Programming Research Group

QUANTUM MEASUREMENTS WITHOUT SUMS

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Abstract

Sums play a prominent role in the formalisms of quantum mechanics, be it for mixing and superposing states, or for composing state spaces. Surprisingly, a conceptual analysis of quantum measurement seems to suggest that quantum mechanics can be done without direct sums, *expressed entirely in terms of the tensor product*. The corresponding axioms define classical spaces as objects that allow copying and deleting data. Indeed, the information exchange between the quantum and the classical worlds is essentially determined by their *distinct capabilities to copy and delete data*. The sums turn out to be an implicit implementation of this capabilities. Realizing it through explicit axioms not only dispenses with the unnecessary structural baggage, but also allows a simple and intuitive *graphical calculus*. In category-theoretic terms, classical data types are *†-compact Frobenius algebras*, and quantum spectra underlying quantum measurements are *Eilenberg-Moore coalgebras* induced by these Frobenius algebras.

1 Introduction

Ever since John von Neumann denounced, back in 1935 [29], his own foundation of quantum mechanics in terms of Hilbert spaces, there has been an ongoing search for a high-level, fully abstract formalism of quantum mechanics. With the emergence of quantum information technology, this quest became more important than ever. The low-level matrix manipulations in quantum informatics are akin to machine programming with bit strings from the early days of computing, which are of course inadequate.[1]

A recent research thread, initiated by Abramsky and the first author [2], aims at recasting the quantum mechanical formalism in *categorical* terms. The upshot of categorical semantics is that it displays concepts in a *compositional* and *typed* framework. In the case of quantum mechanics, it uncovers the *quantum information-flows* [6] which are hidden in the usual formalism. Moreover, while the investigations of quantum structures have so far been predominantly academic, categorical semantics open an alley towards a practical, low-overhead tool for the design and analysis of quantum informatic protocols, versatile enough to capture both quantitative and qualitative aspects of quantum information [2, 7, 10, 13, 30]. In fact, some otherwise complicated quantum informatic protocols

[1] But while computing devices do manipulate strings of 0s and 1s, and high-level modern programming is a matter of providing a convenient interface with that process, the language for quantum information and computation we seek is not a convenient superstructure, but the meaningful infrastructure.

become trivial exercises in this framework [8]. On the other hand, compared with the order-theoretic framework for quantum mechanics in terms of Birkhoff-von Neumann's quantum logic [28], this categorical setting comes with logical derivations, topologically embodied into something as something as simple as "yanking a rope".[2] Moreover, in terms of deductive machanism, it turns out to be some kind of "super-logic" as compared to the Birkhoff-von Neumann "non-logic".

The core of categorical semantics are *†-compact categories*, originally proposed in [2, 3] under the name *strongly compact closed categories*, extending the structure of *compact closed categories*, which have been familiar in various communities since the 1970es [22]. A salient feature of *categorical tensor calculi* of this kind is that they admit sound and complete graphical representations, in the sense that *a well-typed equation in such a tensor calculus is provable from its axioms if and only the graphical interpretation of that equation is valid in the graphical language.*[3] Soundness and completeness of the graphical language of †-compact categories, which can be viewed as a two-dimensional formalization and extension of Dirac's bra-ket notation [8], has been demonstrated by Selinger in [30]. Besides this reference, the interested reader may wish to consult [1, 18, 30] for methods and proofs, and [8, 9] for a more leisurely introduction into †-categories.

An important aspect of the †-compact semantics of quantum protocols proposed in [2, 7, 30] was the interplay of the multiplicative and additive structures of tensor products and direct sums, respectively. The direct sums (in fact biproducts, since all compact categories are self-dual) seemed essential for specifying classical data types, families of mutually orthogonal projectors, and ultimately for defining measurements. The drawback of this was that the additive types do not yield to a simple graphical calculus; in fact, they make it unusable for many practical purposes.

The main contribution of the present paper is a description of quantum measurement entirely in terms of tensor products, with no recourse to additive structure. The conceptual substance of this description is expressed in the framework of †-compact categories through a simple, operationally motivated definition of *classical objects*. A classical object, as a *†-compact Frobenius algebra*, equipped with copying and deleting operations, also provides an abstract counterpart to *GHZ-states*. We moreover expose an intriguing conceptual and structural connection between the classical capabilities to copy and delete data, as compared to quantum [26, 32], and the mechanism of quantum measurement. More precisely, we show how the classical interactions emerge as those morphisms which commute with copying and deleting. While each classical object canonically induces a non-degenerate quantum measurement, we show that general quantum measurements are *coalgebras* for comonads induced by classical objects. Quite remarkably, this coalgebra structure exactly captures von Neumann's projection postulate *in a resource sensitive fashion*. Furthermore, the irreversible probabilistic content of quantum measurements is then captured using the abstract construction of completely positive maps, due to Selinger [30]. With these conceptual components captured in a succinct categorical signature, we

[2] A closely related knot-theoretical scheme has been put forward by Kauffman in [20].

[3] Various graphical calculi have been an important vehicle of computation in physics [27, and subsequent work], and a prominent research topic of category theory e.g. [21, 22, 18].

provide a purely graphical derivation of teleportation and dense coding.

So the present paper provides a purely multiplicative treatment of *projective quantum measurements*. Selinger provides in [30] a multiplicative treatment of *density matrices and completely positive maps*. Paquette and the first author provide in [10] a multiplicative treatment of *POVM*s, a purely graphical proof of *Naimark's theorem*, hence extending the notion of measurement introduced in this paper. The fact that quantum theory can be developed without the additive type constructors seems to shed new light on the question of *parallelism vs. entanglement*. In the final sections of the paper, we show that superposition too can be described as a purely tensorial phenomenon, in contrast with the Hilbert space picture of entanglement as a special case of a superposition. There are also clear structural connections with TQFT [4, 23, and references therein]. Will categorical quantum semantics provide any physical insights about these mathematical structures?

2 Categorical semantics & graphical calculus

We present the mathematical structures of interest here both in the usual category-theoretic form and in a purely graphical calculus; the reader can pick his favorite flavor (and sort of ignore the other one).

†-compact categories. In a *symmetric monoidal category* [25] the objects form a monoid with the tensor $\otimes$ as multiplication and an object I as the multiplicative unit, up to the coherent natural isomorphisms

$$\lambda_A : A \simeq \mathrm{I} \otimes A \qquad \rho_A : A \simeq A \otimes \mathrm{I} \qquad \alpha_{A,B,C} : A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C \,.$$

The fact that a monoidal category is symmetric means that this monoid is commutative, up to the natural transformation

$$\sigma_{A,B} : A \otimes B \simeq B \otimes A$$

coherent with the previous ones. We shall assume that α is strict, i.e. realized by identity, but it will be convenient to carry λ and ρ as explicit structure.[4] Physically we interpret the objects of a symmetric monoidal category as system types, e.g. qubit, two qubits, classical data, qubit + classical data etc. A morphism should be viewed as a *physical operation*, e.g. unitary, or a measurement, classical communication etc. The tensor captures *compoundness* i.e. conceiving two systems or two operations as one. Morphisms of type $\mathrm{I} \to A$ represent *states* conceived through their respective preparations, whereas morphisms of the type $\mathrm{I} \to \mathrm{I}$ capture *scalars* e.g. *probabilistic weights* — cf. complex

[4] *Coherence* means that all diagrams composed of these natural transformations commute. In particular, there is at most one natural isomorphism between any two functors composed from $\otimes$ and I [24]. As a consequence, some functors can be transferred along these canonical isomorphisms, which then become identities. Without loss of generality, one can thus assume that α, λ and ρ are identities, and that the objects form an actual monoid with $\otimes$ as multiplication and I as unit. Such monoidal categories are called *strict*. For every monoidal category, there is an equivalent strict one.

numbers $c \in \mathbb{C}$ are in bijective correspondence with linear maps $\mathbb{C} \to \mathbb{C} :: 1 \mapsto c$. Details of this interpretation are in [9].

A symmetric monoidal category is *compact* [21, 22] if each of its objects has a *dual*. An object B is dual to A when it is given with a pair of morphisms $\eta : \mathrm{I} \to B \otimes A$ and $\varepsilon : A \otimes B \to \mathrm{I}$ often called *unit* and *counit*, satisfying

$$(\varepsilon \otimes 1_A) \circ (1_A \otimes \eta) = 1_A \quad \text{and} \quad (1_B \otimes \varepsilon) \circ (\eta \otimes 1_B) = 1_B\,. \tag{1}$$

It follows that any two duals of A must be isomorphic.[5] A representative of the isomorphism class of the duals of A is usually denoted by A^*. The corresponding unit and counit are then denoted η_A and ε_A.

A *symmetric monoidal* †*-category* $\mathbf{C}$ comes with a contravariant functor $(-)^\dagger : \mathbf{C}^{op} \to \mathbf{C}$, which is identity on the objects, involutive on the morphisms, and preserves the tensor structure [30]. The image $f^\dagger$ of a morphism f is called its (abstract) *adjoint*.

Finally, †-compact categories [2, 3] sum up all of the above structure, subject to the additional coherence requirements that

- every natural isomorphism χ, derived from the symmetric monoidal structure, must be *unitary*, i.e. satisfies $\chi^\dagger \circ \chi = 1$ and $\chi \circ \chi^\dagger = 1$, and
- $\eta_{A^*} = \varepsilon_A^\dagger = \sigma_{A^*A} \circ \eta_A$.

Since in a †-compact category $\varepsilon_A = \eta_{A^*}^\dagger$ some of the structure of the duals becomes redundant. In particular, it is sufficient to stipulate the units $\eta : \mathrm{I} \to A^* \otimes A$, which we call *Bell states*, in reference to their physical meaning. Without the stepwise introduction of the monoidal and the compact structure, †-compact categories thus allow a very succinct presentation on their own [3, 7], as a symmetric monoidal category with

- an involution $A \mapsto A^*$,
- a contravariant identity-on-objects $\otimes$-preserving involution $f \mapsto f^\dagger$,
- for each object a distinct morphism $\eta_A : \mathrm{I} \to A^* \otimes A$,

such that the diagram

$$\begin{tikzcd}
A & \mathrm{I} \otimes A \arrow[l, "\simeq"'] & A \otimes A^* \otimes A \arrow[l, "\eta_{A^*}^\dagger \otimes 1_A"'] \\
A \arrow[u, "1_A"] \arrow[r, "\simeq"'] & A \otimes \mathrm{I} \arrow[r, "1_A \otimes \eta_A"'] & A \otimes A^* \otimes A \arrow[u, "1_{A\otimes A^* \otimes A}"']
\end{tikzcd} \tag{2}$$

[5]If η, ε make B dual to A, while $\widetilde{\eta}, \widetilde{\varepsilon}$ make $\widetilde{B}$ dual to A, then $(1_{\widetilde{B}} \otimes \varepsilon) \circ (\widetilde{\eta} \otimes 1_B) : B \to \widetilde{B}$ and $(1_B \otimes \widetilde{\varepsilon}) \circ (\eta \otimes 1_{\widetilde{B}}) : \widetilde{B} \to B$ make B and $\widetilde{B}$ isomorphic.

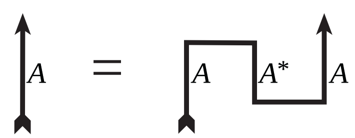
commutes, and subject to the above stated additional coherences. In Hilbert space terms, this definition mainly axiomatizes the Bell-states

$$\eta_{\mathcal{H}} : \mathbb{C} \to \mathcal{H}^* \otimes \mathcal{H} :: 1 \mapsto \sum_{i \in I} |\, i\, i\, \rangle\,,$$

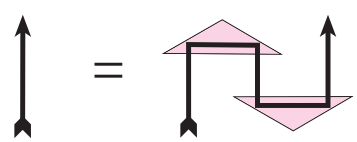
where $\mathcal{H}^*$ is the *conjugate space* to $\mathcal{H}$. Surprisingly, this seemingly very weak axiomatics comes with an amazing amount of typical Hilbert space mathematical machinery [3, 7, 10, 30], such as Hilbert-Schmidt inner-product, completely positive map and POVM, just to mention some.

Graphical calculus. With the above conditions, the structure of †-compact categories becomes coherent, in the sense that it satisfies exactly those equations that can be proven in the corresponding *graphic language*. This fact has been proven by Selinger in [30, **Thm**. 3.9]. We briefly summarize a version of this graphic representation. The objects of a †-compact category are represented by tuples of wires, whereas the morphisms are the I/O-boxes. Sequential composition connects the output wires of one box with the input wires of the other one. The tensor product is the union of the wires, and it places the boxes next to each other. A physicist-friendly introduction to this graphical language for symmetric monoidal categories is in [9]. But the main power of the graphical language lies in its representation of duality. The Bell state (unit) and its adjoint (counit) correspond to a wire from A returning into A^*, with the directions reversed:

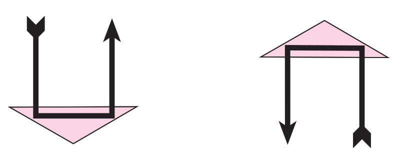
Graphically, the composition of η_A and $\varepsilon_A = \eta^\dagger_{A^*}$ as expressed in commutative diagram (2) boils down to



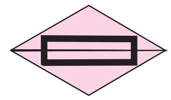
Note that in related papers such as [8] a more involved notation



appears. The triangles witness the fact that in physical terms



respectively stand for a preparations procedure, or state, or *ket*, and for the corresponding *bra*, with an inner-product or *bra-ket*



then yielding a *diamond shaped scalar* (cf. [8]), while the wire itself is now a *loop*. In this paper we will omit these special bipartite triangles.

Given a choice of the duals $A \mapsto A^*$, one can follow the same pattern to define the arrow part $f \mapsto f^*$ of the duality functor $(-)^* : \mathbf{C}^{op} \to \mathbf{C}$ by the commutativity of the following diagram:

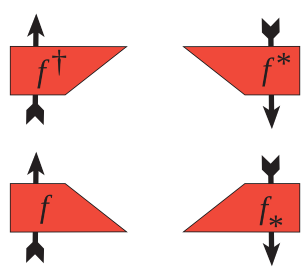
$$\begin{tikzcd}
A^* & A^* \otimes \mathrm{I} \arrow[l, "\simeq"'] & A^* \otimes B \otimes B^* \arrow[l, "1_{A^*} \otimes \eta^\dagger_{B^*}"'] \\
B^* \arrow[u, "f^*"] \arrow[r, "\simeq"'] & \mathrm{I} \otimes B^* \arrow[r, "\eta_A \otimes 1_{B^*}"'] & A^* \otimes A \otimes B^* \arrow[u, "1_{A^*} \otimes f \otimes 1_{B^*}"']
\end{tikzcd}$$

Replacing $f : A \to B$ by $f^\dagger : B \to A$, we can similarly define $f_* : A^* \to B^*$, and thus extend the duality assignment $A \mapsto A^*$ by the morphism assignment $f \mapsto f_*$ to the covariant functor $(-)_* : \mathbf{C} \to \mathbf{C}$. It can be shown [2] that the adjoint decomposes in every $\dagger$-compact category as

$$f^\dagger = (f^*)_* = (f_*)^*$$

with both $(-)^*$ and $(-)_*$ involutive. In finite dimensional Hilbert spaces and linear maps **FdHilb**, these two functors respectively correspond to *transposition* and *complex conjugation*. The functor $(-)_* : \mathbf{C} \to \mathbf{C}$ will thus be called *conjugation*; the image f_* is a conjugate of f. Graphically, the above diagram defining f^*, and the similar one for f_*, respectivelly become

The direction of the arrows is, of course, just relative, and we have chosen to direct the arrows down in order to indicate that both f^* and f_* have the duals as their domain and codomain types. We will use horizontal reflection to depict $(-)^\dagger$ and Selinger's 180° rotation [30] to depict $(-)_*$, resulting in:



Scalars, trace, and partial transpose. One can prove that the monoid $\mathbf{C}(\mathrm{I},\mathrm{I})$ is always commutative [22] and induces a *scalar multiplication*

$$s \bullet f := \lambda_B^{-1} \circ (s \otimes f) \circ \lambda_A : A \to B$$

which by naturality satisfies

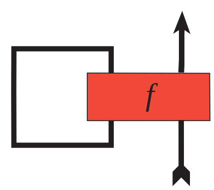
$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g) \quad (s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)\,. \tag{3}$$

As already indicated above, we will depict these scalars by diamonds, and such scalars can arise as loops [22, 1]. The equations (3) show that these diamonds capturing probabilistic weights can be 'freely moved in the pictures'.

A (†-)compact category also always comes with a canonical *trace structure* in the sense of [19], defined by

$$\begin{tikzcd}
B & \mathrm{I}\otimes B \arrow[l, "\simeq"'] & C^*\otimes C\otimes B \arrow[l, "\eta_C^\dagger\otimes 1_B"'] \\
A \arrow[u, "\mathrm{tr}^C_{A,B}(f)"] \arrow[r, "\simeq"'] & \mathrm{I}\otimes A \arrow[r, "\eta_C\otimes 1_A"'] & C^*\otimes C\otimes A \arrow[u, "1_{C^*}\otimes f"']
\end{tikzcd}$$

for every $f : C \otimes A \to C \otimes B$. In a picture $\mathrm{tr}^C_{A,B}(f)$ is:



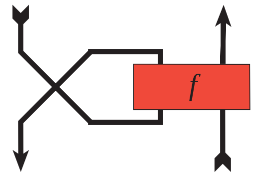
There also is a canonical *partial transpose* structure of type

$$\mathrm{pt}^{C,D}_{A,B} : \mathbf{C}(C \otimes A, D \otimes B) \to \mathbf{C}(D^* \otimes A, C^* \otimes B)$$

and which is defined by setting

$$\begin{tikzcd}
C^*\otimes B & C^*\otimes\mathrm{I}\otimes B \arrow[l, "\simeq"'] & C^*\otimes D^*\otimes D\otimes B \arrow[l, "1_{C^*}\otimes\eta_D^\dagger\otimes 1_B"'] \\
D^*\otimes A \arrow[u, "\mathrm{pt}^{C,D}_{A,B}(f)"] \arrow[r, "\simeq"'] & D^*\otimes\mathrm{I}\otimes A \arrow[r, "1_{D^*}\otimes\eta_C\otimes 1_A"'] & D^*\otimes C^*\otimes C\otimes A \arrow[u, "\sigma_{D^*,C^*}\otimes f"']
\end{tikzcd}$$

given $f : C \otimes A \to D \otimes B$. In a picture $\mathrm{pt}^{C,D}_{A,B}(f)$ is:



This partial transpose can also be seen as a variant on the swap-actions $\sigma_{C,D} \circ -$ and $- \circ \sigma_{C,D}$, which rather than respectively swapping two output types or two input types, swaps an input and an output type.

3 Sums, copying and bases in FdHilb

Before we start the abstract categorical and diagrammatic analysis of quantum measurement, we provide a discussion of the key insights of this paper within the Hilbert space quantum mechanical model. It are these insights which will be exploited in the proceeding abstract development.

Sums in quantum mechanics. Sums occur in the Hilbert space formalism both as a part of the linear structure of states, as well as a part of their projective (convex) structure, through the fundamental theorem of projective geometry and Gleason's theorem [28]. Viewed categorically, these structures lift, respectively, to a vector space enrichment and a projective space enrichment of operators, typically yielding a C^*-algebra. They appear to be necessary because of the specific nature of quantum measurement, and the resulting quantum probabilistic structure. The additive structure permeates not only states, but also state spaces; it is crucial not only for adding vectors, but also for composing and decomposing spaces. In fact, one verifies that operator sums arise from the direct sum:

$$\begin{tikzcd} \mathbb{C}^{\oplus n} \arrow[r, "f+g"] \arrow[d, "d"'] & \mathbb{C}^{\oplus m} \\ \mathbb{C}^{\oplus n} \oplus \mathbb{C}^{\oplus n} \arrow[r, "f \oplus g"'] & \mathbb{C}^{\oplus m} \oplus \mathbb{C}^{\oplus m} \arrow[u, "d^\dagger"'] \end{tikzcd}$$

where $d :: |\,i\rangle \mapsto |\,i\rangle \oplus |\,i\rangle$ is the additive diagonal. As a particular case we have that the vector sums arise from

$$\begin{tikzcd} \mathbb{C} \arrow[r, "|\psi\rangle + |\phi\rangle"] \arrow[d, "d"'] & \mathbb{C}^{\oplus n} \\ \mathbb{C} \oplus \mathbb{C} \arrow[r, "|\psi\rangle \oplus |\phi\rangle"'] & \mathbb{C}^{\oplus n} \oplus \mathbb{C}^{\oplus n} \arrow[u, "d^\dagger"'] \end{tikzcd}$$

where $|\psi\rangle, |\phi\rangle : \mathbb{C} \to \mathbb{C}^{\oplus n}$, recalling that vectors $|\psi\rangle \in \mathbb{C}^{\oplus n}$ are indeed, by linearity, in bijective correspondence with the linear maps

$$\mathbb{C} \to \mathbb{C}^{\oplus n} :: 1 \mapsto |\psi\rangle\,.$$

In addition to this, the direct sum canonically also defines bases (cf. the computational base) in terms of the n canonical injections

$$\mathbb{C} \hookrightarrow \mathbb{C}^{\oplus n} :: 1 \mapsto (0, \ldots, 0, 1, 0, \ldots, 0)\,.$$

No-Cloning and existence of a natural diagonal. The classic No-Cloning theorem [32] states that there exists no unitary operation

$$Clone : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |\psi\rangle \otimes |\,0\rangle \mapsto |\psi\rangle \otimes |\psi\rangle\,. \tag{4}$$

On the other hand, using category-theoretic language, it is also well-known that there exists no *natural diagonal* for the Hilbert space tensor product. Explicitly put, this means that there exists no linear maps

$$\Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}\,,$$

(i.e. one for each choice of Hilbert space $\mathcal{H}$) which make the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "f"] \arrow[d, "\Delta_{\mathcal{H}}"'] & \mathcal{H}' \arrow[d, "\Delta_{\mathcal{H}'}"] \\ \mathcal{H} \otimes \mathcal{H} \arrow[r, "f \otimes f"'] & \mathcal{H}' \otimes \mathcal{H}' \end{tikzcd} \tag{5}$$

commute for any of linear map $f : \mathcal{H} \to \mathcal{H}'$. In particular, for

$$\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |\,i\,\rangle \mapsto |\,ii\,\rangle$$

setting $\mathcal{H} := \mathbb{C}$, $\mathcal{H}' := \mathbb{C} \oplus \mathbb{C}$ and $f : 1 \mapsto |\,0\rangle + |\,1\rangle$ yields a counterexample. Assume now that there is a cloning machine as in (4). Then define

$$\Delta_{\mathcal{H}} := Clone \circ (- \otimes |\,0\rangle) : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle\,.$$

Since $|\psi\rangle$ is arbitrary in (4), we can take it to be $|f(\psi)\rangle$ resulting in

$$\begin{aligned} (\Delta_{\mathcal{H}} \circ f)(|\psi\rangle) &= \Delta_{\mathcal{H}}(|f(\psi)\rangle) = |f(\psi)\rangle \otimes |f(\psi)\rangle = (f \otimes f)(|\psi\rangle \otimes |\psi\rangle) \\ &= (f \otimes f)(\Delta_{\mathcal{H}}(|\psi\rangle)) = ((f \otimes f) \circ \Delta_{\mathcal{H}})(|\psi\rangle) \end{aligned}$$

i.e. we obtain commutation of diagram (5). The careful reader might have observed that commutativity of (5) implies bases-independency of $\Delta_{\mathcal{H}}$, by conceiving the linear map f as a change of base (e.g. [9]). In fact, for Hilbert spaces, category-theoretic naturality of a mathematical concept can very much be thought of as bases-independency of that mathematical concept.

Measurement and bases. Self-adjoint operators, which represent measurements in quantum mechanics comprise two pieces of data, namely eigenvectors and corresponding eigenvalues. For informatic purposes, the eigenvalues are merely token witnesses which discriminate outcomes, so a non-degenerate measurement essentially corresponds to a base, and a degenerate one can also be captured by a base, provided that we also provide an equivalence relation on the base vectors. Now consider again the map $|\,i\,\rangle \stackrel{\delta}{\mapsto} |\,ii\,\rangle$.[6] While it *copies* base vectors it does not do so for other states

$$|\psi\rangle = \sum_i c_i|\,i\,\rangle \;\stackrel{\delta}{\mapsto}\; \sum_i c_i|\,ii\,\rangle \neq |\psi\rangle \otimes |\psi\rangle\,.$$

In fact, this map exactly captures the base $\{|\,i\,\rangle\}_i$. Indeed,

$$\delta :: \sum_{i\in I} c_i|\,i\,\rangle \;\mapsto\; \sum_{i\in I} c_i|\,ii\,\rangle\,,$$

so we obtain a disentangled state under the action of δ if and only if the index set I is a singleton i.e. if and only if $\sum_{i\in I} c_i|\,i\,\rangle$ is itself a base vector. Conversely put, we recover the base by taking the image of pure tensors under the map[7]

$$\delta^\dagger :: \begin{cases} i \neq j & : & |\,ij\,\rangle & \mapsto & \vec{o} \\ else & : & |\,ii\,\rangle & \mapsto & |\,i\,\rangle \end{cases}$$

Since δ is *base-capturing* it is of course not *bases-independent*, and hence not natural in the categorical sense, but it is exactly this un-naturality which allows it to capture a base. Moreover, if we restrict to vectors in this base, δ is a genuine 'classical' *copying* operation, although it drastically fails to be one on the whole Hilbert space. Hence, instead of characterizing a *classical measurement context* by an explicit bases, we can characterize it by an operation which faithfully copies the corresponding classical data, and the base arises as the quantum states which are copy-able under this operation.

Vanishing of non-diagonal elements and deletion. Moreover, the map δ also enables to capture the 'formal decohering' in quantum measurement i.e. the vanishing of the non-diagonal elements in the passage the initial state represented as a density matrix within the measurement base to the density matrix describing the resulting ensemble of possible outcome states.[8] Indeed, non-diagonal elements get erased setting

$$\delta \circ \delta^\dagger :: \begin{cases} i \neq j & : & |\,ij\,\rangle & \mapsto & \vec{o} & \mapsto & \vec{o} \\ else & : & |\,ii\,\rangle & \mapsto & |\,i\,\rangle & \mapsto & |\,ii\,\rangle \end{cases}$$

[6]This map, when assigning agents i.e. $|\,i\,\rangle_A \stackrel{\delta}{\mapsto} |\,i\,\rangle_A \otimes |\,i\,\rangle_B$, has appeared in the literature under the name *coherent bit*, as a 'between classical and quantum'-channel [11, 17].

[7]This operation $\delta^\dagger : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$ has appeared in the quantum informatics literature under the name *fusion*, providing a means for constructing cluster states [5, 31].

[8]See [15] for a discussion why we call this 'formal decohering'.

Note also that δ's adjoint $\delta^\dagger$ doesn't delete classical data, but *compares* its two inputs and only passes on data if they coincide. *Deletion* is

$$\epsilon :: |\, i\,\rangle \mapsto 1 \qquad \text{that is} \qquad 1 \otimes \epsilon :: |\, ij\,\rangle \mapsto |\, i\,\rangle\,.$$

What ϵ and $\delta^\dagger$ do have in common is the fact that

$$\delta^\dagger \circ \delta = (1 \otimes \epsilon) \circ \delta :: |\, i\,\rangle \mapsto |\, ii\,\rangle \mapsto |\, i\,\rangle\,.$$

Also, since in Dirac-notation we have $\delta = \sum_i |\, i\rangle\langle ii\,|$, the (base-dependent) isomorphism $\theta :: |\, i\rangle \mapsto \langle i\,|$ applied to the *bra* turns δ into the generalized GHZ-state $\sum_i |\, iii\rangle$ [16] exposing that δ is 'up to θ' symmetric in all variables.

Canonical bases. While all Hilbert spaces of the same dimension are obviously isomorphic, they are not all equivalent. Indeed, above we already mentioned that the direct sum structure provides the Hilbert space $\mathbb{C}^{\oplus n}$ with a canonical base, from which it also follows that it is canonically isomorphic to its conjugate space $(\mathbb{C}^{\oplus n})^* = (\mathbb{C}^*)^{\oplus n}$, namely for the isomorphism

$$\mathbb{C}^{\oplus n} \to (\mathbb{C}^*)^{\oplus n} :: (c_1, \ldots, c_n) \mapsto (\bar{c}_1, \ldots, \bar{c}_n)\,.$$

In fact, one should not think of $\mathbb{C}^{\oplus n}$ as just being a Hilbert space, but as the pair consisting of a Hilbert space $\mathcal{H}$ and a base $\{|\, i\,\rangle\}_{i=1}^{i=n}$, which by the above discussion boils down to the pair consisting of a Hilbert space $\mathcal{H}$ and a linear map $\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ satisfying certain properties, in particular, its matrix being self-transposed in the canonical base. Below, we will assume the correspondence between $\mathbb{C}^{\oplus n}$ and its dual to be strict, something which can always be established by standard methods. The special status of the objects $\mathbb{C}^{\oplus n}$ in **FdHilb**, in category-theoretic terms, is due to the fact the direct sum is both a product and a coproduct and $\mathbb{C}$ the tensor unit [2].

4 Classical objects

Consider a quantum measurement. It takes a quantum state as its input and produces a measurement outcome together with a quantum state, which is typically different from the input state due to the *collapse*. Hence the type of a quantum measurement should be

$$\mathcal{M} : A \to X \otimes A$$

where A is of the type *quantum state* while X is of the type *classical data*. But how do we distinguish between classical and quantum data types?

We will take a very operational view on this matter, and define classical data types as objects which come together with a copying operation

$$\delta_{(X)} : X \to X \otimes X$$

and a deleting operation

$$\epsilon_{(X)} : X \to \mathrm{I}\,,$$

counterfactually exploiting the fact that such operations do not exist for quantum data. We will refer to these structured objects (X, δ, ϵ) as *classical objects*. The axioms which we require the morphisms δ and ϵ to satisfy are motivated by the operational interpretation of δ and ϵ as copying and deleting operations of classical data. This leads us to introducing the notion of a *special †-compact Frobenius algebra*, which refines the usual topological quantum field theoretic notion of a normalised special Frobenius algebra [23].

Special †-compact Frobenius algebras. An *internal monoid* (X, μ, ν) in a monoidal category $(\mathbf{C}, \otimes, \mathrm{I})$ is a pair of morhisms

$$X \otimes X \xrightarrow{\mu} X \xleftarrow{\nu} \mathrm{I}\,,$$

called the *multiplication* and the *multiplicative unit*, such that

$$\begin{tikzcd}
X & X \otimes X \arrow[l, "\mu"'] \\
X \otimes X \arrow[u, "\mu"] & X \otimes X \otimes X \arrow[l, "\mu \otimes 1_X"] \arrow[u, "1_X \otimes \mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X & \\
\mathrm{I} \otimes X \arrow[ur, "\lambda_X^{-1}"] \arrow[r, "\nu \otimes 1_X"'] & X \otimes X \arrow[u, "\mu"] & X \otimes \mathrm{I} \arrow[l, "1_X \otimes \nu"] \arrow[ul, "\rho_X^{-1}"']
\end{tikzcd}$$

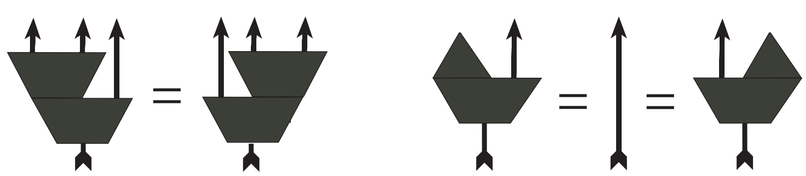
commute. Dually, an *internal comonoid* (X, δ, ϵ) is a pair of morphisms

$$X \otimes X \xleftarrow{\delta} X \xrightarrow{\epsilon} \mathrm{I}\,,$$

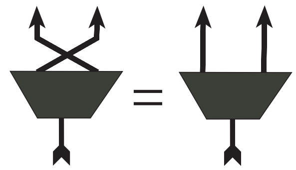
the *comultiplication* and the *comultiplicative unit*, such that

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "\delta"'] & X \otimes X \arrow[d, "1_X \otimes \delta"] \\
X \otimes X \arrow[r, "\delta \otimes 1_X"'] & X \otimes X \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dl, "\lambda_X"'] \arrow[d, "\delta"] \arrow[dr, "\rho_X"] & \\
\mathrm{I} \otimes X & X \otimes X \arrow[l, "\epsilon \otimes 1_X"] \arrow[r, "1_X \otimes \epsilon"'] & X \otimes \mathrm{I}
\end{tikzcd}$$

commute. Graphically these conditions are:



When $(\mathbf{C}, \otimes, \mathrm{I})$ is *symmetric*, the monoid is commutative iff $\mu \circ \sigma_{X,X} = \mu$, and the comonoid is commutative iff $\sigma_{X,X} \circ \delta = \delta$, in a picture:

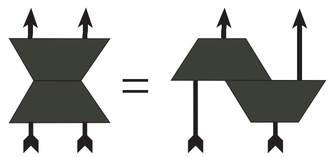


Note that the conditions defining an internal commutative comonoid are indeed what we expect a copying and deleting operation to satisfy.

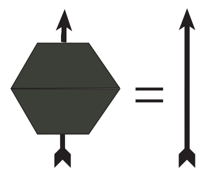
A *symmetric Frobenius algebra* is an internal commutative monoid (X, μ, ν) together with an internal commutative comonoid (X, δ, ϵ) which satisfy

$$\delta \circ \mu = (\mu \otimes 1_X) \circ (1_X \otimes \delta)\,, \tag{6}$$

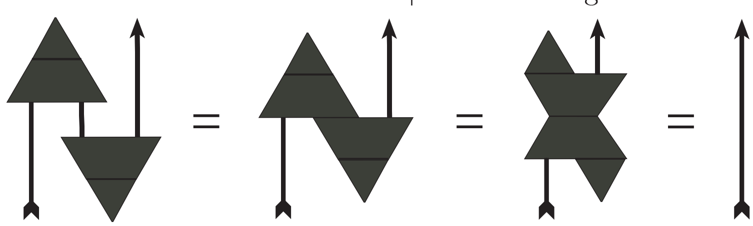
that is, in a picture:



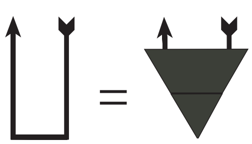
It is moreover *special* iff $\mu \circ \delta = 1_X$, in a picture:



In a symmetric monoidal †-category every internal commutative comonoid (X, δ, ϵ) also defines an internal commutative monoid $(X, \delta^\dagger, \epsilon^\dagger)$, yielding a notion of *†-Frobenius algebra* (X, δ, ϵ) in the obvious manner. In such a †-Frobenius algebra we have:



that is, $\delta \circ \epsilon^\dagger : \mathrm{I} \to X \otimes X$ and $\epsilon \circ \delta^\dagger : X \otimes X \to \mathrm{I}$ satisfy equations (1) of Section 2 and hence canonically provide a unit $\eta = \delta \circ \epsilon^\dagger$ and counit $\varepsilon = \epsilon \circ \delta^\dagger$ which realizes $X^* = X$ (cf. Section 2). In a picture this choice stands for:



One easily verifies that the linear maps δ and ϵ as defined in the previous section indeed yield an internal comonoid structure on the Hilbert space $\mathbb{C}^{\oplus n}$ which satisfies the *Frobenius identity* (6), and that $\delta \circ \epsilon^\dagger$ is the Bell-state.

Definition 4.1 A *classical object* in a †-compact category is a special †-compact Frobenius algebra (X, δ, ϵ) i.e. a special †-Frobenius algebra for which we choose $\eta_X = \delta_X \circ \epsilon_X^\dagger$ in order to realize $X^* = X$.

So typical examples of classical objects are the ones existing in **FdHilb** which were implicitly discussed in Section 3, namely

$$\left(\mathbb{C}^{\oplus n}\ ,\ \delta^{(n)} : \mathbb{C}^{\oplus n} \to \mathbb{C}^{\oplus n} \otimes \mathbb{C}^{\oplus n} :: |\, i\rangle \mapsto |\, ii\rangle\ ,\ \epsilon^{(n)} : \mathbb{C}^{\oplus n} \to \mathbb{C} :: |\, i\rangle \mapsto 1\right).$$

Since the Frobenius identity (6) allows us to set $X^* = X$ we can now compare $\delta_*, \delta : X \to X \otimes X$, and also, $\epsilon_*, \epsilon : X \to \mathrm{I}$, them having the same type. Recalling that in **FdHilb** the covariant functor $(-)_*$ stands for complex conjugation, the structure of a †-compact Frobenius algebra guarantees the highly significant and crucial property that the operations of copying and deleting classical data carry no phase information:

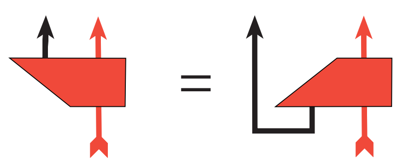
Theorem 4.2 *For a classical object we have $\delta_* = \delta$ and $\epsilon_* = \epsilon$.*

Before we prove this fact we will need to introduce some additional concepts.

Self-adjointness relative to a classical object. From now on we will denote classical objects as X whenever it is clear from the context that we are considering the structured classical data type (X, δ, ϵ) and not the unstructured quantum data type X. Given a classical object X we call a morphism $\mathcal{F} : A \to X \otimes A$ *self-adjoint relative to* X if the diagram

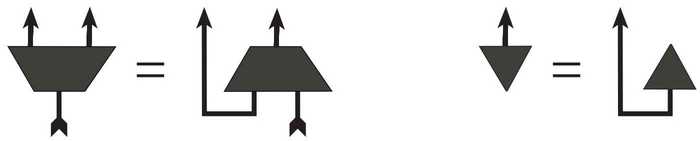
$$\begin{tikzcd}
A \arrow[r, "\mathcal{F}"] \arrow[d, "\lambda_A"'] & X \otimes A \\
\mathrm{I} \otimes A \arrow[r, "\eta_X \otimes 1_A"'] & X \otimes X \otimes A \arrow[u, "1_X \otimes \mathcal{F}^\dagger"']
\end{tikzcd} \tag{7}$$

commutes. In a picture, this is:

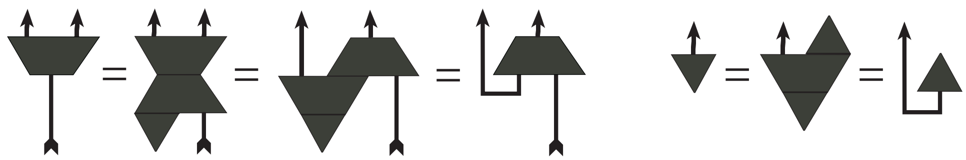


A morphism $\mathcal{F} : X \otimes A \to A$ is self-adjoint relative to X whenever $\mathcal{F}^\dagger$ is. Note furthermore that in every monoidal category, the unit I carries a canonical comonoid structure, with $\delta = \lambda_\mathrm{I} = \rho_\mathrm{I} : \mathrm{I} \to \mathrm{I} \otimes \mathrm{I}$ and $\epsilon = 1_\mathrm{I} : \mathrm{I} \to \mathrm{I}$. In every †-compact category, this comonoid is in fact a degenerate classical object. Self-adjointness in the usual sense of $f^\dagger = f : A \to A$ corresponds to self-adjointness relative to I. For a general classical object X, a morphism $\mathcal{F} : A \to X \otimes A$ can be thought of as an X-indexed family of morphisms of type $A \to A$. Self-adjointness relative to X then means that each of the elements of this indexed family are required to be self-adjoint in the ordinary sense. We abbreviate 'self-adjoint relative to X' to 'X-self-adjoint'. Below and in [10] there are some more analogous generalizations of standard notions e.g. X-scalar, X-inverse, X-unitarity, X-idempotence, X-positivity etc.

Proposition 4.3 *Both the comultiplication δ and the unit ϵ of a classical object X are always X-self-adjoint, that is, in a picture:*

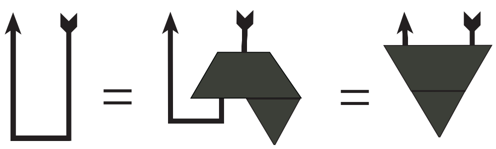


Proof:



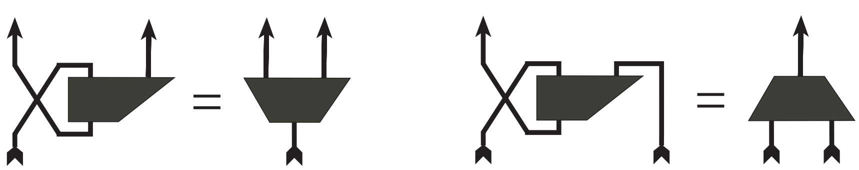
□

Note that X-self-adjointness of ϵ is exactly $\epsilon_* = \epsilon$, already providing part of the proof of Theorem 4.2. In fact, given an internal commutative comonoid (X, δ, ϵ) diagram (7) implicitly stipulates that, of course $X^* = X$, but also that this self-duality of X is realized through $\eta = \delta \circ \epsilon^\dagger$ since we have



Hence it makes sense to speak of an *X-self-adjoint internal comonoid* in a †-compact category. From X-self-adjointness we can straightforwardly derive many other useful properties, including the Frobenius identity itself, hence providing an alternative characterization of classical objects, and also $\delta_* = \delta$, providing the remainder of the proof of Theorem 4.2.

Lemma 4.4 *The comultiplication of an X-self-adjoint commutative internal monoid satisfies the Frobenius identity (6), is partial-transpose-invariant $\mathrm{pt}_{\mathrm{I},X}^{X,X}(\delta) = \delta$, and is self-dual $\delta_* = \delta$ (or $\delta^* = \delta^\dagger$). The latter two depict as:*



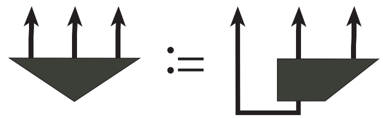
Proof: For the Frobenius identity, apply X-self-adjointness to the lefthandside, use associativity of the comultiplication, and apply X-self-adjointness again, for partial-transpose-invariance apply X-self-adjointness twice, and for self-duality apply X-self-adjointness three times. □

Theorem 4.5 *A classical object can equivalently be defined as a special X-self-adjoint internal commutative comonoid (X, δ, ϵ).*

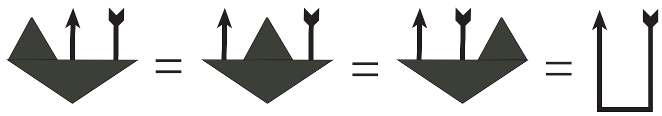
GHZ-states as classical objects. Analogously to the Hilbert space case (cf. Section 3), each classical object X induces an abstract counterpart to generalized GHZ-states, namely

$$\mathsf{GHZ}_X := (1_X \otimes \delta) \circ \eta : \mathrm{I} \to X \otimes X \otimes X\,.$$

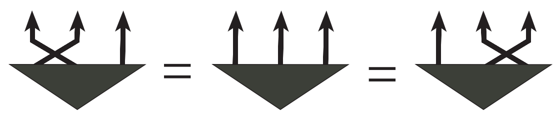
In a picture that is:



The unit property of the comonoid structure, together with the particular choice for the unit of compact closure $\varepsilon = \epsilon \circ \delta^\dagger$ become pleasingly symmetric:



The same is the case for commutativity of the comonoid structure, together with partial-transpose-invariance:



Extracting the classical world. If $\mathbf{C}$ comes with a †-structure then any internal comonoid yields an internal monoid. But there is a clear conceptual distinction between the two structures, in the sense that the comultiplication and its unit admit interpretation in terms of copying and deleting. We wil be able to extract the classical world by defining *morphisms of classical objects* to be those which preserve the copying and deleting operations of these classical objects, or, in other words, by restricting to those morphisms with respect to which the copying and deleting operations *become natural* (cf. Section 3).

Given a †-compact category $\mathbf{C}$, we define two new categories of which the objects are the classical objects. The first one, $\mathbf{C}_\times$, is spanned by the morphisms preserving both δ and ϵ structures, and the second one, $\mathbf{C}_{stoch}$, by the morphisms that preserve ϵ, but not necessarily δ. So $\mathbf{C}_\times$ embeds into $\mathbf{C}_{stoch}$, and both have a forgetful functor into $\mathbf{C}$, i.e.

$$\mathbf{C}_\times \hookrightarrow \ \mathbf{C}_{stoch} \to \ \mathbf{C}\,.$$

In $\mathbf{FdHilb}$, a linear map $f : \mathbb{C}^{\oplus m} \to \mathbb{C}^{\oplus n}$ preserves $\epsilon^{(n)}$ if it is a *stochastic operator* i.e. $\sum_{j=1}^{j=n} f_{ij} = 1$ for all i, and it preserves $\delta^{(n)}$ if $f_{ij}f_{ij} = f_{ij}$ and $f_{ij}f_{ik} = 0$ for $j \neq k$, hence, there is a function $\varphi : m \to n$ such that

$$f(|\, i\rangle) = |\, \varphi(i)\rangle\,.$$

So $\mathbf{FdHilb}_\times = \mathbf{FSet}$, the latter being the category of finite sets and functions, while $\mathbf{FdHilb}_{stoch} = \mathbf{FStoch}$, the latter being the category of finite sets and stochastic maps. Hence morphisms in $\mathbf{C}_\times$ are to be conceived as deterministic manipulations of classical data while morphisms in $\mathbf{C}_{stoch}$ are probabilistic manipulations, i.e. while $\mathbf{C}$ represents the quantum world, $\mathbf{C}_\times$ and $\mathbf{C}_{stoch}$ represent the classical world. The canonical status of $\mathbf{C}_\times$ is exposed by the following result due to Fox [14].

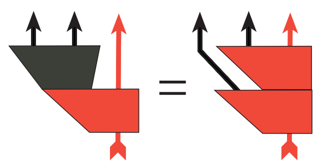
Theorem 4.6 *Let $\mathbf{C}$ be a symmetric monoidal category. The category $\mathbf{C}_\times$ of its commutative comonoids and corresponding morphisms, with the forgetful functor $\mathbf{C}_\times \to \mathbf{C}$, is final among all cartesian categories with a monoidal functor to $\mathbf{C}$, mapping the cartesian product $\times$ to the monoidal tensor $\otimes$.*

5 Quantum spectra

Given a classical object X, a morphism $\mathcal{F} : A \to X \otimes A$ is *idempotent relative to X*, or shorter, *X-idempotent*, if

$$\begin{tikzcd} A \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{F}"'] & X \otimes A \arrow[d, "1_X \otimes \mathcal{F}"] \\ X \otimes A \arrow[r, "\delta \otimes 1_A"'] & X \otimes X \otimes A \end{tikzcd}$$

commutes. In a picture that is:



Continuing in the same vein, an *X-projector* is a morphism $\mathcal{P} : A \to X \otimes A$ which is both X-self-adjoint and X-idempotent. The following proposition shows that an X-projector is not just an indexed family of projectors.

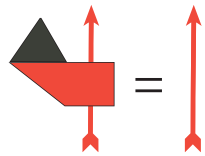
Proposition 5.1 *A $\mathbb{C}^{\oplus n}$-projector in* $\mathbf{FdHilb}$ *exactly corresponds to a family of mutually orthogonal projectors $\{\mathrm{P}_i\}_i$, hence $\sum_{i=1}^{i=k} \mathrm{P}_i \leq 1_{\mathbb{C}^{\oplus n}}$.*

Proof: One easily verifies that from X-idempotence follows idempotence $\mathrm{P}_i^2 = \mathrm{P}_i$ and mutual orthogonality $\mathrm{P}_i \circ \mathrm{P}_{j \neq i} = \mathbf{0}$, and that from X-self-adjointness follows orthogonality of projectors $\mathrm{P}_i^\dagger = \mathrm{P}_i$. $\square$

Definition 5.2 A morphism $\mathcal{P} : A \to X \otimes A$ is said to be X-complete if

$$\lambda_A^\dagger \circ (\epsilon \otimes 1_A) \circ \mathcal{P} = 1_A \,.$$

In a picture that is:



A morphism $\mathcal{P} : A \to X \otimes A$ is a *projector-valued spectrum* if it is an X-projector for some classical object X, and if it is moreover X-complete.

Theorem 5.3 *Projector-valued spectra in* **FdHilb** *exactly correspond to complete families of mutually orthogonal projectors* $\{\mathrm{P}_i\}_i$*, i.e.* $\sum_{i=1}^{i=k} \mathrm{P}_i = 1_{\mathbb{C}^{\oplus n}}$.

Each classical object (X, μ, ϵ) canonically induces a projector-valued spectrum $\mu : X \to X \otimes X$ since associativity of the comultiplication coincides with X-idempotence and the defining property of the comultiplicative unit coincides with completeness. Having in mind the characterization of classical objects of Theorem 4.5, mathematically, projector-valued spectra consitute a generalization of classical objects by admitting *degeneracy*.

Coalgebraic characterization of spectra. Recall that the internal commutative (co)monoid structures over an object X in a monoidal category $\mathbf{C}$ are in one-to-one correspondence with commutative (co)monad structures on the functor $X \otimes - : \mathbf{C} \to \mathbf{C}$. Hence we can attribute a notion of (co)algebra to internal commutative (co)monoids.

Theorem 5.4 *Let* $\mathbf{C}$ *be a* $\dagger$*-compact category. Its projector-valued spectra are exactly the* X*-self-adjoint Eilenberg-Moore coalgebras for the comonads* $X \otimes - : \mathbf{C} \to \mathbf{C}$ *canonically induced by some classical object* X*.*

Proof: The requirements for Eilenberg-Moore coalgebras with respect to the monad $(X \otimes -)$ are exactly X-idempotence and X-completeness. $\square$

We can now rephrase all the above as follows.

Theorem 5.5 X*-self-adjoint coalgebras in* **FdHilb** *exactly correspond to complete families of mutually orthogonal projectors* $\{\mathrm{P}_i\}_i$*.*

Proof: We also rephrase the proof. From the Eilenberg-Moore commuting square we obtain idempotence $\mathrm{P}_i^2 = \mathrm{P}_i$ and mutual orthogonality $\mathrm{P}_i \circ \mathrm{P}_{j \neq i} = \mathbf{0}$, from the Eilenberg-Moore commuting triangle we obtain completeness and from X-self-adjointness follows orthogonality of projectors $\mathrm{P}_i^\dagger = \mathrm{P}_i$. $\square$

6 Quantum measurements

Given projector-valued spectra we are very close to having an abstract notion of quantum measurement. In fact, the type $A \to X \otimes A$ which we attributed to the spectra is indeed the compositional type of a (*non-demolition*) measurement. But what is even more compelling is the following. The fact that a spectum is X-idempotent, or equivalently, that it satisfies the coalgebraic Eilenberg-Moore commuting square, i.e.

$$\begin{tikzcd} A \arrow[r, "Measure"] \arrow[d, "Measure"'] & X \otimes A \arrow[d, "1_X \otimes Measure"] \\ X \otimes A \arrow[r, "Copy \otimes 1_A"'] & X \otimes X \otimes A \end{tikzcd}$$

exactly captures *von Neumann's projection postulate*, stating that repeating a measurement is equivalent to copying the data obtained in its first execution. Note here in particular the manifest *resource sensitivity* of this statement, accounting for the fact that two measurements provide two sets of data, even if this data turns out to be identical.

However, what we get in **FdHilb** is not yet a quantum measurement. For $X, A :\simeq \mathcal{H}$, a non-demolition non-degenerate measurement $\mathcal{M} : A \to X \otimes A$ in the computational base yields

$$|\,\psi\rangle = \sum_i \langle\, i\,|\,\psi\rangle |\, i\rangle_A \xmapsto{\ \mathcal{M}\ } \sum_i \langle\, i\,|\,\psi\rangle \,(|\, i\,\rangle_X \otimes |\, i\,\rangle_A)$$

where $|\, i\,\rangle_X \in X$ is the measurement outcome, $|\, i\,\rangle_A \in A$ is the resulting state of the system for that outcome, and the coefficients $\langle\, i\,|\,\psi\rangle$ in the sum capture the respective probabilities for these outcomes i.e. $|\langle\, i\,|\,\psi\rangle|^2$. This however does not reflect the fact that we cannot retain the relative phase factors present in the probability amplitudes. In other words, the passage from physics to the semantics is not *fully abstract*. But as is well-known, the operation which erases these relative phases does not live in **FdHilb** but is *quadratic* in the state, hence lives in **CPM**(**FdHilb**), the category of Hilbert spaces and completely positive maps. Fortunately, Selinger provided an abstract counterpart for the passage from **FdHilb** to **CPM**(**FdHilb**), as a construction which applies to any †-compact category [30].

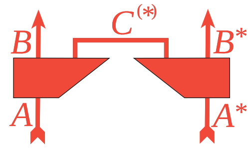
The CPM-construction. This construction takes a †-compact category **C** as its input and produces an 'almost inclusion' (it in fact kills redundant global phases) of **C** into a bigger one **CPM**(**C**). While **C** is to be conceived as containing pure operations with those of type $\mathrm{I} \to A$ being the pure states, **CPM**(**FdHilb**) consists of mixed operations with those of type $\mathrm{I} \to A$ being the mixed states. Explicitly we have the †-compact functor

$$\mathsf{Pure} : \mathbf{C} \to \mathbf{CPM}(\mathbf{C}) :: f \to f \otimes f_* \,.$$

where

$$\mathbf{CPM}(\mathbf{C})(A, B) := \left\{ (1_B \otimes \eta^\dagger_{C^*} \otimes 1_{B^*}) \circ (f \otimes f_*) \,\middle|\, f : A \to B \otimes C \right\}$$

and the †-compact structure on **CPM**(**C**) covariantly inherits its composition, its tensor, its adjoints and its Bell-states from **C**. In a picture the morphisms of **CPM**(**C**) are:



Note in particular that the two copies of each **C**-morphism in these **CPM**(**C**)-morphisms is also present in Dirac's notation when working with density matrices. However, in Dirac notation one considers the pair of a *ket*-vector $|\psi\rangle$ and its adjoint $\langle\psi|$ resulting in the action of an operation being

$$|\psi\rangle\langle\psi| \mapsto f|\psi\rangle\langle\psi|f^\dagger$$

for an ordinary operation, while it becomes

$$|\psi\rangle\langle\psi| \mapsto f(1_C \otimes |\psi\rangle\langle\psi|)f^\dagger$$

for a completely positive map. What we do here is quite similar but now we consider pairs $|\psi\rangle \otimes |\psi\rangle_*$ allowing for more intuitive covariant composition

$$|\psi\rangle \otimes |\psi\rangle_* \mapsto (f \otimes f_*)(|\psi\rangle \otimes |\psi\rangle_*)$$

for an ordinary operation, while it becomes

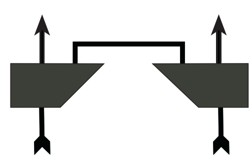
$$|\psi\rangle \otimes |\psi\rangle_* \mapsto (1_B \otimes \eta_C^\dagger \otimes 1_{B^*})(f \otimes f_*)(|\psi\rangle \otimes |\psi\rangle_*)$$

for a completely positive map. The most important benefit of this covariance is 2-dimensional display-ability i.e. it enables graphical calculus.

Formal decoherence. Given a classical object X in a $\dagger$-compact category **C** we define the following morphism

$$\Gamma_X := (1_X \otimes \eta^\dagger \otimes 1_X) \circ (\delta \otimes \delta) \in \mathbf{CPM}(\mathbf{C})(X, X)\,.$$

In a picture that is:

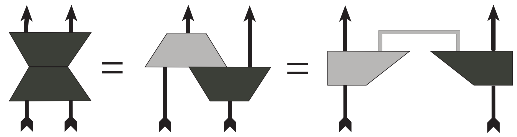


Proposition 6.1 *In $\dagger$-compact category with X a classical object we have*

$$\Gamma_X = \delta \circ \delta^\dagger : X \otimes X \to X \otimes X$$

so in particular is Γ_X idempotent.

Proof: Using the Frobenius identity we have

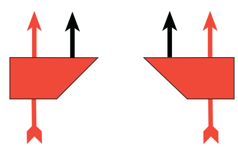


where the highlighted part expresses the use of X-self-adjointness. $\square$

In particular in **FdHilb** we have

$$\begin{tikzcd}
\sum_{ij}\alpha_i\bar{\alpha}_j|i\rangle\otimes|j\rangle_* \arrow[d, "\delta^{(k)}\overset{\top}{\otimes}\delta^{(k)}"'] \arrow[ddr, "\Gamma_{\mathbb{C}^{\oplus n}}"] & \\
\sum_{ij}\alpha_i\bar{\alpha}_j|ii\rangle\otimes|jj\rangle_* \arrow[d, "1_{\mathbb{C}^{\oplus n}}\otimes\eta^{\dagger\top}_{\mathbb{C}^{\oplus n}}\otimes 1_{\mathbb{C}^{\oplus n}}"'] & \\
\sum_{ij}\delta_{ij}\alpha_i\bar{\alpha}_j|i\rangle\otimes|j\rangle_* \arrow[r, equal] & \sum_{i}\alpha_i\bar{\alpha}_i|i\rangle\otimes|i\rangle_*
\end{tikzcd}$$

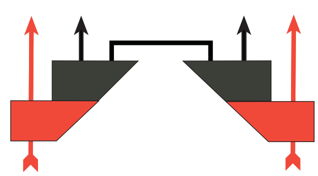
i.e. we obtain the desired effect of elimination of the relative phases. Hence, given a projector-valued spectrum now represented in $\mathbf{CPM}(\mathbf{C})$ through the functor Pure, which depicts as



we obtain a genuine quantum measurement by adjoining Γ_X as in

$$Meas := (1_B\otimes\Gamma_X\otimes 1_B)\circ(\mathcal{M}\otimes\mathcal{M}_*)\,,$$

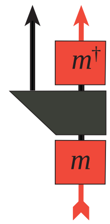
which in a picture becomes:



Demolition measurements. As compared to the type $A\to X\otimes A$ of a non-demolition measurement, a *demolition measurement* has type $A\to X$. We claim that the demolition analogue to a projector-valued spectrum $\mathcal{M}:A\to X\otimes A$ is the adjoint to an *isometry* $m^\dagger:X\to A$, i.e. $m\circ m^\dagger=1_X$ — or equivalently put in our X-jargon, a *normalized* X-*bra*. Indeed, setting

$$\mathcal{M}_m := (1_X\otimes m^\dagger)\circ\delta\circ m : A\to X\otimes A$$

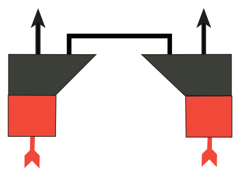
we exactly obtain a projector-valued spectrum since $\mathcal{M}_m$ is trivially X-self-adjoint, and $m \circ m^\dagger = 1_X$ yields X-idempotence. In a picture $\mathcal{M}_m$ is:



The corresponding demolition measurement arises by adjoining Γ_X i.e.

$$DeMeas := \Gamma_X \circ (m \otimes m_*) \,,$$

that is, in a picture:



Such a demolition measurement is *non-degenerate* iff m is unitary.

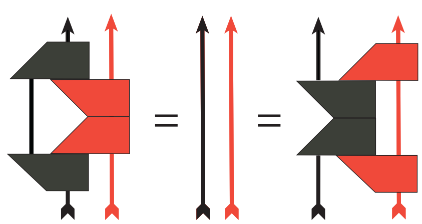
7 Quantum teleportation

The notion of measurements proposed in this paper abstracts over the structure of classical data, and we will show that we can describe and proof correctness of the teleportation protocol without making the classical data structure explicit, nor by relying on the cartesian structure of $\mathbf{C}_\times$.

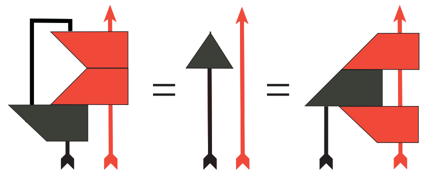
Definition 7.1 Given a classical object X a morphism $\mathcal{U} : X \otimes A \to B$, and at the same time $\mathcal{U}^\dagger$ and $\mathcal{U} \circ \sigma_{X,A}$, are *unitary relative to* X or X*-unitary* iff

$$(1_X \otimes \mathcal{U}) \circ (\delta \otimes 1_A) : X \otimes A \to X \otimes B$$

is unitary in the usual sense i.e. its adjoint is its inverse. In a picture:



A trivial example of such a unitary morphism is $\epsilon \otimes 1_A : X \otimes A \to A$. One easily verifies that the above graphically depicted condition is the same as:

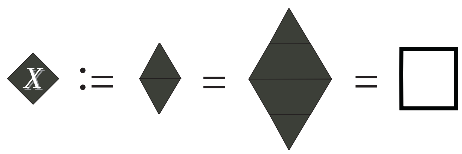


Proposition 7.2 *In* **FdHilb** *morphisms that are* $(\mathbb{C}^{\oplus n}, \delta^{(n)})$*-unitary are in bijective correspondence with* n*-tuples unitary operators of the same type.*

Let the *size of a classical object* be the scalar

$$s_X := \eta_X^\dagger \circ \eta_X = \epsilon_X \circ \epsilon_X^\dagger : \mathrm{I} \to \mathrm{I}$$

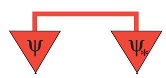
i.e. in a picture:



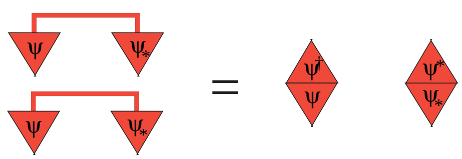
using in the last two steps respectively $\delta^\dagger \circ \delta = 1_X$ and $\eta = \delta \circ \epsilon^\dagger$.

Proposition 7.3 *The positive scalars in the scalar monoid* $\mathbf{C}(\mathrm{I}, \mathrm{I})$*, i.e. those scalars* $s : \mathrm{I} \to \mathrm{I}$ *that can be written as* $s = \psi^\dagger \circ \psi$ *for some* $\psi : \mathrm{I} \to A$*, have self-adjoint square-roots when enbedded in* $\mathbf{CPM}(\mathbf{C})$ *via* Pure*.*

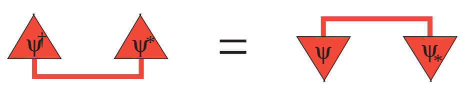
Proof: The image of a positive scalar s under Pure is $s \otimes s_*$. For $t = \eta_{A^*}^\dagger \circ (\psi \otimes \psi_*) \in \mathbf{CPM}(\mathbf{C})(\mathrm{I}, \mathrm{I})$ which we depict in a picture as:



we have $t \circ t = s \otimes s_*$ since



follows from $(f^* \otimes 1_B) \circ \eta_B = (1_A \otimes f) \circ \eta_A$ [2]. Self-adjointness follows from:



Hence $\mathsf{Pure}(s)$ indeed has a scalar in $\mathbf{CPM}(\mathbf{C})(\mathrm{I}, \mathrm{I})$ as a square-root. $\square$

This implies that square-root $\sqrt{s_A} : \mathrm{I} \to \mathrm{I}$ of the *dimension* $s_A := \eta_A^\dagger \circ \eta_A$ of an object A always exist whenever we are within $\mathbf{CPM}(\mathbf{C})$. It can be shown that each $\dagger$-compact category also admits a canonical embedding in another †-compact category in which all scalars have inverses. For scalars

$$s_A \qquad \sqrt{s_A} \qquad \frac{1}{s_A} \qquad \frac{1}{\sqrt{s_A}}$$

respectively we introduce the following graphical notations:

— the reversed symbols representing inverses needn't be confused with the adjoint since these scalar dimensions are always self-adjoint.

Definition 7.4 Let X be a classical object in a †-compact category. A (non-degenerate) *demolition Bell-measurement* is a unitary morphism

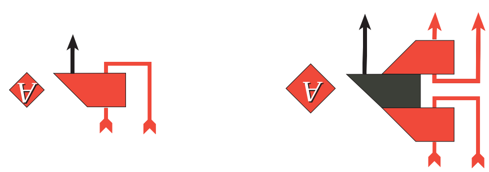
$$DeMeas_{Bell} := \frac{1}{\sqrt{s_A}} \bullet \rho_A^\dagger \circ (1_X \otimes \eta_A^\dagger) \circ (\mathcal{U}^\dagger \otimes 1_A) : A \otimes A \to X$$

which is such that $\mathcal{U} : X \otimes A \to A$ is X-unitary.

The corresponding projector-valued spectrum is

$$\mathcal{M}_{Bell} := (DeMeas_{Bell}^\dagger \otimes 1_X) \circ \delta \circ DeMeas_{Bell} : A \otimes A \to X \otimes A \otimes A\,,$$

from which the corresponding non-demolition Bell-measurement arises by adjoining Γ_X. In a picture the demolition Bell-measurement and corresponding projector-valued spectrum are:



and unitarity of $DeMeas_{Bell}$ is:

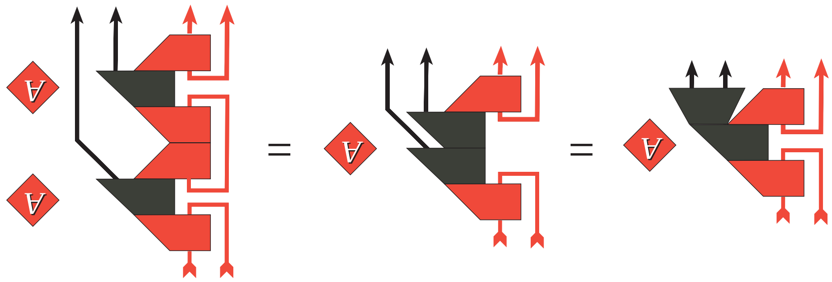
that is, in formulae, respectively,

$$\frac{1}{\sqrt{s_A}} \bullet \mathrm{tr}^A_{X,X}(\mathcal{U}^\dagger \circ \mathcal{U}) = DeMeas_{Bell} \circ DeMeas_{Bell}^\dagger = 1_X \tag{8}$$

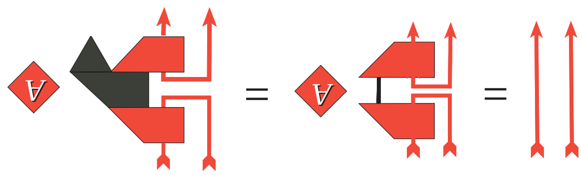
and of course

$$DeMeas_{Bell}^\dagger \circ DeMeas_{Bell} = 1_{A\otimes A}\,. \tag{9}$$

Let us verify that $\mathcal{M}_{Bell}$ is indeed a projector-valued spectrum. Using eq.(8) we obtain X-idempotence:



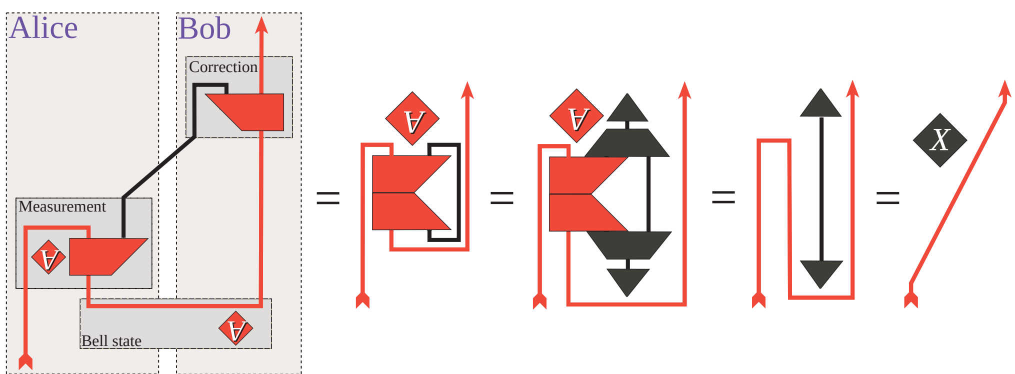
and eq.(9) assures X-completeness:



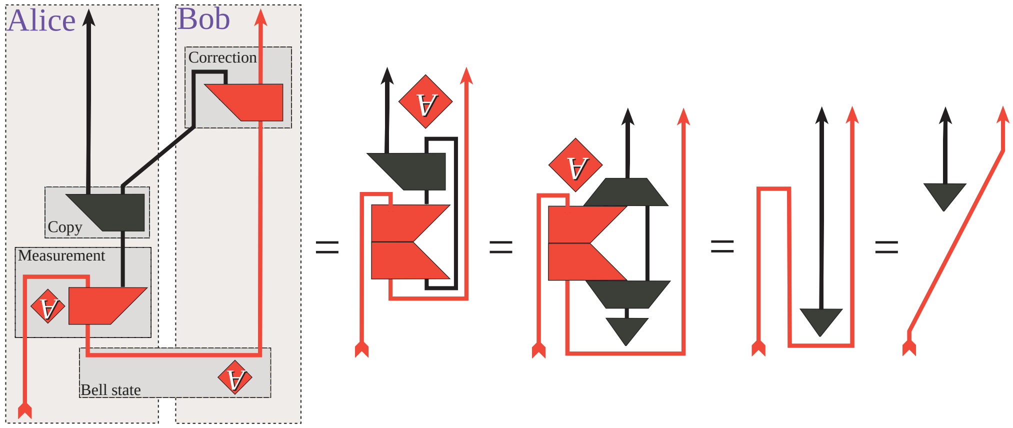
Finally, unitarity of $DeMeas_{Bell}$ yields $X \simeq A \otimes A$, so $DeMeas_{Bell}$ can be conceived as *non-degenerate*. We normalize the *Bell-states* of type A i.e.

$$\frac{1}{\sqrt{s_A}} \bullet \eta_A : \mathrm{I} \to A^* \otimes A\,.$$

Now we will describe the teleportation protocol and prove its correctness. For simplicity we will not explicitly depict Γ_X since it doesn't play an essential role in the topological manipulations of the picture. Here it is:



where the red box in `Measurement` is a unitary morphism $\mathcal{U}_* : A^* \to A^* \otimes X$, hence the red box in `Correction` is the unitary morphism $\mathcal{U} : X \otimes A \to A$, so the bottom red box in the second picture is $\mathcal{U}^* : X \otimes A \to A$, the adjoint to $\mathcal{U}_*$. The presence of the size of X reflects the fact that s_X scenarios have taken place, all leading to the same result. But the reason these scenarios are all equivalent is that we didn't retain the measurement outcomes i.e. the unitary correction 'consumed' them. We could as well have *copied* the result before consuming it, in which case the s_X branches wouldn't be the same, due to the distinct remaining classical data in each of them:



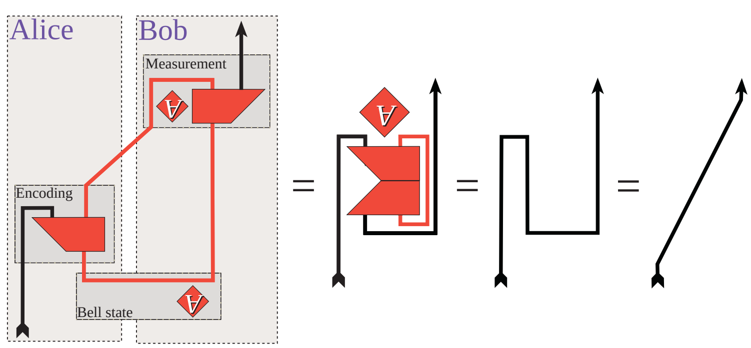
Note that we explicitly used X-self-adjointness. We can of course still choose to delete this data at a later stage using ϵ, nicely illustrating *resource-sensitivity*. Categorically we can fully specify this protocol as[9]

$$\begin{tikzcd}
A \arrow[r, "(1_A\otimes\eta_A)\circ\rho_A"] \arrow[d, "\epsilon^\dagger\otimes 1_A"'] & A\otimes A^*\otimes A \arrow[r, "DeMeas_{Bell}\otimes 1_A"] & X\otimes A \arrow[d, equal] \\
X\otimes A & X\otimes X\otimes A \arrow[l, "1_X\otimes\big((\eta_X\otimes 1_A)\circ(1_X\otimes\mathcal{U}^\dagger)\big)"'] & X\otimes A \arrow[l, "\delta\otimes 1_A"']
\end{tikzcd}$$

The morphism $\epsilon^\dagger\otimes 1_A$ together with commutation of this diagram specifies the intended behavior, i.e. teleporting a state of type A with the creation of classical data as a bi-product, while the other morphisms respectively are: (i) creation of a Bell-state η_A; (ii) a demolition Bell-base measurement $DeMeas_{Bell}$; (iii) copying of classical data using δ; (iv) unitary correction using the X-adjoint to $\mathcal{U}$. The above depicted graphical proof can be converted in an explicit category-theoretic one.

8 Dense coding

We can also give a similar description and proof of dense coding.



[9]The first specification of quantum teleportation as a commutative diagram is in [2].

Note in particular that we rely on a very different property in this derivation than in the derivation of teleportation: here we use (one-sided) unitarity of $DeMeas_{Bell}$ while for teleportation we use X-unitarity of $\mathcal{U}$. Hence it follows that teleportation and dense coding are not as closely related as one usually thinks: they are in fact *axiomatically independent*.

Acknowledgements

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