance precoders.

We next describe the outcome of the optimization procedure for the considered setups. We have chosen not to list the obtained $\boldsymbol{G}$ matrices in this correspondence, but they can be found at www.eit.lth.se/goto/kapetanovicprecoders. From now on, with two different Gram matrices it is meant that they differ only up to scaling, since scaling is of no interest to (6).

A. $N = 2$ With QPSK Inputs

This optimization problem has been solved in [8]. In [8] it is analytically proved that the optimal precoders $\boldsymbol{F}$ produce two different Gram matrices $\boldsymbol{G}$. One of the two Gram matrices has rank 1, while the other has full rank. Which one to use depends on the particular realization of $\mathbf{\Sigma}$.

B. $N = 2$ With 16-QAM Inputs

For 16-QAM inputs, the work in [9] suggests that there should be eight different Gram matrices. One of these is however not optimal; by running the iterative optimization procedure, we obtain a Gram matrix that has a significantly larger minimum distance than one of the eight found in [9]. In terms of symbol error rate (SER), the precoder resulting from the newly found Gram matrix has only minor impact. For channels $\boldsymbol{H}$ with $M = N = 2$, where the channel coefficients are independent zero-mean, unit-variance, complex Gaussian random variables, the channels where the newly found precoder is optimal are scarce so that the effect of the new precoder does almost not show up in simulations. Thus, our improvement of [9] is mainly of theoretical interest, but shows that the iterative optimization is highly efficient, as it can find precoders that the technique in [9] cannot.

C. $N = 3$ With QPSK Inputs

By studying the resulting $\boldsymbol{G}$ matrices for each $\mathbf{\Sigma}$, we identified 14 different $\boldsymbol{G}$ matrices for $N = 3$, as opposed to the 8 proposed in [11]. The 14 different Gram matrices can be characterized into three different classes: I) There are five with full rank; II) eight with rank 2; and III) one with rank 1. The rank deficient precoders are used when some channel eigenvalues are small, so only transmission over the stronger eigenmode occurs.

Unlike Section III-B, the newly found precoder codebook performs slightly better than the codebook proposed in [11]. When it comes to the SER, there is not much gain compared to the precoders from [11]; however, plotting the probability density function (pdf) of $\Delta_{\min}^2$ for the 14 precoders and the precoders from [11], an improvement in $\Delta_{\min}^2$ is observed. The gain in $\Delta_{\min}^2$ is shown in Fig. 1.

D. $N = 4$ With QPSK Inputs

In the case of $N = 4$, our iterative optimization produces a codebook of 77 $\boldsymbol{G}$ matrices. With $M = N = 4$ and complex Gaussian distributed channels, for more than 99.9% of all the channel outcomes, only 31 different Gram matrices from the codebook are optimal for these; hence, the other 46 occur very seldom. Further, in terms of minimum distance, the 31 Gram matrices perform well as there is not much loss compared with using the complete codebook. Out of the 31 Gram matrices, eight have full rank, nineteen have rank 3, three have rank 2, and one has rank 1.

[2]The Riemannian optimization codes are available at http://signal.hut.fi/~tabrudan/unitary_optimization/index.html

[3]The matrix $\exp(+\alpha\mu\mathbf{S}_{1|k})$ is a unitary matrix.

[4]The reason why steepest ascent is used is that the conjugate gradient in [12] would be "too fast" for this purpose.

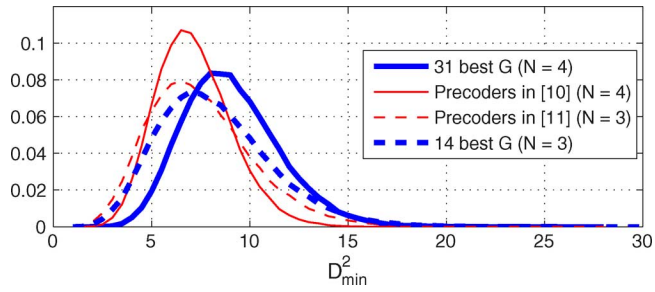

Fig. 1. Probability density functions of $\Delta^2_{\min}$ for QPSK. The comparison is between the 14 new precoders for $N = 3$ and [11], and the 31 new precoders for $N = 4$ and [10], respectively. As seen, there is a significant improvement in $\Delta^2_{\min}$ for $N = 4$, which carries over to SER as illustrated in Fig. 2.

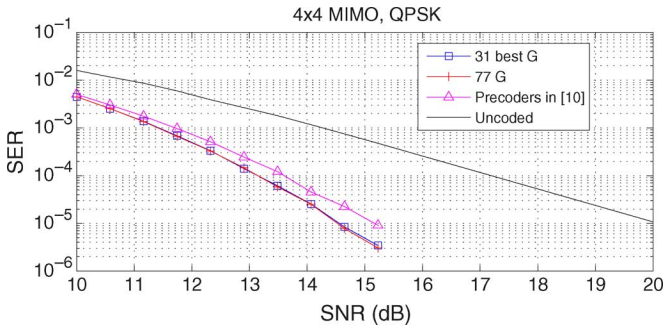

Fig. 2. SER comparison for $N = 4$ and QPSK. The comparison is between the 77 found precoders, the 31 most occuring precoders out of the 77, the precoders in [10] and uncoded transmission. As seen, there is no loss in using only the 31 most frequently occuring precoders, and the gain in SER is significant compared to the precoders in [10].

Fig. 1 shows the probability density function of $\Delta^2_{\min}$ for the 31 precoders as well as for the precoders in [10]. There is a significant gain in minimum distance for the 31 precoders, which predicts a gain in SER. An SER simulation over an independent and identically distributed complex Gaussian channel is provided in Fig. 2. We plot the performance of the 77 precoders, a codebook containing only the 31 most frequently occuring precoders, uncoded transmission, and the suboptimal precoding method from [10]. As can be seen, there is about 2 dB gain of the proposed codebook compared with the competing scheme from [10]. At SER 10^{-3}, there is a 4-dB gain over uncoded systems. There is no performance loss by using the 31 precoders instead of the 77.

IV. Connection With Lattices

An interesting fact about the $\boldsymbol{G}$ matrices obtained from our optimization is the structure of the elements. Namely, with proper scaling for each $\boldsymbol{G}$, the elements $G_{jk} = a_{jk} + b_{jk}i$ are either such that a_{jk} and b_{jk} are rational numbers, or such that $a_{jk} = r_1 + \frac{r_2}{\sqrt{2}}$ and $b_{jk} = r_3 + \frac{r_4}{\sqrt{3}}$, where r_1, r_2, r_3, and r_4 are rational numbers. A similar ordered structure has also been observed for the $\boldsymbol{G}$ matrices in [8] and [9]. This suggests a hidden underlying structure of the precoding problem. It will be demonstrated that there is indeed a profound relationship between the obtained $\boldsymbol{G}$ matrices and standard lattices. For this reason, we now introduce a brief account on lattice theory.

A. Lattices

All matrices and vectors in this subsection are assumed to be real-valued. This covers complex-valued matrices and vectors too, since any complex-valued matrix $\boldsymbol{A}$ and vector $\boldsymbol{v}$ are isomorphic to a real-valued matrix $\boldsymbol{A}_r$ and vector $\boldsymbol{v}_r$ through the transformations

$$\boldsymbol{A}_r = \begin{bmatrix} \mathcal{R}(\boldsymbol{A}) & \mathcal{I}(\boldsymbol{A}) \\ -\mathcal{I}(\boldsymbol{A}) & \mathcal{R}(\boldsymbol{A}) \end{bmatrix}, \quad \boldsymbol{v}_r = \begin{pmatrix} \mathcal{R}(\boldsymbol{v}) \\ \mathcal{I}(\boldsymbol{v}) \end{pmatrix}$$

where $\mathcal{R}, \mathcal{I}$ denote the real and imaginary parts of a matrix/vector, respectively. Note that the dimension of the real-valued space is $2N$.

Let $\boldsymbol{L} = [\boldsymbol{l}_1 \ldots \boldsymbol{l}_N]$ be an $N \times N$ matrix with columns $\boldsymbol{l}_1 \ldots \boldsymbol{l}_N$ and $\boldsymbol{u} = [u_1, \ldots, u_N]^{\mathrm{T}}$ an $N \times 1$ vector where the superscript "T" denotes the transpose operator. A lattice $\Lambda_{\boldsymbol{L}}$ is the set of points

$$\Lambda_L = \{\boldsymbol{L}\boldsymbol{u} \mid u_i \in \mathbb{Z}, 1 \leq i \leq N\} \tag{12}$$

where $\mathbb{Z}$ is the set of integers (i.e., $\boldsymbol{u}$ is an integer vector). The matrix $\boldsymbol{L}$ is called a *generator matrix* for the lattice $\Lambda_{\boldsymbol{L}}$ and the columns of $\boldsymbol{L}$ form a *basis* for the lattice $\Lambda_{\boldsymbol{L}}$. The lattice *minimum distance* of $\Lambda_{\boldsymbol{L}}$ is defined as[5]

$$d^2_{\min}(\Lambda_{\boldsymbol{L}}) = \min_{\boldsymbol{u}_1 \neq \boldsymbol{u}_2} \|\boldsymbol{L}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2 = \min_{\boldsymbol{b} \neq \boldsymbol{0}} \|\boldsymbol{L}\boldsymbol{b}\|^2 = \min_{\boldsymbol{b} \neq \boldsymbol{0}} \boldsymbol{b}^{\mathrm{T}}\boldsymbol{G}_{\boldsymbol{L}}\boldsymbol{b} \tag{13}$$

where $\boldsymbol{u}_1, \boldsymbol{u}_2$, and $\boldsymbol{b} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ are integer vectors, $\boldsymbol{0}$ the zero vector, and $\boldsymbol{G}_{\boldsymbol{L}} = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{L}$ is the Gram matrix of $\boldsymbol{L}$. Given a Gram matrix $\boldsymbol{G}_{\boldsymbol{L}}$, all lattices with Gram matrix $\boldsymbol{G}_{\boldsymbol{L}}$ differ only by a rotation. Hence, up to rotation, there is a unique lattice structure whose Gram matrix is $\boldsymbol{G}_{\boldsymbol{L}}$. Performing a decomposition of the form $\boldsymbol{G}_{\boldsymbol{L}} = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{L}$ on $\boldsymbol{G}_{\boldsymbol{L}}$, one obtains a generator matrix $\boldsymbol{L}$ representing this lattice structure.

A number of lattices are especially interesting and have been given formal symbols in the literature [15]. In particular, the densest lattices in the sense that they maximize the quotient $\frac{d^2_{\min}(\Lambda_{\boldsymbol{L}})}{\det(\boldsymbol{L})}$ are of interest. In 2, 4, 6, and 8 dimensions, the densest lattices are the Hexagonal A_2, Schläfli D_4, E_6, and the Gosset E_8 lattices, respectively. Apart from these, we also encounter the 2-dimensional square lattice Z_2 and the 6-dimensional D_6 lattice.

B. Lattice Identification of Precoders

Transforming the complex-valued Gram matrices obtained from our optimization to real-valued space, we are interested in identifying whether they correspond to well known lattices. Thus, the following discussion is for real-valued matrices. If $\boldsymbol{G}$ is of full rank N, then it corresponds to a certain lattice structure in N dimensions, and with a Cholesky decomposition we obtain a generator matrix for the lattice structure that gives rise to $\boldsymbol{G}$. However, given $\boldsymbol{L}$, it is nontrivial to verify whether it generates a well-known lattice structure up to a rotation. Nevertheless, there are algorithms that can identify whether a given $\boldsymbol{G}$ corresponds to a given lattice structure [16].

If $\boldsymbol{G}$ is of rank K, where $K < N$, it does not necessarily correspond to a lattice structure. The N columns in the $K \times N$ matrix $\boldsymbol{L}$, arising from any decomposition of the form $\boldsymbol{G} = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{L}$, are then linearly dependent, but there exists a set of K linearly independent columns in $\boldsymbol{L}$, $\mathcal{S}_{j_1:j_K} = \{\boldsymbol{l}_{j_1}, \ldots, \boldsymbol{l}_{j_K} : 1 \leq j_1 < \ldots < j_K <= N\}$. If the remaining columns in $\boldsymbol{L}$ are not a rational linear combination of $\mathcal{S}_{j_1:j_K}$ (i.e., a multiplier of some column in $\mathcal{S}_{j_1:j_K}$ is an irrational number), then $\boldsymbol{L}$ will not generate a periodic set of points and is therefore not a lattice structure. On the other hand, if every column in $\boldsymbol{L}$ is a rational combination of $\mathcal{S}_{j_1:j_K}$, then $\boldsymbol{L}$ generates a K-dimensional lattice structure. In the case when $\boldsymbol{L}$ does not generate a lattice structure in K dimensions, the different subsets of columns in $\boldsymbol{L}$ might still generate well-known lattice structures in different dimensions. For example, if $\boldsymbol{L}$ is a 4×6 matrix, it might be the case that two columns in $\boldsymbol{L}$ are orthogonal, which means that they generate the Z_2 lattice in four dimensions, while the other four columns in $\boldsymbol{L}$ might generate the D_4 lattice in four dimensions. If so, then each four-dimensional point generated by $\boldsymbol{L}$ is obtained as a sum between a point from the Z_2 lattice and a point from the D_4 lattice. Hence, even though the set of points generated by $\boldsymbol{L}$ are not a lattice structure, they are generated as a sum between two well-known lattice structures. In general, if it is possible

[5]Note that $\Delta^2_{\min}$ is the minimum distance of the precoded MIMO system while $d^2_{\min}$ refers to the minimum distance of a lattice.

TABLE I
DIFFERENT SUBTABLES PRESENTING THE OUTCOME OF OUR OPTIMIZATION PROCEDURE. EACH SUBTABLE CORRESPONDS TO A CERTAIN SETUP OF OPTIMIZATION PARAMETERS

$N = 2$, QPSK inputs

Rank	Precoders	Classification
4	1	D_4
2	1	$Z_2 \times Z_2$
Total:	2	

$N = 2$, 16-QAM inputs

Rank	Precoders	Classification
4	7	D_4
2	1	$Z_2 \times Z_2$
Total:	8	

$N = 3$, QPSK inputs

Rank	Precoders	Classification
6	5	D_6
4	8	D_4
2	1	$Z_2 \times Z_2 \times Z_2$
Total:	14	

$N = 4$, QPSK inputs

Rank	Precoders	Classification
8	8	E_8
6	2	D_6
6	15	$D_6 \times Z_3$
6	2	$D_4 \times Z_2 \times Z_2$
4	3	D_4
2	1	$Z_2 \times Z_2 \times Z_2 \times Z_2$
Total:	31	

to partition the columns of $\boldsymbol{L}$ in a way so that each subset of columns generates a certain lattice structure $\Lambda_{\mathcal{S}_j}$, $j = 1 \ldots m$, we denote the set of points generated by $\boldsymbol{L}$ as $\mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_m$. Thus, in the 4×6 example, we denote the set of points as $Z_2 \times D_4$.

We summarize our findings in Table I, which is composed of subtables describing the different optimization outcomes. Each table presents, for a specific N and signaling constellation, how many Gram matrices/precoders of a certain rank that were found (now represented in real-valued space), along with the classification of the set of points they generate at the receiver.

V. CONCLUSION

In this correspondence, we have proposed MIMO precoders for a minimum Euclidean distance cost function. MIMO channels with up to 4 transmit antennas have been considered. We have used an iterative optimization technique, which exploits the fact that the power allocation of the precoder can be optimized by means of linear programming. A state-of-the-art technique to optimize the rotation matrix over the unitary group was used as the second component of the iterative optimization method. Altogether, we find the iterative optimization to be an excellent tool for offline precoder optimization. A small, finite codebook of Gram matrices is found, and therefore there is no need to use the optimization technique online. Instead, for each channel realization, one finds the best Gram matrix in the codebook, from which the precoder is constructed. Since the obtained precoders improve upon previous close-to-optimal constructions in the literature, it is believed that for some channels they might even be optimal.

We discovered that the obtained precoders are all instances of well known lattices. For full rank precoders, the optimized precoders can be identified as the Schläfli, D_6, and the Gosset lattice for two, three, and four transmit antennas, respectively. It should especially be noted that with two and four transmit antennas, these lattices are the densest lattices. The optimal precoder for three transmit antennas D_6 is, however, not the densest lattice in six dimensions.

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