DECORATED COSPANS

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Abstract. Let $\mathcal{D}$ be a category with finite colimits, writing its coproduct $+$, and let $(\mathcal{C}, \otimes)$ be a monoidal category. We describe a method of producing a dagger compact category from a lax monoidal functor $F : (\mathcal{D}, +) \to (\mathcal{C}, \otimes)$, and of producing a strict monoidal dagger functor between such categories from a monoidal natural transformation between such functors. The objects of these categories, our so-called 'decorated cospan categories', are simply the objects of $\mathcal{D}$, while the morphisms are pairs comprising a cospan $X \to N \leftarrow Y$ in $\mathcal{D}$ together with an element $1 \to FN$ in $\mathcal{C}$.

1. Introduction

In this article we detail a method for developing composition rules for elements in a monoidal category. Roughly speaking, these composition rules are constructed by specifying an appropriate functor F from a category with finite colimits. This allows us to use cospans $X \to N \leftarrow Y$ in the domain category to equip an element of the image FN of its apex with an 'input', X, and 'output', Y; put another way, we may 'decorate' the apex of the cospan with an element of its image under this functor. The end result is that we can produce monoidal categories from monoidal functors and, in addition, monoidal functors from monoidal natural transformations.

Beyond inherent category theoretic interest, the motivation for such a method lies in developing compositional accounts of semantics associated to topological diagrams. While this has long been a technique associated with topological quantum field theory, dating back to [Atiyah 1988], it has recently had significant influence in the nascent field of categorical network theory, with application to automata and computation [Katis et al 2000, Spivak 2013], electrical circuits [Baez–Fong 2015], signal flow diagrams [Bonchi et al 2014, Baez–Erbele 2015], and Markov processes [Baez–Pollard 2015, Albasini et al, 2011], among others.

It has been recognised for some time that spans and cospans provide an intuitive framework for composing network diagrams [Katis et al 1997], and the material we develop here is a variant on this theme. In the case of finite graphs, the intuition reflected is this: given two graphs, we may construct a third by gluing chosen vertices of the first with chosen vertices of the second. It is our goal in this article to view this process as composition of morphisms in a category, in a way that also facilitates the construction of

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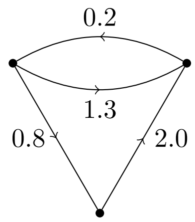
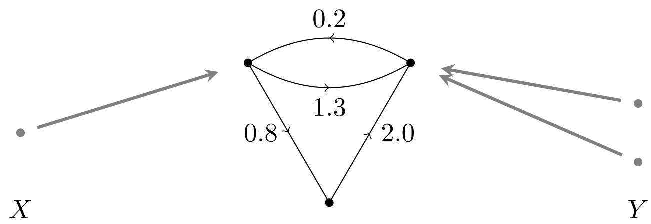
a composition rule for any semantics associated to the diagrams, and a functor between these two resulting categories.

To see how this works, let us start with some graph.

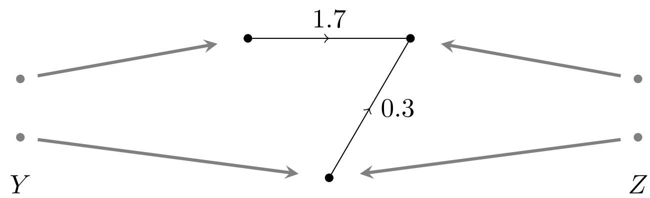


We shall work with labelled, directed graphs, as the additional data helps highlight the relationships between diagrams. Now, for this graph to be a morphism, we must equip it with some notion of input and output. We do this by marking vertices with functions from finite sets:

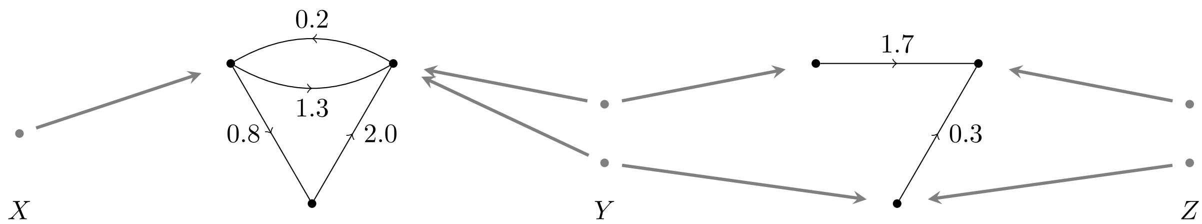


Here the finite sets X, Y, and N comprise one, two, and three elements respectively, drawn as points, and the values of the functions $X \to N$ and $Y \to N$ are indicated by the grey arrows. This forms a cospan in the category of finite sets, one decorated by our given graph.

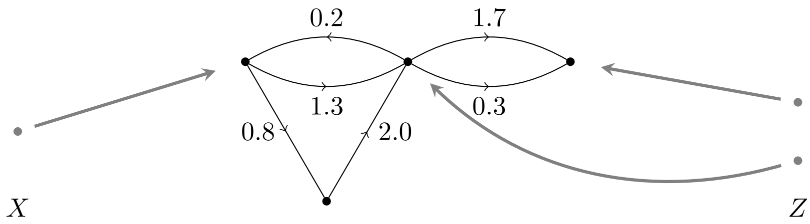
Given another such decorated cospan with input set equal to the output of the above cospan



composition involves gluing the graphs along the identifications



specified by the shared foot of the two cospans. This results in the decorated cospan



The decorated cospans framework generalises this intuitive construction.

More precisely: note that in the case of graphs, given a finite set N, we may talk of the collection of graphs that have N as their set of vertices, and together with a notion of pushforward of graph this gives a lax monoidal functor from $(\mathrm{FinSet}, +)$ to $(\mathrm{Set}, \times)$.[1] Taking any lax monoidal functor $(F, \varphi) : (\mathcal{D}, +) \to (\mathcal{C}, \otimes)$ with $\mathcal{D}$ having finite colimits and coproduct written $+$, the decorated cospan category associated to F has as objects the objects of $\mathcal{D}$, and as morphisms pairs comprising a cospan in $\mathcal{D}$ together with some morphism $1 \to FN$, where 1 is the unit in $(\mathcal{C}, \otimes)$ and N is the apex of the cospan. In the case of our graph functor, this additional data is equivalent to equipping the apex N of the cospan with a graph. We thus think of our morphisms as having two distinct parts: an instance of our chosen structure on the apex, and a cospan describing interfaces to this structure. Our first theorem is that we may further give this data a composition rule, monoidal product, and dagger functor in a way that results in a dagger compact category.

Suppose now we have two such lax monoidal functors; we then have two such decorated cospan categories. Our second theorem is that, given also a monoidal natural transformation between these functors, we may construct a strict monoidal dagger functor between their corresponding decorated cospan categories. These natural transformations can often be specified by some semantics associated to our topological diagrams. A trivial case of such is assigning to a finite graph its the number of vertices, but richer examples abound, including assigning to a directed graph with edges labelled by rates its depicted Markov process, or assigning to an electrical circuit diagram the current–voltage relationship such a circuit would impose.

An advantage of the decorated cospan framework is that the resulting categories are dagger compact, and the resulting functors respect this structure. Dagger compact categories themselves have a rich diagrammatic nature, as exposited by [Selinger 2011], and as has been exploited in graphical calculi for categorical quantum mechanics [Abramsky–Coecke, 2004]. In cases when our decorated cospan categories are inspired by diagrammatic applications, the dagger compact structure provides language to describe natural operations on our topological diagrams, such as juxtaposing, rotating, and reflecting them.

We also note that, following strictification, decorated cospans provide a framework for constructing PROPs, or one-sorted symmetric monoidal theories, and their algebras.

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[1] Here $(\mathrm{FinSet}, +)$ is the monoidal category of finite sets and functions with coproduct as monoidal product, and $(\mathrm{Set}, \times)$ is the category of sets and functions with product as monoidal product.

The structure of this paper is straightforward: in the following section we review some basic background material, which then allows us to give the constructions of decorated cospan categories and their functors in Sections 3 and 4 respectively. We then explicate these definitions through some examples in Section 5, and prove they indeed have the properties we claim in Section 6.

2. Background

2.1. Cospan categories. Recall that a **cospan** from X to Y in a category $\mathcal{C}$ is an object C in $\mathcal{C}$ with a pair of morphisms $f : X \to C$, $g : Y \to C$:

$$\begin{tikzcd} & C & \\ X \arrow[ur, "f"] & & Y. \arrow[ul, "g"'] \end{tikzcd}$$

We shall refer to X and Y as the **feet**, and C as the **apex** of the cospan. When such pushouts exist, cospans may be composed using the pushout from the common foot: given cospans $X \xrightarrow{f} C \xleftarrow{g} Y$ from X to Y and $Y \xrightarrow{f'} C' \xleftarrow{g'} Z$ from Y to Z, their composite cospan is $X \xrightarrow{i \circ f} P \xleftarrow{i' \circ g'} Z$ where P, $i : C \to P$, and $i' : C' \to P$ form the top half of the pushout square:

$$\begin{tikzcd} & & P & & \\ & C \arrow[ur, "i"] & & C' \arrow[ul, "i'"'] & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "g"'] \arrow[ur, "f'"] & & Z. \arrow[ul, "g'"'] \end{tikzcd}$$

A **map of cospans** is a morphism $h : C \to C'$ in $\mathcal{C}$ between the apices of two cospans $X \xrightarrow{f} C \xleftarrow{g} Y$ and $X \xrightarrow{f'} C' \xleftarrow{g'} Y$ with the same feet, such that

$$\begin{tikzcd} & C \arrow[dd, "h"] & \\ X \arrow[ur, "f"] \arrow[dr, "f'"'] & & Y \arrow[ul, "g"'] \arrow[dl, "g'"] \\ & C' & \end{tikzcd}$$

commutes. Given a category $\mathcal{C}$ with pushouts, we may define a category $\mathbf{Cospan}(\mathcal{C})$ with objects the objects of $\mathcal{C}$ and morphisms isomorphism classes of cospans [Benabou 1967]. We will often abuse our terminology and refer to cospans themselves as morphisms in some cospan category Cospan($\mathcal{C}$); we of course refer instead to the isomorphism class of the said cospan. Note that such cospan categories come equipped with a so-called dagger functor, which maps a cospan $X \xrightarrow{f} C \xleftarrow{g} Y$ to its reflection $Y \xrightarrow{g} C \xleftarrow{f} X$.

2.2. Dagger compact categories. We remind ourselves that a **dagger functor** is an involutive, contravariant endofunctor that is the identity on objects. That is, given a category $\mathcal{C}$, a dagger functor is a contravariant functor $\dagger : \mathcal{C} \to \mathcal{C}$ such that $\dagger(A) = A$ for all objects $A \in \mathrm{Ob}\mathcal{C}$ and $\dagger(\dagger(f)) = f$ for all morphisms f in $\mathcal{C}$. A dagger functor expresses the idea that the direction of morphisms can be reversed: through a dagger functor each morphism specifies a map from its codomain to its domain, in addition to the map it is from its domain to its codomain.

When other structure is present, we prefer this dagger to play nice with it. We say that a morphism f is **unitary** if its dagger provides it with an inverse morphism; that is, if $f \circ f^\dagger$ and $f^\dagger \circ f$ are both identity morphisms. A **dagger symmetric monoidal category** is a symmetric monoidal category equipped with a dagger symmetric monoidal functor—that is, a dagger functor that coherently preserves the symmetric monoidal structure. Concretely, this requires that the dagger functor preserve the tensor product, and that the associator, unitors, and braiding of the symmetrical monoidal category be unitary. Furthermore, letting L and R be dual objects of a dagger symmetric monoidal category, with monoidal unit I, braiding $\sigma_{L,R} : L \otimes R \to R \otimes L$, and unit $\eta : I \to R \otimes L$ and counit $\epsilon : L \otimes R \to I$, we say that L and R are **dagger dual** if $\eta = \sigma \circ \epsilon^\dagger$. A **dagger compact category** is a dagger symmetric monoidal category in which every object has a dagger dual.

Dagger compact categories were first introduced in the context of categorical quantum mechanics [Abramsky–Coecke, 2004], under the name strongly compact closed category.

2.3. Example. Of particular interest is the category $\mathrm{Cospan}(\mathcal{D})$ of cospans in a category $\mathcal{D}$ with finite colimits. We think of this category as a dagger compact category, with the monoidal product given by coproducts in the category $\mathcal{D}$, the structural isomorphisms for the symmetric monoidal structure inherited from viewing $(\mathcal{D}, +)$ as a wide subcategory, the dagger functor the aforementioned, and each object X dual to itself with unit and counit

$$
\begin{tikzcd}
 & X & \\
0 \arrow[ur, "!"] & & X+X \arrow[ul, "{[1_X,1_X]}"']
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
 & X & \\
X+X \arrow[ur, "{[1_X,1_X]}"] & & 0 \arrow[ul, "!"']
\end{tikzcd}
$$

respectively, where 0 is the initial object and hence monoidal unit of $\mathcal{D}$, where ! is the unique function of the given type, and we write $[f, g]$ for the coproduct of morphisms f and g. It is a simple computation to check that this monoidal product is functorial, and that the dagger functor, symmetric monoidal structure, and the duals for objects interact well to indeed define a dagger compact category [Stay 2015].

3. Decorated cospan categories

We now detail our central construction and state the main theorem.

3.1. Definition. *Let*

$$(F, \varphi) : (\mathcal{D}, +) \longrightarrow (\mathcal{C}, \otimes)$$

be a lax monoidal functor. We may define a category $\boldsymbol{F}$**Cospan**, *the category of F-decorated cospans, with objects the objects of $\mathcal{D}$, and morphisms equivalence classes of pairs*

$$(X \xrightarrow{i} N \xleftarrow{o} Y, \;\; 1 \xrightarrow{s} FN)$$

comprising a cospan $X \xrightarrow{i} N \xleftarrow{o} Y$ in $\mathcal{D}$ together with an element $1 \xrightarrow{s} FN$ of the F-image FN of the apex of the cospan. We shall call the element $1 \xrightarrow{s} FN$ the **structure element** *of the decorated cospan. Equivalence is defined up to isomorphism of cospans; an isomorphism of cospans induces a one-to-one correspondence between structure elements of their apices.*

Composition in this category is given by pushout of cospans in $\mathcal{D}$

$$\begin{tikzcd}
 & & N +_Y M & & \\
 & N \arrow[ur, "j_N"] & & M \arrow[ul, "j_M"'] & \\
X \arrow[ur, "i_X"] & & Y \arrow[ul, "o_Y"'] \arrow[ur, "i_Y"] & & Z \arrow[ul, "o_Z"']
\end{tikzcd}$$

paired with the pushforward

$$1 \xrightarrow{\lambda^{-1}} 1 \otimes 1 \xrightarrow{s \otimes t} FN \otimes FM \xrightarrow{\varphi_{N,M}} F(N+M) \xrightarrow{F[j_N, j_M]} F(N +_Y M)$$

of the tensor product of the structure elements along the F-image of the coproduct of the pushout maps.

The key insight of the construction is contained in this last sentence: although at first glance it might seem surprising that we can construct a composition rule for structure elements $s : 1 \to FN$ and $t : 1 \to GM$ just from an understanding of how structure elements pushforward along morphisms, the monoidality of F means we can take the product of their structure elements $s \otimes t$, up to isomorphism an element of $F(N + M)$, and then push it along the coproduct of the pushout maps, a function $N + M \to N +_Y M$, to get an element of the $F(N +_Y M)$. This pushforward encodes the identification of the image of Y in N with the image of the same in M, and so describes merging the 'overlap' of the two structure elements.

3.2. Theorem. *Let $(F, \varphi) : (\mathcal{D}, +) \to (\mathcal{C}, \otimes)$ be a lax monoidal functor. Then F*Cospan *is a well-defined category, and moreover a dagger compact category.*

We also write $+$ for the monoidal product in FCospan; on objects it is also given by the coproduct in $\mathcal{D}$. We defer giving further details about the monoidal and dagger structure, and proof of the well-definedness of these structures, until Section 6, in favour of first also defining a class of functors on these categories and illustrating the constructions with some examples.

4. Functors between decorated cospan categories

Decorated cospans provide a setting for formulating various operations that we might wish to enact on instances of the decorating structure, including the composition of these instances, both sequential and monoidal, as well as dagger and dualising operations. We now observe that these operations are formulated in a systematic way, so that transformations of the decorating structure—that is, monoidal transformations between lax monoidal functors defining decorated cospan categories—respect these operations.

When the decorating structure comprises some notion of topological diagram, such as a graph, such transformations include semantic interpretation of the decorating structure, and thus construct functorial semantics for the decorated cospan category.

4.1. Definition. *Let*

$$(F,\varphi),(G,\gamma) : (\mathcal{D},+) \longrightarrow (\mathcal{C},\otimes)$$

be lax monoidal functors, and let

$$\theta : (F,\varphi) \Longrightarrow (G,\gamma)$$

be a monoidal natural transformation between them. Then we may define a functor

$$T : F\mathrm{Cospan} \longrightarrow G\mathrm{Cospan}$$

between the corresponding decorated cospan categories by letting objects of $F\mathrm{Cospan}$ *map to the same object as an object of* $G\mathrm{Cospan}$*, and letting morphisms*

$$(X \xrightarrow{i} N \xleftarrow{o} Y, 1 \xrightarrow{s} FN)$$

map to that with the element precomposed with the natural transformation $\theta_N : FN \to GN$ *on the apex* N

$$(X \xrightarrow{i} N \xleftarrow{o} Y, 1 \xrightarrow{\theta_N \circ s} GN).$$

As mentioned, these induced functors preserve the dagger and monoidal structure.

4.2. Theorem. *The above functor* T *is a well-defined functor, and moreover a strict monoidal dagger functor.*

Again we defer the further details regarding the monoidal structure of this functor, as well as proof of well-definedness, until Section 6. We first give some examples of such constructions.

5. Examples

We outline three decorated cospans constructions, the first based on labelled graphs, and the remaining two on Markov processes and electrical circuits respectively. We shall see that the decorated cospan framework allows us to take a notion of closed system and construct a corresponding notion of open or composable system, together with functorial semantics for these systems.

5.1. Labelled graphs. To begin we return to the example of this paper's introduction.

Define a **$(0, \infty)$-graph** (N, E, s, t, r) to comprise a finite set N of vertices (or *nodes*), a finite set E of edges, functions $s, t : E \to N$ describing the source and target of each edge, and a function $r : E \to (0, \infty)$ of labels.[2] The decorated cospans framework allows us to construct a category with, roughly speaking, these graphs as morphisms. More precisely, our morphisms will consist of these graphs, together with subsets of the nodes marked, with multiplicity, as 'input' and 'output' connection points.

Consider the functor

$$\mathrm{Graph} : (\mathrm{FinSet}, +) \longrightarrow (\mathrm{Set}, \times)$$

taking a finite set N to the set $\mathrm{Graph}(N)$ of $(0, \infty)$-graphs (N, E, s, t, r) with set of nodes N. On morphisms let it take a function $f : N \to M$ to the function that pushes labelled graph structures on a set N forward onto the set M:

$$\begin{aligned} \mathrm{Graph}(f) : \mathrm{Graph}(N) &\longrightarrow \mathrm{Graph}(M); \\ (N, E, s, t, r) &\longmapsto (M, E, f \circ s, f \circ t, r). \end{aligned}$$

As this map simply acts by post-composition, our map Graph is indeed functorial.

We then arrive at a lax monoidal functor (Graph, ρ) by equipping this functor with the natural transformation

$$\begin{aligned} \rho_{N,M} : \mathrm{Graph}(N) \times \mathrm{Graph}(M) &\longrightarrow \mathrm{Graph}(N + M); \\ \big((N, E, s, t, r), (M, F, s', t', r')\big) &\longmapsto \big(N + M, E + F, s + s', t + t', [r, r']\big), \end{aligned}$$

together with the unit map

$$\begin{aligned} \rho_1 : 1 &\longrightarrow \mathrm{Graph}(\varnothing); \\ \bullet &\longmapsto (\varnothing, \varnothing, \varnothing, \varnothing, \varnothing), \end{aligned}$$

where we use $\varnothing$ to denote both the empty set and the unique function of the appropriate codomain with domain the empty set, and we remind ourselves that we write $[r, r']$ for the coproduct of the functions r and r'. The naturality of this collection of morphisms, as well as the coherence laws for lax monoidal functors, follow from the universal property of the coproduct.

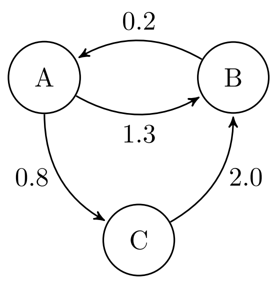
Definition 3.1 thus constructs a category GraphCospan. For an intuitive visual understanding of the morphisms of this category and its composition rule, see this paper's introduction.

[2] The data (N, E, s, t) is commonly termed a *directed multigraph*; we simply refer to such structures as graphs.

5.2. Markov processes. As a second example we consider finite, continuous-time Markov chains—that is, we consider continuous-time stochastic processes on a finite state space, in which the future behaviour of the process depends only on its current state. This is easily generalised to variants of this structure, such as stochastic Petri nets or chemical reaction networks. More details can be found in [Baez–Pollard 2015].

Define a **Markov process** to be a finite set N together with a transition function $t : (N \times N) \setminus \Delta_N \to [0, \infty)$, where $\Delta_N = \{(n, n) \in N \times N \mid n \in N\}$. We interpret N as the set of states for the Markov process, and the function t as a transition function describing the rates through which the system probabilistically transitions from one state to another.[3]

It is common to depict Markov processes graphically, using $(0, \infty)$-graphs. We do this by viewing a $(0, \infty)$-graph (N, E, s, t, r) as specifying a Markov process with N the set of states, and the rate of transition between two distinct nodes $n, n' \in N$ equal to the sum of the labels $r(e)$ for all edges with source $s(e)$ equal to n and target $t(e)$ equal to n'. As an example, the following diagram depicts a Markov process with three states, A, B, and C, and with transitions rates $t(A, B) = 1.3$, $t(A, C) = 0.8$, $t(B, A) = 0.2$, $t(B, C) = 0$, $t(C, A) = 0$, and $t(C, B) = 2.0$:



In fact this process of interpretation is natural, and through Definition 4.1 we may construct a functor from the category GraphCospan to a category of cospans decorated by Markov processes.

To see how this works, define a lax monoidal functor

$$(\mathrm{Markov}, \mu) : (\mathrm{FinSet}, +) \longrightarrow (\mathrm{Set}, \times)$$

as follows. On objects let Markov take a finite set N to the set $\mathrm{Markov}(N)$ comprising all transition functions $t : (N \times N) \setminus \Delta_N \to [0, \infty)$ with set of states N. On morphisms let this functor take a function $f : N \to M$ to the function that pushes transition functions on a set N forward onto the set M; that is let $\mathrm{Markov}(f)$ map a transition function

$$t : (N \times N) \setminus \Delta_N \longrightarrow [0, \infty)$$

[3] Note that some definitions of Markov process also include an 'initial' probability distribution over the set of states N; we omit this part here.

to the transition function

$$\begin{aligned}\mathrm{Markov}(f)(t) : (M \times M) \setminus \Delta_M &\longrightarrow [0, \infty);\\ (m, m') &\longmapsto \sum_{\substack{(n,n') \in (N \times N) \setminus \Delta_N \\ f(n)=m,\, f(n')=m'}} t(n, n').\end{aligned}$$

Finally, for monoidal products let

$$\begin{aligned}\mu_{N,M} : \mathrm{Markov}(N) \times \mathrm{Markov}(M) &\longrightarrow \mathrm{Markov}(N + M);\\ \big(t, u\big) &\longmapsto tu,\end{aligned}$$

where $t \times u$ is the function

$$\begin{aligned}t \times u : ((N + M) \times (N + M)) \setminus \Delta_{N+M} &\longrightarrow [0, \infty);\\ (n, n') &\longmapsto \begin{cases} t(n, n') & \text{if } n, n' \in N \\ u(n, n') & \text{if } n, n' \in M \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

acting as t on pairs of elements of N, u on pairs of elements of M, and as the zero function otherwise. Also let

$$\begin{aligned}\mu_1 : 1 &\longrightarrow \mathrm{Markov}(\varnothing);\\ \bullet &\longmapsto \big(\varnothing : \varnothing \to [0, \infty)\big)\end{aligned}$$

where again we use $\varnothing$ to denote both the empty set and the unique function of the given codomain with domain the empty set. It is straightforward to check this gives a well-defined lax monoidal functor, and hence a strict symmetric monoidal dagger category MarkovCospan.

The interpretation of $(0, \infty)$-graphs as Markov processes is then described by the monoidal natural transformation

$$\theta_N : \mathrm{Graph}(N) \longrightarrow \mathrm{Markov}(N)$$

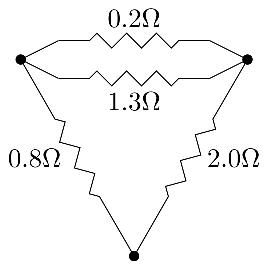
mapping a $(0, \infty)$-graph (N, E, s, t, r) to the function

$$\begin{aligned}(N \times N) \setminus \Delta_N &\longrightarrow [0, \infty);\\ (n, n') &\longmapsto \sum_{\substack{e \in E \\ s(e)=n \\ t(e)=n'}} r(e).\end{aligned}$$

Definition 4.1 hence uses this data to construct a functor GraphCospan $\to$ MarkovCospan expressing a semantic interpretation of the morphisms of GraphCospan. By Theorem 4.2, this functor is a strict monoidal dagger functor, and so this semantic interpretation retains the algebraic structure of the graph-decorated cospans.

5.3. Electrical circuits. We quickly remark upon the motivating application for the decorated cospans construction, that of electrical circuits and their diagrams. Specialising to the case of networks of linear resistors, this provides an alternate semantics for the category GraphCospan, in terms of linear relations. Further details can be found in [Baez–Fong 2015].

Our first observation here is that electrical circuit diagrams may also be viewed as labelled graphs, and vice versa. For example, after choosing a unit of resistance, say ohms (Ω), each $(0, \infty)$-graph can be viewed as a network of linear resistors, with the $(0, \infty)$-graph of the introduction now more commonly depicted as



GraphCospan may then be viewed as a category with morphisms circuits of linear resistors equipped with chosen input and output terminals.

The suitability of this language is seen in the way the different categorical structures of GraphCospan capture different operations that can be performed with circuits. To wit, the sequential composition expresses the fact that we can connect the outputs of one circuit to the inputs of the next, while the monoidal composition models the placement of circuits side-by-side. Furthermore, the symmetric monoidal structure allows us reorder input and output wires, the compactness captures the interchangeability between input and output terminals of circuits—that is, the fact that we can choose any input terminal to our circuit and consider it instead as an output terminal, and vice versa—and the dagger functor expresses the fact that we may reverse the orientation of entire circuit components.

Theorem 4.2 also finds application. Each node in a network of resistors can be assigned an electric current and potential, and so the set N of vertices of a $(0, \infty)$-graph can be seen as generating a $2|N|$-dimensional real vector space. Via a principle known as Ohm's law, a network of linear resistors imposes constraints on these currents and potentials, selecting a linear subspace of this vector space as physically realisable states. This defines a monoidal natural transformation from the monoidal functor (Graph, ρ) to the monoidal functor assigning to each finite set N the collection of linear subspaces of the vector space generated by $N + N$. The latter functor generates a decorated cospan category which maps densely onto the category of finite vector spaces and linear relations, and allows us to define functorial semantics of circuit diagrams as linear relations.

6. Proofs

We conclude this paper by checking the proposed constructions are well-defined and have the properties claimed. We first consider decorated cospan categories, with the intent to prove Theorem 3.2: that decorated cospan categories are indeed categories, and moreover dagger compact categories.

To begin, we must observe that $F\mathrm{Cospan}$ is indeed a category.

6.1. Lemma. *Let $(F, \varphi) : (\mathcal{D}, +) \to (\mathcal{C}, \otimes)$ be a lax monoidal functor. Then* $F\mathrm{Cospan}$ *is a well-defined category.*

Proof. We check that we have a well-defined category. For this we need to check that this composition rule is associative, and that each object has an identity morphism.

Associativity: Suppose we have morphisms

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ \ 1 \xrightarrow{s} FN),$$

$$(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z,\ \ 1 \xrightarrow{t} FM),$$

$$(Z \xrightarrow{i_Z} P \xleftarrow{o_W} W,\ \ 1 \xrightarrow{u} FP).$$

As pushouts are unique up to unique isomorphism, we know that the composition of the cospans is associative. We must check that the pushforward of the elements is also an associative process. Write

$$\tilde{\alpha} : (N +_Y M) +_Z P \longrightarrow N +_Y (M +_Z P)$$

for the unique isomorphism between the two pairwise pushouts constructions from the above three cospans. Consider then the following diagram, with top row the element obtained by taking the composite of the first two morphisms first, and the bottom row the element obtained by taking the composite of the last two morphisms first.

$$
\begin{tikzcd}
F((N+_Y M)+_Z P) \arrow[rrrrrr, "F\tilde{\alpha}"] & & & & & & F(N+_Y(M+_Z P)) \\
F((N+_Y M)+P) \arrow[u, "{F[j_{N+_Y M},j_P]}"] & & & (6) & & & F(N+(M+_Z P)) \arrow[u, "{F[j_N,j_{M+_Z P}]}"'] \\
& F((N+M)+P) \arrow[ul, "{F([j_N,J_M]+1_P)}"'] \arrow[rrrr, "F\alpha"] & & & & F(N+(M+P)) \arrow[ur, "{F(1_N+[j_M,j_P])}"] & \\
F(N+_Y M)\otimes FP \arrow[uu, "\varphi_{N+_Y M,P}"] & & & (3) & & & FN\otimes F(M+_Z P) \arrow[uu, "\varphi_{N,M+_Z P}"'] \\
& F(N+M)\otimes FP \arrow[ul, "{F[j_N,j_M]\otimes 1_{FP}}"] \arrow[uu, "\varphi_{N+M,P}"'] & & & & FN\otimes F(M+P) \arrow[uu, "\varphi_{N,M+P}"] \arrow[ur, "{1_{FN}\otimes F[j_M,j_P]}"'] & \\
& & (FN\otimes FM)\otimes FP \arrow[ul, "\varphi_{N,M}\otimes 1_{FP}"] \arrow[rr, "\alpha"] & & FN\otimes(FM\otimes FP) \arrow[ur, "1_{FN}\otimes\varphi_{M,P}"'] & & \\
& & (1\otimes 1)\otimes 1 \arrow[u, "(s\otimes t)\otimes u"] \arrow[rr, "\alpha"] & & 1\otimes(1\otimes 1) \arrow[u, "s\otimes(t\otimes u)"'] & & \\
& & & 1\otimes 1 \arrow[ul, "\rho^{-1}\otimes 1"] \arrow[ur, "1\otimes\lambda^{-1}"'] & & & \\
& & & 1 \arrow[u, "\lambda^{-1}"] & & &
\arrow[from=2-1, to=5-2, phantom, "(4)"]
\arrow[from=2-7, to=5-6, phantom, "(5)"]
\arrow[from=6-3, to=7-5, phantom, "(2)"]
\arrow[from=8-4, to=7-4, phantom, "(1)"]
\end{tikzcd}
$$

This diagram commutes as (1) is the triangle coherence equation for the monoidal category $(\mathcal{C}, \otimes)$, (2) is naturality for the associator α, (3) is the associativity condition for the monoidal functor F, (4) and (5) commute by the naturality of φ, and (6) commutes as it is the F-image of a hexagon describing the associativity of the pushout. This shows that the two structure elements obtained by the two different orders of composition of our three morphisms are equal up to the unique isomorphism $\tilde{\alpha}$ between the two different pushouts that may be obtained. Our composition rule is hence associative.

Identity morphisms: The morphism $X \to X$ given by the pair

$$(X \xrightarrow{1_X} X \xleftarrow{1_X} X,\ 1 \xrightarrow{F! \circ \varphi_1} FN)$$

acts as an identity for the composition we have defined, where $\varphi_1 : 1 \to F0$ is the unit coherence map for the monoidal functor. We shall show that it is an identity for composition on the left; the case for composition on the right is similar. Observe that the cospan in this pair is known to be the identity cospan in $\mathrm{Cospan}(\mathcal{D})$. We thus need to check that, given a morphism

$$(X \xrightarrow{i} N \xleftarrow{o} Y,\ 1 \xrightarrow{s} FN),$$

the pushforward of the product $(F! \circ \varphi_1) \otimes s$ along the F-image of the coproduct $[i_X, 1_N] : X + N \to N$ of the pushout maps is again the element s; this pushforward being, by definition, the structure element of the composite of the given morphism and the claimed identity map. This is shown by the commutativity of the diagram below, with the path along the lower edge equal to the aforementioned pushforward.

$$\begin{tikzcd}
1 \arrow[rrrrr, "s"] \arrow[dr, "\lambda_1^{-1}"'] \arrow[drr, phantom, "(1)"] & & & & & FN \\
& 1 \otimes 1 \arrow[r, "1 \otimes s"] \arrow[drr, "(F! \circ \varphi_1) \otimes s"'] & 1 \otimes FN \arrow[urrr, "\lambda_{FN}"] \arrow[r, "\varphi_1 \otimes 1_{FN}"] \arrow[d, phantom, "(3)"] & F0 \otimes FN \arrow[r, "\varphi_{0,N}"] \arrow[d, "F! \otimes 1_{FN}"] \arrow[dr, phantom, "(4)"] & F(0+N) \arrow[ur, "F\lambda_N"] \arrow[d, "F(! + 1_N)"] \arrow[u, phantom, "(2)"] \arrow[r, phantom, "(5)"] & \\
& & & FX \otimes FN \arrow[r, "\varphi_{X,N}"'] & F(X+N) \arrow[uur, "{F[i_X, 1_N]}"'] &
\end{tikzcd}$$

This diagram commutes as each subdiagram commutes: (1) commutes by the naturality of λ, (2) by the unit monoidality of the functor F, (3) by the interchange law, (4) by the naturality of φ, and (5) as it is the F-image of the commutative triangle

$$\begin{tikzcd}
0 + N \arrow[r, "{[!, 1_N] = \lambda_N}"] \arrow[d, "! + 1_N"'] & N \\
X + N \arrow[ur, "{[i_X, 1_N]}"'] &
\end{tikzcd}$$

in $\mathcal{D}$. ∎

To define the monoidal and dagger compact structure, we first observe that a category of 'undecorated' cospans lies inside each decorated cospan category.

6.2. Lemma. *Let $(F,\varphi) : (\mathcal{D},+) \to (\mathcal{C},\otimes)$ be a lax monoidal functor. Then there is a wide embedding*

$$\mathrm{Cospan}(\mathcal{D}) \hookrightarrow F\mathrm{Cospan}.$$

Proof. Given an object N, call $1 \xrightarrow{\varphi_1} F0 \xrightarrow{F!} FN$ the **empty structure** on N. We then let this embedding send objects to themselves, and decorate the cospans $X \xrightarrow{i} N \xleftarrow{o} Y$ of $\mathcal{D}$ with the empty structure, giving the decorated cospan

$$(X \xrightarrow{i} N \xleftarrow{o} Y,\ 1 \xrightarrow{F!\circ\varphi_1} FN).$$

This functor, should it be well-defined, is evidently bijective on objects and faithful.

To check that this functor is well-defined, we must check the composite of empty structures is again an empty structure. This involves a now routine diagram chase of the above sort, based on the observation that the monoidality of the functor F allows us to factor the composite of empty structures

$$1 \xrightarrow{\lambda^{-1}} 1\otimes 1 \xrightarrow{\varphi_1\otimes\varphi_1} F0\otimes F0 \xrightarrow{F!\otimes F!} FN\otimes FM \xrightarrow{\varphi_{N,M}} F(N+M) \xrightarrow{F[j_N,j_M]} F(N+_Y M)$$

via $F0 \cong F(0+0)$. ■

We give FCospan the dagger and symmetric monoidal structure required for this embedding to be a dagger monoidal functor. With this data we may complete the proof that FCospan is a dagger compact category.

Proof of Theorem 3.2. To be more precise, we define the monoidal product of objects X and Y of FCospan to be their coproduct $X+Y$, and define the monoidal product of decorated cospans $(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN)$ and $(X' \xrightarrow{i_{X'}} N' \xleftarrow{o_{Y'}} Y',\ 1 \xrightarrow{t} FN')$ to be

$$\left(\begin{tikzcd} & N+N' & \\ X+X' \arrow[ur, "i_X+i_{X'}"] & & Y+Y' \arrow[ul, "o_Y+o_{Y'}"'] \end{tikzcd},\quad \begin{tikzcd} F(N+N') \\ 1 \arrow[u, "\varphi_{N,N'}\circ(s\otimes t)\circ\lambda^{-1}"'] \end{tikzcd}\right).$$

The functoriality of this monoidal product, as far as the structure elements are concerned, follows from the isomorphism of pushouts

$$(N+N') +_{Y+Y'} (M+M') \cong (N+_Y M) + (N' +_{Y'} M').$$

The functoriality for the cospan part follows from the properties of the coproduct.

As we now have a monoidal embedding, compactness for FCospan then follows immediately from the compact structure of Cospan(FinSet).

The dagger functor also mimics exactly that on $\mathcal{D}$; we define the dagger to reflect the cospan part of a decorated cospan, keeping the same structure element:

$$\dagger(X \xrightarrow{i} N \xleftarrow{o} Y,\ 1 \xrightarrow{s} FN) = (Y \xrightarrow{o} N \xleftarrow{i} X,\ 1 \xrightarrow{s} FN).$$

The commutativity of the required coherence diagrams is then an immediate consquence of Lemma 6.2. This proves Theorem 3.2. ■

The following fact is an easy consequence of Lemma 6.2.

6.3. Proposition. *Let* $1_{\mathcal{D}} : (\mathcal{D}, +) \to (\mathcal{D}, +)$ *be the identity functor on a monoidal category* $\mathcal{D}$*, where* $\mathcal{D}$ *has finite colimits and* $+$ *is the product. Then the categories* $\mathrm{Cospan}(\mathcal{D})$ *and* $1_{\mathcal{D}}\mathrm{Cospan}$ *are isomorphic dagger compact categories.*

Finally, we turn our attention to the induced functors between decorated cospan categories, checking that they are indeed monoidal dagger functors as claimed by Theorem 4.2. Note that as we claim the induced functors are strict monoidal, we need not specify any extra data.

Proof of Theorem 4.2. We check that this proposed functor preserves identities, composition, monoidal composition, and dagger functors.

Identities: Let

$$(X \xrightarrow{1_X} X \xleftarrow{1_X} X,\ 1 \xrightarrow{F! \circ \varphi_1} FX)$$

be the identity morphism on some object X in the category of F-decorated cospans. Now this morphism has T-image

$$(X \xrightarrow{1_X} X \xleftarrow{1_X} X,\ 1 \xrightarrow{\theta_X \circ F! \circ \varphi_1} GX).$$

But we have the following diagram

$$\begin{tikzcd}
 & F0 \arrow[r, "F!"] \arrow[d, "\theta_0"] \arrow[dr, phantom, "(2)"] & FX \arrow[d, "\theta_X"] \\
1 \arrow[ur, "\varphi_1"] \arrow[dr, "\gamma_1"'] \arrow[r, phantom, "(1)"] & G0 \arrow[r, "G!"'] & GX
\end{tikzcd}$$

where (1) commutes by the unitality condition for the monoidal natural transformation θ, and (2) commutes by the naturality of θ. Thus we have the equality of structure elements $\theta_X \circ F! \circ \varphi_1 = G! \circ \gamma_1 : 1 \mapsto GX$, and so T sends identity morphisms to identity morphisms.

Composition: Let

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN) \qquad \text{and} \qquad (Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z,\ 1 \xrightarrow{t} FM),$$

be morphisms in $F\mathrm{Cospans}$. As the composition of the cospan part is by pushout in $\mathcal{D}$ in both cases, and as T acts as the identity on these cospans, it is clear that T preserves composition of cospans. To see that T preserves composition of the structure elements, observe that the composite of s and t is given by the top line in the following diagram, while the composite of their images under T is given by the bottom line:

$$\begin{tikzcd}
& & FN \otimes FM \arrow[r, "\varphi_{N,M}"] \arrow[d, "\theta_N \otimes \theta_M"] \arrow[dr, phantom, "(2)"] & F(N+M) \arrow[r, "F{[j_N, j_M]}"] \arrow[d, "\theta_{N+M}"] \arrow[dr, phantom, "(3)"] & F(N +_Y M) \arrow[d, "\theta_{N+_Y M}"] \\
1 \arrow[r, "\lambda^{-1}"] & 1 \otimes 1 \arrow[ur, "s \otimes t"] \arrow[dr, "(\theta_N \circ s)\otimes(\theta_M \circ t)"'] \arrow[r, phantom, "(1)"] & {} & & \\
& & GN \otimes GM \arrow[r, "\gamma_{N,M}"'] & G(N+M) \arrow[r, "G{[j_N, j_M]}"'] & G(N +_Y M)
\end{tikzcd}$$

As (1) commutes by the interchange law, (2) by monoidality of the natural transformation θ, and (3) by its naturality, we see that the composite of the T-images of s and t is equal to the T-image of their composite, and so that T preserves composition.

Monoidal product: To show T is strict monoidal, it is enough to observe that for any pair of F-decorated cospans $\hat{s} = (X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN)$ and $\hat{t} = (X' \xrightarrow{i_{X'}} N' \xleftarrow{o_{Y'}} Y',\ 1 \xrightarrow{t} FN')$, we have the equality of G-decorated cospans

$$T\hat{s} + T\hat{t} = T(\hat{s} + \hat{t}).$$

This is trivially true for the cospan part, and true for structure elements by the monoidality of θ.

Dagger: The dagger functor in both FCospans and GCospans simply reflects the cospan part of each morphism. As T acts as the identity on each such part, we trivially have that $T \circ \dagger = \dagger \circ T$, as required. ■

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