Programming Research Group

Wronksi brackets and the ferris wheel

Keye Martin

PRG-RR-04-05

Oxford University Computing Laboratory
Wolfson Building, Parks Road, Oxford OX1 3QD

Abstract

We connect the Bayesian order on classical states to a certain Lie algebra on $C^\infty[0,1]$. This special Lie algebra structure, made precise by an idea we introduce called a *Wronski bracket*, suggests new phenomena the Bayesian order naturally models. We then study Wronski brackets on associative algebras, and in the commutative case, discover the beautiful result that they are equivalent to derivations.

1 Introduction

By changing our focus from $x \sqsubseteq y$ to the processes that may cause $x \sqsubseteq y$, here modeled by curves on a domain that connect x to y, we discover the connection between the Bayesian order on classical states and the Lie algebra $C^\infty[0,1]$ whose bracket is the beautiful $[x,y] = \dot{x}y - x\dot{y}$. This we name the *Wronski bracket.*

The Wronski bracket arises in so many different contexts that by combining them in a meaningful way we discover new phenomena naturally modeled by the Bayesian order. One of these, the postdoc ferris wheel, teaches us what the quintessential monotone process in the Bayesian order is like. Seeing the effectiveness and beauty of this bracket causes us to wonder about the origin of its magic, so we press further.

We end up introducing and studying Wronski brackets on associative algebras. In the case of a commutative algebra like $C^\infty[0,1]$, the property that serves to distinguish Wronski brackets is that they satisfy

$$[ax, by] = (ab)[x,y] + [a,b](xy)$$

which turns out to be a stronger form of the Jacobi identity,

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

From an algebraic viewpoint this is interesting because we carry two distinct multiplications, and our axiom serves to relate them. We discover the beautiful one-to-one correspondence between Wronski brackets and derivations, which implies for example that if M is a smooth manifold then a Wronski bracket on $C^\infty(M)$ is exactly a *vector field.*

At the end we go back and explain what all this algebra has to do with domain theory, and suggest some new ideas that may be worth exploring.

2 Differentiable curves

For an integer $n \geq 2$, the *classical states* are

$$\Delta^n := \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = 1 \right\}.$$

A classical state $x \in \Delta^n$ is *pure* when $x_i = 1$ for some $i \in \{1, \ldots, n\}$; we denote such a state by e_i. Pure states $\{e_i\}_i$ are the actual states a system can be in, while general mixed states x and y are epistemic entities. If we know x and by some means determine outcome i is not possible, our knowledge improves to

$$p_i(x) = \frac{1}{1 - x_i}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \in \Delta^n,$$

where $p_i(x)$ is obtained by first removing x_i from x and then renormalizing. The partial mappings which result, $p_i : \Delta^{n+1} \rightharpoonup \Delta^n$ with $\mathrm{dom}(p_i) = \Delta^{n+1} \setminus \{e_i\}$, are called the *Bayesian projections* and lead one to the following relation on classical states.

Definition 2.1 For $x, y \in \Delta^{n+1}$,

$$x \sqsubseteq y \equiv (\forall i)(x, y \in \mathrm{dom}(p_i) \Rightarrow p_i(x) \sqsubseteq p_i(y)).$$

For $x, y \in \Delta^2$,

$$x \sqsubseteq y \equiv (y_1 \leq x_1 \leq 1/2) \text{ or } (1/2 \leq x_1 \leq y_1)\,.$$

The relation $\sqsubseteq$ on Δ^n is called the *Bayesian order.*

The Bayesian order was introduced in [4] where the following is proven:

Theorem 2.2 *$(\Delta^n, \sqsubseteq)$ is a domain with least element $\bot := (1/n, \ldots, 1/n)$ and $\max(\Delta^n) = \{e_i : 1 \leq i \leq n\}$.*

The Bayesian order has a more direct description: The *symmetric formulation* [4]. Let $S(n)$ denote the group of permutations on $\{1, \ldots, n\}$ and

$$\Lambda^n := \{x \in \Delta^n : (\forall i < n)\, x_i \geq x_{i+1}\}$$

denote the collection of *monotone* decreasing classical states.

Theorem 2.3 *For $x, y \in \Delta^n$, we have $x \sqsubseteq y$ iff there is a permutation $\sigma \in S(n)$ such that $x \cdot \sigma, y \cdot \sigma \in \Lambda^n$ and*

$$(x \cdot \sigma)_i (y \cdot \sigma)_{i+1} \leq (x \cdot \sigma)_{i+1} (y \cdot \sigma)_i$$

for all i with $1 \leq i < n$.

Thus, $(\Delta^n, \sqsubseteq)$ can be thought of as $n!$ many copies of the domain $(\Lambda^n, \sqsubseteq)$ identified along their common boundaries, where $(\Lambda^n, \sqsubseteq)$ is

$$x \sqsubseteq y \equiv (\forall i < n)\, x_i y_{i+1} \leq x_{i+1} y_i.$$

It should be remarked though that the problems of ordering Λ^n and Δ^n are very different, with the latter being far more challenging [7] if one also wants to consider quantum mixed states.

The first thing we are going to prove is that movement in the Bayesian order implies differentiability (a.e.). A curve $\pi : [0,1] \to \Lambda^n$ can be written as $\pi = (\pi_1, \ldots, \pi_n)$ where the functions $\pi_i : [0,1] \to [0,1]$ satisfy $\pi_i \geq \pi_{i+1}$ for $i < n$ and

$$\sum_{i=1}^{n} \pi_i(t) = 1$$

for all $t \in [0,1]$.

Lemma 2.4 *If $x \sqsubseteq y$ in Λ^n, then*

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$$

for all $1 \leq k \leq n$.

Proof. Because $x \sqsubseteq y$ in Λ^n, there is an integer $1 < m \leq n$ such that $x_i \leq y_i$ for $i < m$ and $x_i \geq y_i$ for $i \geq m$. Then for $k < m$ the claim is obvious, while for $k \geq m$ we have

$$\sum_{i=1}^{k} x_i = 1 - \sum_{i=k+1}^{n} x_i \leq 1 - \sum_{i=k+1}^{n} y_i = \sum_{i=1}^{k} y_i,$$

using $x_i \geq y_i$ for $i \geq m$. $\square$

The property of the Bayesian order expressed by Lemma 2.4 defines an order $\leq$ on Λ^n in its own right,

$$x \leq y \equiv (\forall k \in \{1, \ldots, n\}) \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$$

called *Majorization* [2][8][9]. It too yields a domain $(\Lambda^n, \leq)$ as noted in [6]; however, it has no natural extension to all of Δ^n. Despite this sharp difference, a monotone curve in either order has a decent amount of analytic structure.

Proposition 2.5 *If $\pi = (\pi_1, \ldots, \pi_n) : [0,1] \to (\Lambda^n, \leq)$ is a monotone curve, then each π_i is of bounded variation. Thus,*

(i) *The curve π has a 'length.'*

(ii) *It is continuous except on a countable set.*

(iii) *It is differentiable except on a set of measure zero.*

Proof. Define $f_k : [0,1] \to [0,1]$ by

$$f_k(t) = \sum_{i=1}^{k} \pi_i(t)$$

for $k \geq 0$. By the definition of $\leq$, the f_k are monotone increasing functions, where we note that $f_k = 0$ for $k = 0$. Since $\pi_i = f_i - f_{i-1}$, the map π_i is the difference of monotone increasing maps and thus of bounded variation. This establishes (i), (ii) and (iii). $\square$

From the qualitative *follows* differentiability. We study differentiable curves, which as we shall see, will take us naturally to the Wronski bracket.

3 The Wronski bracket

Let X denote the set of differentiable real valued functions defined on $[0,1]$. For $x, y \in X$,

$$[x, y] := \dot{x}y - \dot{y}x.$$

This is called the *Wronski bracket.*

Lemma 3.1 *Let* $x, y, z \in X$. *Then*

(i) $y[x, z] = z[x, y] + x[y, z]$,

(ii) $\dot{y}[x, z] = \dot{z}[x, y] + \dot{x}[y, z]$,

(iii) $y[x \dot{,} z] = z[x \dot{,} y] + x[y \dot{,} z]$.

In addition, $[x, 1] = \dot{x}$.

Proof. (iii) Use the identity $[x \dot{,} y] = \ddot{x}y - \ddot{y}x$. $\square$

The perfection of form noted in the last lemma suggests that X with $[\,,\,]$ has more structure than the usual Lie algebra. Nevertheless:

Lemma 3.2 $[\,,\,]$ *is a Lie bracket.*

(i) $[\,,\,]$ *is bilinear,*

(ii) $[x, y] = -[y, x]$,

(iii) *The Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

holds.

Thus, the smooth maps in X *with* $[\cdot, \cdot]$ *form a Lie algebra.*

Proof. (iii) What is beautiful here is that the Jacobi identity follows from (ii) and (iii) of Lemma 3.1. □

In the next proof, most details center on resolving the cases of boundary points: This is far simpler if the functions have *continuous* derivatives; then we can just take limits.

Proposition 3.3 *Let $f, g, h : [0,1] \to [0,\infty)$ be differentiable maps with $f \geq g \geq h$.*

(i) *If $[f,g] \geq 0$ and $[g,h] \geq 0$, then $[f,h] \geq 0$.*

(ii) *If $[f,g] = 0$ and $[g,h] = 0$, then $[f,h] = 0$.*

Proof. (i) Let $s \in [0,1]$. If $g(s) > 0$, then the equation

$$g[f,h] = h[f,g] + f[g,h]$$

of Lemma 3.1(i) applies, which immediately yields $[f,h](s) \geq 0$. Thus, we need only verify the assertion for $g(s) = 0$.

But if $g(s) = 0$, then we must also have $h(s) = 0$. For $s \in (0,1)$, we have $\dot{h}(s) = 0$, which immediately gives $[f,h](s) = 0$. Likewise, $f(s) = 0$ also gives $[f,h](s) = 0$, so we may assume that (i) $s \in \{0,1\}$, (ii) $f(s) > 0$, and (iii) $g(s) = h(s) = 0$. If $s = 1$, then

$$\dot{h}(s) = \lim_{t \to 1^-} \frac{h(t)}{t-1} \leq 0,$$

so $[f,h](s) = -\dot{h}(s)f(s) \geq 0$. If $s = 0$, then $\dot{h}(s) \geq 0$. Since $g(s) = 0$, $[f,g](s) = -\dot{g}(s)f(s) \geq 0$, so $f(s) > 0$ yields $\dot{g}(s) \leq 0$. But $g \geq h$ gives

$$\dot{h}(s) = \lim_{t \to 0^+} \frac{h(t)}{t} \leq \lim_{t \to 0^+} \frac{g(t)}{t} = \dot{g}(s) \leq 0,$$

and hence $\dot{h}(s) = 0$. Then $[f,h](s) = 0$, which finishes the proof.

(ii) Applying (i) to $f \geq g \geq h$ shows $[f,h] \geq 0$. But we can also apply (i) to $f \geq g \geq g - h$ since $[g, g-h] = 0$, which gives

$$[f, g-h] \geq 0$$

and that is exactly $[f,h] \leq 0$. □

The *Wronski bracket* $[x,y] = \dot{x}y - \dot{y}x$ determines when two solutions of a second order equation are independent, the area swept out by a curve in the plane, angular momentum in mechanics, its sign classifies the direction of motion as being either clockwise or counterclockwise, and it arises in model theory as the only known example of a 'stable' infinite dimensional Lie algebra. It is also intimately connected to the Bayesian order.

Theorem 3.4 *A differentiable curve $\pi = (\pi_1, \ldots, \pi_n) : [0,1] \to (\Lambda^n, \sqsubseteq)$ is increasing iff $[\pi_i, \pi_{i+1}] \geq 0$ for all $i < n$.*

Proof. (i) $\Rightarrow$ (ii): To prove $[\pi_i, \pi_{i+1}](s) \geq 0$, we can assume $\pi_i(s) > 0$ (otherwise, we trivially have $[\pi_i, \pi_{i+1}](s) = 0$). By the continuity of π_i, there must be an open interval $(a, b) \subseteq \mathbb{R}$ containing s such that $\pi_i > 0$ on $U := (a, b) \cap [0, 1]$. By the monotonicity of π, the map π_{i+1}/π_i is monotone decreasing on U. Since it is also differentiable, its derivative cannot be positive. But

$$\dot{\left(\frac{\pi_{i+1}}{\pi_i}\right)} = \frac{[\pi_{i+1}, \pi_i]}{\pi_i^2} \leq 0,$$

which makes it clear that $[\pi_i, \pi_{i+1}] = -[\pi_{i+1}, \pi_i] \geq 0$.

(ii) $\Rightarrow$ (i): We need to show that

$$\pi_i(s)\pi_{i+1}(t) \leq \pi_{i+1}(s)\pi_i(t)$$

whenever $s \leq t$ and $1 \leq i < n$. We can assume $s < t$. In the simple case, $\pi_i > 0$ on $[s, t]$, and then as before

$$\dot{\left(\frac{\pi_{i+1}}{\pi_i}\right)} = \frac{[\pi_{i+1}, \pi_i]}{\pi_i^2} = -\frac{[\pi_i, \pi_{i+1}]}{\pi_i^2} \leq 0$$

which means that π_{i+1}/π_i is monotone decreasing on $[s, t]$. In particular, $(\pi_{i+1}/\pi_i)(s) \geq (\pi_{i+1}/\pi_i)(t)$, which finishes this case.

Suppose now that $\pi_i(c) = 0$ for some $c \in [s, t]$. We claim that $\pi_i(t) = 0$. For the proof, a simple induction based on Prop. 3.3 gives $[\pi_i, \pi_j] \geq 0$ for $1 \leq i \leq j \leq n$. This means that $f_k : [0, 1] \to [0, 1]$ given by

$$f_k = \sum_{i=1}^{k} \pi_i$$

is monotone increasing since

$$\dot{f}_k = [f_k, 1] = \left[f_k, f_k + \sum_{j>k} \pi_j \right] = 0 + \sum_{j>k} [f_k, \pi_j] = \sum_{j>k} \sum_{i=1}^{k} [\pi_i, \pi_j] \geq 0.$$

Because $\pi(c)$ is a monotone state, $\pi_i(c) = 0$ implies that $f_i(c) = 1$. But if $\pi_i(t) > 0$, then the monotonicity of f_i gives $1 = f_i(c) < f_i(t)$, which contradicts the fact that $\pi(t)$ is a classical state. Then $\pi_i(t) = 0$, so $\pi_i \geq \pi_{i+1}$ gives $\pi_{i+1}(t) = 0$, which proves the desired inequality holds. $\square$

The previously mentioned uses of the Wronski bracket provide many new ways to explain the Bayesian order. We can also characterize it analytic terms.

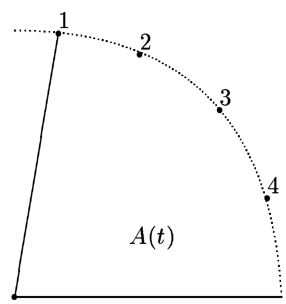
Corollary 3.5 *For $x, y \in \Lambda^n$, $x \sqsubseteq y$ iff there is a differentiable curve $\pi : [0, 1] \to \Lambda^n$ from $x = \pi(0)$ to $y = \pi(1)$ with $[\pi_i, \pi_{i+1}] \geq 0$ for all $i < n$.*

Proof. Given $x \sqsubseteq y$, let $\pi : [0,1] \to \Lambda^n$ be the straight line path from x to y. The other direction follows from the last theorem. $\square$

An interesting point is that for a differentiable curve $\pi : [0,1] \to \Lambda^n$, we have $[\pi_i, \pi_j]$ constant for all $1 \leq i, j \leq n$ iff $\pi(t) = (1-t) \cdot \pi(0) + t \cdot \pi(1)$. Finally, all results in this section apply equally well to curves $\pi : [a,b] \to \Lambda^n$. We simply took $a = 0$ and $b = 1$ because we like them.

4 The postdoc ferris wheel

The connection between the Bayesian order and the Wronski bracket suggests the quintessential example of a monotone process. Imagine four postdocs on a ferris wheel at the start of *a new revolution*:



The question we want to ask is

What is the probability that postdoc i wins?

where by 'wins' we mean that postdoc i completes the revolution before all others. This seems like a strange and uninteresting question until we learn more about the postdocs:

Postdoc 1: Believes there's only one place to go from the top.

Postdoc 2: Just wants to enjoy the ride while it lasts.

Postdoc 3: Thinks that every *revolution* needs at least one martyr.

Postdoc 4: His funding was just renewed for *two* more months – wow!

So for various reasons, it is possible that some of the postdocs may jump from the ferris wheel before the current revolution is completed. How then can we calculate the probability that postdoc i wins?

First, let's calculate the probability that one of them jumps. The only information we have is that based on observation (literally "watching" the wheel move), so the probability that i jumps is determined by the percentage of total area swept out by the line joining the origin to i:

$$P(i \text{ jumps})(t) = 1 - a_i(t) := 1 - \frac{A(t - (i-1)\varepsilon)}{A}$$

where A is the total area swept out by one revolution of the wheel, $A(t)$ is the area swept out by postdoc 1 after t units,

$$A(t) = \frac{1}{2}\int_0^t [y, x]\, ds,$$

and the coordinates of postdoc 1 are $(x(t), y(t))$. Notice that we take the coordinates of postdoc i to be $(x(t-(i-1)\varepsilon), y(t-(i-1)\varepsilon))$ where $0 < \varepsilon < 1$ is a constant. Using that the probability of Y given X is

$$P(Y|X) = \frac{P(Y\&X)}{P(X)},$$

we get

$$\begin{aligned} P(i \text{ wins}) &= P(i \text{ does not jump } \& \text{ all } j \text{ jump for } j < i) \\ &= P(i \text{ does not jump } \mid \text{ all } j < i \text{ jump}) \cdot P(\text{all } j < i \text{ jump}) \\ &= a_i \cdot P(\text{all } j < i \text{ jump}) \\ &= a_i \cdot \prod_{j=1}^{i-1} (1 - a_j) \end{aligned}$$

assuming for the last equality that the postdocs jump independent of one another. These probabilities, when normalized by $P := \sum P(i \text{ wins})$, define a curve π into Λ^4 given by $\pi_i = P(i \text{ wins})/P$. After enough time elapses, π is *monotone*!

To see this, suppose that the coordinates of postdoc 1 have been parameterized over $[0, 1]$ as

$$x(t) = r\cos(2\pi t) \ : \ y(t) = r\sin(2\pi t)$$

where r is the radius of the ferris wheel. Then π is now defined on $[3\varepsilon, 1]$.

Lemma 4.1 *The curve π is monotone increasing on $[2\varepsilon + \sqrt{\varepsilon}, 1]$.*

Proof. Using the identity $[ax, ay] = a^2[x, y]$, bilinearity and Theorem 3.4, π will be monotone if $[a_i, a_{i+1}] + a_i^2 \dot{a}_{i+1} \geq 0$ for all i. Since each a_i is a translation of a_1, we first consider the case $i = 1$. We get

$$a_1(t) = t \ \ \& \ \ a_2(t) = t - \varepsilon \ \ \& \ \ A = \pi r^2$$

so $[a_1, a_2] + a_1^2 \dot{a}_2 \geq 0$ iff $t \geq \sqrt{\varepsilon}$. Applying this result to a_i and a_{i+1}, we get $[a_i, a_{i+1}] + a_i^2 \dot{a}_{i+1} \geq 0$ iff $t \geq (i-1)\varepsilon + \sqrt{\varepsilon}$. Setting $i = 3$ finishes the proof. □

Now suppose the wheel is turning when at some point $t \in [2\varepsilon + \sqrt{\varepsilon}, 1]$, postdoc i jumps. At this instant, our knowledge of who will win becomes

$$p_i \circ \pi : [t, 1] \to \Lambda^3$$

where

$$p_i(x) = \frac{1}{1-x_i}(x_1, \ldots, \hat{x_i}, \ldots, x_4) \in \Delta^3$$

is a Bayesian projection that first removes x_i from x and then renormalizes. This new curve $p_i \circ \pi$ *is also monotone increasing* because π is monotone increasing in the *Bayesian order.* Why doesn't π increase over the entire interval $[3\varepsilon, 1]$?

We don't experience an increase in information until $t = 2\varepsilon + \sqrt{\varepsilon}$, which means that if you were watching the ferris wheel, then from the moment that postdoc 4 crossed the x-axis ($t = 3\varepsilon$), you would have to wait

$$(2\varepsilon + \sqrt{\varepsilon}) - 3\varepsilon = \sqrt{\varepsilon} - \varepsilon$$

units of time until you experienced an increase of information (according to the Bayesian order). This it seems should be regarded as "the amount of time required for information to be converted into credible belief," on the grounds that it takes time for credible belief to be established.

Another example in the same spirit as the ferris wheel would be the state of a queue of processes waiting to exploit some resource. At any moment, a process may get tired of waiting and elect to remove itself from the queue. For instance, if a number of users are waiting to download a large file. What is important, though, is to specify that a user has no knowledge of their position in the queue – otherwise, taking the probabilities for jumping to be independent may not make much sense.

The Wronski bracket arises in many different contexts and combining these (conservation of angular momentum, the Bayesian order) has led to new phenomena modeled by the Bayesian order, such as a ferris wheel of maladjusted postdocs, or a queue of impatient processes. What we want to do now is identify some property it has that can help explain why it is so 'special.' A hint is provided by the formula we encountered in the study of the ferris wheel:

$$[ax, ay] = a^2[x, y].$$

Pragmatically this tells us that renormalization does not affect monotonicity in the Bayesian order. This is a special case of a more general property

$$[ax, by] = (ab)[x, y] + [a, b](xy)$$

which serves to *characterize* the Wronski bracket.

5 Wronski brackets and derivations

An *algebra* over the field of real numbers is a real vector space $(A, +)$ with a bilinear multiplication $\cdot : A^2 \to A$. An algebra is associative if $\cdot$ is associative, commutative if $\cdot$ is commutative and has an identity if $\cdot$ has an identity. Let A be an associative algebra with identity. Its commutator is

$$\langle x, y \rangle := xy - yx$$

for $x, y \in A$.

Definition 5.1 A *Wronski bracket* is a bilinear map $[\,,\,] : A^2 \to A$ that is antisymmetric and satisfies

$$[ax, by] = (ab)[x, y] + [a, b](xy) + a\langle b, x\rangle[y, 1] + [1, a]\langle b, x\rangle y$$

for all $a, b, x, y \in A$.

An equivalent definition is to replace antisymmetry above by the axiom $[x, 1] + [1, x] = 0$.

Definition 5.2 A *derivation* is a linear map $d : A \to A$ such that

$$d(xy) = dx \cdot y + x \cdot dy$$

for all $x, y \in A$.

Proposition 5.3 *For each derivation d with $\langle dx, y\rangle = \langle x, dy\rangle$,*

$$[x, y] := dx \cdot y - x \cdot dy$$

is a Wronski Bracket.

Proof. It is simple to see that $[\,,\,]$ is linear in its first argument. It is antisymmetric because

$$[x, y] = -[y, x] \iff \langle dx, y\rangle = \langle x, dy\rangle.$$

These two together imply linearity in the second argument – bilinearity. For the last axiom,

$$\begin{aligned}
[ax, by] &= d(ax)(by) - (ax)d(by) \\
&= a[x, b]y + (da \cdot x)(by) - (ax)(b \cdot dy) \\
&= a(-[b, x])y + (da \cdot x)(by) - (ax)(b \cdot dy) \\
&= (ab \cdot dx \cdot y - axb \cdot dy) + (da \cdot xby - a \cdot db \cdot xy) \\
&= (ab[x, y] + a(bx - xb)dy) + (da(xb - bx)y + [a, b](xy)) \\
&= ab[x, y] + a\langle b, x\rangle[y, 1] + [1, a]\langle b, x\rangle y + [a, b](xy)
\end{aligned}$$

which finishes the proof. □

Proposition 5.4 *For each Wronski bracket $[\,,\,]$*

$$dx := [x, 1]$$

is a derivation with $\langle dx, y\rangle = \langle x, dy\rangle$ and

$$[x, y] = dx \cdot y - x \cdot dy$$

for all $x, y \in A$.

Proof. For elements x, y we get

$$\begin{aligned} d(xy) &= [xy, 1 \cdot 1] \\ &= (x \cdot 1)[y, 1] + [x, 1](y \cdot 1) + 0 + 0 \\ &= x \cdot dy + dx \cdot y \end{aligned}$$

To show that d determines the bracket, the trick is to write

$$\begin{aligned} [x, y] &= [x \cdot 1, 1 \cdot y] \\ &= (x \cdot 1)[1, y] + [x, 1](1 \cdot y) + 0 + 0 \\ &= dx \cdot y - x \cdot dy \end{aligned}$$

Finally, the explicit expression for the bracket makes it clear that its antisymmetry is equivalent to $\langle dx, y \rangle = \langle x, dy \rangle$. □

So there is a map from derivations to Wronski brackets $d \mapsto [\,,\,]_d$ and another from Wronski brackets to derivations $[\,,\,] \mapsto d_{[\,,\,]}$.

Theorem 5.5 *There is a one-to-one correspondence between Wronski brackets and derivations $d : A \to A$ that satisfy $\langle dx, y \rangle = \langle x, dy \rangle$.*

For the case we are most interested in, let A be a commutative algebra with identity. Because the commutator is now identically zero, the axiom for the Wronski bracket simplifies to

$$[ax, by] = (ab)[x, y] + [a, b](xy)$$

and $\langle dx, y \rangle = \langle x, dy \rangle$ holds for all derivations $d : A \to A$. The axiom for Wronski brackets can be used to derive the equations shown valid in the case $A = C^\infty[0, 1]$, including the Jacobi identity.

Definition 5.6 A *Lie bracket* is a bilinear, antisymmetric $[\,,\,] : A^2 \to A$ with

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in A$.

Given a Wronski bracket $[\,,\,]$ on A, we write

$$\dot{x} := [x, 1]$$

for its associated derivation. Then

$$[x, y] = \dot{x}y - x\dot{y}$$

for all $x, y \in A$.

Proposition 5.7 *A Wronski bracket* $[\,,\,]$ *satisfies*

(i) $y[x, z] = z[x, y] + x[y, z]$

(ii) $\dot{y}[x, z] = \dot{z}[x, y] + \dot{x}[y, z]$

(iii) $y\dot{[x, z]} = z\dot{[x, y]} + x\dot{[y, z]}$

Thus, a Wronski bracket is a Lie bracket.

Proof. We use

$$\dot{[x, y]} = \ddot{x}y - \ddot{y}x$$

to prove the first three. For the Jacobi identity, note

$$\begin{aligned}
[x, [y, z]] &= \dot{x}[y, z] - x\dot{[y, z]} \\
[y, [z, x]] &= \dot{y}[z, x] - y\dot{[z, x]} \\
[z, [x, y]] &= \dot{z}[x, y] - z\dot{[x, y]}
\end{aligned}$$

When we add these three together, the left terms on the right side sum to zero using (ii), while the right terms on the right side give zero by (iii). □

An important point: It was never clear before why the Lie structure of the Wronski bracket mattered. From the present point of view, the Jacobi identity is a generalization of the Leibniz identity (the product rule for derivatives). To the best of our knowledge, this connection between the two has not been made.

Example 5.8 *Vector fields.* If M is a smooth manifold, a vector field is a derivation on the algebra of smooth real-valued functions $C^\infty(M)$. Thus, Wronski brackets on $C^\infty(M)$ are precisely vector fields.

A natural question is whether there is a connection between Wronski brackets and the crucial *Poisson brackets*: Lie brackets $[,]$ with the property that each $[x, \cdot]$ defines a derivation, for $x \in A$. The Wronski bracket $[x, y] = \dot{x}y - x\dot{y}$ is not a Poisson bracket, so there does not seem to be any connection between these two ideas.

6 An algebraic formulation of monotone process

It may seem that we have wandered off course pretty far from the domain of classical states, so we should point out before ending that the extra structure in an algebra combined with what we have learned about Wronski brackets allows one to adopt an abstract view of the Bayesian order that may be applicable in other settings. First, if A is an algebra, we can define the *spectrum* of $x \in A$ to be

$$\sigma x := \{\alpha \in \mathbb{R} : x - \alpha \cdot 1 \text{ has no multiplicative inverse}\}$$

and then say $x \geq 0$ iff $\sigma x \subseteq [0, \infty)$. Using this, we can define a relation

$$x \leq y \equiv (y - x) \geq 0.$$

In the case $A = C^\infty(M)$, we will get $\sigma x = x(M)$, and $\leq$ is the usual pointwise order on functions. Given a commutative algebra A and a derivation $d : A \to A$, we take $\leq$ and call a vector $v \in A^n$ with $v_i \geq v_{i+1}$ *monotone* if

$$(\forall i)\, [v_i, v_{i+1}] \geq 0$$

where $[,]$ is the canonical Wronski bracket associated to d. We can also require $\sum v_i = 1$ in general, though in some cases this may be difficult to justify (perhaps something like $\sum dv_i = 0$ is better). In the case $A = C^\infty[0, 1]$, a monotone $v \in V^n$ with $\sum v_i = 1$ is then a monotone curve in the Bayesian order. It is possible to distinguish states from curves in this way as well, by calling $v \in A^n$ a *state* if $dv_i = 0$. Properties like those in Prop. 3.3 are probably the kind one needs to develop these notions more fully.

One thing ideas along this line seem to offer is a completely different way of thinking about (the Bayesian) order, an *implicit* description of it. Instead of $x \sqsubseteq y$, we characterize events that begin with x and end at y, i.e., *processes* that cause a change of state from x to y. In some cases, it may not be necessary to know x and y, but only processes which connect them.

7 Closing remarks

Is every Wronski bracket a Lie bracket, in general? This might be a good way to test a noncommutative definition of Wronski bracket. It may be possible to obtain more pleasing results in the noncommutative case, but in the case we are most interested in here (the commutative one), we do not believe that it is. It would be good to consider an algebra of operators on a Hilbert space.

A more realistic model of the ferris wheel might assume the postdocs are well-adjusted and instead that there is a person nearby with a control panel having buttons labelled 1, 2, 3, 4. The person, called "professor," has the option of pressing button i, and if he does, this results in postdoc i being immediately thrown from the ferris wheel. Unfortunately, the present author was born without the ability to take this model seriously, so we have assumed that the postdocs are governed by free will.

We don't have to study maladjusted postdocs on ferris wheels of course. It could be a ferris wheel of unhappy professors concerned that one of their favorite postdocs is going to commit academic suicide before realizing his true potential. But, regardless of the choice, the crucial point should always be not to bore the reader (or the writer) to the desperate point of sleep. Those who have trouble keeping us awake will not like this.

References

[1] S. Abramsky and A. Jung. *Domain theory.* In S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, editors, Handbook of Logic in Computer Science, vol. III. Oxford University Press, 1994.

[2] P. M. Alberti and A. Uhlmann. *Stochasticity and partial order: doubly stochastic maps and unitary mixing.* Dordrecht, Boston, 1982.

[3] G. Birkhoff and J. von Neumann. *The logic of quantum mechanics.* Annals of Mathematics, **37**, 823–843, 1936.

[4] B. Coecke and K. Martin. *A partial order on classical and quantum states.* 2002. Abstract at `http://web.comlab.ox.ac.uk/oucl/work/keye.martin`

[5] L. K. Grover. *Quantum mechanics helps in searching for a needle in a haystack.* Physical Review Letters, **78**:325, 1997.

[6] K. Martin. *Entropy as a fixed point.* Oxford University Computing Laboratory, Research Report PRG-RR-03-05, February 2003, `http://web.comlab.ox.ac.uk/oucl/work/keye.martin`

[7] K. Martin. *Epistemic motion in quantum searching.* Oxford University Computing Laboratory, Research Report PRG-RR-03-06, March 2003, `http://web.comlab.ox.ac.uk/oucl/work/keye.martin`

[8] A. W. Marshall and I. Olkin. *Inequalities: Theory of majorization and its applications.* Academic Press Inc., 1979.

[9] R. F. Muirhead. *Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters.* Proc. Edinburgh Math. Soc., 21:144-157, 1903.