Detecting Contextuality: Sheaf Cohomology and All vs Nothing Arguments

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Abstract

The main topic of this thesis is the study of contextuality, a key characteristic feature of quantum mechanics that represents one of the most valuable resources for quantum computation. We will adopt the sheaf-theoretic approach to non-locality and contextuality introduced in [AB11]. This elegant mathematical theory will enable us to consider contextuality on a higher scale, as a general, abstract mathematical property. We will specifically focus on mathematical methods to detect the contextuality of empirical models. In particular, the first part of the thesis is dedicated to *All vs Nothing (AvN) arguments* [ABK+15], a generalisation of a proof originally formulated by Mermin to demonstrate the strong contextuality of the GHZ model [Mer90]. We develop an algorithm capable of identifying a large number of quantum-realisable states that admit such proofs of contextuality, thus providing a significant amount of concrete examples of strongly contextual models. The second part of the project inspects an algebraic-topological method based on sheaf cohomology [AMSB12, ABK+15]. Contributions in this field include counterexamples to some recently advanced hypothesis (e.g. Conjecture 8.1. in [AMSB12]) as well as new implications and characterisations concerning SC and CSC models based on the injectivity of the cohomological obstructions homomorphisms, and an alternative description of the first cohomology group based on torsors of presheaves. This discussion naturally leads us to a new compact and natural approach to obstructions in higher cohomology groups. We define the higher counterpart of cohomological obstructions and investigate its properties. Of particular interest is the fact that these obstructions are organised in a hierarchy of logical implications, which unfortunately turns out to be not suitable for the study of no-signaling models. In the final chapter, we bring together cohomology and AvN arguments in a unique discussion following the ideas already expressed in [ABK+15].

CONTENTS

INTRODUCTION

Since the dawn of quantum physics, the philosophical debate around its foundations has brought to light inconsistencies and paradoxes, some of which remain unsolved today. Although quantum mechanics has been repeatedly proved to be the most accurate physical theory available to explain the vast majority of physical phenomena, it is undoubtedly one of the most controversial. Numerous generations of scientists, including some of the most revolutionary minds of the 20th century, have attempted to provide a solid foundation for quantum physics. Yet the very nature of the theory seems to refuse any sort of axiomatisation, and even from an experimental point of view, it is sometimes difficult to believe in its complete truthfulness. It is thus no surprise that even Albert Einstein, arguably the most influential physicist in history, has custed doubt on the completeness of quantum mechanics many times, in primis with the well known EPR paradox [EPR35]. In the article, he argued that quantum theory cannot provide a complete and consistent description of reality, since it fails to verify what he defined as the *criterion of reality*, which essentially requires a physical theory to assign well-defined predetermined values to every physical quantity [Noro5]. It is somehow curious that Einstein's attempted rebuttal to quantum physics would eventually end up triggering the discovery of two of the most characteristic key features of the theory: non-locality and, ultimately, contextuality. These two phenomena revolutionised our way of conceiving physics, precisely because we have to abandon the obsolete necessity of assigning predetermined values to physical entities. The theorems of Bell [Bel64] and Bell-Kochen-Specker [Bel66, KS75] showed in fact that the counterintuitive features of quantum mechanics highlighted by the EPR paradox are in fact unavoidable components of any theory that agrees with the (correct) predictions of the theory. The EPR paradox turned out not to be a paradox at all. It only showed how astonishingly unsuitable our intuition and our classical conception of physics are when dealing with the complex phenomena explained by quantum mechanics.

Nonetheless, even after the formulation of Bell's theorem, non-locality and contextuality have been considered an obscure aspect of quantum physics, a paradoxical trait that threatened the foundations of the theory itself. Physicists seemed to be too satisfied with the undoubted power of prediction of quantum mechanics to care about understanding in details these features. In the 1980s, the birth of quantum computation utterly changed this attitude. Scientists started to consider non-locality and contextuality as valuable resources to break through the limits of classical computation and cryptography. The study of quantum foundations gained new relevance, and it now represents one of the most active domains in theoretical computer science.

Recent work has shown that contextuality provides the "magic" for quantum computation [HWVE14]. This result (and many others previously announced) has elected the study of contextuality one of the main goals for the development of the branch. Until 2010, this research domain has always been carried out in a rather heterogeneous way, where each interesting example has been analysed as a separate entity, and the whole discipline was considered as a mere sub-branch of quantum physics.

The unified sheaf-theoretic approach to non-locality and contextuality developed in [AB11] represents a milestone in this research area as it solidly proves that these phenomena can be fully described abstractly via general mathematical structures. This new abstract perspective allows a unified study of all the related concepts and shows that contextuality is not a feature specific to quantum mechanics, but rather a general mathematical law. As such, it can independently be applied to other areas of computer science, even to non-quantum-related domains such as constraint satisfaction [AGK13] and relational databases [Abr12]. It also allows the development of purely mathematical methods to detect and study the contextuality of physical systems [AB11, ABK+15, AMSB12, AH12]. With our thesis, we aim to specifically contribute to this area by analysing and, hopefully, implementing some of these methods.

In particular, we will consider two possible ways of detecting contextuality based on *All vs Nothing arguments*, and *sheaf cohomology*. The first method generalizes a class of proofs that has been used in the past to show the contextuality of specific models, while the second introduces a completely new way of approaching the problem by studying the topological properties of sheaves associated to empirical models.

OUTLINE AND CONTRIBUTIONS

We now outline the structure of the document, as well as mention the main contributions. The most important results will be highlighted by a bullet.

In the first part of Chapter 1 we introduce the sheaf theoretic structure of non-locality and contextuality following the ideas presented in [AB11]. The discussion includes several digressions that give full details on the notions introduced by our reference articles, as well as a number of concrete examples. Then, we show how measurement covers can be described using simplicial complexes, and we apply this viewpoint to introduce a method of graphically representing simple empirical models. This representation will be extensively used throughout the thesis and represents one of its characterising flavors. The last section is dedicated to Vorob'ev's theorem and its relation to the study of contextuality.

In Chapter 2, we present All vs Nothing arguments via a formal mathematical study that provides full details on the notions presented in [ABK+15]. A first small interesting new fact we show concerns the Galois correspondence between subgroups of the Pauli n-group and their stabilisers. In particular, we show that this correspondence is induced by a Galois connection between subgroups of $\mathcal{P}_n$ and sub-vector-spaces of $\mathbb{C}^n$. We also give an alternative proof to Theorem 4.2 in [ABK+15].

- The theory of stabilisers and its connection to quantum mechanics is used to define an algorithm capable of identifying all AvN triples contained in the Pauli n-group $\mathcal{P}_n$ for a sufficiently small n (Section 2.2). This allows us to identify a large number of quantum states admitting AvN arguments, thus strongly contextual. Until now, we only had rather a few concrete examples of quantum-realisable strongly contextual models. Our algorithm gives us a significant amount of examples of models of this kind. We implement the program in *Mathematica* obtaining 19224 different quantum states admitting AvN triples (216 for $n = 3$ and 19008 for $n = 4$). We conclude the chapter by providing some randomly-chosen examples (Example 2.2.4). Note that, if the AvN triple conjecture 2.1.11 holds, then the states identified by the algorithm are the only possible states admitting AvN arguments.

In Chapter 3 we present the method of detection based on sheaf cohomology outlined in [ABK+15, AMSB12]. Also in this case, we give full details on every aspect considered. Of particular interest is the proof of the equivalence of the two distinct definitions provided by our reference articles. We then proceed to give explicit examples of computation of the cohomological obstructions. We also show that the Hardy model is a false positive in a more intuitive way using the graphical representation of the model presented in Chapter 1. This representation also plays a key role in the following result

- An example of a strongly contextual model which is cohomologically non-contextual is provided in Example 3.3.2. It is a particularly badly-behaved false positive, and it proves that

$$\mathrm{SC}(\mathcal{S}) \not\Rightarrow \mathrm{CLC}_{\mathrm{R}}(\mathcal{S}).$$

  The only example of a strongly contextual false positive we had so far was the Kochen-Specker model for a specific cover which "does not satisfy any reasonable criterion for symmetry, nor does it satisfy any strong form of connectedness" [AMSB12], which lead to the formulation of Conjecture 8.1 in [AMSB12]. Our model is defined on a simple bipartite Bell-type scenario, which verifies all sorts of nice symmetry and connectedness property. Therefore, it looks like Conjecture 8.1 of [AMSB12] is false.

Some new developments concerning the injectivity of the connecting homomorphisms γ defining cohomological obstructions are presented in Section 3.4.

- It has been recently advanced the hypothesis that different non extendable sections of an empirical model would give rise to different non-zero cohomological obstructions. We show that this statement characterises cohomologically strongly contextual models in Proposition 3.4.4, and it is thus not true in general. We also provid a number of concrete examples of this fact (Examples 3.4.6 and 3.4.7).

- We show that the injectivity of one single γ^{C} relative to a context C is a sufficient condition for the strong contextuality of the model (Proposition 3.4.8).

- Another hypothesis we aimed to prove states that the first cohomology group $\check{H}^1(\mathcal{M}, \mathcal{F}\mid_{C_0})$ associated to an abelian presheaf $\mathcal{F}$ and relative to a context C_0 is exactly the group of all cohomological obstructions to the existence of compatible families extending local sections at C_0. Unfortunately, we show in 3.4.11 that this conjecture does not hold with an explicit counterexample.

- As a result, an alternative description of the first cohomology group associated to an empirical model is developed in Section 3.5. This viewpoint allows us to see elements of the group $\check{H}^1(\mathcal{M}, \mathcal{F}\mid_{C_0})$ as $\mathcal{F}$-torsors.

In Chapter 4 we generalise cohomological obstructions to higher cohomology groups. Although this has been attempted before [Ji13], we give a completely different viewpoint on the subject which will enable us to obtain stronger results.

- A definition of higher cohomological obstruction is provided both in the conceptual way similar to [ABK+15] and in the more concrete one of [AMSB12]. It is arguably the most natural generalisation of cohomological obstruction to higher cohomology groups.
- A hierarchy of cohomological obstructions is established in Theorem 4.2.1. This result suggests the existence of an infinite amount of "levels" of contextuality, all organised in a hierarchy of logical implications. However,
- We show that the natural generalisation of cohomological obstructions cannot be used in the study of no-signaling models since they are always q-non-contextual (Section 4.3). Possible alternatives are briefly discussed.

Eventually, Chapter 5.3 establishes a connection between AvN arguments and sheaf cohomology. We followed closely the guidelines of [ABK+15] to prove this result.

1 SHEAF THEORY AND CONTEXTUALITY

In this chapter we will introduce and discuss the main ideas of [AB11] on how sheaf theory can give us a mathematical model for non-locality and contextuality. In particular, we are interested in distinguishing the different possible types of contextuality (simple, possibilistical and strong) as we will later present methods to detect them. The last part of the chapter is dedicated to a more detailed study of measurement covers and their properties.

1.1 EVENTS AND CONTEXTS

Suppose we have a system on which we can perform a finite amount of measurements. Every measurement is labelled by an element of a set X, and when it is carried out it produces an element of the *set of outcomes* O as a result. For each subset $U \subseteq X$, a *section over* U is a function $s \in O^U$. This function abstractly describes the event in which we choose to perform the measurements labelled by indices in U that resulted in outcome $s(m)$ for each $m \in U$. It is convenient to consider X as a finite discrete topological space as this allows us to define the *sheaf of events*

Definition 1.1.1. Let X be a finite, discrete measurement set and O a set of outcomes. The associated *presheaf of events* is the presheaf defined as

$$\begin{array}{rccc} \mathcal{E}: & \mathcal{P}(X)^{op} = \mathbf{Open}(X)^{op} & \longrightarrow & \mathbf{Set} \\ & U & \longmapsto & \mathcal{E}(U) := O^U \\ & U \subseteq U' & \longmapsto & \rho_U^{U'}, \end{array}$$

where $\rho_U^{U'} : \mathcal{E}(U') \to \mathcal{E}(U) :: s \mapsto s\,|_U$.

It is easy to see that this presheaf is actually a sheaf. In fact, let $C \subseteq X$ and let $\{C_i\}_{i\in I}$ be such that $\bigcup_{i\in I} C_i = C$. Consider a compatible family $\{s_i \in \mathcal{E}(C_i)\}_{i\in I}$, i.e. such that

$$s_i\,|_{C_i\cap C_j} = s_j\,|_{C_i\cap C_j} \quad \forall i,j \in I$$

then there exists a unique section $s \in \mathcal{E}(C)$ such that $s\,|_{C_i} = s_i$ for all $i \in I$, namely the one defined piecewise to be s_i inside C_i. We can thus refer to $\mathcal{E}$ as the *sheaf of events*.

An important feature of quantum physics is that we may not be able to carry out different measurements at the same time. For instance, this is the case for two non commutative observables like position and momentum of a particle (by the Heisenberg uncertainty priciple). We refer to such measurements as *mutually non-compatible*, and they represent one of the key ingredients of non-locality and contextuality. To capture this behavior, we will introduce the concept of *measurement cover*, a collection of subsets of X whose elements represent the maximal sets of measurements that can be performed jointly.

Definition 1.1.2. Let X be a measurement set. A *measurement cover* for X is a family $\mathcal{M} \subseteq \mathcal{P}(X)$ such that

- $\bigcup_{M \in \mathcal{M}} M = X$;
- if $C, C' \in \mathcal{M}$ and $C \subseteq C'$, then $C = C'$.

the second condition can be interpreted as the fact that $\mathcal{M}$ is an *anti-chain*. This is necessary to capture the fact that we are interested in the *maximum* sets of jointly performable measurements. We call the elements $C \in \mathcal{M}$ *measurement contexts*.

1.2 EMPIRICAL MODELS

Now, suppose we have a system, a set of measurements X, a set of outcomes O, and a measurement cover $\mathcal{M}$. We are interested in the following experiment: we choose a context $C \in \mathcal{M}$, we carry out every measurement m in the context, and we store the outcome of m. Suppose we repeat this experiment over and over. We will obtain a frequency distribution on each context of $\mathcal{M}$. If we let the number of repetitions of the experiment tend to infinity, we will obtain a probability distribution for every C in $\mathcal{M}$. Such a distribution is a function that assigns to each section over C (i.e. to each event) in $\mathcal{E}(C)$ a probability $p_C(s)$ for that event to happen, based on *empirical observation*.

Example 1.2.1. Let us mention a concrete physical example to clarify the concepts introduced so far. More conceptual ones will follow once we will have introduced the distribution presheaf. We will consider a typical scenario used in many basic quantum mechanics courses. Suppose we have a hydrogen atom and we want to measure the square module of its angular momentum L^2 as well as its components L_x, L_y, L_z in the x, y, z axis respectively. It can be shown that

$$[\hat{L}^2, \hat{L}_i] = 0 \quad \forall i \in \{x, y, z\}$$
$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z; \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x; \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

i.e. that L^2 is compatible with each component of L, but components of L are not compatible with each other [AF, Section 3.5]. This means that we have a measurement cover $\mathcal{M} = \{C_1, C_2, C_3\}$ made up of three measurement contexts, namely

$$C_1 := \{L^2, L_x\};$$
$$C_2 := \{L^2, L_y\};$$
$$C_3 := \{L^2, L_z\}.$$

We now start to do measurements in the way described above: we choose a context, we carry out its measurements and record the outcomes, and we repeat this procedure for a very large amount of times. Consider the following section at C_1 as an example

$$\begin{array}{rccc} s: & C_1 & \longrightarrow & O \\ & L^2 & \longmapsto & \lambda_1 \\ & L_x & \longmapsto & \lambda_2. \end{array}$$

This correspond to the event *"the measurement of* L^2 *has produced outcome* λ_1 *and the measurement of* L_x *gave outcome* λ_2*"*. Suppose that all the repetitions we performed, out of every time we chose C_1, one third of the times the event described took place. This means that the *empirical probability* of s is $\frac{1}{3}$. This discussion is valid for every section s at some C_i, thus we come up with a probability distribution on each $C_i, i = 1, 2, 3$.

Let us formalize the concept of an empirical probability distribution. We define the set of distributions over a general semiring R as follows

$$\mathcal{D}_R(X) := \left\{ d : X \to R : |\text{supp}(d)| < \infty; \sum_{x \in X} d(x) = 1 \right\}.$$

The reason why we want to relax the usual definition, where $R = \mathbb{R}_{\geqslant 0}$, is that different choices for R are suitable to study different types of scenarios. An important example is the case where R is the booleans $\mathbb{B}$. In this case we don't have information on the *probability* of a certain event to take place, but only on the *possibility* of it to occur.

Given a morphism $f : X \to Y$ in **Set**, we can define the map

$$\begin{array}{rccc} \mathcal{D}_R(f) : & \mathcal{D}_R(X) & \longrightarrow & \mathcal{D}_R(Y) \\ & d & \longmapsto & \left(D_f^d :: y \mapsto \sum_{x : f(x) = y} d(x) \right) \end{array}$$

This assignment on morphisms is functorial, in fact, let $d \in \mathcal{D}_R(X)$ and $z \in Z$, we have

$$\begin{aligned} (\mathcal{D}_R(g) \circ \mathcal{D}_R(f))(d)(z) = \mathcal{D}_R(g)(D_f^d)(z) = D_g^{D_f^d}(z) &= \sum_{g(y) = z} D_f^d(y) \\ &= \sum_{g(y)=z} \sum_{f(x)=y} d(x) = \sum_{g(f(x))=z} d(x) \\ &= D_{g \circ f}^d(d)(z) = \mathcal{D}_R(g \circ f)(d)(z). \end{aligned}$$

Moreover,

$$\mathcal{D}_R(\mathrm{id}_X)(d)(x) = D_{\mathrm{id}_X}^d(x) = \sum_{\mathrm{id}_X(y) = x} d(x) = d(x) = \mathrm{id}_{\mathcal{D}_R(X)}(d)(x).$$

Thus we can define the functor

$$\begin{array}{rccc} \mathcal{D}_R : & \textbf{Set} & \longrightarrow & \textbf{Set} \\ & X & \longmapsto & \mathcal{D}_R(X) \\ & f & \longmapsto & \mathcal{D}_R(f). \end{array}$$

We can compose this functor with the event sheaf $\mathcal{E}$ for a measurement X to obtain the presheaf

$$\mathcal{D}_R \mathcal{E} : \mathcal{P}(X)^{op} = \mathrm{Open}(X)^{op} \longrightarrow \textbf{Set}$$

Explicitly, we have $\mathcal{D}_R\mathcal{E}(U) = \mathcal{D}_R(O^U)$, and

$$\mathcal{D}_R\mathcal{E}(U \subseteq U') = \mathcal{D}_R(\rho_U^{U'}) : \mathcal{D}_R(U') \to \mathcal{D}_R(U) :: d \mapsto d\,|_U,$$

where

$$d\,|_U\,(s) = \sum_{s' \in \mathcal{E}(U') : s'|_U = s} d(s').$$

We can now introduce the concept of *empirical model*.

Definition 1.2.2. Let X be a set of measurement, O a set of outcomes, $\mathcal{M}$ a measurement cover for X and R a semiring. An *empirical model* for $\langle X, \mathcal{M}, O\rangle$ over R is a compatible family for the cover $\mathcal{M}$ with respect to the presheaf $\mathcal{D}_R\mathcal{E}$. More explicitly, it is a set of distributions $\{e_C\}_{C\in\mathcal{M}}$, one for each measurement context, such that

$$e_C\mid_{C\cap C'} = e_{C'}\mid_{C\cap C'} \quad \forall C, C' \in \mathcal{M}$$

If $R := \mathbb{R}_{\geqslant 0}$ the model is called *probabilistic*, if $R := \mathbb{B}$ it is called *possibilistic*

The following remark formalizes mathematically a rather simple intuition: every probabilistic empirical model generates a possibilistic empirical model given by its support. Although the abstraction introduced here is not really necessary as we have a clear intuition of the result, we think it is worth giving a full explanation of an argument that is repeatedly used in our reference articles, but never completely formalized.

Remark 1.2.3. Every probabilistic empirical model $\{e_C\}_{C\in\mathcal{M}}$ generates a possibilistic empirical model $\{\tilde{e}_C\}_{C\in\mathcal{M}}$ defined by

$$\begin{array}{rccc} \tilde{e}_C : & \mathcal{E}(C) & \longrightarrow & \mathbb{B} \\ & x & \longmapsto & \chi_{supp(e_C)}(s), \end{array}$$

where $\chi_{supp(e_C)}$ denotes the characteristic function of $supp(e_C)$. It is easy to show that this is indeed a possibilistic empirical model. First of all $\tilde{e}_C$ is a distribution over $\mathbb{B}$, in fact $supp(\tilde{e}_C) = supp(e_C)$ which is finite by definition of e_C, and

$$\bigvee_{s\in\mathcal{E}(C)} \tilde{e}_C(s) = \bigvee_{s\in\mathcal{E}(C)} \chi_{supp(e_C)}(s) = 1$$

since there must be at least one possible outcome for each context (otherwise e_C would not be a probability distribution). In order to show compatibility for $\{\tilde{e}_C\}_{C\in\mathcal{M}}$, let $s'' \in \mathcal{E}(C\cap C')$. We have

$$\begin{aligned} \tilde{e}_C\mid_{C\cap C'}(s'') &= \bigvee_{s:s|_{C\cap C'}=s''} \chi_{supp(e_C)}(s) \\ &= \begin{cases} 1 & \text{if } \exists s \in supp(e_C) : s\mid_{C\cap C'} = s'' \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{1.1}$$

and since

$$\begin{aligned} \exists s \in supp(e_C) : s\mid_{C\cap C'} = s'' &\Leftrightarrow e_C\mid_{C\cap C'}(s'') \neq 0 \\ &\Leftrightarrow e_{C'}\mid_{C\cap C'}(s'') \neq 0 \\ &\Leftrightarrow \exists s' \in supp(e_{C'}) : s'\mid_{C\cap C'} = s''. \end{aligned}$$

(1.1) must be equal to

$$\bigvee_{s':s'|_{C\cap C'}} \chi_{supp(e'_C)}(s') = \tilde{e}_{C'}\mid_{C\cap C'}(s'').$$

Thus we conclude that $\{\tilde{e}_C\}$ is indeed a possibilistic empirical model. Notice that, on the other hand, every possibilistic model can be seen as the support of a probabilistic model.

This model has also the following property: $\tilde{e}_C\mid_{C'} = \widetilde{e_C\mid_{C'}}$, in fact given a section $s' \in \mathcal{E}(C')$,

$$\begin{aligned}\tilde{e}_C\mid_{C'}(s') &= \bigvee_{s:s|_{C'}=s'} \chi_{supp(e_C)}(s') = \begin{cases}1 & \text{if } \exists s \in supp(e_C) : s\mid_{C'} = s' \\ 0 & \text{otherwise}\end{cases} \\ &= \chi_{supp(e_C|_{C'})}(s') = \widetilde{e_C\mid_{C'}}(s')\end{aligned} \tag{1.2}$$

It is time to introduce some concrete examples of empirical models.

Example 1.2.4.

- *The Bell Model*. The experiment is the following: we have a system and two agents Alice and Bob who can choose to perform one measurement each. Alice can choose between a_1 and a_2, and Bob between b_1 and b_2. All the measurement can give either 0 or 1 as an outcome, thus $O := \{0, 1\}$. We have $X := \{a_1, a_2, b_1, b_2\}$, and since measurements for the same agent cannot be performed jointly, we have a measurement cover composed by the sets $C_1 := \{a_1, b_1\}, C_2 := \{a_1, b_2\}, C_3 := \{a_2, b_1\}, C_4 := \{a_2, b_2\}$. Suppose that the successive repetition of the experience multiple times gives us the probability distributions described by Table 1. The reader can check that these probability distributions constitute in fact an empirical model. The model is called Bell's model, and it is very important since it is a formal version of Bell's argument, which lead to prove that non-locality and contextuality are indeed features of quantum mechanics. In the next pages we will present a formal way to show how to detect the contextuality in Bell's model.

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	$\frac{1}{2}$	0	0	$\frac{1}{2}$
a_1	b_2	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
a_2	b_1	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
a_2	b_2	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Table 1: Bell's model

- *Possibilistic Bell's model* Following the idea presented in Remark 1.2.3, we can consider the possibilistic empirical model induced by Bell's model. The experiment is exactly the same as before, but now we are only interested in the *possibility* of a certain event to happen, rather than the *probability* for it to occur. The resulting possibilistic model is summarized in Table 2.

- *Possibilistic Hardy model* A more interesting possibilistic empirical model is the one induced by Hardy's probabilistic model [Har93] and summarized in Table 6

The experiment underlying the examples of empirical models presented so far (i.e. two agents with two possible choices of measurements) is in fact a member of a more general class of scenarios, called *Bell-type scenarios*. The general description is the following: Suppose we have a disjoint family

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	1	0	0	1
a_1	b_2	1	1	1	1
a_2	b_1	1	1	1	1
a_2	b_2	1	1	1	1

Table 2: Possibilistic Bell's model

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	1	1	1	1
a_1	b_2	0	1	1	1
a_2	b_1	0	1	1	1
a_2	b_2	1	1	1	0

Table 3: Possibilistic Hardy model

$\{X_i\}_{i\in I}$. We can think of the index set I as labelling the parts of a system. Each X_i is the set of measurements that can be carried out at part i. The measurement cover $\mathcal{M}$ is defined to contain contexts of the form $\{x_i\}_i \in I$, where $x_i \in X_i$ for all $i \in I$. This corresponds to perform one and only one measurement for each part of the system. In general, we will call a (n, k, l) scenario a Bell-type scenario with n parts, each having k possible measurements, each with l possible outcomes. It is easy to see that both the Bell's model and the Hardy model are bipartite Bell-type scenarios (they both are $(2, 2, 2)$ scenarios), where the "parts" of the system are constituted by the one where Alice chooses a measurement, and the one where Bob chooses a measurement.

The following example refers to the GHZ model, whose underlying scenario is a tripartite Bell-type $(3, 2, 2)$ scenario (and can be generalized to a general n-partite model). This example will be crucial in Chapter 2.

- *GHZ model* We start by defining the GHZ state, a tripartite state of qubits, defined as
$$\mathsf{GHZ} := \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}$$
Each party $i = 1, 2, 3$ can perform Pauli measurements $\{X_i, Y_i\}$, where
$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
are the observables corresponding to measuring spin in the x and y axis respectively. Each one of these observables has eigenvalues (i.e. outcomes) -1 (spin down) or $+1$ (spin up). Thus we can set $\mathcal{M} := \{(A_1, A_2, A_3) \mid A_i \in \{X_i, Y_i\}\}$ (a more formal version of this cover will be given in Chapter 2). In Table 4 we have summarized a portion of the GHZ possibilistic model, where $- = -1$ and $+ = 1$. It will become clear later why this portion is the most relevant.

The compatibility condition satisfied by an empirical model is in fact quite important, and it is equivalent to a general form of *no-signaling*. To prove this, suppose we have a bipartite Bell-type scenario, with $C_1 := \{a, b_1\}$ and

1	2	3	$---$	$--+$	$-+-$	$-++$	$+--$	$+-+$	$++-$	$+++$
X_1	X_2	X_3	1	0	0	1	0	1	1	0
X_1	Y_2	Y_3	0	1	1	0	1	0	0	1
Y_1	X_2	Y_3	0	1	1	0	1	0	0	1
Y_1	Y_2	X_3	0	1	1	0	1	0	0	1

Table 4: GHZ possibilistic model

$C_2 := \{a, b_2\}$ (it is the same scenario as in the Bell model, but now Alice has only one possible choice) and an empirical model $\{e_{C_i}\}_{i=1,2}$. Let us fix a section $s_0 \in \mathcal{E}(\{m_a\})$. Compatibility for the empirical model implies

$$\sum_{s_1 \in \mathcal{E}(C_1): s_1|_{m_a} = s_0} e_{C_1}(s_1) = \sum_{s_2 \in \mathcal{E}(C_2): s_2|_{m_a} = s_0} e_{C_2}(s_2).$$

In words, this means that the probability for alice to have outcome s_0 for her measurement m_a is *independent of the choice of Bob*, which is exactly the statement of the no-signaling theorem of quantum mechanics.

1.3 CONTEXTUALITY

We now have everything we need to define contextuality in terms of this sheaf-theoretic approach. We have shown that the presheaf of events $\mathcal{E}$ is in fact a sheaf. We might wonder whether this is also true for the presheaf $\mathcal{D}_R\mathcal{E}$ defined with respect to a measurement cover $\mathcal{M}$. Let us translate the sheaf condition for $\mathcal{D}_R\mathcal{E}$ in words. Since empirical models are defined as compatible families for $\mathcal{D}_R\mathcal{E}$, to say that the sheaf condition holds for an empirical model $\{e_C\}_{C\in\mathcal{M}}$ is equivalent to say that there exists a *global* distribution $d \in \mathcal{D}_R\mathcal{E}(X)$, such that $d_C = e_C$ for all $C \in \mathcal{M}$. This means that there exists a distribution defined on all measurements, which marginalizes to explain the empirically observed probabilities. In the case of a bipartite Bell-type scenario, this means that whenever Alice and Bob choose their measurements, they are just looking at a portion of a *predetermined* set of outcomes, which is therefore *independent of their choice*. From a classical point of view of the above experiment, this seems to be the only possible explanation of the empirical distribution found. However, it turns out the we have examples of the failure of the sheaf condition for $\mathcal{D}_R\mathcal{E}$ for some empirical models. We will show this in detail in the next section.

1.3.1 Detecting contextuality: logical Bell's inequalities

We will now show that Bell's model 1 does not admit a global section. To prove this, we will use a very elegant logical and probabilistic argument, known as the *Bell's inequalities* [AH12]. Suppose we have N propositional formulas $\phi_1, \dots, \phi_N$. We think of the boolean variables appearing in each formula as empirically testable quantities. Thus each ϕ_i corresponds to a certain statement on the results of an experiment involving these quantities. Suppose we have an empirical probability distribution for the outcomes of the experiment. Then we can assign to each formula ϕ_i a probability p_i for it to be satisfied by the experiment. Let $\Phi := \bigwedge_{i=1}^{N} \phi_i$ and $P := \mathbb{P}[\Phi]$. We have

$$1 - P = \mathbb{P}[\neg\Phi] = \mathbb{P}\left[\bigvee_{i_1}^{N} \neg\phi_i\right] \leqslant \sum_{i=1}^{N} \mathbb{P}[\neg\phi] = \sum_{i=1}^{N} (1 - p_i) = N - \sum_{i=1}^{N} p_i \tag{1.3}$$

Now suppose ϕ_i cannot be all satisfied at the same time, then $\mathbb{P}[\Phi] = 0$. Thus inequality (1.3) becomes

$$\sum_{i=1}^{N} p_i \leqslant N - 1 \tag{1.4}$$

Let us consider again Bell's model presented in Table 1. We can actually restrict our analysis to the following entries

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$\frac{1}{2}$			$\frac{1}{2}$
a_1	b_2	$\frac{3}{8}$			$\frac{3}{8}$
a_2	b_1	$\frac{3}{8}$			$\frac{3}{8}$
a_2	b_2		$\frac{3}{8}$	$\frac{3}{8}$	

Table 5: Some entries of the possibilistic Bell's model

We associate to each section of the table a formula that describes it, where we take 0=false, 1=true. For instance, the first top left entry (which corresponds to the section $\{a_1 \mapsto 0, b_1 \mapsto 0\}$) represents the fact that both measurements a_1 and b_1 have returned a false result, thus the event can be represented by the formula $\neg a_1 \wedge \neg b_1$. With the same idea, we can associate to each row of Table 5 a formula that describes the entries in the row. We obtain

$$\begin{aligned}
\phi_1 &:= (\neg a_1 \wedge \neg b_1) \vee (a_1 \wedge b_1) = a_1 \Leftrightarrow b_1 \\
\phi_2 &:= (\neg a_1 \wedge \neg b_2) \vee (a_1 \wedge b_2) = a_1 \Leftrightarrow b_2 \\
\phi_3 &:= (\neg a_2 \wedge \neg b_1) \vee (a_2 \wedge b_1) = a_2 \Leftrightarrow b_1 \\
\phi_4 &:= (a_2 \wedge \neg b_2) \vee (\neg a_2 \wedge b_2) = a_2 \oplus b_2
\end{aligned}$$

It is straightforward to see that these formulas are jointly contradictory, in fact

$$b_2 \overset{\phi_2}{\Longleftrightarrow} a_1 \overset{\phi_1}{\Longleftrightarrow} b_1 \overset{\phi_3}{\Longleftrightarrow} a_2$$

and this contradicts ϕ_4. Thus, equation (1.4) tells us that

$$\sum_{i=1}^{4} p_i = 1 + \frac{6}{8} + \frac{6}{8} + \frac{6}{8} = \frac{13}{4} \overset{(1.4)}{\leqslant} 4 - 1 = 3$$

and this is clearly not true! In practical terms, this means that we cannot find a global probablity distribution that marginalizes to explain the one observed in the table, i.e. the empirical model *does not admit a global section*. We might simply conclude that such an empirical model cannot therefore exist. However, the core of the argument of Bell's theorem, is the proof of the fact that this model is actually realizable, and witnesses the feature of *contextuality* [Bel64]. Thus, it now makes sense to introduce the following definition

Definition 1.3.1. An empirical model on $\langle X, \mathcal{M}, O\rangle$ is called *contextual* if it does not admit a global section.

1.3.2 Classifying contextuality

When dealing with possibilistic empirical models we can refine our definition of contextuality by classifying it into different levels. Let us consider a probabilistic empirical model $\{e_C\}_{C\in\mathcal{M}}$ on $\langle X, \mathcal{M}, O\rangle$. The possibilistic empirical model generated by the support of $\{e_C\}_{C\in\mathcal{M}}$ can equivalently be described by a subpresheaf $\mathcal{S}$ of $\mathcal{E}$, where for each subset $U \subseteq X$, $\mathcal{S}(U) \subseteq O^U$ is the set of all possible local sections at U. Explicitly,

$$\mathcal{S}(U) := \{s \in \mathcal{E}(U) \mid s\,|_{U\cap C} \in \operatorname{supp}(e_C\,|_{U\cap C})\} \tag{1.5}$$

Remark 1.3.2. In Remark 1.2.3 we have given a more formal definition of the associated possibilistic empirical model $\{\tilde{e}_C\}_{C\in\mathcal{M}}$. Using this definition, the set of all possible local sections at U is

$$\{s \in \mathcal{E}(U) \mid \tilde{e}_C\,|_{U\cap C}\,(s\,|_{U\cap C}) = 1\}.$$

Notice that this definition is equivalent to (1.5), in fact

$$\begin{aligned}
&\{s \in \mathcal{E}(U) \mid \tilde{e}_C\,|_{U\cap C}\,(s\,|_{U\cap C}) = 1\} \\
&\overset{(1.2)}{=} \{s \in \mathcal{E}(U) \mid \widetilde{e_C\,|_{U\cap C}}(s\,|_{U\cap C}) = 1\} \\
&= \{s \in \mathcal{E}(U) \mid \chi_{\operatorname{supp}(e_C|_{U\cap C})}(s\,|_{U\cap C})) = 1\} \\
&= \{s \in \mathcal{E}(U) \mid s\,|_{U\cap C} \in \operatorname{supp}(e_C\,|_{U\cap C})\} \\
&= \mathcal{S}(U).
\end{aligned}$$

Inspired by this discussion, we can reformulate the notion of possibilistic empirical model in a more axiomatic way.

Definition 1.3.3. A possibilistic empirical model on $\langle X, \mathcal{M}, O\rangle$ is a subpresheaf $\mathcal{S}$ of $\mathcal{E}$ that satisfies the following properties

1. $\mathcal{S}(C) \neq \varnothing$ for all $C \in \mathcal{M}$.

2. $\mathcal{S}$ is *flasque beneath the cover*, i.e. the map $\mathcal{S}(U \subseteq U')$ is surjective whenever $U \subseteq U' \subseteq C$ for some $C \in \mathcal{M}$.

3. A *compatible family* for $\mathcal{M}$ is a family $\{s_C\}_{C\in\mathcal{M}}$ with $s_C \in \mathcal{S}(C)$ and such that
$$s_C\,|_{C\cap C'} = s_{C'}\,|_{C\cap C'} \quad \forall C, C' \in \mathcal{M}.$$
We assume that such a family induces a global section in $\mathcal{S}(X)$ (notice that this global section must be unique, since $\mathcal{S}$ is a subpresheaf of $\mathcal{E}$).

Remark 1.3.4. It is important to show that this definition is actually compatible with the previous ones.

- Condition 1. of Definition 1.3.3 translates in the fact that for each measurement context, there is at least one possible outcome. This is equivalent to the $\tilde{e}_C$s being probability distributions over the booleans.

- Condition 2. is a possibilistic version of no-signaling. In fact, consider again a bipartite Bell-type scenario, where Alice has only one option a, and Bob has two options $\{b_1, b_2\}$. Let $U := \{a\}$ and $U' := \{a, b_1\} = C$

(we chose to denote it by C since this set is a context of the cover). Let $s \in \mathcal{S}(\{a\})$ be a possible section. By condition 2. we know

$$\exists t_1 \in \mathcal{S}(\{a, b_1\}) : t_1 \mid_a = s \quad \text{and} \quad \exists t_2 \in \mathcal{S}(\{a, b_2\}) : t_2 \mid_a = s$$

This means that the fact that s is possible is independent of the choice of Bob (since we have two possible sections that restrict to s in both cases).

We can now further characterize contextuality as follows

Definition 1.3.5. Let $\mathcal{S}$ be a possibilistic empirical model on $\langle X, \mathcal{M}, O \rangle$.

- For all $C \in \mathcal{M}$ and $s \in \mathcal{S}(C)$, we say that $\mathcal{S}$ is *logically contextual* or *possibilistically contextual at* s (denoted $LC(\mathcal{S}, s)$) if s is not a member of any compatible family. We say that $\mathcal{S}$ is *logically contextual* (denoted $LC(\mathcal{S})$) if $LC(\mathcal{S}, s)$ for some s.
- We say that $\mathcal{S}$ is *strongly contextual* (denoted by $SC(\mathcal{S})$) if $LC(\mathcal{S}, s)$ for all s. By condition 3. this is equivalent to say that $\mathcal{S}(X) = \emptyset$.

Remark 1.3.6. These definitions can be extended to a probabilistic empirical model by simply considering its induced possibilistic model. In this case the subpresheaf is defined exactly as in (1.5).

We will now show that

$$\text{Strong contextuality} \Rightarrow \text{Possibilistic contextuality} \Rightarrow \text{Contextuality} \quad (1.6)$$

and that these implications are strict.

- The first implication is clearly true by definition of the concepts involved. It is also strict since the Hardy model (cf. Table 6) is possibilistically contextual but not strongly contextual. In fact, consider the following entries. By interpreting 0 as true and 1 as false, and following the same idea presented for the proof of the contextuality of Bell's model, we obtain the following four formulas

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	1			
a_1	b_2	0	1	1	1
a_2	b_1	0	1	1	1
a_2	b_2	1	1	1	0

Table 6: Some entries of the possibilistic Hardy model

$$\phi_1 := a_1 \wedge b_1 \quad \phi_2 := \neg(a_1 \wedge b_2) \quad \phi_3 := \neg(a_2 \wedge b_1) \quad \phi_4 := a_2 \vee b_2.$$

They are jointly contradictory, in fact

$$a_1 \wedge b_1 \overset{\phi_2}{\Longrightarrow} \neg b_2 \overset{\phi_4}{\Longrightarrow} a_2 \overset{\phi_3}{\Longrightarrow} \neg b_1$$

is an evident contradiction. Notice that for $i = 2, 3, 4$, each ϕ_i describes the full support of the Hardy model for the corresponding row. Thus we must have $p_2 = p_3 = p_4 = 1$, and we have

$$\sum_{i=1}^{4} p_i = 3 + p_1 \nleq 4 - 1 = 3.$$

This certainly proves that the model is contextual, but it actually tells us more, i.e. that it is logically contextual, as it implies for example that the section $s = \{a_1 \mapsto 0, b_1 \mapsto 0\}$ is not a member of any compatible family. In fact, if we try to find a compatible family by starting with s and picking the only possible compatible section in each context we end up with

$$s = \{a_1 \mapsto 0, b_1 \mapsto 0\} \to \{a_2 \mapsto 1, b_1 \mapsto 0\} \to \{a_2 \mapsto 1, b_2 \mapsto 0\} \\ \to \{a_1 \mapsto 0, b_2 \mapsto 0\}$$

which is not a possible section according to the model. On the other hand, the Hardy model is *not* strongly contextual, in fact the global assignment

$$\{a_1 \mapsto 1, a_2 \mapsto 0, b_1 \mapsto 1, b_2 \mapsto 0\}$$

is in $\mathcal{S}(X)$.

- In order to prove the second implication, we will consider the opposite statement. If a probabilistic empirical model $\{e_C\}_{C \in \mathcal{M}}$ is non-contextual, then there exists a global distribution d such that $d\,|_C = e_C$ for all $C \in \mathcal{M}$. According to what discussed in Remark 1.2.3, we have

$$\tilde{d}\,|_C \overset{(1.2)}{=} \widetilde{d\,|_C} = \tilde{e}_C.$$

Thus the support of the global distribution of the probabilistic empirical model is a global boolean distribution for the induced possibilistic model. Therefore, the latter must be logically non-contextual.

In order to show that the implication is strict we claim that Bell's model (cf. Table 1) is contextual (already proved) but not logically contextual. In fact, consider its possibilistic version (cf. Table 2). The first and last columns only contain ones, thus all the sections in those columns are members of a compatible family formed of sections of the column. The only sections we need to consider are thus the ones in red in Table 7.

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	1	0	0	1
a_1	b_2	1	1	1	1
a_2	b_1	1	1	1	1
a_2	b_2	1	1	1	1

Table 7: Possibilistic Bell's model

It is easy to check that each one of them is contained in a compatible family. In fact, since all the sections of rows 3 and 4 are possible, we only need to find a compatible possible section in the first row. Since both $\{a_1 \mapsto 0, b_1 \mapsto 0\}$ and $\{a_1 \mapsto 1, b_1 \mapsto 1\}$ are possible, we can always find such a compatible section.

Our dissertation mainly focuses on studying possibilistic and strong contextuality. For this reason, from now on, every model we will define will be a possibilistic model, unless otherwise specified.

1.4 REPRESENTING COVERS AND EMPIRICAL MODELS

As mentioned before, an essential ingredient for the existence of contextuality phenomena is the incompatibility of measurements. If all the measurements of a scenario could be performed at the same time, the whole discussion carried out so far would be meaningless for that particular scenario, as it would obviously be non-contextual. This suggests that the cover $\mathcal{M}$ actually plays an important role in determining what kind of contextuality we can witness on it. In the following paragraphs, we will introduce a formal way of representing measurements covers that will be useful in the study of their properties. We will also see that we can use this approach to graphically represent empirical models defined on simple covers, allowing us to intuitively analyse their properties and construct explicit examples. We will take full advantage of this approach in Chapter 3.

1.4.1 Abstract simplicial complexes

Abstract simplicial complexes can accurately represent both the combinatorial and geometrical properties of measurement covers. We will adopt the notations and general guidelines of [Bar15].

Definition 1.4.1. An *abstract simplicial complex* Σ on a set X (called the *set of veritices* $V(\Sigma)$) is a non-empty downwards closed family of finite subsets of X containing all the singletons. Explicitly, Σ is a subset of $\mathcal{P}(X)_{<\infty} := \{S \subseteq X \mid |S| < \infty\}$ such that

1. The empty set is contained in Σ.
2. For all $x \in X$, we have $\{x\} \in \Sigma$
3. We have $\Sigma = \downarrow \Sigma$, where
$$\downarrow \Sigma := \{\tau \in \mathcal{P}(X)_{<\infty} \mid \exists \sigma \in \Sigma : \tau \subseteq \sigma\}.$$

We call each element $\sigma \in \Sigma$ a *face* of the complex. A *maximal face* (in the inclusion order) is called a *facet*.

Let $\mathcal{M}$ be a measurement cover on a set of measurements X. We can describe the cover $\mathcal{M}$ as an abstract simplicial complex $\Sigma_{\mathcal{M}}$ whose faces are sets of compatible measurements, and whose facets are the measurement contexts $C \in \mathcal{M}$. The antichain condition of Definition 1.1.2 insures that $\Sigma_{\mathcal{M}}$ is well-defined.

Example 1.4.2.

- Suppose we have a scenario where all the measurements can be jointly performed. In other words, $\mathcal{M} = \{X\}$. Then, the associated simplicial complex is called the *trivial complex* or *simplex* on X and it is denoted by $\Delta_X := \mathcal{P}(X)$. For an $n \in \mathbb{N}$, we will use the abbreviation Δ_n to denote $\Delta_{\{1,\dots,n\}}$ (called the $n-1$ *simplex*). Note that every trivial complex is isomorphic to Δ_n for some $n \in \mathbb{N}$. In Figure 1 and 2 we have illustrated some examples.
- The antipoidal situation is represented by a scenario where all the measurements are mutually non-compatible (i.e. we can perform only one

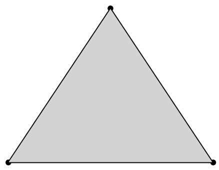


Figure 1: The simplex Δ_3 (filled triangle).

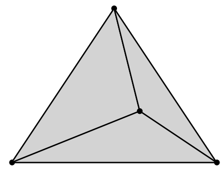


Figure 2: The simplex Δ_4 (solid tetrahedron).

measurement at a time). This corresponds to a cover $\mathcal{M} := \{\{x\} \mid x \in X\}$, which determines the so-called *discrete simplicial complex* defined by

$$\mathcal{D}_X := \{\varnothing\} \cup \{\{x\} \mid x \in X\}.$$

Once again, we denote by $\mathcal{D}_n$ the discrete complex $\mathcal{D}_{\{i,\dots,n\}}$. Note that every discrete simplicial complex is isomorphic to $\mathcal{D}_n$ for some $n \in \mathbb{N}$.

$$\underbrace{\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet}_{n}$$

Figure 3: The complex $\mathcal{D}_n$

- A more interesting type of scenario is the one where every j measurements are compatible, where $1 \leqslant j \leqslant |X|$ is a natural number. This corresponds to the cover $\mathcal{M} := \{S \subseteq X \mid |S| = j\}$ and gives rise to the complex

  $$\Delta_X^{(\leqslant j)} := \{\sigma \in \Delta_X \mid |\sigma| \leqslant j\}..$$

  Once again, we denote $\Delta_n^{(\leqslant j))} := \Delta_{\{1,\dots,n\}}^{(\leqslant j)}$.

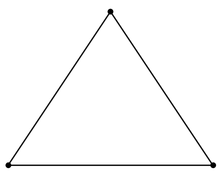


Figure 4: The complex $\Delta_3^{(\leqslant 2)}$ (unfilled triangle).

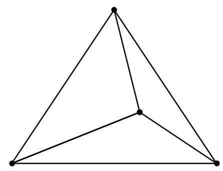


Figure 5: The complex $\Delta_4^{(\leqslant 2)}$ (edges of a tetrahedron).

Composing scenarios

In order to model experimental scenarios involving more parties (like in the case of Bell-type scenario) using abstract simplicial complexes, we need to be able to compose complexes representing the available set of measurements at each part. The natural way to perform this operation is to use the simplicial join.

Definition 1.4.3. Let Σ_1 and Σ_2 be simplicial complexts. Their *simplicial join* $\Sigma_1 * \Sigma_2$ is the simplicial complex on the vertices $V(\Sigma_1) \sqcup V(\Sigma_2)$ with faces

$$\begin{aligned}\Sigma_1 * \Sigma_2 &:= \{\sigma \subseteq V(\Sigma_1) \sqcup V(\Sigma_2) \mid \sigma \cap V(\Sigma_1) \in \Sigma_1 \wedge \sigma \cap V(\Sigma_2) \in \Sigma_2\} \\ &= \{\sigma_1 \sqcup \sigma_2 \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\} \\ &\cong \Sigma_1 \times \Sigma_2\end{aligned}$$

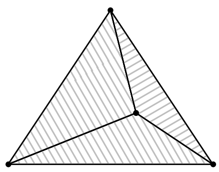


Figure 6: The complex $\Delta_4^{(\leqslant 3)}$ (hollow tetrahedron).

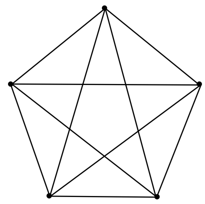


Figure 7: The complex $\Delta_5^{(\leqslant 2)}$.

Using this definition, we can for instance describe the usual bipartite Bell-type scenario with Alice and Bob as $\mathcal{D}_A * \mathcal{D}_B$, where A and B are the set of available measurements for Alice and Bob respectively. The fact that each party is represented via a discrete complex is due to the fact that the two agents can only choose 1 single measurement at a time. In general, we can describe an n-partite scenario as a complex of the form

$$\Sigma_1 * \cdots * \Sigma_n,$$

where each Σ_i describes the compatibility relations between measurements available to the i-th party of the experiment. In particular, we can see that an n-partite scenario is a Bell-type n-partite scenario if and only if it is of the form

$$\mathcal{D}_{A_1} * \cdots * \mathcal{D}_{A_n},$$

where A_i is the set of measurements available at part i. Most of the time, we will be dealing with the special case of (n, k, l) scenarios. While the parameter l is defined by the outcome set O, the cover of such a scenario can be written as

$$\mathcal{D}_k^{*n} := \mathcal{D}_k * \cdots * \mathcal{D}_k,$$

since every part has k possible measurements. In Figure 8, we represented the cover of the Bell model and the Hardy model, in Figure 10 the cover of the GHZ model is shown. Figure 9 represents a scenario with two agents, the first having four possible measurements, and the second only two.

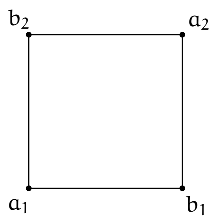


Figure 8: A $(2, 2, l)$ Bell-Type scenario ($\mathcal{D}_2 * \mathcal{D}_2$).

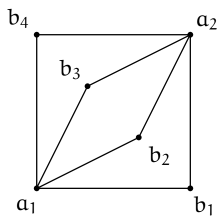


Figure 9: A Bell-type scenario $\mathcal{D}_2 * \mathcal{D}_4$.

We can summarise the idea behind these representations by simply stating that an edge connects two vertices whenever the corresponding measurements are compatible (i.e. they can be carried out jointly). Contexts are represented by maximal faces.

1.4.2 A graphical representation of empirical models

As briefly mentioned in the introduction to the section, simplicial complexes turn out to be very useful in representing empirical models diagrammatically. If the measurement cover is simple enough, we can add a representation of the outcome set which allows us to illustrate empirical models

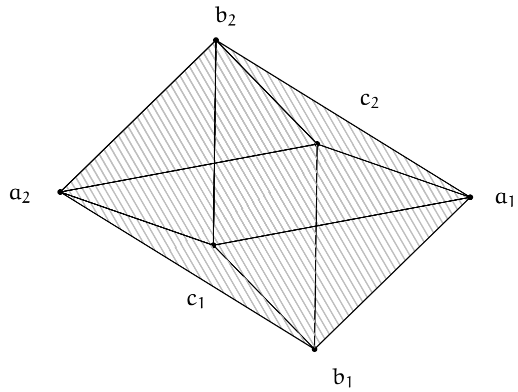


Figure 10: A $(3, 2, l)$ Bell-type scenario (hollow octahedron) $(\mathcal{D}_2 * \mathcal{D}_2 * \mathcal{D}_2)$

on the scenario. For instance, in Figure 11 (left), we represented a $(2, 2, 2)$ Bell-type model. At the base we can see the complex relative to the cover (cf.

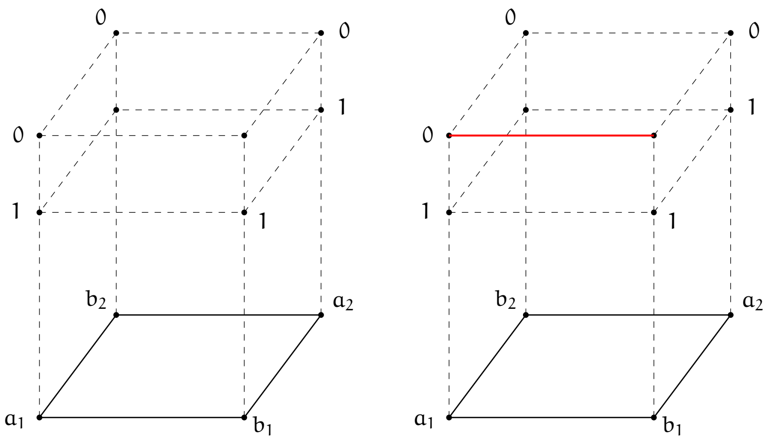


Figure 11: Diagram of a $(2, 2, 2)$ Bell-type scenario with $O := \{0, 1\}$

Figure 8), and on top of each vertex we added the possible outcomes (0 or 1) for the corresponding measurement. When necessary, we will refer to these new vertices added on top as *outcome-vertices* (as opposed to *measurement-vertices*). Given this setting, we can represent a local section of a model at a context as an edge connecting the outcome-vertices corresponding to the measurements involved according to the section. As an example, on the right hand side of Figure 11 we highlighted the section $(a_1, b_1) \mapsto (0, 0)$. We will say that an edge is *above* a context C, if it represents a local section at C.

With this idea in mind, we can give the graphical representation of some of our example models.

Notice that, if we add more "floors" of outcome-vertices to this prism-shaped diagram, we can model $(2, 2, l)$ scenarios. We will often use this representation to intuitively analyse empirical models and to give new examples. Therefore, it is important to understand how the various definitions translate in diagrammatic terms.

- A diagram representing an empirical model has to verify the following rules
  - Each context must have at least one edge above it. This is due to condition 1. of Definition 1.3.3. For instance, the right hand side

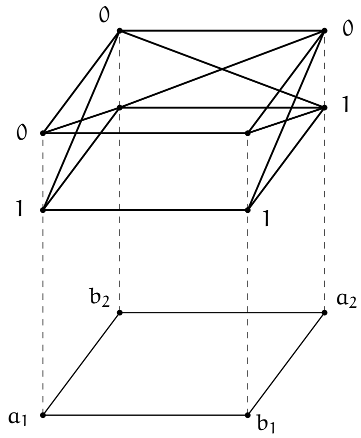


Figure 12: Diagram of the possibilistic Bell model

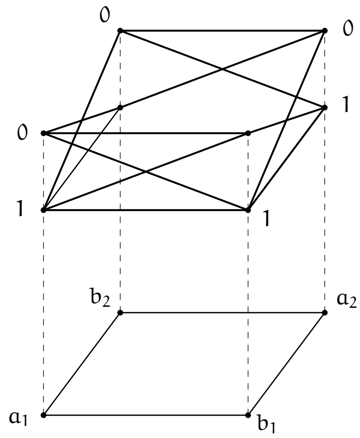


Figure 13: Diagram of the Hardy model

of Figure 11 does not represent an empirical model since three contexts do not have any edge above them.

– If an outcome-vertex v is touched by an edge above a context C, then it must be touched by at least one edge above every other context C' involving v. It is easy to see that this is the diagrammatic equivalent to no-signaling.

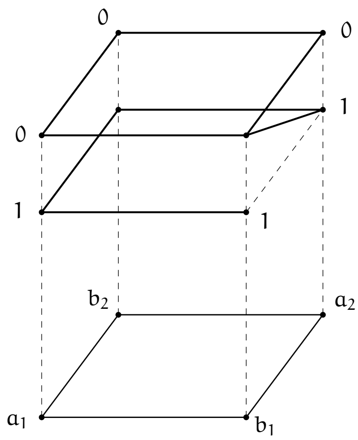


Figure 14: This model does not verify no-signaling

For instance, the diagram of Figure 14 does not represent an empirical model since it fails to verify no-signaling. In fact, we can see that the outcome-vertex corresponding to $b_1 \mapsto 1$ is touched by the edge representing the section $(a_1, b_1) \mapsto (1, 1)$ but is not touched by any edge above the context (a_2, b_1). This means that the choice of Alice influences the possibility for Bob to have outcome 1 for measurement b_1. In fact, it is possible for Bob to obtain $b_1 \mapsto 1$ only if Alice chooses a_1.

- A compatible family is represented diagrammatically by a taking a single edge above each context in such a way that they all touch each other at outcome-vertices (cf. Figure 15).

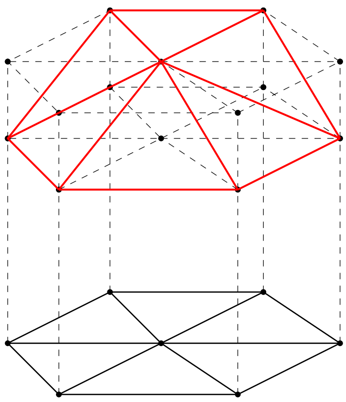


Figure 15: Example of a compatible family on a general scenario

In the case of a scenario whose cover can be represented by a polygonal graph, this corresponds to a loop running above the polygon one time (and only one!), as shown in Figure 16.

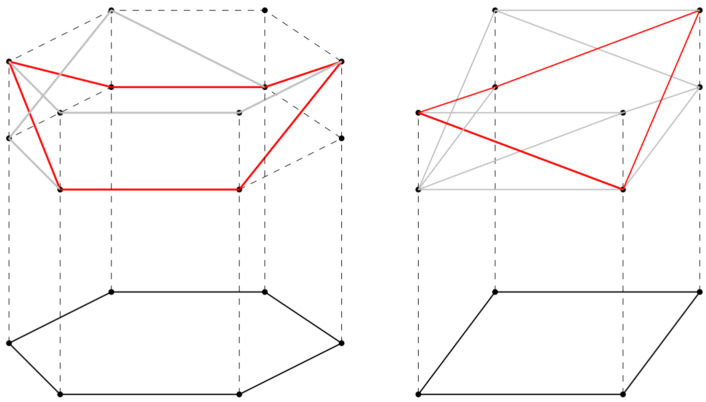


Figure 16: If the cover is a polygon, a compatible family is a loop

We will show with a simple example how we can intuitively prove the contextuality of an empirical model.

Example 1.4.4. Consider the PR-Box empirical model on a $(2, 2, 2)$ Bell-type scenario shown in Table 8.

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

Table 8: PR-Box model

We can show that none of its sections are extendable to a compatible family by simply looking at its diagram. In Figure 17 we highlighted in red the section $(a_1, b_1) \mapsto (0, 0)$. We can clearly see that this section is not a part of a loop. In fact, if we try to extend by proceeding, say, counterclockwise and picking the only compatible edge at each outcome-vertex, we do not obtain a full loop (cf. black sections on the right panel of Figure 17)

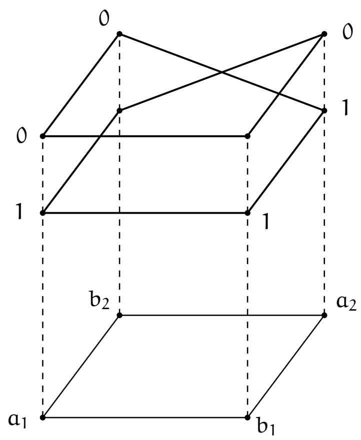


Figure 17: Diagram of the PR-Box model

We can apply the same graphical argument to any other possible section in the model, and conclude that the PR-box model is strongly contextual. In this example we can see particularly well the fact that we are not allowed to travel more than once around the cover.

The PR-Box model is one of the most important examples of strongly contextual models. It is actually the only strongly contextual model on a $(2,2,2)$ scenario [AB11, Lal13, SM13] (cf. proposition 2.6.4 of [Man13] for a proof).

1.5 VOROB'EV'S THEOREM

We already mentioned that the compatibility structure of a scenario is an important factor of the contextuality phenomenon. In this section we will give more details on this particular subject. We will take inspiration from a result due to Vorob'ev [Vor62] that – appropriately rephrased to fit our discussion – determines with precision which measurements covers admit contextual models defined on them. More specifically, Vorob'ev's theorem tells us that a sufficient and necessary condition for the cover to admit contextual no-signaling models is non-*acyclicity*. We will follow the approach proposed in [Bar15] to define this notion.

1.5.1 Acyclicity of measurement covers

We can intuitively think of an acyclic measurement scenario as a cover which can be constructed inductively by starting with the empty set and adding a new measurement at a time, in such a way that the new measurement is a member of one and only one context. This intuition can be captured using the Graham-reduction.

Definition 1.5.1. Let $\langle X, \mathcal{M}, O\rangle$ be a measurement scenario. Consider the complex $\Sigma_{\mathcal{M}}$ associated to the cover $\mathcal{M}$. For each context $C \in \mathcal{M}$ (i.e. each facet σ_C of $\Sigma_{\mathcal{M}}$), we denote by π_C the set of vertices of $\Sigma_{\mathcal{M}}$ which belong to σ_C and not to any other facet.

$$\pi_C := \{x \in V(\Sigma) \mid (x \in \tau \Rightarrow \tau \subseteq \sigma_C), \forall \tau \in \Sigma_{\mathcal{M}}\}$$

If $\pi_C \neq \varnothing$ for some $C \in \mathcal{M}$, we say that there is a *Graham-reduction* step from $\Sigma_{\mathcal{M}}$ to the subcomplex

$$\Sigma' := \Sigma_{\mathcal{M}} |_{V(\Sigma_{\mathcal{M}})\setminus\pi_C} = \{\sigma \in \Sigma_{\mathcal{M}} \mid \sigma \cap \pi_C = \varnothing\} = \{\sigma\setminus\pi_C \mid \sigma \in \Sigma_{\mathcal{M}}\}$$

consituted by all the vertices except the ones in π_C. In this case, the Graham-reduction from $\Sigma_{\mathcal{M}}$ to Σ' is denoted by $\Sigma_{\mathcal{M}} \rightsquigarrow \Sigma'$. The cover $\mathcal{M}$ (and the whole scenario) is said to be *acyclic* is there is a series of Graham-reduction steps

$$\Sigma_{\mathcal{M}} =: \Sigma_0 \rightsquigarrow \Sigma_1 \rightsquigarrow \cdots \rightsquigarrow \Sigma_n = \Delta_0.$$

In Figure 18 we illustrate an example of Grahm reduction in the case of both a cyclic and acyclic cover. In red it is highlighted the vertex remove at each step.

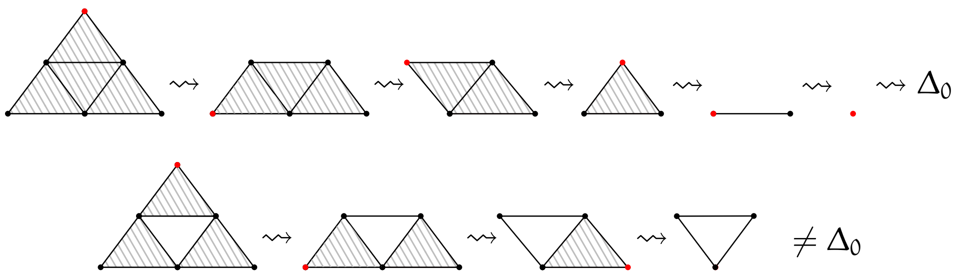


Figure 18: Example of an acyclic (top) and cyclic (bottom) cover.

We finally present Vorob'ev's theorem.

Theorem 1.5.2 (Vorob'ev)**.** *Let $\langle X, \mathcal{M}, O\rangle$ be a scenario. Any probabilistic empirical model defined on $\langle X, \mathcal{M}, O\rangle$ is non-contextual if and only if $\Sigma_{\mathcal{M}}$ is acyclic.*

Proof. We will only give a formal proof for the backward direction. Suppose $\Sigma_{\mathcal{M}}$ is acyclic. If $\Sigma_{\mathcal{M}} = \{*\}$, then any model defined on $\mathcal{M}$ is clearly noncontextual since we only have one measurement available. We then proceed by induction on a Grahm reduction. Suppose $\Sigma_{\mathcal{M}}$ can be Grahm-reducted via the following steps

$$\Sigma_{\mathcal{M}} =: \Sigma_0 \rightsquigarrow \Sigma_1 \rightsquigarrow \cdots \rightsquigarrow \Sigma_n = \Delta_0,$$

we will show that if every model is non-contextual on the cover represented by Σ_i than the same statement is true for Σ_{i-1}. Suppose Σ_i only admits non-contextual models and let C be the context of Σ_i whose vertices got removed in the Grahm-reduction process. Recall that probabilistic empirical models are compatible families for the presheaf $\mathcal{D}_R\mathcal{E}$, and they are non-contextual if and only if there exists a global sections that marginalises to the model. Let $\{e_\sigma \in \mathcal{D}_R\mathcal{E}(\sigma)\}_{\sigma\in\Sigma_{i-1}}$ be an empirical model (to be precise, the empirical model is defined only on the maximal faces, which correspond to contexts, but this description is equivalent). Then, restricting the model to the simplices in Σ_i, we obtain a compatible family $\{e_{\sigma'} \in \mathcal{D}_R\mathcal{E}(\sigma')\}_{\sigma'\in\Sigma_i}$. By inductive hypothesis, there exists a global section $g \in \mathcal{D}_R\mathcal{E}(X\setminus\pi_C)$ such that $g|_{\sigma'} = e_{\sigma'}$ for all $\sigma' \in \Sigma_i$.

Note that $g|_{C\setminus\pi_C} = e_C|_{C\setminus\pi_C}$, i.e. $e_C \in \mathcal{D}_R\mathcal{E}(C)$ and $g \in \mathcal{D}_R\mathcal{E}(X\setminus\pi_C)$ agree on their restriction to $C \cap (X\setminus\pi_C) = C\setminus\pi_C$. This allows us to define the following distribution on $\mathcal{E}(C \cup (X\setminus\pi_C)) = \mathcal{E}(X)$: for each $\nu \in \mathcal{E}(X)$,

$$d(\nu) := \begin{cases} \dfrac{e_C(\nu|_C)\cdot g\left(\nu|_{X\setminus\pi_C}\right)}{e_C|_{C\setminus\pi_C}\left(\nu|_{C\setminus\pi_C}\right)} = \dfrac{e_C(\nu|_C)\cdot g\left(\nu|_{X\setminus\pi_C}\right)}{g|_{C\setminus\pi_C}\left(\nu|_{C\setminus\pi_C}\right)} & \text{if } e_C|_{C\setminus\pi_C}\left(\nu|_{C\setminus\pi_C}\right) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We leave to the reader the verification of the fact that d is indeed a distribution on $\mathcal{E}(X)$ (cf. [Bar15, Proposition VI.15] for deeper insights). This distribution has the following property

$$d\,|_C = e_C \quad\text{and}\quad d\,|_{X\setminus\pi_C} = g,$$

in fact, if $\nu \in C$, we have

$$\begin{aligned}
d\,|_C\,(\nu) &= \frac{e_C\,(\nu)\cdot g\left(\nu\,|_{X\setminus\pi_C}\right)}{g\,|_{C\setminus\pi_C}\left(\nu\,|_{C\setminus\pi_C}\right)} = \frac{e_C\,(\nu)\cdot g\left((\nu\,|_C)\,|_{X\setminus\pi_C}\right)}{g\,|_{C\setminus\pi_C}\left(\nu\,|_{C\setminus\pi_C}\right)} \\
&= \frac{e_C\,(\nu)\cdot g\left(\nu\,|_{C\setminus\pi_C}\right)}{g\,|_{C\setminus\pi_C}\left(\nu\,|_{C\setminus\pi_C}\right)} = \frac{e_C\,(\nu)\cdot g\,|_{C\setminus\pi_C}\left(\nu\,|_{C\setminus\pi_C}\right)}{g\,|_{C\setminus\pi_C}\left(\nu\,|_{C\setminus\pi_C}\right)} = e_C(\nu).
\end{aligned}$$

and if ν is in $X\setminus\pi_C$, then

$$\begin{aligned}
d\,|_{X\setminus\pi_C}\,(\nu) &= \frac{e_C\,(\nu\,|_C)\cdot g\,(\nu)}{e_C\,|_{C\setminus\pi_C}\left(\nu\,|_{C\setminus\pi_C}\right)} = \frac{e_C\left(\left(\nu\,|_{X\setminus\pi_C}\right)|_C\right)\cdot g\,(\nu)}{e_C\,|_{C\setminus\pi_C}\,(\nu)} \\
&= \frac{e_C\left(\nu\,|_{C\setminus\pi_C}\right)\cdot g\,(\nu)}{e_C\,|_{C\setminus\pi_C}\,(\nu)} = \frac{e_C\,|_{C\setminus\pi_C}\left(\nu\,|_{C\setminus\pi_C}\right)\cdot g\,(\nu)}{e_C\,|_{C\setminus\pi_C}\,(\nu)} = g(\nu).
\end{aligned}$$

This means that d is a global section for the original model on Σ_i. In fact, each $\sigma \in \Sigma_i$ is either in Σ_{i-1} or contained in C. In the first case we have

$$d\,|_\sigma = d\,|_{X\setminus\pi_C}|_\sigma = g\,|_\sigma = e_\sigma,$$

and in the second

$$d\,|_\sigma = d\,|_C|_\sigma = e_C\,|_\sigma = e_\sigma.$$

By induction, we conclude that if the cover is acyclic, then it is impossible to define a contextual model on it. □

Notice that the hierarchy of contextuality established in (1.6) implies that acyclicity of the cover implies also the impossibility to define possibilistic or strongly contextual models on the cover.

2 ALL VS NOTHING ARGUMENTS

In this Chapter, we will introduce All vs Nothing arguments as proofs of the contextuality of empirical models. We will start by a simple scenario, and we will then generalise it mathematically taking full advantage of the theory of stabilisers and their connection with quantum physics.

2.1 INTRODUCTION

In the previous Chapter we have seen a first method to detect contextuality using the logical implications induced by an empirical model. In [AB11], a more algebraic method is presented. We now want to focus our attention on a recently developed criterion that involves *All vs Nothing arguments* (AvN in short). The motivating example is Mermin's proof of the strong contextuality of the GHZ model [Mer90]. We already introduced a partial table for this model in Example 1.2.4, Table 4. We can summarize Mermin's argument by only considering this partial table. Since each observable involved has eigenvalues either -1 or $+1$, and the result of joint measurement of three observables must be the product of the eigenvalues corresponding to their outcomes, Table 4 yields the following equations

$$\begin{aligned} \overline{X}_1 \cdot \overline{X}_2 \cdot \overline{X}_3 &= -1 \\ \overline{X}_1 \cdot \overline{Y}_2 \cdot \overline{Y}_3 &= 1 \\ \overline{Y}_1 \cdot \overline{X}_2 \cdot \overline{Y}_3 &= 1 \\ \overline{Y}_1 \cdot \overline{Y}_2 \cdot \overline{X}_3 &= 1, \end{aligned} \tag{2.1}$$

where $\overline{A}$ denotes the eigenvalue observed after performing measurement $A \in \{X, Y\}$. If we multiply the left hand side of (2.1), we obtain 1 (since every variable occurs twice), whereas the right hand side gives -1 (this motivates the term *All vs Nothing*). Thus system (2.1) is inconsistent. Therefore, we can conclude that the model is strongly contextual, since we cannot find any global assignment of outcomes to observables consistent with the local ones.

Motivated by this example, we can generalize this argument to a much larger class of states. Let us introduce the basic mathematical setting. We refer to Section 10.5 of [NC11] for some of the following definitions.

Definition 2.1.1. The *Pauli n-group* $\mathcal{P}_n$ is the group whose elements are n-tuples $(A_i)_{i=1^n}$ of Pauli operators (i.e. $A_i \in \{X_i, Y_i, Z_i, I_i\}$), with global phase contained in $\{\pm 1, \pm i\}$. The multiplication is componentwise matrix multiplication, and the unit is $(I_i)_{i=1}^n$.

The group $\mathcal{P}_n$ acts on $\mathbb{C}^n$ as follows

$$\begin{array}{ccc} \mathcal{P}_n \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ ((A_i)_{i=1}^n, |\psi\rangle) & \longmapsto & (A_i)_{i=1}^n \bullet |\psi\rangle := \left(\prod_{i=1}^n A_i\right) |\psi\rangle \end{array}$$

This is indeed an action since $\left(\prod_{i=1}^{n} I_i\right)|\psi\rangle = I|\psi\rangle = |\psi\rangle$, and for each $A := (A_i)_{i=1}^{n}$, $B := (B_i)_{i=1}^{n}$ in $\mathcal{P}_n$,

$$A \bullet (B \bullet |\psi\rangle) = A \bullet \left(\prod_{i=1}^{n} B_i\right)|\psi\rangle = \left(\prod_{i=1}^{n} A_i \cdot B_i\right)|\psi\rangle = (A \cdot B) \bullet |\psi\rangle.$$

Definition 2.1.2. let $S \leqslant \mathcal{P}_n$ be a subgroup of $\mathcal{P}_n$. The *stabilizer* of S is defined as the sub-vector space

$$V_S := \{|\psi\rangle \in \mathbb{C}^n \mid A \bullet |\psi\rangle \ \forall A \in S\}$$

It is easy to see that V_S is a sub-vector space of $\mathbb{C}^n$, in fact we have $0 \in V_S$ and, given $|\phi\rangle, |\psi\rangle \in V_S$ and $\lambda, \mu \in \mathcal{C}$, we have

$$A \bullet (\lambda|\phi\rangle + \mu|\psi\rangle) = \lambda A \bullet |\phi\rangle + \mu A \bullet |\psi\rangle = \lambda|\phi\rangle + \mu|\psi\rangle.$$

The following lemma gives a simple characterization of the stabilizers.

Lemma 2.1.3. *For any subgroup S of the Pauli n- group, we have*

$$V_S = \bigcap_{A \in S} V_{\{A\}}.$$

Proof.

$$\begin{aligned} |\psi\rangle \in V_S &\Leftrightarrow A \bullet |\psi\rangle = |\psi\rangle \ \forall A \in S \Leftrightarrow |\psi\rangle \in V_{\{A\}} \ \forall A \in S \\ &\Leftrightarrow |\psi\rangle \in \bigcap_{A \in S} V_{\{A\}}. \end{aligned}$$

□

Let us recall the notions of *Galois connection* and *Galois correspondence*

Definition 2.1.4. Let A and B be posets. An *antitone Galois connection* between A and B is a pair of order reversing maps $f : A \to B$, $g : B \to A$ such that $a \leqslant g(f(a))$ for all $a \in A$, and $b \leqslant f(g(b))$ for all $b \in B$. We say that the connection is an *antitone Galois correspondence* if $a = g(f(a))$ for all $a \in A$, and $b = f(g(b))$ for all $b \in B$.

Let $\mathcal{SG}(\mathcal{P})$ be the set of all subgroups of the Pauli n-group, the definition of stablizer gives us the following bijection

$$F : \mathcal{SG}(\mathcal{P}) \longrightarrow \{V_S \mid S \in \mathcal{SG}(\mathcal{P}_n)\} :: S \longmapsto V_S.$$

We can think of $\mathcal{SG}(\mathcal{P})$ and $\{V_S \mid S \in \mathcal{SG}(\mathcal{P}_n)\}$ as posets with order induced by inclusion. We can then formulate the following result.

Proposition 2.1.5. *The maps F and $G := F^{-1}$ form an antitone Galois correspondence between $(\mathcal{SG}(\mathcal{P}), \subseteq)$ and $(\{V_S \mid S \in \mathcal{SG}(\mathcal{P}_n)\}, \subseteq)$.*

Proof. Since an antitone Galois correspondence is essentially a pair of order reversing maps that are inverse of each other, and we already know that $G = F^{-1}$, all we need to check is that F is order-reversing (then its inverse G will also automatically be order-reversing). This is easily proved using Lemma 2.1.3 (and we can see it clearly even without it), in fact suppose $S \subseteq T$ in $\mathcal{SG}(\mathcal{P})$, then

$$V_T \overset{(2.1.3)}{=} \bigcap_{A \in T} V_{\{A\}} \subseteq \bigcap_{A \in S} V_{\{A\}} \overset{(2.1.3)}{=} V_S.$$

□

In the Appendix, Section 6.1, we give some more details on the matter. In particular we show how this Galois correspondence is induced by a Galois connection between $\mathcal{SG}(\mathcal{P})$ and $\mathcal{SS}(\mathbb{C}^n)$, the set of all sub-vector spaces of $\mathbb{C}^n$.

2.1.1 AvN arguments for subgroups of $\mathcal{P}_n$

We will now generalize Mermin's argument. Let us start by considering a general observable A. Recall the formula for the expected value of A on a state $|\psi\rangle$ in quantum mechanics

$$\langle A \rangle_\psi := \langle \psi | A | \psi \rangle.$$

Note that we must have

$$\langle \psi | A | \psi \rangle = 1 \Leftrightarrow A|\psi\rangle = |\psi\rangle. \tag{2.2}$$

Suppose A is a dichotomic observable with eigenvalues ± 1 (like for instance an observable in $\mathcal{P}_1$). As reported in [ABK+15], we can relabel $+1, -1, \times$ as $0, 1, \oplus$ respectively. Since eigenvalues of a joint measurement $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ are the products of the eigenvalues of the measurements at each part of the system, they are also ± 1. Therefore, joint measurements remain dichotomic and can only distinguish joint outcomes up to parity. This means that if A is dichotomic and $|\psi\rangle$ is stabilised by A, due to (2.2), the support of the distribution on joint outcomes obtained by measuring A on $|\psi\rangle$ must contain only outcomes of even parity (whereas if $-A$ stabilises $|\psi\rangle$, then it will contain only outcomes of odd parity).

Let us now bring back $\mathcal{P}_n$ in the discussion and suppose $A = (A_i)_{i=1}^n \in \mathcal{P}_n$. The argument above tells us that if A stabilises $|\psi\rangle$ then the following equation must hold

$$\bigoplus_{i=1}^{n} x_i = 0,$$

where we have associated the variable x_i to A_i (as its outcome). On the other hand, if $-A$ stabilises $|\psi\rangle$, we must have

$$\bigoplus_{i=1}^{n} x_i = 1.$$

Now, consider a subgroup S of $\mathcal{P}_n$ and let $|\psi\rangle \in V_S$. Since every $A \in S$ stabilises $|\psi\rangle$, we can associate to each $A \in S$ an equation similar to the ones above. We thus obtain a system of $|S|$ equations. Inspired by Mermin's argument we formulate the following definition

Definition 2.1.6. We say that a subgroup S of $\mathcal{P}_n$ *gives rise to an All vs Nothing argument (AvN)* for a state $|\psi\rangle$ in V_S if the associated system of equations is inconsistent (thus showing the strong contextuality of the empirical model obtained by selecting the elements of S as measurement contexts).

Let us give an example.

Example 2.1.7 (Cluster States)**.** The *cluster states* are fundamental resources in measurement-based quantum computation. We define the 4-qubit 1-dimensional cluster state as the state stabilised by the subgroup S of $\mathcal{P}_4$ generated by $\{e, f, g, h\}$, where

$$e := (X_1, Z_2, I_3, I_4), \quad f := (Z_1, X_2, Z_3, I_4),$$
$$g := (I_1, Z_2, X_3, Z_4), \quad h := (I_1, I_2, Z_3, X_4)$$

We claim that S gives rise to an AvN argument for every state in V_S. To show this, let $|\psi\rangle$ be a state in V_S. Since e, f, g, h stabilise $|\psi\rangle$, the following equations must hold

$$\begin{aligned} x_1 \oplus z_2 \oplus i_3 \oplus i_4 &= 0 \\ z_1 \oplus x_2 \oplus z_3 \oplus i_4 &= 0 \\ i_1 \oplus z_2 \oplus x_3 \oplus z_4 &= 0 \\ i_1 \oplus i_2 \oplus z_3 \oplus x_4 &= 0 \end{aligned} \tag{2.3}$$

Let us compute

$$\begin{aligned} e \cdot f \cdot g \cdot h &= (X_1 Z_1, Z_2 X_2 Z_2, Z_3 X_3 Z_3, Z_4 X_4) = (-iY_1, -X_2, -X_3, -iY_4) \\ &= -(Y_1, X_2, X_3, Y_4) \end{aligned}$$

This element is member of S, thus it also stabilises $|\psi\rangle$, yielding the additional equation

$$y_1 \oplus x_2 \oplus x_3 \oplus y_4 = 1 \tag{2.4}$$

The system of equations obtained by adding (2.4) to (2.3) is inconsistent, in fact summing on the left hand side we obtain 0 (since every variable appears twice), while the right hand side gives 1. Therefore, S gives rise to an AvN argument for $|\psi\rangle$.

2.1.2 AvN triples

Now that we have defined AvN arguments more rigorously , we are interested in characterizing them. We introduce here the concept of AvN triple, which will give us a sufficient condition for a subgroup S of $\mathcal{P}_n$ to give rise to an AvN argument.

Definition 2.1.8. An *AvN triple* in $\mathcal{P}_n$ is a triple $\langle e, f, g\rangle$ elements of $\mathcal{P}_n$ with global phases $+1$, which pairwise commute, and which satisfy the following conditions:

1. For each $i = 1, \dots, n$, at least two of e_i, f_i, g_i are equal.
2. The number of i such that $e_i = g_i \neq f_i$, all distinct from I, is odd.

Example 2.1.9. Consider again Mermin's original setting. The triple

$$\langle e := (X_1, Y_2, Y_3), f := (Y_1, X_2, Y_3), g := (Y_1, Y_2, X_3)\rangle$$

used to prove the strong contextuality of the system is clearly an AvN triple. In fact, it clearly satisfies the conditions of Definition 2.1.8, and the elements involved pairwise commute since $e \cdot f = (iZ_1, -iZ_2, I_3) = (Z_1, Z_2, I_3) = (-iZ_1, iZ_2, I_3) = f \cdot e$ (and similarly for the others).

The following Theorem (Theorem 4.2 in [ABK+15]) tells us that if a subgroup S is generated by an AvN triple, then it gives rise to an AvN argument. We give here an alternative proof of the result, which shows rather clearly how each of the properties of AvN triples plays a crucial role.

Theorem 2.1.10. *Let S be the subgroup of $\mathcal{P}_n$ generated by an AvN triple, and V_S the subspace stabilised by S. For every state $|\psi\rangle$ in V_S, the empirical model realised by $|\psi\rangle$ under the Pauli measurements admits an All vs Nothing argument.*

Proof. Let $\langle e, f, g\rangle$ be the AvN triple generating S and $|\psi\rangle$ a state in V_S. Since e stabilises $|\psi\rangle$, by (2.2) we know that $\langle e\rangle_{|\psi\rangle} = 1$, which implies

$$1 = \langle e\rangle_{|\psi\rangle} = \langle\psi| \prod_{i=1}^{n} e_i |\psi\rangle = \prod_{i=1}^{n} \overline{e}_i \langle\psi|\psi\rangle = \prod_{i=1}^{n} \overline{e}_i$$

where $\overline{e}_i$ denotes the eigenvalue for the measurement of the i-th component of e. We can apply the same reasoning to f and g and obtain altogether the following three equations

$$\prod_{i=1}^{n} \overline{e}_i = 1; \qquad \prod_{i=1}^{n} \overline{f}_i = 1; \qquad \prod_{i=1}^{n} \overline{g}_i = 1 \tag{2.5}$$

(note that these equations correspond to $0 = \bigoplus_i \overline{e}_i = \bigoplus_i \overline{f}_i = \bigoplus_i \overline{g}_i$ if we interpret $\{-1, +1, \times\} \sim \{1, 0, \oplus\}$).

The first condition of Definition 2.1.8 tells us that, for all i,

$$e_i f_i g_i = \begin{cases} e_i & \text{if } f_i = g_i \\ -f_i & \text{if } e_i = g_i \neq f_i \\ g_i & \text{if } e_i = f_i \end{cases} \tag{2.6}$$

Let us now consider the eigenvalues of these observables. Since they are either -1 or $+1$, by (2.6) we have

$$\overline{e_i f_i g_i} = \begin{cases} \overline{e_i} \cdot \overline{f_i} \cdot \overline{g_i} & \text{if } f_i = g_i \text{ or } e_i = f_i \\ -\overline{e_i} \cdot \overline{f_i} \cdot \overline{g_i} & \text{if } e_i = g_i \neq f_i. \end{cases}$$

Therefore

$$\prod_{i=1}^{n} \overline{e_i f_i g_i} = (-1)^{|\{i : e_i = g_i \neq f_i\}|} \cdot \prod_{i=1}^{n} \overline{e_i} \cdot \overline{f_i} \cdot \overline{g_i} \overset{(2.5)}{=} (-1)^{|\{i : e_i = g_i \neq f_i\}|}$$

By the second condition of Definition 2.1.8, we know that $|\{i : e_i = g_i \neq f_i\}|$ is odd. Thus we obtain

$$\prod_{i=1}^{n} \overline{e_i f_i g_i} = -1 \tag{2.7}$$

On the other hand, since $e \cdot f \cdot g$ is in S, it stabilises $|\psi\rangle$, thus we must have

$$\prod_{i=1}^{n} \overline{e_i f_i g_i} = 1 \tag{2.8}$$

Clearly equations (2.7) and (2.8) are inconsistent. Thus S gives rise to an AvN argument for $|\psi\rangle$. (notice that, under theinterpretation $\{-1, +1, \times\} \sim \{1, 0, \oplus\}$, equations (2.7) and (2.8) can be written as $0 = \bigoplus_i \overline{e_i f_i g_i} = 1$, which is an evident contradiction). □

It has recently been advanced the hypothesis that the existence of an AvN triple is not only a sufficient but also *necessary* condition for a model to admit an AvN argument.

Conjecture 2.1.11. *The presence of an AvN triple in a stabiliser subgroup S is a necessary as well as sufficient condition for S to admit an AvN argument.*

In the following section we will consider some computational aspects that could help us prove this conjecture. We aim to develop this viewpoint in future work.

2.2 FINDING AVN TRIPLES

The main goal of this section is to introduce an algorithm capable of counting all AvN triples contained in $\mathcal{P}_n$. To address this problem, we will have to convert the conditions of the definition of an AvN triple in a way that is understandable by computers. We start by applying a very useful way of representing an element of $\mathcal{P}_n$, i.e. its associated *check vector* (cf. [NC11, Section 10.5]).

Definition 2.2.1. To each element $e := (e_i)_{i=1}^n \in \mathcal{P}_n$, we associate its *check vector*, a $2n$-vector

$$r(e) := (x_1, \dots, x_n, z_1, \dots, z_n) \in \mathbb{Z}_2^{2n}$$

whose entries are defined as follows

$$(x_i, z_i) = \begin{cases} (0,0) & \text{if } e_i = I \\ (1,0) & \text{if } e_i = X \\ (1,1) & \text{if } e_i = Y \\ (0,1) & \text{if } e_i = Z \end{cases}$$

We can see that every check vector $r(e)$ completely determines e up to phase (i.e. $r(e) = r(\alpha e)$ for all $\alpha \in \{\pm 1, \pm i\}$).

For a finitely generated subgroup $S := \langle g_1, \dots, g_l \rangle$ of $\mathcal{P}_n$, we say that its generators $g_1, \dots, g_l$ are *independent* if removing any generator g_i makes the group generated smaller. Since we are interested in counting AvN triples, we want to impose the condition that the triples are indeed constituted of independent elements of $\mathcal{P}_n$. We would also like to have a computable way to check whether three elements of $\mathcal{P}_n$ are independent. To address this problem, notice that we can associate to each finitely generated subgroup S of $\mathcal{P}_n$ a *check matrix* $C(S)$ composed of all the check vectors of the generators. We have the following proposition (the proof is adapted from [NC11], Proposition 10.3).

Proposition 2.2.2. *Let $S = \langle g_1, \dots, g_l \rangle$ be a finitely generated subgroup of $\mathcal{P}_n$ such that V_S is not trivial. Then the generators $g_1, \dots, g_l$ are independent if and only if the rows of $C(S)$ are linearly independent (i.e. $C(S)$ has full rank).*

Proof. First of all, consider a general element $e \in \mathcal{P}_n$ and its check vector $r(e) = (x_1, \dots, x_n, z_1, \dots z_n)$. Notice that if we ignore phase, we can write $e_i = X^{x_i} Z^{z_i}$. Thus, since $X^2 = Z^2 = I$ and X, Z commute up to a phase factor, we can see that for each $e, f \in \mathcal{P}_n$ we have $r(ef) = r(e) \oplus r(f)$ (i.e. addition of check vectors corresponds to multiplication in S, up to phase). Suppose the rows of $C(S)$ are linearly dependent, then there exist $\{\lambda_1, \dots \lambda_l\}$ with at least one $\lambda_j \neq 0$, such that $\bigoplus_i \lambda_i r(g_i) = 0$. By the discussion above, this is true if and only if $\prod_i g_i^{\lambda_i} = I$ up to a phase still to determine. In [NC11], it is proven that $V_S \neq 0 \Leftrightarrow -I \notin S$. Thus by hypothesis we know that $-I \notin S$ and hence the phase must be 1. Thus the last condition corresponds to $g_j = g_j^{-1} = \prod_{i \neq j} g_i^{\lambda_i}$ and therefore $g_1, \dots, g_l$ are not independent. □

This proposition gives us a fast way of checking whether the elements of an AvN triple are independent (it is sufficient to show that the rank of the check matrix is 3).

Another important result, on the other hand, gives us a computable way of checking whether two group elements commute (this is Exercise 10.33 in [NC11]).

Proposition 2.2.3. *Let $g, g' \in \mathcal{P}_n$ with global phase 1. Then*

$$gg' = g'g \Leftrightarrow r(g)\Lambda r(g')^T = 0,$$

where $\Lambda = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Proof. For each i, g_i either commutes or anticommutes with g_i'. Let

$$m := |\{i : g_i g_i' = -g_i' g_i\}|,$$

then $gg' = (-1)^m g'g$, which implies that g and g' commute if and only if m is even. We can show that g_i and g_i' anticommute if and only if $x_i z_i' \oplus z_i x_i' = 1$, where $r(g) = (x_1, \dots, x_n, z_1, \dots, z_n)$ and $r(g') = (x_1', \dots, x_n', z_1', \dots, z_n')$. Therefore,

$$gg' = g'g \Leftrightarrow m \text{ is even } \Leftrightarrow \bigoplus_{i=1}^{n} x_i z_i' + z_i x_i' = 0 \Leftrightarrow r(g)\Lambda r(g)^T = 0.$$

□

Notice that we can also translate condition 1. of Definition 2.1.8 in terms of check vectors of the elements e, f, g that constitute the triple. Let S be generated by an AvN triple $\langle e, f, g \rangle$. Then Condition 1. can be characterized as follows: For all $j = 1, \dots, n$, consider the columns $C(S)_j$ and $C(S)_{n+j}$. These two column must be equal for at least two row indices. More explicitly

$$\exists i, k \in \{1, 2, 3\} \text{ s.t. } \begin{cases} C(S)_{i,j} = C(S)_{k,j} \\ C(S)_{i,n+j} = C(S)_{k,n+j} \end{cases} \tag{2.9}$$

This allows us to create an algorithm to count and find how many AvN triples there are for an arbitrary n. In the following section, we have provided an example using *Mathematica*.

2.2.1 Identifying AvN triples with Mathematica

We finally provide here the algorithm designed to identify AvN triples. The construction takes full advantage of the general theory developed in the last section. It will allow us to determine all the AvN triples contained in $\mathcal{P}_n$ for a small enough n.

We choose an `n` and start by generating all possible $2n \times 3$ matrices.

```
triples= Tuples[{1,0}, {3, 2*n}];
```

Our idea is to implement functions for each of the conditions we need to check, and finally select the matrices in `triples` satisfying all the conditions. We start with the condition that the matrix has to have rank 3:

```
condition1[x_] :=
    If[MatrixRank[x] == 3, Return[True], Return[False]];
```

Then we implement the fact that we want at least two of e_i, f_i, g_i to be equal. This is slightly more complicated and we use two distinct functions to do it. The first function, `cond2` checks whether the condition is satisfied for a single column of a matrix (according to (2.9)) . The function `condition2` gives us the full characterization of the condition.

```
cond2[x_,j_]:=
If[{x[[1]][[j]], x[[1]][[n + j]]} != {x[[2]][[j]], x[[2]][[n + j]]}
&& {x[[1]][[j]], x[[1]][[n + j]]} != {x[[3]][[j]], x[[3]][[n + j]]}
&& {x[[2]][[j]], x[[2]][[n + j]]} != {x[[3]][[j]], x[[3]][[n + j]]},
Return[False], Return[True]];
```

The function `condition2` gives us the full characterization of the condition.

```
condition2[x_]:=Module[{k},k=0;
   (For[i = 1, i <= n, i++,
       If[!cond2[x, i], k++] ]; If[k>0, Return[False],
         Return[True]])];
```

The third condition we need to check is the fact that the number of cases where $e_i = g_i \neq f_i$ and they are all different from I is odd.

```
condition3[x_] := Module[{k}, k = 0;
    (For[i = 1, i <= n, i++,
      If[x[[All, i]][[1]] == x[[All, i]][[3]]
         && x[[All, n + i]][[1]] == x[[All, n + i]][[3]]
         && (x[[All, i]][[1]] != x[[All, i]][[2]] ||
            x[[All, n + i]][[1]] != x[[All, n + i]][[2]])
         && !((x[[All, i]][[1]] == 0 && x[[All, n + i]][[1]] == 0)
            || (x[[All, i]][[2]] == 0 &&x[[All, n + i]][[2]] == 0)),
     k++]];
    If[Mod[k, 2] == 1, Return[True], Return[False]])];
```

Finally, the last condition we need to check is pairwise commutativity. Thanks to Propostion 2.2.3, we can define the matrix Λ as `lambdan` and implement the characterizing function as follows.

```
condition4[x_] := If[x[[1]].lambdan.x[[2]] == 0
           && x[[1]].lambdan.x[[3]] == 0
           && x[[2]].lambdan.x[[3]] == 0, Return[True],
           Return[False]];
```

Now that we have everything we need, we can define the function that checks whether a triple in `triples` is an AvN triple.

```
avntriple[x_] := If[condition1[x] && condition2[x] && condition3[x]
&& condition4[x], Return[True], Return[False]];
```

Finally, we can use this function to select all the AvN triples out of `triples` as follows

```
avn = Select[triples, avntriple];
```

The variable `avn` stores all the AvN triples in $\mathcal{P}_n$. We have to keep in mind that the algorithm does not take into account the order in which the elements of a triple are taken. This means that AvN triples $\langle e, f, g\rangle$ and $\langle f, e, g\rangle$, for instance, are counted as two different entities. Thus, if we want to count how many AvN triples there are in $\mathcal{P}_n$ we have to divide the variable `Length[avn]` by $3! = 6$. Using this method we find out, for example, that we have $1296 \div 6 = 216$ AvN triples in $\mathcal{P}_3$ and $114048 \div 6 = 19008$ AvN triples in $\mathcal{P}_4$. To give an idea of the magnitude of these numbers, we can observe that the total number of triples composed of random elements of $\mathcal{P}_n$ is $2^{3 \cdot 2n}$. Thus the percentage of AvN triples among random triples in $\mathcal{P}_3$ is approximately 0.54%, whereas for $n = 4$ we have approximately 0.68%.

More importantly, by Theorem 2.1.10, this algorithm gives us an enormous number of examples of strongly contextual quantum states similar to the Cluster States we saw in Example 2.1.7 (so far, we only had rather a few concrete examples of such states). We list some of them in the following example.

Example 2.2.4. As mentioned before, the program implemented in *Mathematica* gives us 216 distinct quantum states stabilised by AvN triples in $\mathcal{P}_3$ (thus strongly contextual). We list here two randomly chosen such examples (it is sufficient to run the code to find them all).

- Consider the state stabilised by the subgroup S of $\mathcal{P}_3$ generated by $\{e, f, g\}$, where

$$\begin{aligned} e &:= (Y_1, Y_2, Y_3), \\ f &:= (Y_1, X_2, Y_3), \\ g &:= (Y_1, X_2, X_3). \end{aligned}$$

It is easy to verify that S gives rise to an AvN argument for every state in V_S. In fact, suppose $|\psi\rangle$ is a state in V_S. Since e, f, g stabilise $|\psi\rangle$, the following equations must hold

$$\begin{aligned} y_1 \oplus y_2 \oplus y_3 &= 0 \\ y_1 \oplus x_2 \oplus y_3 &= 0 \\ y_1 \oplus x_2 \oplus x_3 &= 0 \end{aligned} \tag{2.10}$$

We compute

$$\begin{aligned} e \cdot f \cdot g &= (Y_1 Y_1 Y_1, Y_2 X_2 Y_2, Y_3 X_3 X_3) = (Y_1, -X_2, Y_3) \\ &= -(Y_1, X_2, Y_3). \end{aligned}$$

This element is in S, thus it also stabilises $|\psi\rangle$, yielding the equation

$$y_1 \oplus x_2 \oplus y_3 = 1. \tag{2.11}$$

The system of equations obtained by attaching (2.11) to (2.10) is inconsistent. In fact, summing on the left hand side gives 0 (every variable appears twice), and summing on the right hand side gives 1.

- Consider the state stabilised by the subgroup S of $\mathcal{P}_3$ generated by $\{e, f, g\}$, where

$$\begin{aligned} e &:= (Y_1, X_2, Z_3), \\ f &:= (X_1, Z_2, Z_3), \\ g &:= (X_1, X_2, Y_3). \end{aligned}$$

Once again, we compute

$$\begin{aligned} e \cdot f \cdot g &= (Y_1X_1X_1, X_2Z_2X_2, Z_3Z_3Y_3) = (Y_1, -Z_2, Y_3) \\ &= -(Y_1, Z_2, Y_3), \end{aligned}$$

and we obtain the following system of equations

$$\begin{aligned} y_1 \oplus x_2 \oplus z_3 &= 0 \\ x_1 \oplus z_2 \oplus z_3 &= 0 \\ x_1 \oplus x_2 \oplus y_3 &= 0 \\ y_1 \oplus z_2 \oplus y_3 &= 1, \end{aligned}$$

which is inconsistent (summing on the left gives 0 since every variable appears twice, and summing on the right gives 1).

The algorithm detects 19008 distinct quantum states stabilised by AvN triples in $\mathcal{P}_4$. Once again, we randomly chose two of them to give a concrete example.

- Consider the state stabilised by the subgroup S of $\mathcal{P}_4$ generated by $\{e, f, g\}$, where

$$\begin{aligned} e &:= (X_1, Z_2, Y_3, I_4), \\ f &:= (Y_1, X_2, Y_3, I_4), \\ g &:= (X_1, X_2, X_3, Y_4). \end{aligned}$$

  Suppose $|\psi\rangle$ is a state in V_S. Since e, f, g stabilise $|\psi\rangle$, the following equations must hold

$$\begin{aligned} x_1 \oplus z_2 \oplus y_3 \oplus i_4 &= 0 \\ y_1 \oplus x_2 \oplus y_3 \oplus i_4 &= 0 \\ x_1 \oplus x_2 \oplus x_3 \oplus y_4 &= 0 \end{aligned} \tag{2.12}$$

  We compute

$$\begin{aligned} e \cdot f \cdot g &= (X_1Y_1X_1, Z_2X_2X_2, Y_3Y_3X_3, I_4I_4Y_4) = (-Y_1, Z_2, X_3, Y_4) \\ &= -(Y_1, Z_2, X_3, Y_4). \end{aligned}$$

  This element is in S, thus it also stabilises $|\psi\rangle$, yielding the equation

$$y_1 \oplus z_2 \oplus x_3 \oplus y_4 = 1. \tag{2.13}$$

  The system of equations obtained by attaching (2.13) to (2.12) is inconsistent. In fact, summing on the left hand side gives 0 (every variable appears twice), and summing on the right hand side gives 1.

- Consider the state stabilised by the subgroup S of $\mathcal{P}_4$ generated by $\{e, f, g\}$, where

$$\begin{aligned} e &:= (X_1, I_2, X_3, Z_4), \\ f &:= (Z_1, I_2, X_3, X_4), \\ g &:= (X_1, Y_2, Y_3, X_4). \end{aligned}$$

  Once again, we compute

$$\begin{aligned} e \cdot f \cdot g &= (X_1Z_1X_1, I_2I_2Y_2, X_3X_3Y_3, Z_4X_4X_4) = (-Z_1, Y_2, Y_3, Z_4) \\ &= -(Z_1, Y_2, Y_3, Z_4), \end{aligned}$$

and we obtain the following system of equations

$$\begin{aligned} x_1 \oplus i_2 \oplus x_3 \oplus z_4 &= 0 \\ z_1 \oplus i_2 \oplus x_3 \oplus x_4 &= 0 \\ x_1 \oplus y_2 \oplus y_3 \oplus x_4 &= 0 \\ z_1 \oplus y_2 \oplus y_3 \oplus z_4 &= 1, \end{aligned}$$

which is inconsistent (summing on the left gives 0 since every variable appears twice, and summing on the right gives 1).

2.3 DISCUSSION

Following the guidelines of [ABK+15], we introduced All vs Nothing arguments in a formal mathematical way that successfully generalises the one used by Mermin in his first proof of the contextuality of the GHZ model. This abstraction has allowed us to define a sufficient condition for the existence of AvN arguments for quantum states (cf. Theorem 2.1.10).

- It remains unanswered the question of whether this condition is also necessary (AvN conjecture 2.1.11)

However, with the ultimate goal of proving this conjecture, we took full advantage of the general theory of stabilisers and their relations to quantum physics to develop an algorithm that finds all AvN triples contained in the Pauli n-group for a sufficiently small n. This result gives us a large number of previously unknown examples of strongly contextual quantum-realisable models and it brings us one step closer to intuitively understand whether the conjecture is true or not. We would also like to mention that we worked on a general formula to count AvN triples for a general n. The final result we obtained is that the number of AvN triples contained in $\mathcal{P}_n$ is given by the number of partitions of n into natural numbers $r_{0,1,2}, s_{0,1,2}, t_{0,1,2}, q$ such that

$$\begin{cases} \sum_i r_i + \sum_i s_i + \sum_i t_i + q = n \\ r_0 + s_0 + t_0 \equiv 1 \pmod 2 \\ r_{[i]_3} + s_{[i+1]_3} = s_{[i]_3} + r_{[i+1]_3}, \quad \forall i = 0, 1, 2, \end{cases}$$

where $[i]_3$ denotes the class of i in $\mathbb{Z}_3$. However, since we believe this formula can be strongly implemented, and due to time constraints, we decided not to introduce it in our thesis. We will hopefully give a more definite result in future work.

3 SHEAF COHOMOLOGY

In this chapter we will present another approach to detect contextuality of empirical models via methods of algebraic topology.

3.1 ČECH COHOMOLOGY

Let us start by introducing the Čech cohomology associated to the support of a probabilistic empirical model.

Let X be a topological space, $\mathcal{M}$ an open cover of X and $\mathcal{F} : \mathbf{Open}(X)^{op} \to \mathbf{Ab}$ a presheaf of abelian groups on X. In accordance with the definitions of Chapter 1 we are particularly interested in the case where X is a finite discrete set, $\mathcal{M}$ is a measurement cover and $\mathcal{F}$ is the subpresheaf associated to an empirical model, somehow modified to make it a subpresheaf of abelian groups (cf. Section 3.2.2).

Definition 3.1.1. A q-*simplex of the nerve* of $\mathcal{M}$ is a tuple $\sigma = (C_0, \dots, C_q)$ of elements of $\mathcal{M}$ such that

$$|\sigma| := \bigcap_{i=0}^{q} C_i \neq \varnothing.$$

We also define the set $\mathcal{N}(\mathcal{M})^q$ of q-simplices.

Let $\sigma := (C_0, \dots, C_{q+1})$. For all $0 \leqslant j \leqslant q$ we can define the maps $\partial_j : \mathcal{N}(\mathcal{M})^{q+1} \to \mathcal{N}(\mathcal{M})^q$ given by the expression

$$\partial_j(\sigma) := (C_0, \dots, C_{j-1}, \hat{C}_j, C_{j+1}, \dots, C_{q+1}).$$

Definition 3.1.2. The *augmented Čech cochain complex* is defined as the sequence

$$0 \xrightarrow{0} C^0(\mathcal{M}, \mathcal{F}) \xrightarrow{\delta^0} C^1(\mathcal{M}, \mathcal{F}) \xrightarrow{\delta^1} \dots,$$

where

- for each $q \geqslant 0$,

  $$C^q(\mathcal{M}, \mathcal{F}) := \bigoplus_{\sigma \in \mathcal{N}(\mathcal{M})^q} \mathcal{F}(|\sigma|)$$

  is the abelian group of q-*cochains*. We can see its elements as sequences $(\omega(\sigma))_{\sigma \in \mathcal{N}(\mathcal{M})^q}$, where $\omega : \mathcal{N}(\mathcal{M})^q \to \bigoplus_{\sigma \in \mathcal{N}(\mathcal{M})^q}$ is such that $\omega(\sigma) \in \mathcal{F}(|\sigma|)$ for all $\sigma \in \mathcal{N}(\mathcal{M})^q$.

- for each $q \geqslant 0$, the q-*th coboundary map* $\delta^q : C^q(\mathcal{M}, \mathcal{F}) \to C^{q+1}(\mathcal{M}, \mathcal{F})$ is defined as

  $$\delta^q(\omega)(\sigma) := \sum_{j=0}^{q+1} (-1)^j \rho_{|\sigma|}^{|\partial_j(\sigma)|}(\omega(\partial_j \sigma)),$$

  where $\rho_U^{U'} :: s \mapsto s\,|_{U'}$ denotes $\mathcal{F}(U \subseteq U')$.

A straightforward calculation shows the following proposition.

Proposition 3.1.3. *For each* q, $\delta^{q+1} \circ \delta^q = 0$.

Thus the object of the definition is indeed a cochain complex.

We define cohomology in analogy with the classical case: we start by defining q-*cocycles* and q-*coboundaries* as

$$\begin{aligned} Z^q(\mathcal{M}, \mathcal{F}) &:= \ker(\delta^q) \\ B^q(\mathcal{M}, \mathcal{F}) &:= \mathrm{im}(\delta^{q-1}). \end{aligned}$$

Since $\delta \circ \delta = 0$, we have that $B^q(\mathcal{M}, \mathcal{F})$ is a (normal) subgroup of $Z^q(\mathcal{M}, \mathcal{F})$, thus we can formulate the following definition

Definition 3.1.4. We define the q-*th Čech cohomology group* as

$$\check{H}^q(\mathcal{M}, \mathcal{F}) := Z^q(\mathcal{M}, \mathcal{F}) \big/ B^q(\mathcal{M}, \mathcal{F})$$

Remark 3.1.5. Since the first map of the Čech complex is the 0 map, we have $B^0(\mathcal{M}, \mathcal{F}) = 0$, thus $\check{H}^0(\mathcal{M}, \mathcal{F}) \cong Z^0(\mathcal{M}, \mathcal{F})$.

Definition 3.1.6. Let $\mathcal{M} := \{C_i\}_{i \in I}$ be an open cover. A *compatible family* with respect to $\mathcal{M}$ is a family $\{r_i \in \mathcal{F}(C_i)\}_{i \in I}$ such that, for all i, j,

$$r_i \mid_{C_i \cap C_j} = r_j \mid_{C_i \cap C_j} .$$

Proposition 3.1.7. *Compatible families correspond bijectively to elements of* $\check{H}^0(\mathcal{M}, \mathcal{F})$.

Proof. By Remark 3.1.5, we know that $\check{H}^0(\mathcal{M}, \mathcal{F}) \cong Z^0(\mathcal{M}, \mathcal{F})$. To every $\omega \in C^0(\mathcal{M}, \mathcal{F})$ we can associate the family

$$\{\omega(C_i) \in \mathcal{F}(C_i)\}_{i \in I}.$$

This clearly defines a bijection (with inverse $\{r_i\}_{i \in I} \mapsto \sigma$, where σ is defined as $\sigma(C_i) := r_i$). We have

$$\begin{aligned} r_i \mid_{C_i \cap C_j} = r_j \mid_{C_i \cap C_j} &\Leftrightarrow \rho^{C_i}_{C_i \cap C_j}(\omega(C_i)) = \rho^{C_j}_{C_i \cap C_j}(\omega(C_j)) \\ &\Leftrightarrow \rho^{C_j}_{C_i \cap C_j}(\omega(C_j)) - \rho^{C_i}_{C_i \cap C_j}(\omega(C_i)) = 0 \\ &\Leftrightarrow \sum_{k=0}^{1} (-1)^k \rho^{|\partial_k \sigma|}_{|\sigma|}(\omega(\partial_k \sigma)) = 0, \quad \forall \sigma = (C_i, C_j) \in \mathcal{N}(\mathcal{M})^2 \\ &\Leftrightarrow \delta^0(\omega) = 0 \Leftrightarrow \omega \in Z^0(\mathcal{M}, \mathcal{F}) \cong \check{H}^0(\mathcal{M}, \mathcal{F}). \end{aligned}$$

□

Remark 3.1.8. Note that the bijection here is valid only if we assume that the cover $\mathcal{M}$ is *connected*, meaning that, given two contexts C and C' in $\mathcal{M}$, there always exists a sequence of contexts

$$C = C_0, C_1, \ldots, C_n = C',$$

such that $C_i \cap C_{i+1} \neq \varnothing$ for all i. In fact, there is no condition that requires compatibility over restrictions to the empty context. This will not cause any problem in our study, since we are only interested in the cases where the cover is connected (since we can always study the connected components of every scenario). For this reason, from now on we will always suppose our cover to be connected (cf. [ABK+15] for deeper insights).

3.2 RELATIVE SHEAF COHOMOLOGY

We will also need to define the relative cohomology of $\mathcal{F}$ with respect to an open subset $U \subseteq X$. This is due to the fact that our goal is to understand whether we can find a global section that extends a compatible family. To do this let $U \subseteq X$ be an open subset, we will define two other presheafs related to $\mathcal{F}$ with respect to U. The first one is the following

$$\begin{array}{rccc} \mathcal{F}\,|_U\colon & \mathbf{Open}(X)^{op} & \longrightarrow & \mathbf{Ab} \\ & V & \longmapsto & \mathcal{F}(U \cap V) \\ & V \subseteq V' & \longmapsto & \tilde{\rho}_V^{V'} := \rho_{U\cap V}^{U \cap V'}. \end{array}$$

The relation with $\mathcal{F}$ is given by the following *morphism of sheaves* (i.e. a natural transformation) $p^U : \mathcal{F} \Rightarrow \mathcal{F}\,|_U$, where for any open $V \subseteq X$, we have

$$p_V^U : \mathcal{F}(V) \longmapsto \mathcal{F}\,|_U\,(V) :: r \mapsto r\,|_{U\cap V}\,.$$

Notice that this is indeed a natural transformation, in fact we can see that the following diagram commutes (where we have highlighted in blue the diagram chase, starting from the circled element and ending at the top right one).

$$\begin{tikzcd}
& \mathcal{F}(V) \arrow[rr, "p_V^U"] & & \mathcal{F}\,|_U\,(V) \\
\rho_V^W(r) \arrow[ur, dashed, no head] \arrow[rr, mapsto] & & \rho_{U\cap V}^W(r) \arrow[ur, dashed, no head] & \\
& \mathcal{F}(W) \arrow[uu, "\rho_V^W"] \arrow[rr, "p_W^U"] & & \mathcal{F}\,|_U\,(W) \arrow[uu, "\tilde{\rho}_V^W"'] \\
r \arrow[uu, mapsto] \arrow[ur, dashed, no head] \arrow[rr, mapsto] & & \rho_{U\cap W}^W(r) \arrow[uu, mapsto] \arrow[ur, dashed, no head] &
\end{tikzcd}$$

The diagram commutes since the top right element of the diagram chase gives $\rho_{U\cap V}^V \circ \rho_V^W(r) = \rho_{U\cap V}^W(r)$ coming from the left and $\rho_{U\cap V}^{U\cap W} \circ \rho_{U\cap W}^W(r) = \rho_{U\cap V}^W(r)$ coming from below.

We can now define the second presheaf as follows

$$\mathcal{F}_{\tilde{U}} : \mathbf{Open}(X)^{op} \longrightarrow \mathbf{Ab} :: V \longmapsto \ker(p_V).$$

Thanks to these presheaves, we have the following exact sequence of presheaves

$$\mathbf{0} \overset{0}{\Longrightarrow} \mathcal{F}_{\tilde{U}} \overset{\text{incl.}}{\Longrightarrow} \mathcal{F} \overset{p^U}{\Longrightarrow} \mathcal{F}\,|_U \tag{3.1}$$

Definition 3.2.1. The *relative cohomology of* $\mathcal{F}$ *with respect to* U is defined to be the cohomology of the presheaf $\mathcal{F}_{\tilde{U}}$.

3.2.1 Cohomological obstructions

After having introduced the general setting, we focus now on the particular case where X is a finite, discrete measurement set, O is an outcome set, $\mathcal{M}$ is a measurement cover, and $\mathcal{F}$ is the subpresheaf associated to a possibilistic empirical model on $\langle X, \mathcal{M}, O\rangle$ according to Definition 1.3.3, where some modification has to be done in order to make it a subpresheaf of abelian groups (in [AMSB12] this is done linearly by taking free groups generated by sets. We will discuss this in the next section).

Let us consider the correstriction

$$\tilde{\delta}^0 := \delta^0 \,|^{Z^1(\mathcal{M},\mathcal{F})} : C^0(\mathcal{M},\mathcal{F}) \longrightarrow Z^1(\mathcal{M},\mathcal{F}).$$

This map has kernel $Z^0(\mathcal{M},\mathcal{F}) \cong \check{H}^0(\mathcal{M},\mathcal{F})$ and cokernel

$$Z^1(\mathcal{M},\mathcal{F})/B^1(\mathcal{M},\mathcal{F}) \cong \check{H}^1(\mathcal{M},\mathcal{F}).$$

Thus we have an exact sequence

$$\check{H}^0(\mathcal{M},\mathcal{F}) \cong Z^0(\mathcal{M},\mathcal{F}) \xrightarrow{\text{incl.}} C^0(\mathcal{M},\mathcal{F}) \xrightarrow{\tilde{\delta}^0} Z^1(\mathcal{M},\mathcal{F}) \xrightarrow{\text{quot.}} \check{H}^1(\mathcal{M},\mathcal{F}).$$

Consider again the exact sequence (3.1). For each $C \in \mathcal{M}$, it yields an exact sequence on objects

$$0 \xrightarrow{0} \mathcal{F}_{\tilde{U}}(C) := \ker(p^U_C) \xrightarrow{\text{incl.}_C} \mathcal{F}(C) \xrightarrow{p^U_C} \mathcal{F}\,|_U\,(C) := \mathcal{F}(U \cap C)$$

We can sum these morphisms for every $C \in \mathcal{M}$ and "lift" exactness to the chain level:

$$0 \xrightarrow{0} C^0(\mathcal{M},\mathcal{F}_{\tilde{U}}) \xrightarrow{\bigoplus_{C\in\mathcal{M}} \text{incl.}_C} C^0(\mathcal{M},\mathcal{F}) \xrightarrow{\bigoplus_{C\in\mathcal{M}} p^U_C} C^0(\mathcal{M},\mathcal{F}\,|_U) \quad (3.2)$$

Notice that, since $\mathcal{F}$ is associated to an empirical model, it is flasque beneath the cover, meaning that p^U_C is surjective for all $C \in \mathcal{M}$. Thus $\bigoplus_{C\in\mathcal{M}} p^U_C$ is also surjective. Thus (3.2) is in fact a short exact sequence

$$0 \longrightarrow C^0(\mathcal{M},\mathcal{F}_{\tilde{U}}) \longrightarrow C^0(\mathcal{M},\mathcal{F}) \longrightarrow C^0(\mathcal{M},\mathcal{F}\,|_U) \longrightarrow 0$$

Summarizing the situation, we have

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{M},\mathcal{F}_{\tilde{U}}) \arrow[r] \arrow[d, "\tilde{\delta}^0"] & C^0(\mathcal{M},\mathcal{F}) \arrow[r] \arrow[d, "\tilde{\delta}^0"] & C^0(\mathcal{M},\mathcal{F}\,|_U) \arrow[r] \arrow[d, "\tilde{\delta}^0"] & 0 \\
0 \arrow[r] & Z^1(\mathcal{M},\mathcal{F}_{\tilde{U}}) \arrow[r] & Z^1(\mathcal{M},\mathcal{F}) \arrow[r] & Z^1(\mathcal{M},\mathcal{F}\,|_U) &
\end{tikzcd}$$

We can therefore apply the snake lemma (cf. [Wei94, Lemma 1.3.2]) to obtain the connecting morphism $\gamma : \check{H}^0(\mathcal{M},\mathcal{F}\,|_U) \to \check{H}^1(\mathcal{M},\mathcal{F}_{\tilde{U}})$ as part of a long exact sequence as follows

$$\begin{tikzcd}
& \check{H}^0(\mathcal{M},\mathcal{F}_{\tilde{U}}) \arrow[r] \arrow[d] & \check{H}^0(\mathcal{M},\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^0(\mathcal{M},\mathcal{F}\,|_U) \arrow[d] \arrow[dddll, rounded corners, "\gamma"] & \\
0 \arrow[r] & C^0(\mathcal{M},\mathcal{F}_{\tilde{U}}) \arrow[r] \arrow[d] & C^0(\mathcal{M},\mathcal{F}) \arrow[r] \arrow[d] & C^0(\mathcal{M},\mathcal{F}\,|_U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^1(\mathcal{M},\mathcal{F}_{\tilde{U}}) \arrow[r] \arrow[d] & Z^1(\mathcal{M},\mathcal{F}) \arrow[r] \arrow[d] & Z^1(\mathcal{M},\mathcal{F}\,|_U) \arrow[d] & \\
& \check{H}^1(\mathcal{M},\mathcal{F}_{\tilde{U}}) \arrow[r] & \check{H}^1(\mathcal{M},\mathcal{F}) \arrow[r] & \check{H}^1(\mathcal{M},\mathcal{F}\,|_U) &
\end{tikzcd} \quad (3.3)$$

Now, suppose U is an element $C_0 \in \mathcal{M}$

Lemma 3.2.2. *If* C_0 *is a fixed element of* $\mathcal{M}$,

$$\check{H}^0(\mathcal{M}, \mathcal{F}|_{C_0}) \cong \mathcal{F}(C_0).$$

Proof. We already know that $\check{H}^0(\mathcal{M}, \mathcal{F}|_{C_0}) \cong Z^0(\mathcal{M}, \mathcal{F}|_{C_0})$. Recall that, by Proposition 3.1.7, elements of $Z^0(\mathcal{M}, \mathcal{F}|_{C_0})$ are compatible families for the presheaf $\mathcal{F}|_{C_0}$. Thus by condition 3. of Definition 1.3.3 for each element $(s_{C_i})_{i\in I} \in Z^0(\mathcal{M}, \mathcal{F}|_{C_0})$ there exists a unique global section $\mathtt{glob}((s_{C_i})_i) \in \mathcal{F}(C_0)$ (global with respect to $\mathcal{F}|_{C_0}$) that restricts locally to the elements of $(s_{C_i})_{i\in I}$. We can now define the following functions.

$$\begin{array}{llcccccc} \psi^0: & \mathcal{F}(C_0) & \longrightarrow & Z^0(\mathcal{M}, \mathcal{F}|_{C_0}) & :: & r_{C_0} & \longmapsto & (r_{C_0}|_{C_0\cap C_i})_{i\in I} \\ \phi^0: & Z^0(\mathcal{M}, \mathcal{F}|_{C_0}) & \longrightarrow & \mathcal{F}(C_0) & :: & (s_{C_i})_{i\in I} & \longmapsto & \mathtt{glob}((s_{C_i})_i), \end{array}$$

By unicity of the global section, we clearly have that these two maps are inverse of each other. Moreover, since $r_{C_0}|_{C_0\cap C_i} =: \rho^{C_0}_{C_0\cap C_i}(r_{C_0})$, and the ρs are homomorphisms of groups by definition of $\mathcal{F}$, we have for all $r_{C_0}, s_{C_0} \in \mathcal{F}(C_0)$, using additive notation,

$$\begin{aligned} \psi^0(r_{C_0} + s_{C_0}) &= (\rho^{C_0}_{C_0\cap C_i}(r_{C_0} + s_{C_0}))_{i\in I} = (\rho^{C_0}_{C_0\cap C_i}(r_{C_0}) + \rho^{C_0}_{C_0\cap C_i}(s_{C_0}))_{i\in I} \\ &= (\rho^{C_0}_{C_0\cap C_i}(r_{C_0}))_{i\in I} + (\rho^{C_0}_{C_0\cap C_i}(s_{C_0}))_{i\in I} = \psi^0(r_{C_0}) + \psi^0(s_{C_0}). \end{aligned}$$

Thus we conclude that ψ is an isomorphism of abelian groups.

□

Remark 3.2.3. The reason why we added the index 0 to ψ will be clear in section 4, when we will show a generalisation of ψ to groups of higher order.

It now makes sense to introduce the following definition.

Definition 3.2.4. Let C_0 be an element of the cover $\mathcal{M}$ and $r_0 \in \mathcal{F}(C_0)$. Then, the *cohomological obstruction* of r_0 is the element $\gamma(r_0)$ of $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$.

Remark 3.2.5. It is important to point out that, for a presheaf $\mathcal{F}$ of abelian groups defined in relation to an empirical model on a cover $\mathcal{M}$, we have as many connecting homomorphisms γ as elements of the cover $\mathcal{M}$. In fact, each γ is defined with respect to a single context in $\mathcal{M}$ (C_0 in the discussion). Sometimes we will need to specify the underlying context of γ. In this case we will explicitly denote the homomorphism as γ^{C_0}.

Remark 3.2.6. Notice that this definition is the one from [ABK+15]. Our other main reference, [AMSB12] gives us a more concrete definition. We will show here in detail that they coincide.

For a fixed $C_0 \in \mathcal{M}$, we can summarize the definition of the obstruction of $r_0 \in \mathcal{F}(C_0)$ according to [AMSB12] in the following steps, where after each step we give a commentary on how to translate it in the more abstract viewpoint we have used in our project. (here we take $I = \{0, 1, \ldots, n\}$).

1. By no-signaling of the empirical model, there exists a family $\{r_i \in \mathcal{F}(C_i)\}_{i\in I}$ such that $r_0|_{C_0\cap C_i} = r_i|_{C_0\cap C_i}$ for all i.

We have already shown that no-signaling is equivalent to the property of flaccidity beneath the cover. Thus, formally, this step is exploiting the fact that the maps $p^{C_0}_{C_i} : \mathcal{F}(C_i) \to \mathcal{F}|_{C_0}(C_i)$ are surjective.

2. Define $c := (r_0, \ldots, r_n) \in C^0(\mathcal{M}, \mathcal{F})$ and let $z := \delta^0(c)$.

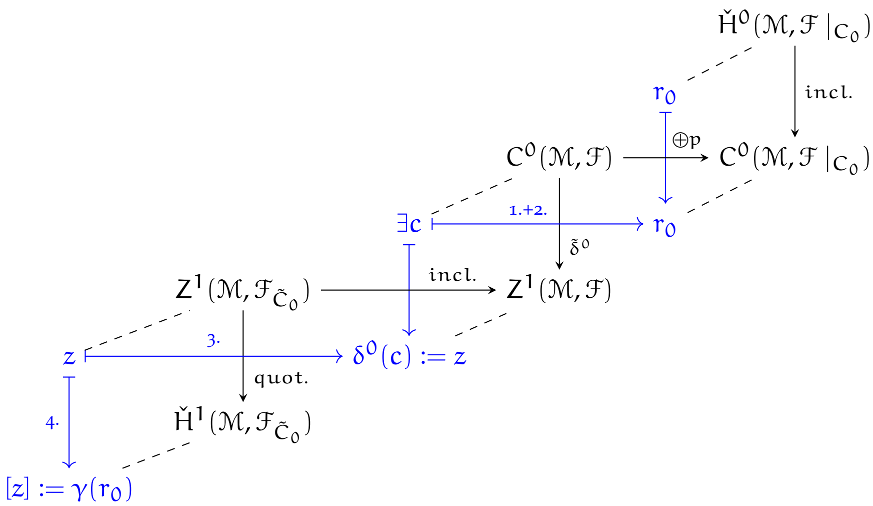
Here we use the "lift" of the flaccidity property to the chain level. Formally it is equivalent to using surjectivity of $\bigoplus_{i\in I} p^{C_0}_{C_i} : C^0(\mathcal{M},\mathcal{F}) \to C^0(\mathcal{M},\mathcal{F}|_{C_0})$ to obain an element c such that it gets mapped to r_0.

3. By Proposition 4.1 in [AMSB12], conclude that $z \in Z^1(\mathcal{M},\mathcal{F}_{\tilde{C}_0})$.

4. Define the obstruction of r_0 as the class $[z] \in \check{H}^1(\mathcal{M},\mathcal{F}_{\tilde{C}_0})$

These last two steps are self-explanatory.

Summarizing these steps, we can see that the more concrete definition of [AMSB12] corresponds to taking a zig-zag line from $\check{H}^0(\mathcal{M},\mathcal{F}|_{C_0})$ to $\check{H}^1(\mathcal{M},\mathcal{F}_{\tilde{C}_0})$ in (3.3) instead of the direct way, using the snake connecting morphism γ. The following diagram chase shows this more graphically.



The following proposition will give us the criterion to detect contextuality.

Proposition 3.2.7. *Let $\mathcal{M}$ be a connected cover (cf. Remark 3.1.8). Let $C_0 \in \mathcal{M}$ and $r_0 \in \mathcal{F}(C_0)$. Then, $\gamma(r_0) = 0$ if and only if there is a compatible family $\{r_C \in \mathcal{F}(C)\}_{C\in\mathcal{M}}$ such that $r_{C_0} = r_0$.*

Proof. Since the morphism $\gamma : \check{H}^0(\mathcal{M},\mathcal{F}|_{C_0}) \to \check{H}^1(\mathcal{M},\mathcal{F}_{\tilde{C}_0})$ is defined via the Snake Lemma, it is part of a long exact sequence. Therefore,

$$\ker(\gamma) = \mathrm{im}\left(P : \check{H}^0(\mathcal{M},\mathcal{F}) \longrightarrow \check{H}^0(\mathcal{M},\mathcal{F}|_{C_0})\right).$$

Since the cover is connected, by Proposition 3.1.7 and Remark 3.1.8, the elements of $\check{H}^0(\mathcal{M},\mathcal{F})$ are all the compatible families. We have

$$\begin{array}{rccc} P: & \check{H}^0(\mathcal{M},\mathcal{F}) & \longrightarrow & \check{H}^0(\mathcal{M},\mathcal{F}|_{C_0}) \\ & \{r_C \in \mathcal{F}(C)\}_{C\in\mathcal{M}} & \longmapsto & \{r_C|_{C\cap C_0} \in \mathcal{F}|_{C_0}(C)\}_{C\in\mathcal{M}} \end{array}$$

But since $r_C|_{C_0\cap C} = r_{C_0}|_{C_0\cap C}$, we can see that the isomorphism ϕ of Lemma 3.2.2 sends $\{r_C|_{C\cap C_0} \in \mathcal{F}|_{C_0}(C)\}_{C\in\mathcal{M}}$ to $r_{C_0} \in \mathcal{F}(C_0)$. Therefore, the image of P is exactly those local sections at C_0 that belong to a compatible family. □

As a consequence, we infer that every section $r_0 \in \mathcal{F}(C_0)$ which cannot be extended to a compatible family gives rise to a non-identity cohomology class $\gamma(r_0)$.

Remark 3.2.8. We insist recalling that in order to study the cohomological properties of an empirical model $\mathcal{S}$, we need to somehow modify it to obtain a presheaf of abelian groups $\mathcal{F}$. We say that $\mathcal{F}$ is the presheaf of abelian groups *relative* to the empirical model $\mathcal{S}$, and that $\mathcal{S}$ is the *underlying* empirical model of $\mathcal{F}$. In the next section we will show how this procedure is done in practice.

3.2.2 Detecting contextuality

We shall now show how we can use this fact to detect the contextuality of an empirical model. We have briefly mentioned erlier that the theory developed so far only works if $\mathcal{F}$ is a subpresheaf of abelian groups, whereas the subpresheaf of an empirical model is simply defined on sets. We thus need to find a way to obtain such a presheaf of abelian groups from an empirical model, possibly without modifying the information carried by the original subpresheaf of sets. We propose here the most intuitive and natural solution.

Let R be a ring and let us define the functor

$$\begin{array}{rccc} F_R: & \textbf{Set} & \longrightarrow & R-\textbf{Mod} \\ & X & \longmapsto & \{\phi : X \to R \mid |supp(\phi)| < \infty\} \\ & f & \longmapsto & F_R f :: \phi \mapsto \lambda y. \sum_{f(x)=y} \phi(x). \end{array} \tag{3.4}$$

We can see every function $\phi \in F_R(X)$ as a formal linear combination $\sum_{x \in X} \phi(x) \cdot x$ of elements of X. Note that we can naturally embed X into $F_R(X)$ via the map $x \mapsto 1 \cdot x$ (this operation will be carried out implicitly). Actually, we can now see that $F_R(X)$ is the free R-module generated by X. Thus, in particular, $F_{\mathbb{Z}}(X)$ is the free abelian group generated by X. This suggests that whenever we wish to study an empirical model given by a subpresheaf S cohomologically, we use instead its approximation $F_R\mathcal{S}$ for some ring R. Motivated by the theory developed in the previous section, and by Definition 1.3.5, we have

Definition 3.2.9. Let $s \in \mathcal{S}(C)$ be a local section for an empirical model $\mathcal{S}$. We associate to s the cohomological obstruction $\gamma_{F_R\mathcal{S}}(s)$ of Definition 3.2.4.

- If there exists a local section $s_0 \in \mathcal{S}(C_0)$ such that $\gamma_{F_R\mathcal{S}}(s_0) \neq 0$ we say that $\mathcal{S}$ is *cohomologically logically* (or *possibilistically*) *contextual*, or $CLC_R(\mathcal{S}, s_0)$ (or simply $CLC_R(\mathcal{S})$ if the section is unimportant).
- If $\mathcal{S}$ is such that $CLC_R(\mathcal{S}, s)$ for all local sections s, then we say that the model is *cohomologically strongly contextual*, or $CSC_R(\mathcal{S})$.

The following proposition is key: it tells us that the the cohomological approach devloped in the last section gives us a sufficient condition to witness contextuality of an empirical model.

Proposition 3.2.10. *Let $\mathcal{S}$ be an empirical model. We have*

$$CLC_R(\mathcal{S}) \Longrightarrow LC(\mathcal{S})$$
$$CSC_R(\mathcal{S}) \Longrightarrow SC(\mathcal{S})$$

Proof. Suppose $\mathcal{S}$ is not logically contextual. Then for every context $C_0 \in \mathcal{M}$ and each $s_0 \in \mathcal{S}(C_0)$, there exists a compatible family in $\{s_C \in \mathcal{S}(C)\}_{C \in \mathcal{M}}$ such that $s_{C_0} = s_0$. This family yields a compatible family $\{s_C \in F_R\mathcal{S}(C)\}_{C \in \mathcal{M}}$ for $F_R\mathcal{S}$ since $\mathcal{S}(C)$ embeds into $F_R\mathcal{S}$. By Proposition 3.2.7 we conclude that $\gamma(s) = 0$. This argument shows also the second implication, as it is sufficient to repeat it for a single section. ☐

Remark 3.2.11. Going back to Chapter 1, and particularly to Section 1.5, we would like to highlight a curious analogy. Vorob'ev's theorem 1.5.2 shows that a cover only admits non-contextual models if and only if it is acyclic. Proposition 3.2.10, tells us that a non-contextual model is also chomologically non-contextual, thus acyclic in cohomology.

Acyclicity of the measurement cover translates in acyclicity in cohomology.

It would be interesting to see whether this connection is purely coincidental, or if it can be further implemented.

It is important to note that the implications of Proposition 3.2.10 are strict. This will be formally shown in Section 3.3 using the Hardy model, which features a section that is not a member of any compatible family of sections in the support of the model (thus showing non-locality), but whose obstruction vanishes, giving rise to a false positive. The reader might wonder where we lost such information in the abstraction from the model subpresheaf to its cohomology. It appears clear that the necessity of modifying the original subpresheaf of sets into a presheaf of abelian groups is the answer. We can clearly see that the free-group approach allows some flaws in the consequent study of the cohomology. Future research on how to make this operation smoother could potentially give us a full cohomological characterization of contextuality.

3.3 COHOMOLOGY IN PRACTICE

Let us apply the cohomological approach to some concrete examples

Example 3.3.1.

- We consider the PR-box empirical model on a bipartite Bell-type scenario, defined in Table 1.

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

Table 1: PR-Box model

We numerate all the sections in the following order

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	s_1	s_2	s_3	s_4
a_1	b_2	s_5	s_6	s_7	s_8
a_2	b_1	s_9	s_{10}	s_{11}	s_{12}
a_2	b_2	s_{13}	s_{14}	s_{15}	s_{16}

Table 2: Enumeration of the sections

We choose $R = \mathbb{Z}$ and denote $\mathcal{M} := \{C_1, C_2, C_3, C_4\}$, where C_i corresponds to the i-th row of the table.

Recall that $F_{\mathbb{Z}}\mathcal{S}(C)$ is the set of all formal linear combinations of *possible* sections. Thus, for instance, an element $r_1 \in F_{\mathbb{Z}}\mathcal{S}(C_1)$ must be of the form $a \cdot s_1 + b \cdot s_4$, for $a, b \in \mathbb{Z}$, whereas an element $r_4 \in F_{\mathbb{Z}}\mathcal{S}(C_4)$ must be of the form $g \cdot s_{14} + h \cdot s_{15}$, for $g, h \in \mathbb{Z}$. Therefore, in order to generate elements of a family $\{r_C \in F_{\mathbb{Z}}\mathcal{S}(C)\}_{C \in \mathcal{M}}$ we need to determine each coefficient of Table 3.

A	B	(0, 0)	(1, 0)	(0, 1)	(1, 1)
a_1	b_1	a	0	0	b
a_1	b_2	c	0	0	d
a_2	b_1	e	0	0	f
a_2	b_2	0	g	h	0

Table 3: Possible coefficients for a compatible family

In order to define a *compatible* family $\{r_i \in F_{\mathbb{Z}}\mathcal{S}(C_i)\}_{i=1}^4$, we need to add the condition

$$r_i |_{C_i \cap C_j} = r_j |_{C_i \cap C_j} \quad \forall i, j.$$

This imposes some constraints on the coefficients. For instance, the coefficient a must be equal to c since $\{a1 \mapsto 0, b_2 \mapsto 0\}$ (i.e. the section corresponding to c) is the only possible section at C_2 that is compatible with $\{a_1 \mapsto 0, b_1 \mapsto 0\}$ (which corresponds to a). With the same idea, we obtain the following constraints.

$$a = c, \quad b = d, \quad a = e, \quad b = f, \quad c = h, \quad d = g, \quad e = g, \quad f = h,$$

which altogether imply

$$a = b = c = d = e = f = g = h. \tag{3.5}$$

For a possible section s_i among the ones of Table 2, s_i is part of a compatible family $\{r_i \in F_{\mathbb{Z}}\mathcal{S}(C_i)\}_{i=1}^4$ if we can assign 1 to the corresponding coefficient in table 3, and 0 to every other coefficient in the same row (for instance $s_1 = 1 \cdot s_1 + 0 \cdot s_4$). In the case of the PR-Box, this is clearly impossible, since all the coefficients are equal by (3.5) and there are 2 coefficients for each row (if it was possible, we would have $1 = 0$). Therefore, there cannot by a compatible family, and the obstruction does not vanish for any possible section. We conclude that the PR-box model is cohomologically strongly contextual, and, by Proposition 3.2.10, that it is strongly contextual. In Section 3.4 we will show that every local section of a CSC model gives rise to a *distinct* cohomological obstruction (Proposition 3.4.4). Therefore, we can give a lower bound to the group $\check{H}^1(\mathcal{M}, F_{\mathbb{Z}}\mathcal{S}_{\tilde{C}_0})$: the elements of the first cohomology group relative to a context C are at least $|\mathrm{supp}(e_C)|$, where $\{e_C\}_{C \in \mathcal{M}}$ in this case is the PR-Box model. Hence

$$|\check{H}^1(\mathcal{M}, F_{\mathbb{Z}}\mathcal{S}_{\tilde{C}_i})| \geqslant |\mathrm{supp}(e_{C_i})| = 2, \quad \forall 1 \leqslant i \leqslant 4.$$

- We look again at the GHZ model of Example 1.2.4, Table 4. With the same notations as in the previous example, we introduce the table of coefficients 4.

	1	2	3	$---$	$--+$	$-+-$	$-++$	$+--$	$+-+$	$++-$	$+++$
$C_1 :=$	X_1	X_2	X_3	a	0	0	b	0	c	d	0
$C_2 :=$	X_1	Y_2	Y_3	0	e	f	0	g	0	0	h
$C_3 :=$	Y_1	X_2	Y_3	0	i	j	0	k	0	0	l
$C_4 :=$	Y_1	Y_2	X_3	0	m	n	0	o	0	0	p

Table 4: Coefficients for families in GHZ

Let us consider the compatibility condition for this model. For instance consider the possible sections in the first row with outcome "$-$" for X_1, i.e. the ones corresponding to coefficients a and b in the table. To assure compatibility with C_2 we must take coefficients in the second row corresponding to sections with outcome "$-$" for X_1, i.e. the ones corresponding to e and f. Thus yielding the equation $a + b = e + f$. By applying the same reasoning to the entire table, we obtain the following conditions

$$\begin{array}{cc} a+b=e+f & c+d=g+h \\ a+c=i+k & b+d=j+l \\ a+d=n+o & b+c=m+p \\ f+g=j+k & e+h=i+l \\ e+g=m+o & f+h=n+p \\ i+j=m+n & k+l=o+p \end{array}$$

Once again, checking that a specific section is in a compatible family amounts to give value 1 to the corresponding coefficient, and 0 to the rest of the coefficients in the same row. It is actually sufficient to show that these equations are not satisfiable for coefficients in $\mathbb{Z}_2$, in fact if we had a solution in $\mathbb{Z}$, it would induce a solution in $\mathbb{Z}_2$ via the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}_2$. It has been shown computationally that non of the sections give rise to solutions in $\mathbb{Z}_2$, thus showing once again the strong contextuality of the model. For the same reason as before, this in particular implies that, if we denote by $\{e_C\}_{C \in \mathcal{M}}$ the GHZ empirical model,

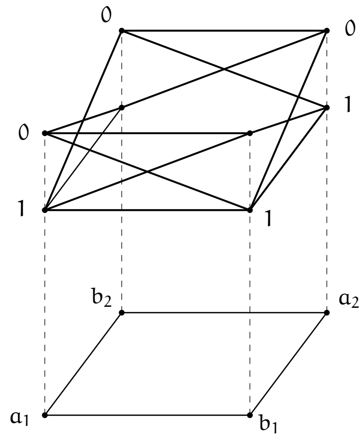
$$|\check{H}^1(\mathcal{M}, F_{\mathbb{Z}}\mathcal{S}_{\tilde{C}_i})| \geqslant |\mathrm{supp}(e_{C_i})| = 4.$$

We also want to explicitly show that the implications of Proposition 3.2.10 are strict by providing examples of *false positives*.

Example 3.3.2 (False positives)**.**

1. Consider the Hardy model (Table 5). We already showed in Section 1.3 that it is contextual. More specifically, we argued that section $s_1 := (a_1, b_1) \mapsto (0, 0)$ is not contained in any compatible family. This can be visually checked in the left panel of Figure 1. The section s_1 is highlighted in red, while in black we have the family obtained when trying to extend s_1 by choosing the only possible compatible section at each context (proceeding counterclockwise). We can clearly see that the black family is not compatible (at $\{a_1\}$, the black path does not touch the red line), proving the non-extendability of s_1.

   However, when we consider the presheaf of abelian groups $\mathcal{F} := F_{\mathbb{Z}}\mathcal{S}$ associated to the Hardy model $\mathcal{S}$, we are allowed to take linear combinations of sections. Therefore, in the context (a_1, b_2), we are allowed



A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_1	b_1	1	1	1	1
a_1	b_2	0	1	1	1
a_2	b_1	0	1	1	1
a_2	b_2	1	1	1	0

Table 5: Hardy model

to travel on the blue path displayed in the right panel of figure 1, obtained by taking the linear combination

$$(a_1, b_1) \mapsto (0, 1) - (a_1, b_1) \mapsto (1, 1) + (a_1, b_1) \mapsto (1, 0).$$

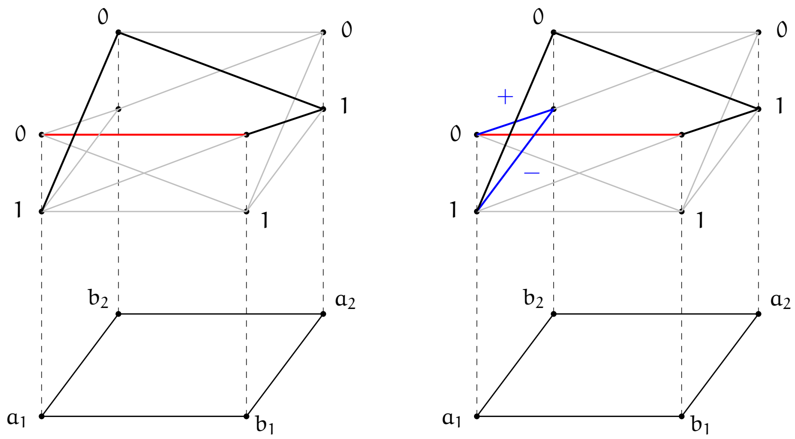


Figure 1: The Hardy model is LC but not CLC

This allows us to find a compatible family of $\mathcal{F}$ extending s_1 as a section in $\mathcal{F}((a_1, b_1))$, showing that the model is not cohomologically logically contextual at s_1 (nor at any other section, as we can easily verify in the same way).

In order to give a more formal proof, we can enumerate the sections as in Table 6. Then the family $\{r_i\}$ defined by

$$r_1 := s_1, \quad r_2 := s_6 + s_7 - s_8, \quad r_3 := s_{11}, \quad r_4 := s_{15}$$

A	B	(0,0)	(0,1)	(1,0)	(1,1)
a_1	b_1	s_1	s_2	s_3	s_4
a_1	b_2	s_5	s_6	s_7	s_8
a_2	b_1	s_9	s_{10}	s_{11}	s_{12}
a_2	b_2	s_{13}	s_{14}	s_{15}	s_{16}

Table 6: Possibilistic Hardy model

is compatible. In fact,

$$r_2|_{a_1} = 1 \cdot (a_1 \mapsto 0) + 1 \cdot (a_1 \mapsto 1) - 1 \cdot (a_1 \mapsto 1) = r_1|_{a_1}$$
$$r_2|_{b_2} = 1 \cdot (b_2 \mapsto 1) + 1 \cdot (b_2 \mapsto 0) - 1 \cdot (b_2 \mapsto 1) = r_4|_{b_2}$$

and the other equations

$$r_1|_{b_1} = 1 \cdot (b_1 \mapsto 0) = r_3|_{b_1}$$
$$r_3|_{a_2} = 1 \cdot (a_2 \mapsto 1) = r_4|_{a_2}$$

are trivially satisfied. Since $r_1 := s_1$ this is a compatible family of $\mathcal{F}$ extending s_1, showing that the model is not cohomologically logically contextual at s_1. We conclude that the Hardy model gives rise to a *false positive*.

2. The following model is an example of an even worse false positive. Its peculiarity is that it is a *strongly contextual model* that is *cohomologically non-contextual*. In other words, although all the sections of the model are impossible to extend to a compatible family, each one of them can be extended to a compatible family of linear combinations of sections.

   Let us call this model $\mathcal{S}$. It is based on a $(2,2,4)$ Bell-type scenario with $O := \{0,1,2,3\}$, and it is graphically represented in Figure 2 (we don't explicitly give the full table as it would be too large and does not give any significant additional information).

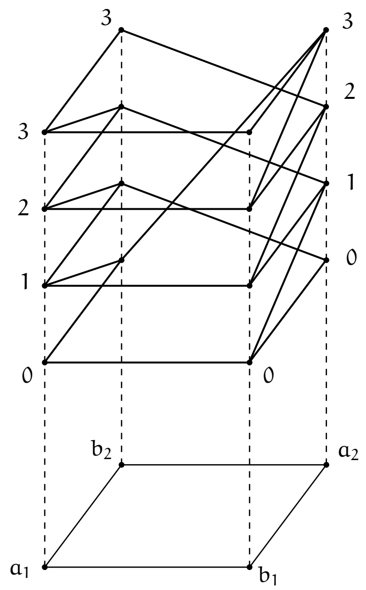


Figure 2: This model is SC but not CLC at any section

We will show the result for two sections and will leave the rest to the reader (the verification can be carried out exactly in the same way).

Consider the section $s_0 := (a_1, b_1) \mapsto (0,0)$, highlighted in red in Figure 3. On the left diagram of Figure 3 we marked in black all the possible paths we obtain when trying to extend s_0 to a compatible family proceeding counterclockwise around the parallelepiped. We can clearly see that none of them is compatible as they all fail to touch the red line at $\{a_1\}$. This means that s_0 is not extendable to a compatible family of sections of the model.

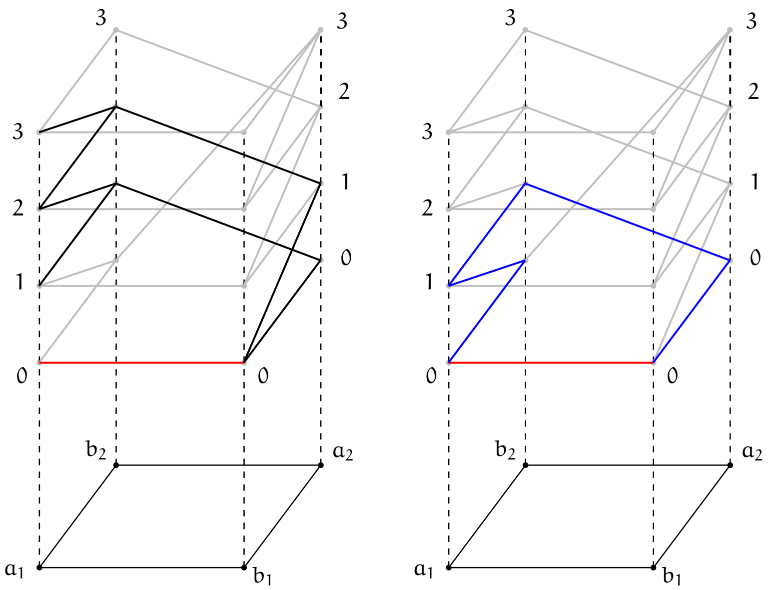


Figure 3: The red section can only be extended in $\mathcal{F}$

However, consider the family $\{r_i\}_{i=0}^{3}$ defined as

$$r_0 := s_0 \quad r_1 := (a_1, b_2) \mapsto (1,1) - (a_1, b_2) \mapsto (1,0) + (a_1, b_2) \mapsto (0,0)$$
$$r_2 := (a_2, b_1) \mapsto (0,0) \quad r_3 := (a_2, b_2) \mapsto (0,1)$$

It is a compatible family for $\mathcal{F} := F_{\mathbb{Z}}\mathcal{S}$, in fact

$$r_1 |_{a_1} = 1 \cdot (a_1 \mapsto 1) - 1 \cdot (a_1 \mapsto 1) + 1 \cdot (a_1 \mapsto 0) = a_1 \mapsto 0 = r_0 |_{a_1}$$
$$r_1 |_{b_2} = 1 \cdot (b_2 \mapsto 1) - 1 \cdot (b_2 \mapsto 0) + 1 \cdot (b_2 \mapsto 0) = b_2 \mapsto 1 = r_3 |_{b_2},$$

and the other equations

$$r_0 |_{b_1} = 1 \cdot (b_1 \mapsto 0) = r_2 |_{b_1}$$
$$r_2 |_{a_2} = 1 \cdot (a_2 \mapsto 0) = r_3 |_{a_2}$$

are trivially verified. Since $r_0 := s_0$, we conclude that $\{r_i\}$ is a compatible family of $\mathcal{F}$ extending s_0. This shows that the model is not cohomologically logically contextual at s_0. This can be dinstinctly visualised in the right diagram of Figure 3, where the path representing $\{r_i\}$ is marked in blue and it is clearly compatible since it is a loop.

We can apply this same argument for any other section of the model (in Figure 4 we give another example concerning a local section on a different context. Once again, the blue path corresponds to a compatible family for $\mathcal{F}$).

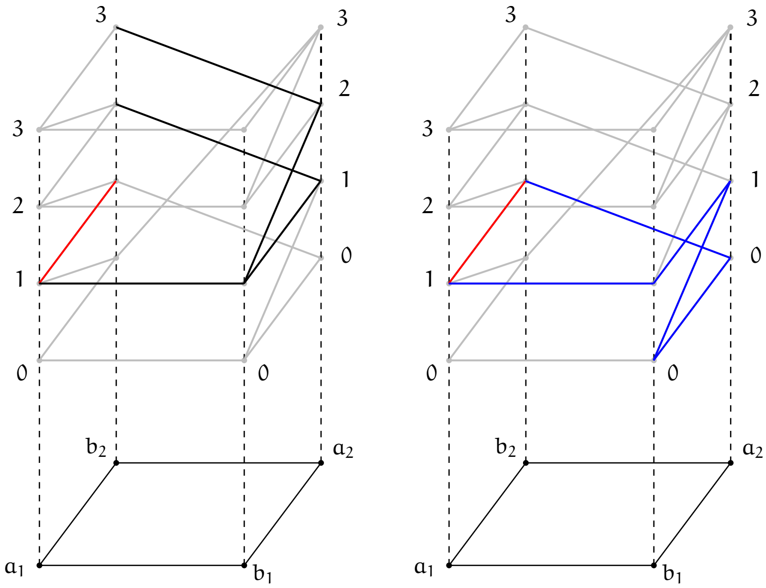


Figure 4: The red section can only be extended in $\mathcal{F}$

This last example is in fact quite important as it shows that, given an empirical model $\mathcal{S}$,

$$\mathrm{SC}(\mathcal{S}) \not\Rightarrow \mathrm{CLC}_{\mathbb{R}}(\mathcal{S}),$$

proving that cohomology can even fail to detect strong contextuality.

Until now, the only strongly contextual model known to give rise to a false positive is the Kochen-Specker model (cf. [KS75]) for the cover

$$\{A, B, C\}, \{B, D, E\}, \{C, D, E\}, \{A, D, F\}, \{A, EG\},$$

which "does not satisfy any reasonable criterion for symmetry, nor does it satisfy any strong form of connectedness" [ABK+15]. Due to these limitations it has been advanced the hypothesis that, "under suitable assumptions of symmetry and connectedness of the cover, the cohomology obstruction is a complete invariant for strong contextuality" (Conjecture 8.1 of [ABK+15]). Notice that the model depicted in Figure 2 is defined on a very simple $(2, 2, 4)$ scenario that has all sorts of nice symmetry and connectedness properties. We can therefore conclude that it is a counterexample to the conjecture.

The existence of such badly behaved false positives deeply compromises the accuracy of the cohomological approach to the detection of contextuality. For this reason, we aim to understand whether the false positiveness of a model can be detected elsewhere in the structure of its cohomology groups. In Chapter 4 we will discuss this possibility by providing a generalisation of the cohomological obstruction to higher cohomology groups.

3.4 THE HOMOMORPHISM γ

In this section we will introduce some new results and examples concerning the properties of the connecting homomorphism γ. In particular, we will give an answer to some recent hypothesis concerning the injectivity and surjectivity of γ and consider their consequences on the structure of the first relative cohomology group $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$.

In Section 3.2.1 we highlighted the importance of the first cohomology group in detecing the contextuality of empirical models, yet a full understanding of the nature of its elements is still to be achieved. It was recently advanced the hypothesis that cohomological obstructions to the existence of global sections actually *coincide* with the elements of $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$ [Abr15]. This result would give us an elegant characterisation of the first cohomology group that could potentially be used to better understand other sheaf cohomology-related domains.

Conjecture 3.4.1. *The first relative cohomology group* $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$ *is exactly the group of all cohomological obstructions to the existence of compatible families extending sections of* $\mathcal{F}(C_0)$.

It is easy to see that this statement is equivalent to the surjectivity of γ^C for each context $C \in \mathcal{M}$ (cf. Remark 3.2.3 for notation).

Another hypothesis involving cohomological obstructions concerns on the other hand the injectivity of γ. Proposition 3.2.7 shows that every section that cannot be extended to a compatible family gives rise to a non-identity cohomology class. The conjecture proposed is that these obstructions have to be all distinct.

Conjecture 3.4.2. *For any context* C_0*, every section* $s_0 \in \mathcal{F}(C_0)$ *which cannot be extended to a compatible family gives rise to a* distinct *non-identity cohomology class* $\gamma^{C_0}(s_0)$ *in* $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$.

Notice that this conjecture is trivially satisfied by cohomologically non-contextual models since all their sections are extendable. On the other hand, if $\mathcal{F}$ is contextual, it is easy to see that the statement of Conjecture 3.4.2 is equivalent to the injectivity of γ^C for each context $C \in \mathcal{M}$.

Unfortunately, it turns out that none of these conjectures are true. In the next sections we will provide explicit counterexamples which show that, in general, γ is neither injective nor surjective. The discussion will nonetheless bring to light some interesting implications of the injectivity/surjectiviy of the connecting homomorphisms in terms of properties of the corresponding empirical model.

3.4.1 The injectivity of the connecting homomorphisms γ characterises CSC models.

In the following discussion we will use Definition 3.2.9 in a slighlty broader sense in order to make our agrument more general. Given a presheaf of abelian groups $\mathcal{F}$ relative to an empirical model $\mathcal{S}$, we will call $\mathcal{S}$ cohomologically logically/strongly contextual (CLC/CSC) if it satisfies the conditions of Definition 3.2.9 *even in the case where* $\mathcal{F} \neq F_R\mathcal{S}$.

We start our analysis by pointing out that Conjecture 3.4.2 holds for cohomologically strongly contextual models (CSC).

Proposition 3.4.3. *Suppose we have a presheaf* $\mathcal{F}$ *of abelian groups relative to a cohomologically strongly contextual model. Let* C_0 *be a context. Then each section* $\mathcal{F}(C_0)$ *gives rise to a* distinct *non-identity cohomology class in* $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$.

Proof. Since the model is cohomologycally strongly contextual, $\gamma^{C_0}(s_0) \neq 0$ for all $0 \neq s_0 \in \mathcal{F}(C_0)$. Thus $\ker(\gamma^{C_0}) = 0$, which means that γ^{C_0} is injective. In other words, every non extendable local sections in $\mathcal{F}(C_0)$ gives rise to a distinct cohomology class.

□

Therefore, in cohomologically strongly contextual models, γ^C is injective for all $C \in \mathcal{M}$. It turns out that this last statement actually *characterises* CSC models.

Proposition 3.4.4. *The empirical model underlying $\mathcal{F}$ is cohomologically strongly contextual if and only if, for each context $C \in \mathcal{M}$, the connecting homomorphism γ^C is injective.*

Proof. The first implication has been proved in Proposition 3.4.3. Now, suppose γ^C is injective for every context C. This means that $\ker(\gamma^C) = 0$ for all contexts $C \in \mathcal{M}$. In other words, every nonzero local section of $\mathcal{F}$ has a nonzero cohomological obstruction. This is equivalent to the cohomological strong contextuality of the empirical model. □

Remark 3.4.5. It is clear from the proof of this proposition that γ^C is injective if and only if every local section at C is non-extendable to a compatible family.

We conclude that Conjecture 3.4.2 is true if and only if the model in question is either cohomologically strongly contextual or cohomologically noncontextual. In particular, this means that $\mathsf{CLC} \wedge \neg\mathsf{CLC}$ models always have at least one context in which different non-extendable sections give rise to the same cohomology class.

We can give a more explicit proof of this fact.Consider a presheaf $\mathcal{F}$ relative to a $\mathsf{CLC} \wedge \neg\mathsf{CLC}$ model. Since the model is CLC, there exists a context C_0 and a section $s_0 \in \mathcal{F}(C_0)$ which is not extendable to a compatible family. By Proposition 3.2.7 this implies $\gamma(s_0) \neq 0$.
On the other hand, since the model is not cohomologically strongly contextual, we can find a

$$0 \neq r_0 \in \mathcal{F}(C_0) \ \text{ such that } \ \gamma(r_0) = 0.$$

Let $t_0 := s_0 + r_0$. Then $t_0 \neq s_0$ since $r_0 \neq 0$. However

$$\gamma(t_0) = \gamma(s_0 + r_0) = \gamma(s_0) + \underbrace{\gamma(r_0)}_{0} = \gamma(s_0).$$

Therefore, we *can* find two *different* non-extendable sections that give rise to the same cohomology class, showing that Conjecture 3.4.2 is indeed false. We will now apply this general argument to find a concrete example of a model violating Conjecture 3.4.2.

Example 3.4.6. Consider the following model on a bipartite Bell-type scenario with set of outcomes $O := \{0, 1, 2\}$ (i.e. the scenario is $(2, 2, 3)$). Table 7 shows the possible sections and Figure 5 gives a graphically representation of the model.

	00	01	10	11	02	20	22	12	21
$C_0 := (a_1, b_1)$	1	0	0	1	0	0	1	0	0
$C_1 := (a_1, b_2)$	1	0	0	0	0	0	1	1	0
$C_2 := (a_2, b_1)$	1	1	0	0	0	0	1	0	0
$C_3 := (a_2, b_2)$	1	0	0	0	0	0	1	0	0

Table 7: Example model

We take $\mathcal{F} := F_{\mathbb{Z}}\mathcal{S}$, where $\mathcal{S}$ is the presheaf of sets of all possible sections. We can already see from the diagram that $s_0 := (a_1, b_1) \mapsto (1, 1) \in$

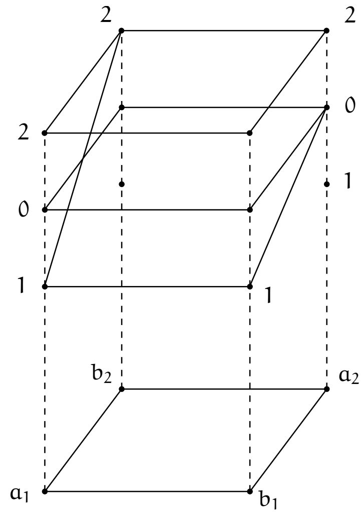


Figure 5: Graphical representation of the model

$\mathcal{F}(C_0)$ is not contained in any compatible family (red path in Figure 6), thus $\gamma(s_0) \neq 0$. On the other hand, $r_0 := (a_1, b_1) \mapsto (2, 2) \in \mathcal{F}(C_0)$ is contained in the compatible family

$$\{r_0, (a_1, b_2) \mapsto (2, 2), (a_2, b_1) \mapsto (2, 2), (a_2, b_2) \mapsto (2, 2)\},$$

(blue path in Figure 6). Thus $r_0 \neq 0$ and $\gamma(r_0) = 0$ and we can apply the argument above to conclude that the two distinct sections s_0 and $s_0 + r_0$ give rise to the same non-identity cohomology class.

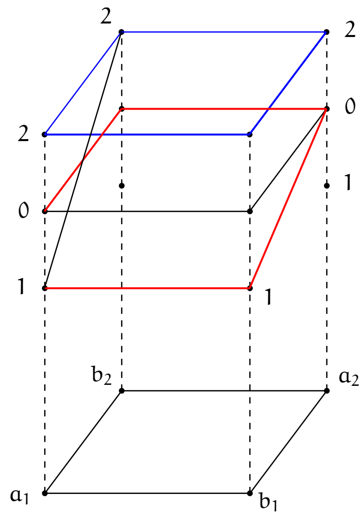


Figure 6: Model 7 is CLC but not CSC

We can prove the fact that $\gamma(s_0) \neq 0$ more formally, following the same procedure as in Section 3.3. We display in the following table the coefficients for a candidate compatible family $\{s_i\}_i$ containing s_0.

Notice that the coefficient relative to the section s_0 is b.

The relations imposed by compatibility (i.e. $s_i|_{C_i \cap C_j} = s_j|_{C_i \cap C_j}$) are

$$b = f, \quad a = d, \quad c = e, \quad a = g, \quad b = h,$$
$$c = i, \quad d = l, \quad e + f = m, \quad g + h = l, \quad j = m.$$

	00	01	10	11	02	20	22	12	21
$C_0 := (a_1, b_1)$	a	0	0	b	0	0	c	0	0
$C_1 := (a_1, b_2)$	d	0	0	0	0	0	e	f	0
$C_2 := (a_2, b_1)$	g	h	0	0	0	0	i	0	0
$C_3 := (a_2, b_2)$	l	0	0	0	0	0	m	0	0

Thus we have

$$a = d = l = g + h = a + h \Rightarrow h = 0 \Rightarrow \underline{b = 0}$$

Therefore, we cannot find a compatible family that contains s_0, which means $\gamma(s_0) \neq 0$.

Although this example clearly shows that different non-extendable sections of $\mathcal{F}$ can give rise to the same cohomological obstruction, we have to keep in mind that $\mathcal{F}$ is a mere abelian counterpart to an original empirical model $\mathcal{S}$. The general argument used to prove that CLC $\wedge \neg$CSC models violate Conjecture 3.4.2 constructs the second section t_0 from a sum of two other sections (recall $t_0 := s_0 + r_0$ in the discussion). This is only allowed because $\mathcal{F}$ is a presheaf of abelian groups and does not have any meaning in the empirical model underlying $\mathcal{F}$. In other words, the section t_0 does not exist in the original empirical model. The reader might wonder whether Conjecture 3.4.2 is true if we only consider sections of the original empirical model.

We give the following example specifically to show that it is not the case.

Example 3.4.7. Let us consider the following empirical model.

	00	01	10	11	02	20	22	12	21
$C_0 := (a_1, b_1)$	1	0	1	1	0	0	1	0	0
$C_1 := (a_1, b_2)$	1	0	0	0	0	0	1	1	0
$C_2 := (a_2, b_1)$	1	1	0	0	0	0	1	0	0
$C_3 := (a_2, b_2)$	1	0	0	0	0	0	1	0	0

Table 8: Possible sections of the model

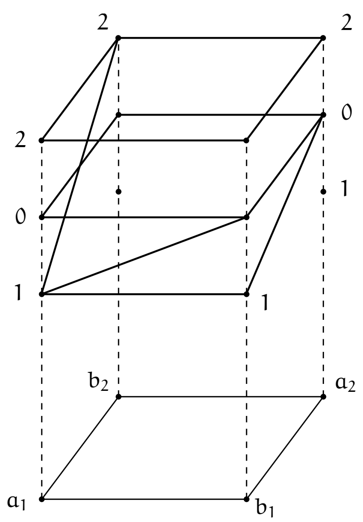


Figure 7: Graphical representation of model 8

Notice that this exactly the same model of Example 3.4.6, except we added section $(a_1, b_1) \mapsto (1, 0)$ to the possible sections at C_0.

Now, consider sections $s_0 := (a_1, b_1) \mapsto (1, 0)$ (blue in the left panel of Figure 8) and $t_0 := (a_1, b_1) \mapsto (1, 1)$ (red in the left panel of Figure 8). It is important to remark that these two sections are possible sections of the empirical model 8. We will leave to the reader the verification of the fact that neither s_0 nor t_0 are extendable to a compatible family (this can be quite easily visually verified in the diagram of the model).

On the right hand side of Figure 8 we highlighted in blue a no-signaling family s for the section s_0 and in red a no-signaling family t for t_0.

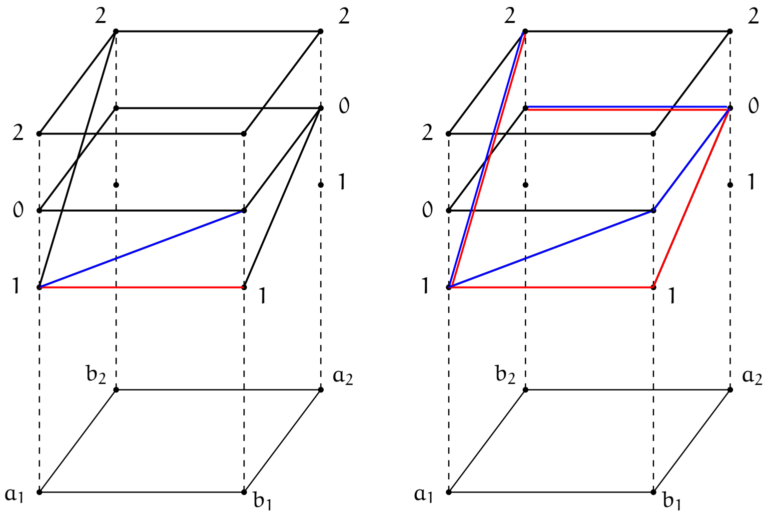


Figure 8: No signaling families for the sections s_0 (s in blue) and t_0 (t in red).

Notice that both families are compatible everywhere with the exception of the intersection $C_1 \cap C_3$, thus $\delta^0(s)(\sigma) = \delta^0(t)(\sigma) = 0$ whenever $\sigma \neq (C_1, C_3)$ or $\sigma \neq (C_3, C_1)$. In the case where $\sigma = (C_1, C_3)$ we have

$$\begin{aligned}\delta^0(s)(C_1, C_3) &= s_1 |_{C_1 \cap C_3} - s_3 |_{C_1 \cap C_3} = t_1 |_{C_1 \cap C_3} - t_3 |_{C_1 \cap C_3} \\ &= \delta^0(t)(C_1, C_3).\end{aligned}$$

and similarly for $\sigma = (C_3, C_1)$. We conclude that $\delta^0(s) = \delta^0(t)$.

Therefore, using the concrete definition of cohomological obstruction of [AMSB12] discussed in Remark 3.2.6,

$$0 \overset{3.2.7}{\neq} \gamma(s_0) = [\delta^0(s)] = [\delta^0(t)] = \gamma(t_0) \overset{3.2.7}{\neq} 0.$$

We thus showed the existence of two distinct local sections of an empirical model that give rise to the same non-identity cohomology class.

The following proposition gives us a sufficient condition for the strong contextuality of an empirical model using the injectivity property of the connecting homomorphisms γ.

Proposition 3.4.8. *Let R be a ring and suppose $\mathcal{F} := F_R \mathcal{S}$ is the presheaf of abelian groups relative to an empirical model $\mathcal{S}$ on a cover $\mathcal{M}$. If there exists a context $C_0 \in \mathcal{M}$ such that γ^{C_0} is injective, then $\mathcal{S}$ is strongly contextual ($SC(\mathcal{S})$).*

Proof. Suppose there exists a $C_0 \in \mathcal{M}$ such that γ^{C_0} is injective and suppose by contradiction that $\mathcal{S}$ is *not* strongly contextual. Then, there exists a context

$C_i \in \mathcal{M}$ and a section $s_i \in \mathcal{S}(C_i)$ that is extendable to a compatible family $\{s_i \in \mathcal{S}(C_i)\}_{i=0}^n$. Consider the element $s_0 \in \mathcal{S}(C_0)$ of this family. It is an extendable local section at C_0. Thus, the model is *not* logically contextual at s_0. By Proposition 3.2.10, this implies that the model is not cohomologically logically contextual at s_0 either. Therefore, $\gamma^{C_0}(s_0) = 0$. Since $s_0 \in \mathcal{S}(C_0)$, it is nonzero in $\mathcal{F}(C_0)$. We conclude that $\ker(\gamma^{C_0}) \neq 0$, and this implies that γ^{C_0} is not injective, which is a contradiction. □

Remark 3.4.9. Note that the assumption $\mathcal{F} := F_R\mathcal{S}$ is only needed to make sure that a section $s_0 \in \mathcal{S}(C_0)$ is nonzero in $\mathcal{F}(C_0)$. We can actually relax it and simply suppose that $\mathcal{F}$ is such that a local section at a context C_0 of the original empirical model is never the zero section of $\mathcal{F}(C_0)$.

Not only this proposition gives us a useful method of detecting the strong contextuality of an empirical model, but it highlights the difference between CSC models and SC in a rather concrete way, showing more conceptually the strictness of the implications of Proposition 3.2.10.

- We need *all* the connecting homomorphisms $\{\gamma^C\}_{C\in\mathcal{M}}$ to be injective in order to conclude that a model is CSC.
- It is sufficient to have *one* injective connecting homomorphisms γ^C to conclude that a model is SC.

3.4.2 The connecting homomorphism γ is not surjective in general

In this section we will disprove Conjecture 3.4.1 showing that some models admit non-surjective connecting homomorphisms γ. The most simple case to study is a non-contextual empirical model. If Conjecture 3.4.1 was valid, such a model would have trivial first cohomology group $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$. We will show that this is not true. In order to prove it, we will firstly need to enumerate the characterising properties of the elements of $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$.

A straightforward description of $\check{H}^1$.

We will give here the most straightforward interpretation to the elements of $\check{H}^1(\mathcal{M}, \mathcal{F})$, i.e. the one directly inferred by the definition of the Čech cohomology groups for a presheaf $\mathcal{F}$ relative to an empirical model on a cover $\mathcal{M} := \{C_i\}_{i=0}^n$. First of all, we need to understand what 1-cocycles are. A 1-cocycle $z \in Z^1(\mathcal{M}, \mathcal{F})$ is characterised by the equation

$$\begin{aligned}\delta^1(z)(C_i, C_j, C_k) = z(C_j, C_k)\,|_{C_{i,j,k}} - z(C_i, C_k)\,|_{C_{i,j,k}} \\ + z(C_i, C_j)\,|_{C_{i,j,k}}, \quad \forall 1 \leqslant i, j, k \leqslant n.\end{aligned} \tag{3.6}$$

where $C_{i,j,k} := C_i \cap C_j \cap C_k$. Notice that this implies

$$z(C_i, C_i) = 0 \quad \forall 0 \leqslant i \leqslant n, \tag{3.7}$$

in fact,

$$z(C_i, C_i) = z(C_i, C_i) - z(C_i, C_i) + z(C_i, C_i) = \delta^1(z)(C_i, C_i, C_i) = 0.$$

As a consequence, we also have

$$z(C_i, C_j) = z(C_j, C_i) \quad \forall 0 \leqslant i, j \leqslant n, \tag{3.8}$$

in fact

$$\begin{aligned} z(C_i, C_j) - z(C_j, C_i) = z(C_i, C_j)\,|_{C_i \cap C_j} - \underbrace{z(C_j, C_j)}_{0}\,|_{C_i \cap C_j} \\ - z(C_j, C_i)\,|_{C_i \cap C_j} \\ = \delta^1(z)(C_i, C_j, C_i) = 0. \end{aligned}$$

Remark 3.4.10. It is important to note that, if we consider special cases of equation (3.6) (e.g. cases where i, j, k are not distinct like $\delta^0(z)(C_i, C_i, C_j) = 0$), the only conditions we obtain on z are (3.7) and (3.8).

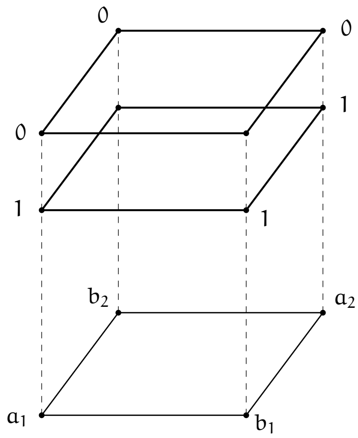
We can see that two 1-cocycles z and z' are such that $[z] = [z'] \in \check{H}^1(\mathcal{M}, \mathcal{F})$ (i.e. they are cohomologous) if and only if there exists a family $g := \{g_i \in \mathcal{F}(C_i)\}_{i=0}^n$ such that $\delta^0(g) = z - z'$, i.e.

$$g_i\,|_{C_i \cap C_j} - g_j\,|_{C_i \cap c_j} = z(C_i, C_j) - z'(C_i, C_j).$$

Thus, $\check{H}^1(\mathcal{M}, \mathcal{F})$ is simply the group of cohomology classes of elements of $C^1(\mathcal{M}, \mathcal{F})$ satisfying (3.6).

We can now introduce the counterexample to Conjecture 3.4.1.

Example 3.4.11. Consider the model $\mathcal{S}$ on a $(2, 2, 2)$ scenario presented in Table 9. Let $R := \mathbb{Z}_2$ and let $\mathcal{F} := F_R\mathcal{S}$ (we can construct a vimilar counterexample with $R = \mathbb{Z}$ but the argument is more complicated).



	$(0, 0)$	$(1, 0)$	$(0, 1)$	$(1, 1)$
$C_0 := (a_1, b_1)$	1	0	0	1
$C_1 := (a_1, b_2)$	1	0	0	1
$C_2 := (a_2, b_1)$	1	0	0	1
$C_3 := (a_2, b_2)$	1	0	0	1

Table 9: A non-contextual empirical model on a $(2, 2, 2)$ scenario.

The model $\mathcal{S}$ is clearly non-contextual since every possible section is a member of a compatible family. We will show that $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0}) \neq 0$, disproving Conjecture 3.4.1. We start by defining a relative 1-cocycle $z \in Z^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$ as follows

$$z(C_i, C_j) := \begin{cases} (b_2 \mapsto 1) \in \mathcal{F}(C_1 \cap C_3) & \text{if } \{i, j\} = \{1, 3\} \\ (a_2 \mapsto 0) \in \mathcal{F}(C_2 \cap C_3) & \text{if } \{i, j\} \in \{2, 3\} \\ 0 & \text{otherwise.} \end{cases}$$

Let us verify that z is indeed a relative 1-cocycle. First of all, notice that

$$z(C_i, C_j)\,|_{C_i \cap C_j \cap C_0} = 0 \quad \forall 0 \leqslant i, j \leqslant 3,$$

showing that

$$z(C_i, C_j) \in \mathcal{F}_{\tilde{C}_0} \quad \forall 0 \leqslant i, j \leqslant 3.$$

Now, if i, j, k are all distinct, equation (3.6) is always verified since $C_i \cap C_j \cap C_k = \varnothing$. Moreover, z satisfies (3.7) by definition, since $z(C_i, C_i) = 0$ for all i. Finally, (3.8) is also verified since

$$\begin{cases} z(C_1, C_3) = (b_2 \mapsto 1) = -(b_2 \mapsto 1) = -z(C_3, C_1) \\ z(C_2, C_3) = (a_2 \mapsto 0) = -(a_2 \mapsto 0) = -z(C_3, C_2) \\ z(C_i, C_j)\,|_{C_i \cap C_j} = 0 = -z(C_j, C_i)\,|_{C_i \cap C_j} & \text{if } \{i, j\} \neq \{1, 3\}, \{2, 3\} \end{cases}$$

By Remark 3.4.10 these are the only conditions we need to check to conclude that z is indeed a member of $Z^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$.

Our claim is that $[z] \neq 0 \in \check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$. In other words, $z \neq \delta^0(t)$ for any $t \in C^0(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$. To show this, suppose that there exists a family $t \in C^0(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$ such that $\delta^0(t) = z$. We will denote the possible sections of $\mathcal{S}$ as follows

$$s_i^j := (C_i \mapsto (j, j), \quad \forall 0 \leqslant i \leqslant 3, \forall j = 0, 1.$$

Since $t_i \in \mathcal{F}_{\tilde{C}_0}(C_i)$, we have $t_0 = 0$. Moreover, if we write $t_i \in \mathcal{F}(C_i)$ in its general form

$$t_i := \lambda_i^0 \cdot s_i^0 + \lambda_i^1 \cdot s_i^1, \quad \lambda_i^0, \lambda_i^1 \in \mathbb{Z}_2,$$

it is clear that the condition $t_i \in \mathcal{F}_{\tilde{C}_0}(C_i)$ imposes $\lambda_i^0, \lambda_i^1 = 0$ for all $i \neq 3$. This means that t_3 is the only non-zero t_i. Therefore,

$$\delta^0(t)(C_i, C_j) = \begin{cases} t_3\,|_{\{b_2\}} & \text{if } \{i, j\} = \{1, 3\} \\ t_3\,|_{\{a_2\}} & \text{if } \{i, j\} = \{2, 3\} \\ 0 & \text{otherwise.} \end{cases}$$

The condition $\delta^0(t) = z$ imposes the following equations

$$t_3\,|_{\{b_2\}} = (b_2 \mapsto 1)$$
$$t_3\,|_{\{a_2\}} = (a_2 \mapsto 0),$$

which cannot be both true since the first one forces $t_3 = (C_3 \mapsto (1, 1))$ and the second one gives $t_3 = (C_3 \mapsto (0, 0))$. We conclude that such a $t \in C^0(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$ cannot exist. Therefore,

$$0 \neq [z] \in \check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0}),$$

contradicting Conjecture 3.4.1.

Unfortunately, due to time constraints, unlike the previous section, we were not able to give a full explanation of what the surjectivity of the connecting homomorphism actually means for an empirical model. We could not find with any concrete example of a model with this feature. We hope to study this property in more details in future research.

3.5 AN ALTERNATIVE DESCRIPTION OF THE FIRST COHOMOLOGY GROUP

The failure of Conjecture 3.4.1 has left unanswered the question of understanding the elements of the first cohomology group. Solving this problem would significantly improve our grasp of the cohomological structure of empirical models. In the following paragraphs, we will introduce an alternative description of the first cohomology group in terms of $\mathcal{F}$-torsors. Although this approach is still at a developing stage, we are confident it will help us understand in detail what sort of information the first cohomology group can give us on the contextuality of empirical models. The following general discussion is inspired and readapted to fit the study of contextuality from [Sk001, Chap. 2] and [GW10, Chap. 11].

3.5.1 $\mathcal{F}$-torsors and their relation to $\check{H}^1$.

Let $\mathcal{F} : \mathbf{Open}(X)^{op} \to \mathbf{Ab}$ be a presheaf of abelian groups relative to a topological space X.

Definition 3.5.1.

- An *$\mathcal{F}$-presheaf* is a presheaf
$$T : \mathbf{Open}(X)^{op} \to \mathbf{Ab}$$
equipped with a map of presheaves (i.e. a natural transformation)
$$\varphi : \mathcal{F} \times T \Rightarrow T,$$
such that, for each open $U \subseteq X$, the map
$$\mathcal{F}(U) \times T(U) \to T(U) :: (g, t) \mapsto g \bullet t$$
is a left action of $\mathcal{F}(U)$ on $T(U)$.
- Given two $\mathcal{F}$-presheaves T and T', a *morphism of $\mathcal{F}$-presheaves* from T to T' is a morphism of presheaves (i.e. a natural transformation)
$$\psi : T \Rightarrow T'$$
such that ψ_U is equivariant for all open $U \subseteq X$. Explicitly,
$$g \bullet \psi_U(t) = \psi_U(g \bullet t), \quad \forall g \in \mathcal{F}(U), \ \forall t \in T(U).$$

These notions define the *category $\mathcal{F}$-**PSh** of $\mathcal{F}$-presheaves*.

It is easy to verify that $\mathcal{F}$-**PSh** is indeed a category. The identities are given by the identity natural transformations $T \Rightarrow T$ (which are clearly equivariant). The composition law is induced by the usual composition of natural transformations. In fact, suppose we have two morphisms of $\mathcal{F}$-presheaves.
$$T \overset{\psi}{\Rightarrow} T' \overset{\psi'}{\Longrightarrow} T'',$$
then $\psi' \circ \psi \in \mathbf{PSh}(T, T'')$, in fact, given a $g \in \mathcal{F}(U)$ and a $t \in T(U)$, we have
$$\begin{aligned} g \bullet (\psi' \circ \psi)_U(t) &= g \bullet \psi'_U(\psi_U(t)) = \psi'_U(g \bullet \psi_U(t)) = \psi'_U(\psi_U(g \bullet t)) \\ &= (\psi' \circ \psi)_U(g \bullet t) \end{aligned}$$

We can now define the notion of *$\mathcal{F}$-torsor*.

Definition 3.5.2. Let $T \in \mathcal{F}$-**PSh**, we say that T is an *$\mathcal{F}$-torsor* if

1. For all open $U \subseteq X$, we have $T(U) \neq \varnothing$.
2. The action ϕ_U is simply transitive for each open $U \subseteq X$. Explicitly, for each pair of elements $t, t' \in T(U)$, there exists a unique element $g \in \mathcal{F}(U)$ such that $g \bullet t = t'$.

We define the *category $\mathcal{F}$-**Tor** of $\mathcal{F}$-torsors* on X as the subcategory of $\mathcal{F}$-**PSh** containing only $\mathcal{F}$-torsors.

Example 3.5.3. A very simple example of an $\mathcal{F}$-torsor is the presheaf $\mathcal{U}\mathcal{F}$ (where $\mathcal{U}$: **Ab** $\to$ **Set** denotes the forgetful funcor). The actions ϕ_U are simply given by left multiplication

$$\begin{array}{rccc} \phi_U : & \mathcal{F}(U) \times \mathcal{U}\mathcal{F}(U) & \longrightarrow & \mathcal{U}\mathcal{F}(U) \\ & (g, \mathcal{U}(h)) & \mapsto & g \bullet \mathcal{U}(h) = \mathcal{U}(g \cdot h), \end{array}$$

where $g, h \in \mathcal{F}(U)$. Given two elements $h, k \in \mathcal{F}(U)$, $g := k \cdot h^{-1}$ is clearly the unique element in $\mathcal{F}(U)$ such that $g \bullet \mathcal{U}(h) = \mathcal{U}(k)$, in fact

$$g \bullet \mathcal{U}(h) = \mathcal{U}(g \cdot h) = \mathcal{U}(k \cdot h^{-1} \cdot h) = \mathcal{U}(k).$$

This torsor is called the *trivial* $\mathcal{F}$-torsor.

Proposition 3.5.4. *An $\mathcal{F}$-torsor T is isomorphic to the trivial $\mathcal{F}$-torsor if and only if $T(X) \neq \varnothing$.*

Proof.

- Suppose there exists a $t \in T(X)$. Then, for each open $U \subseteq X$ we have an isomorphism

$$f : \mathcal{U}\mathcal{F} \xrightarrow{\cong} T(U) :: \mathcal{U}(g) \mapsto g \bullet t\mid_U .$$

  In fact, given $g, h \in \mathcal{F}(U)$, such that $f(\mathcal{U}(g)) = f(\mathcal{U}(h))$ we have $g \bullet t\mid_U = h \bullet t\mid_U$, and this forces $g = h$ by simple transitivity of the action.

- Suppose for all U there is an isomorphism $\mathcal{U}\mathcal{F} \xrightarrow{\cong} T(U)$. Then, in particular, $\varnothing \neq \mathcal{U}\mathcal{F}(X) \cong T(X)$.

$\square$

Definition 3.5.5. Given a cover an open $\mathcal{U}$, we say that $\mathcal{U}$ *trivialises* an $\mathcal{F}$-torsor T it $T(U) \neq \varnothing$ for all $U \in \mathcal{U}$.

Now we will adapt these general definitions to our discussion on empirical models. Suppose we have a set of measurements X, and a presheaf of abelian groups $\mathcal{F}$ relative to an empirical model on a cover $\mathcal{M} := \{C_i\}_{i=0}^n$. We denote by $\mathrm{Trs}_{\mathcal{F}}$ the set of isomorphism classes of $\mathcal{F}$-torsors. Let

$$\mathrm{Trs}(\mathcal{M}, \mathcal{F}) := \{T \in \mathrm{Trs}_{\mathcal{F}} \mid T \text{ is trivialised by } \mathcal{M}\}.$$

We can see $\mathrm{Trs}(\mathcal{M}, \mathcal{F})$ as a pointed set, where the distinguished element is the trivial $\mathcal{F}$-torsor.

Theorem 3.5.6. *We have a bijection of pointed sets*

$$\mathrm{Trs}(\mathcal{M}, \mathcal{F}) \cong \check{H}^1(\mathcal{M}, \mathcal{F}),$$

where the distinguished elements of $\check{H}^1(\mathcal{M}, \mathcal{F})$ is obviously its neutral elements 0.

Proof. Let $T \in \mathrm{Trs}(\mathcal{M}, \mathcal{F})$. We start by arbitrarily choosing a collection $\{t_i \in T(C_i)\}_{i=0}^n$ (this is possible since $\mathcal{M}$ trivialises T). Since T is an $\mathcal{F}$-torsor, by simple transitivity, there exists a unique $g_{ij} \in \mathcal{F}(C_i \cap C_j)$ such that $g_{ij} \bullet t_j |_{C_i \cap C_j} = t_i |_{C_i \cap C_j}$, for all $0 \leqslant i, j \leqslant n$. We have

$$\begin{aligned}(g_{kj} + g_{ji}) \bullet t_i |_{C_i \cap C_j \cap C_k} &= g_{kj} \bullet \left(g_{ji} \bullet t_i |_{C_i \cap C_j}\right) |_{C_i \cap C_j \cap C_k} \\ &= g_{kj} \bullet \left(t_j |_{C_i \cap C_j}\right) |_{C_i \cap C_j \cap C_k} \\ &= \left(g_{kj} \bullet t_j |_{C_j \cap C_k}\right) |_{C_i \cap C_j \cap C_k} \\ &= t_k |_{C_i \cap C_j \cap C_k} \\ &= g_{ki} \bullet t_i |_{C_i \cap C_j \cap C_k},\end{aligned} \tag{3.9}$$

for all $0 \leqslant i, j, k \leqslant n$. By simple transitivity, (3.9) implies

$$g_{kj} + g_{ji} = g_{ki},$$

Thus, if we define $f(T)(C_i, C_j) := g_{ij}$, by (3.6) we know that $f(T)$ is a 1-cocycle. Let us define

$$\begin{aligned} f : \quad \mathrm{Trs}(\mathcal{M}, \mathcal{F}) &\longrightarrow \check{H}^1(\mathcal{M}, \mathcal{F}) \\ T &\longmapsto f(T). \end{aligned}$$

In order to show that this map is well-defined, we need to prove that $f(T)$ is independent of the choice of the family $\{t_i\}$. Suppose we choose $\{t'_i \in T(C_i)\}_{i=0}^n$ instead, then we obtain a family $\{g'_{ij} \in \mathcal{F}(C_i \cap C_j)\}$ like before. By simple transitivity, for each $0 \leqslant i \leqslant n$ there exists an element $g_i \in \mathcal{F}(C_i)$ such that $g_i \bullet t'_i = t_i$. We thus obtain a family $g := \{g_i \in \mathcal{F}(C_i)\}_{i=0}^n$. We have

$$\begin{aligned}\left(g_i |_{C_i \cap C_j} + g'_{ij}\right) \bullet t'_j |_{C_i \cap C_j} &= g_i |_{C_i \cap C_j} \bullet \left(g'_{ij} \bullet t'_j |_{C_i \cap C_j}\right) \\ &= g_i |_{C_i \cap C_j} \bullet t'_i |_{C_i \cap C_j} = t_i |_{C_i \cap C_j}\end{aligned}$$

On the other hand,

$$\begin{aligned}\left(g_{ij} + g_j |_{C_i \cap C_j}\right) \bullet t'_j |_{C_i \cap C_j} &= g_{ij} \bullet \left(g_j |_{C_i \cap C_j} \bullet t'_j |_{C_i \cap C_j}\right) \\ &= g_{ij} \bullet t_j |_{C_i \cap C_j} = t_i |_{C_i \cap C_j}.\end{aligned}$$

Again, by simple transitivity, we must then have

$$g_i |_{C_i \cap C_j} + g'_{ij} = g_{ij} + g_j |_{C_i \cap C_j},$$

and this is equivalent to say

$$\delta^0(g)(C_i, C_j) = g'_{ij} - g_{ij},$$

and, consequently, that it does not matter whether we define $f(T)(C_i, C_j) := g_{ij}$ or $f(T)(C_i, C_j) := g'_{ij}$ since these two 1-cocycles are cohomologous.

We now have to show that the map f is indeed a bijection. We will define an inverse map $g : \check{H}^1(\mathcal{M}, \mathcal{F}) \to \mathrm{Trs}(\mathcal{M}, \mathcal{F})$. Given a $[z] \in \check{H}^1(\mathcal{M}, \mathcal{F})$, we define the presheaf $g([z])$ as follows: given an open set $U \subseteq X$, we define $g([z])(U)$ as

$$\left\{(t_i)_{i=0}^n \in \bigoplus_{i=0}^n \mathcal{F}(C_i \cap U) \mid t_i |_{C_{i,j} \cap U} - t_j |_{C_{i,j} \cap U} = z(C_i, C_j) |_{C_{i,j} \cap U}\right\}$$

The restriction maps are defined as follows

$$g([z])(U \subseteq U') :: (t'_i)_{i=0}^n \mapsto (t'_i |_{U \cap U'})_{i=0}^n.$$

We define the $\mathcal{F}$-action on $g([z])$ as follows: given a $g \in \mathcal{F}(U)$,

$$g \bullet (t_i)_{i=0}^n := (t_i - g |_{U \cap C_i})_i.$$

For any context $C_j \in \mathcal{M}$, there exists an isomorphism of $\mathcal{F} |_{C_j}$-presheaves $\mathcal{F} |_{C_j} \Rightarrow T |_{C_j}$. To see this, take an open $U \subseteq C_j$, then the map

$$\begin{array}{rccc} h_U^j : & \mathcal{F}(U) & \longrightarrow & g([z])(U) \\ & g & \longmapsto & \Big(z(C_i, C_j) - g_{C_i \cap C_j}\Big)_{i=0}^n \end{array}$$

is an isomorphism with inverse

$$\begin{array}{rccc} k_U^j : & g([z])(U) & \longrightarrow & \mathcal{F}(U) \\ & (t_i)_{i=0}^n & \longmapsto & -t_j. \end{array}$$

In fact, h_U^j is equivariant since

$$\begin{aligned} g \bullet h_U^j(h) &= g \bullet \Big(z(C_i, C_j) - h_{C_i \cap C_j}\Big)_{i=0}^n \\ &= \Big(z(C_i, C_j) - h_{C_i \cap C_j} - g_{C_i \cap C_j}\Big) \\ &= h_U^j(g + h) = h_U^j(g \bullet h). \end{aligned}$$

and k_U^j is indeed the inverse of h_U^j since

$$\begin{aligned} h_U^j\Big(k_U^j\,((t_i)_{i=0}^n)\Big) &= h_U(-t_j) = \Big(z(C_i, C_j) + t_j |_{C_i \cap C_j}\Big)_{i=0}^n \\ &= \Big(t_i |_{C_i \cap C_j} - t_j |_{C_i \cap C_j} + t_j |_{C_i \cap C_j}\Big)_{i=0}^n \\ &= (t_i)_{i=0}^n \end{aligned}$$

,

and

$$\begin{aligned} k_U^j(h_U^j(g)) &= k_U\left(\Big(z(C_i, C_j) - g_{C_i \cap C_j}\Big)_{i=0}^n\right) \\ &= -\underbrace{z(C_j, C_j)}_{0} + g = g. \end{aligned}$$

Thanks to this observation, we now that $g([z])$ is an $\mathcal{F}$-torsor trivialised by $\mathcal{M}$.

Finally, we show that the definition of g is independent of the choice of the representative z of the 1-cocycle $[z]$. Suppose we take a cohomologous 1-cocycle z'. Then there exists a family $h := \{h_i \in \mathcal{F}(C_i)\}_{i=0}^n$ such that

$$z'(C_i, C_j) - z(C_i, C_j) = \delta^0(h).$$

Then we can define an isomorphism of $\mathcal{F}$-torsors $g([z]) \cong g([z'])$ induced by the maps

$$\psi_U : g([z])(U) \longrightarrow g([z'])(U) :: (t_i)_{i=0}^n \longmapsto (h_i + t_i)_{i=0}^n.$$

In fact, this map is equivariant since

$$\begin{aligned} g \bullet \psi_U\left((t_i)_{i=0}^n\right) &= g \bullet \left((h_i + t_i)_{i=0}^n\right) \\ &= \left(h_i + t_i - g\mid_{U \cap C_i}\right)_{i=0}^n \\ &= \psi_U\left(\left(t_i - g\mid_{C_i \cap C_j}\right)_{i=0}^n\right) \\ &= \psi_U\left(g \bullet (t_i)_{i=0}^n\right), \end{aligned}$$

and its inverse is clearly

$$g([z'])(U) \longrightarrow g([z])(U) :: (t_i')_{i=0}^n \longmapsto (t_i' - h_i)_{i=0}^n.$$

□

We can actually go further and show that this bijection induces a group structure on $\mathsf{Trs}(\mathcal{M}, \mathcal{F}$. The addition of two $\mathcal{F}$-torsors is defined componentwise at each subset $U \subseteq X$ as follows. Given $z, t \in Z^1(\mathcal{M}, \mathcal{F}\mid_{C_0})$, we have

$$g([z])(U) + g([w])(U) := g([z] + [w])(U),$$

which clearly verifies all the necessary conditions for the composition law of a group. Moreover, with this definition it is straightforward to see that g is a homomorphism of groups. Thus we have proved

Theorem 3.5.7. *We have an isomorphism of groups*

$$\mathsf{Trs}(\mathcal{M}, \mathcal{F}) \cong \check{H}^1(\mathcal{M}, \mathcal{F}),$$

i.e. we can see the elements of the first cohomology group as $\mathcal{F}$-torsors trivialised by the cover $\mathcal{M}$.

. □

Due to time constraints, we were not able to develop this viewpoint any further. We are confident it can be a valuable alternative to the conventional study of contextuality via cohomological obstructions, and we aim to study it in more details in future work.

3.6 DISCUSSION

We successfully introduced cohomological obstructions as a valuable method of detecting contextuality. However, we also showed that cohomology is not a full invariant for contextuality since we can find false positives such as the Hardy model. We actually contributed to make the situation even worse with our new example of a strongly contextual but cohomologically non-contextual model on a simple bipartite Bell-type scenario, which seems to put a definite end to the efforts on finding suitable conditions of the cover that would make cohomology a full invariant for strong contextuality. However, a limit of this example is that it is most likely not realisable in quantum mechanics. Therefore, it remains an open question to understand whether Conjecture 8.1 of [AMSB12] is still true for quantum-realisable models.

Another interesting open question is the following

- Can we give a better refinement of the abelian counterpart of the set-presehaf of an empirical model $\mathcal{S}$ than the canonical $F_R\mathcal{S}$?

It may be possible that such a refinement could end up giving us a full invariant for contextuality.

Our results on the injectivity of the connecting homomorphisms have helped us better understand these properties and their implications on the contextuality of empirical models. On the other hand, even though we proved Conjecture 3.4.1 false, we did not achieve a full grasp of the meaning of the surjectivity of γ.

- What does it mean for an empirical model to have all surjective homomorphisms γ^C?

Moreover, the failure of Conjecture 3.4.1 has left unanswered the following question

- How can we concretely describe the elements of the first cohomology group $\check{H}^1(\mathcal{M}, \mathcal{F})$?

Our alternative decription of this group via $\mathcal{F}$-torsors is a first partial answer to this question. However, it is still unclear how this approach can be more satisfactory than the most straightforward one of Section 3.4.2. Although it is definitely a more compact and elegant definition, the nature of $\mathcal{F}$-torsors does not appear to be more easily understandable in the terms we need for the study of empirical models.

4 HIGHER COHOMOLOGY GROUPS

In this chapter, we will propose a generalisation of the cohomological obstruction to higher cohomology groups and shed light on the information they can provide on the contextuality of empirical models. More specifically, we will show that, for each $q \geqslant 0$, to every section $s_0 \in \mathcal{F}(C_0)$ we can associate a well-defined q-obstruction $\gamma^q(s_0) \in \check{H}^{2q+1}(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$, and that these obstructions can be organised in a hierarchy of logical implications.

A first attempt to generalise the cohomological obstruction to higher cohomology groups has been made by Ji [Ji13]. We will essentially introduce a more natural and compact way to define higher obstructions. This approach will lead us to establish a hierarchy between different levels of contextuality. Eventually, we will show that this method cannot be used to collect additional information on the properties of no-signaling empirical models.

4.1 DEFINING HIGHER COHOMOLOGY OBSTRUCTIONS

In this section we introduce the definition of higher cohomological obstructions. The construction is essentially inspired by the base case discussed in Section 3.2.1. Each step is simply the most natural way to generalise the corresponding passage in the base case, making this definition arguably the most natural way to describe higher obstructions.

First of all we need to generalise Lemma 3.2.2 to the higher case in order to obtain a representation of a section $s_0 \in \mathcal{F}(C_0)$ inside higher groups of cocycles $Z^q(\mathcal{M}, \mathcal{F}_{C_0})$, $q > 0$. This procedure is analogous to the $q = 0$ case, where we used the isomorphism ψ^0 to obtain a cocycle in $Z^0(\mathcal{M}, \mathcal{F}\mid_{C_0})$ representing $s_0 \in \mathcal{F}(C_0)$.

Let $q \geqslant 0$ be an integer. To each section $s_0 \in \mathcal{F}(C_0)$ we associate a q-relative cochain $c^q_{s_0} \in C^q(\mathcal{M}, \mathcal{F}\mid_{C_0})$ defined as follows:

$$c^q_{s_0}(\omega) := s_0\mid_{C_0 \cap |\omega|}, \quad \forall \omega \in N(\mathcal{M})^q$$

This determines the following homomorphism

$$\begin{aligned} \psi^q : \quad \mathcal{F}(C_0) &\longrightarrow C^q(\mathcal{M}, \mathcal{F}\mid_{C_0}) \\ s_0 &\longmapsto c^q_{s_0} \end{aligned} \tag{4.1}$$

Notice that, if $q = 0$, this definition coincides with the one of ψ^0 given in Lemma 3.2.2.

Lemma 4.1.1. *For each $q \geqslant 0$, the homomorphism ψ^q is injective.*

Proof. Let $s_0 \in \ker(\psi^q)$. Then $c^q_{s_0} = 0$, thus in particular

$$0 = c^q_{s_0}(\underbrace{C_0, \ldots, C_0}_{q+1 \text{ times}}) = s_0.$$

Therefore $\mathrm{Ker}(\psi^q) = 0$ and the homomorphism is injective. $\square$

We showed that the image of ψ^0 is contained in $Z^0(\mathcal{M}, \mathcal{F}|_{C_0})$ (it actually coincides with $Z^0(\mathcal{M}, \mathcal{F}|_{C_0})$). We will now prove that this fact remains true for ψ^q only in the case where q is even.

Lemma 4.1.2.
For each $q \geqslant 0$, the image of ψ^{2q} is contained in $Z^{2q}(\mathcal{M}, \mathcal{F}|_{C_0})$. If $q = 0$ the inclusion is an equality, thus ψ^0 is an isomorphism.

Proof. Let $c^{2q}_{s_0} \in \mathrm{im}(\psi^{2q})$. For any $\omega \in \mathcal{N}(\mathcal{M})^{2q+1}$ we have

$$\begin{aligned}
\delta^{2q}\left(c^{2q}_{s_0}\right)(\omega) &= \sum_{k=0}^{2q+1}(-1)^k \rho^{|\partial_k\omega|}_{|\omega|}\left(c^{2q}_{s_0}(\partial_k\omega)\right) \\
&= \sum_{k=0}^{2q+1}(-1)^k \rho^{|\partial_k\omega|}_{|\omega|}\left(s_0|_{C_0\cap|\partial_k\omega|}\right) \\
&= \sum_{k=0}^{2q+1}(-1)^k s_0|_{C_0\cap|\omega|} = 0,
\end{aligned}$$

where the last equality comes from the fact that it is a alternating sum with an even number of terms (here we can see the importance of having to distinguish in parity). The fact that ψ^0 is an isomorphism has already been proved in Lemma 3.2.2. □

Given an integer $q \geqslant 0$, we can now apply the same arguments used in Section 3.2.1 to define the higher cohomological obstructions. Recall the general exact sequence of presheaves (3.1) at the base of the definition of the relative cohomology of $\mathcal{F}$ with respect to C_0.

$$\mathbf{0} \xRightarrow{0} \mathcal{F}_{\tilde{C}_0} \xRightarrow{\mathrm{incl.}} \mathcal{F} \xRightarrow{p^{C_0}} \mathcal{F}|_{C_0}$$

For each $\sigma \in \mathcal{N}(\mathcal{M})^{2q}$, it yields an exact sequence on objects

$$0 \xrightarrow{0} \mathcal{F}_{\tilde{C}_0}(|\sigma|) := \ker(p^{C_0}_{|\sigma|}) \xrightarrow{\mathrm{incl.}_{|\sigma|}} \mathcal{F}(|\sigma|) \xrightarrow{p^{C_0}_{|\sigma|}} \mathcal{F}|_{C_0}(|\sigma|)$$

We can sum these morphisms for every $\sigma \in \mathcal{N}(\mathcal{M})^{2q}$ and "lift" exactness to the chain level:

$$0 \xrightarrow{0} C^{2q}(\mathcal{M}, \mathcal{F}_{\tilde{C}_0}) \xrightarrow{\bigoplus_\sigma \mathrm{incl.}_{|\sigma|}} C^{2q}(\mathcal{M}, \mathcal{F}) \xrightarrow{\bigoplus_\sigma p^{C_0}_{|\sigma|}} C^{2q}(\mathcal{M}, \mathcal{F}|_{C_0}) \tag{4.2}$$

Once again, by flaccidity beneath the cover of the presheaf $\mathcal{F}$, we know that $p^{C_0}_{|\sigma|}$ is surjective for all $\sigma \in \mathcal{N}(\mathcal{M})^{2q}$. Thus

$$\bigoplus_{\sigma\in\mathcal{N}(\mathcal{M})^{2q}} p^{C_0}_{|\sigma|}$$

is also surjective. Hence, (4.2) is in fact a short exact sequence

$$0 \longrightarrow C^{2q}(\mathcal{M}, \mathcal{F}_{\tilde{C}_0}) \longrightarrow C^{2q}(\mathcal{M}, \mathcal{F}) \longrightarrow C^{2q}(\mathcal{M}, \mathcal{F}|_{C_0}) \longrightarrow 0$$

We can now use the boundary correstriction

$$\tilde{\delta}^{2q} : C^{2q}(\mathcal{M}, \mathcal{F}) \longrightarrow Z^{2q+1}(\mathcal{M}, \mathcal{F})$$

of the boundary maps δ^{2q} (and their relative versions) in the same way as in the $q = 0$ case and obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^{2q}(\mathcal{M},\mathcal{F}_{\check{C}_0}) \arrow[r] \arrow[d, "\delta^{2q}"] & C^{2q}(\mathcal{M},\mathcal{F}) \arrow[r] \arrow[d, "\delta^{2q}"] & C^{2q}(\mathcal{M},\mathcal{F}\mid_{C_0}) \arrow[r] \arrow[d, "\delta^{2q}"] & 0 \\
0 \arrow[r] & Z^{2q+1}(\mathcal{M},\mathcal{F}_{\check{C}_0}) \arrow[r] & Z^{2q+1}(\mathcal{M},\mathcal{F}) \arrow[r] & Z^{2q+1}(\mathcal{M},\mathcal{F}\mid_{C_0})
\end{tikzcd}$$

Eventually, we apply the snake lemma again to obtain the connecting homomorphism $\tilde{\gamma}^q : Z^{2q}(\mathcal{M},\mathcal{F}\mid_{C_0}) \to \check{H}^{2q+1}(\mathcal{M},\mathcal{F}_{\check{C}_0})$.

$$\begin{tikzcd}
 & & & \mathcal{F}(C_0) \arrow[d, "\psi^{2q}"] & \\
 & Z^{2q}(\mathcal{M},\mathcal{F}_{\check{C}_0}) \arrow[r] \arrow[d] & Z^{2q}(\mathcal{M},\mathcal{F}) \arrow[r] \arrow[d] & Z^{2q}(\mathcal{M},\mathcal{F}\mid_{C_0}) \arrow[d] \arrow[dddll, rounded corners, "\tilde{\gamma}^q"] & \\
0 \arrow[r] & C^{2q}(\mathcal{M},\mathcal{F}_{\check{C}_0}) \arrow[r] \arrow[d] & C^{2q}(\mathcal{M},\mathcal{F}) \arrow[r] \arrow[d] & C^{2q}(\mathcal{M},\mathcal{F}\mid_{C_0}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^{2q+1}(\mathcal{M},\mathcal{F}_{\check{C}_0}) \arrow[r] \arrow[d] & Z^{2q+1}(\mathcal{M},\mathcal{F}) \arrow[r] \arrow[d] & Z^{2q+1}(\mathcal{M},\mathcal{F}\mid_{C_0}) \arrow[d] & \\
 & \check{H}^{2q+1}(\mathcal{M},\mathcal{F}_{\check{C}_0}) \arrow[r] & \check{H}^{2q+1}(\mathcal{M},\mathcal{F}) \arrow[r] & \check{H}^{2q+1}(\mathcal{M},\mathcal{F}\mid_{C_0}) &
\end{tikzcd} \tag{4.3}$$

Definition 4.1.3 (Higher cohomological obstruction)**.** Let $s_0 \in \mathcal{F}(C_0)$. We define the q-*th cohomological obstruction* of s_0 to be the element

$$\gamma^q(s_0) := \tilde{\gamma}^q(\psi^{2q}(s_0)) \in \check{H}^{2q+1}(\mathcal{M},\mathcal{F}_{\check{C}_0}).$$

Notice that, due to the parity arguments needed to achieve this definition, the cohomological obstruction is generalisable only to *odd-dimensional cohomology groups.*

In the case $q = 0$, Proposition 3.2.7 tells us that the vanishing of the cohomological obstruction of a local section s_0 is equivalent to the existence of a compatible family containing s_0. The analogous higher case result is given in the following proposition.

Proposition 4.1.4. *Given a* $q \geqslant 0$*, a context* $C_0 \in \mathcal{M}$ *and a local section* $s_0 \in \mathcal{F}(C_0)$, $\gamma^q(s_0) = 0$ *if and only if there exists a family* $s \in Z^{2q}(\mathcal{M},\mathcal{F})$ *such that*

$$p^{C_0}_{|\sigma|}(s(\sigma)) = c^{2q}_{s_0}(\sigma) = s_0\mid_{C_0 \cap |\sigma|} . \tag{4.4}$$

Proof. Since $\gamma^q(s_0) = 0$, we know that $\tilde{\gamma}(c^{2q}_{s_0}) = 0$. Therefore $c^{2q}_{s_0} \in \ker(\tilde{\gamma}^q)$. Now, $\tilde{\gamma}^q$ is defined using the snake lemma, thus it is part of a long exact sequence. Hence there exists a family $s \in Z^{2q}(\mathcal{M},\mathcal{F})$ such that (4.4) is verified. □

We can see from this proposition that the higher analogous of a compatible family is a member of the group $Z^{2q}(\mathcal{M}, \mathcal{F})$.

An alternative, more concrete definition.

Note that, in analogy with the case $q = 0$, we can achieve an equivalent definition of the higher cohomology obstruction in a more concrete way following the guidelines given in [AMSB12] and already discussed in the previous sections.

Given an integer $q \geqslant 0$ and a section $s_0 \in \mathcal{F}(C_0)$, we can find, by flaccidity, a cochain $c \in C^q(\mathcal{M}, \mathcal{F})$ such that

$$c(\sigma)\mid_{C_0 \cap |\sigma|} = p^{C_0}_{|\sigma|}(c(\sigma)) = c^q_{s_0}(\sigma) = s_0 \mid_{C_0 \cap |\sigma|} \quad \forall \sigma \in \mathcal{N}(\mathcal{M})^q. \tag{4.5}$$

Let us denote by $z := \delta^q(c)$ the q-coboundary of the cochain c. We prove here the counterpart Proposition 4.1 of [AMSB12] for the general $q \geqslant 0$ case. Note that, as we would expect, the proposition is generalisable only in the case where q is even.

Proposition 4.1.5. *Suppose* q *is even. Then the* q*-coboundary* z *of* c *vanishes under restriction to* C_0*, and hence is a* q*-cocycle in the relative cohomology with respect to* C_0*.*

Proof. It is sufficient to show that

$$p^{C_0}_{|\omega|}(z(\omega)) = z(\omega)\mid_{|\omega| \cap C_0} = 0 \quad \forall \omega \in \mathcal{N}(\mathcal{M})^{q+1}.$$

We have

$$\begin{aligned}
z(\omega)\mid_{|\omega| \cap C_0} &= \rho^{|\omega|}_{|\omega| \cap C_0}(\delta^q(c)(\omega)) \\
&= \rho^{|\omega|}_{|\omega| \cap C_0}\left(\sum_{k=0}^{q+1}(-1)^k \rho^{|\partial_k \omega|}_{|\omega|}(c(\partial_k \omega))\right) \\
&= \sum_{k=0}^{q+1}(-1)^k \rho^{|\omega|}_{|\omega| \cap C_0} \circ \rho^{|\partial_k \omega|}_{|\omega|}(c(\partial_k \omega)) \\
&= \sum_{k=0}^{q+1}(-1)^k \rho^{|\partial_k \omega|}_{|\omega| \cap C_0}(c(\partial_k \omega)) \\
&= \sum_{k=0}^{q+1}(-1)^k c(\partial_k \omega)\mid_{|\omega| \cap C_0} \\
&= \sum_{k=0}^{q+1}(-1)^k \left(c(\partial_k \omega)\mid_{|\partial_k \omega| \cap C_0}\right)\mid_{|\omega|} \\
&\overset{(4.5)}{=} \sum_{k=0}^{q+1}(-1)^k \left(s_0 \mid_{|\partial_k \omega| \cap C_0}\right)\mid_{|\omega|} \\
&= \sum_{k=0}^{q+1}(-1)^k \left(s_0 \mid_{|\omega| \cap C_0}\right) = 0,
\end{aligned}$$

where the last equality is due to the fact that the sum is alternating and has an even number of terms since q is even. □

We can eventually formulate the concrete definition of cohomological obstruction

Definition 4.1.6 (Alternative definition of higher cohomological obstruction)**.** Given an integer $q \geqslant 0$ and a section $s_0 \in \mathcal{F}(C_0)$ we define the q*-th cohomological obstruction* $\gamma^q(s_0)$ of s_0 to be the cohomology class $[z] \in \check{H}^{2q+1}(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$.

We still need to check that this alternative definition of γ^q is independent of the choice of the cochain c.

Proposition 4.1.7. *Definition 4.1.6 is independent of the choice of* c.

Proof. Let c' be such that

$$c'(\sigma)\mid_{C_0\cap|\sigma|} =: p^{C_0}_{|\sigma|}(c'(\sigma)) = c'^{q}_{s_0}(\sigma) := s_0\mid_{C_0\cap|\sigma|} \quad \forall\sigma \in \mathcal{N}(\mathcal{M})^q.$$

Consider $c - c' \in C^{2q}(\mathcal{M}, \mathcal{F})$. We have

$$\begin{aligned}(c-c')(\sigma)\mid_{C\cap|\sigma|} &= c(\sigma)\mid_{C\cap|\sigma|} - c'(\sigma)\mid_{C\cap|\sigma|} = s_0\mid_{C_0\cap|\sigma|} - s_0\mid_{C_0\cap|\sigma|} \\ &= 0 \quad \forall\sigma \in \mathcal{N}(\mathcal{M})^q.\end{aligned}$$

Thus, $c - c' \in C^{2q}(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$, which implies

$$\delta^{2q}(c) - \delta^{2q}(c') = \delta^{2q}(c - c') \in B^{2q+1}(\mathcal{M}, \mathcal{F}_{\tilde{C}_0}),$$

which means $[\delta^2 q(c)] = [\delta^2 q(c')]$ and thus γ^q is well-defined. □

The same argument used to show the equivalence of Definition 3.2.4 and the one of [AMSB12] in Section 3.2.1 can be applied here to show the equivalence of Definition 4.1.3 and Definition 4.1.6.

4.2 A HIERARCHY OF COHOMOLOGICAL OBSTRUCTIONS

We will now show the main result of this chapter: the existence of a hierarchy of different types of contextuality, based on the order of the first vanishing higher cohomological obstruction.

Theorem 4.2.1. *Let* $q \geqslant 0$ *and* $s_0 \in \mathcal{F}(C_0)$*. If* $\gamma^q(s_0) = 0$*, then* $\gamma^{q'}(s_0) = 0$ *for all* $q' \geqslant q$.

Proof. Since $\gamma^q(s_0) = 0$, by Proposition 4.1.4 there exists a family $s \in Z^{2q}(\mathcal{M}, \mathcal{F})$ such that

$$\bigoplus_{\sigma\in\mathcal{N}(\mathcal{M})^{2q}} p^{C_0}_{|\sigma|}(s(\sigma)) = c^{2q}_{s_0}. \tag{4.6}$$

We will now show that we can find a family $f(s) \in Z^{2q+1}(\mathcal{M}, \mathcal{F})$ such that

$$\bigoplus_{\sigma\in\mathcal{N}(\mathcal{M})^{2q+2}} p^{C_0}_{|\sigma|}(f(s)(\sigma)) = c^{2q+2}_{s_0}. \tag{4.7}$$

By Proposition 4.1.4, this will imply that $\gamma^{q+1}(s_0) = 0$ and, by recursion, the final result.

Let us start by defining, for all $\tau \in \mathcal{N}(\mathcal{M})^{2q+2}$,

$$\begin{aligned}f(s)(\tau) &:= \rho^{|\partial_{2q+1}\partial_{2q+2}\tau|}_{|\tau|}(s(\partial_{2q+1}\partial_{2q+2}\tau)) \\ &= s(\partial_{2q+1}\partial_{2q+2}\tau)\mid_{|\tau|}.\end{aligned} \tag{4.8}$$

Notice that $f(s)(\tau) \in \mathcal{F}(|\tau|)$, thus $f(s) \in C^{2q+2}(\mathcal{M}, \mathcal{F})$. We can actually show that $f(s) \in Z^{2q+2}(\mathcal{M}, \mathcal{F})$ as follows. Given an arbitrary $\nu \in \mathcal{N}(\mathcal{M})^{2q+3}$, we have

$$\begin{aligned}
\delta^{2q+2}(f(s))(\nu) &:= \sum_{k=0}^{2q+3} (-1)^k \rho_{|\nu|}^{|\partial_k \nu|}(f(s)(\partial_k \nu)) \\
&\overset{(4.8)}{=} \sum_{k=0}^{2q+3} (-1)^k \rho_{|\nu|}^{|\partial_k \nu|} \rho_{|\partial_k \nu|}^{|\partial_{2q+1}\partial_{2q+2}\partial_k \nu|}(s(\partial_{2q+1}\partial_{2q+2}\partial_k \nu)) \\
&= \sum_{k=0}^{2q+3} (-1)^k \rho_{|\nu|}^{|\partial_{2q+1}\partial_{2q+2}\partial_k \nu|}(s(\partial_{2q+1}\partial_{2q+2}\partial_k \nu)) \\
&= \sum_{k=0}^{2q+1} (-1)^k \rho_{|\nu|}^{|\partial_{2q+1}\partial_{2q+2}\partial_k \nu|}(s(\partial_{2q+1}\partial_{2q+2}\partial_k \nu)) \\
&+ \rho_{|\nu|}^{|\partial_{2q+1}\partial_{2q+2}\partial_{2q+2} \nu|}(s(\partial_{2q+1}\partial_{2q+2}\partial_{2q+2} \nu)) \\
&- \rho_{|\nu|}^{|\partial_{2q+1}\partial_{2q+2}\partial_{2q+3} \nu|}(s(\partial_{2q+1}\partial_{2q+2}\partial_{2q+3} \nu))
\end{aligned} \tag{4.9}$$

Notice that the last two terms cancel out, in fact

$$\partial_{2q+1}\partial_{2q+2}\partial_{2q+2}\nu = \partial_{2q+1}\partial_{2q+2}\partial_{2q+3}\nu \tag{4.10}$$

since on the left we cancel the penultimate element of ν, then the last one, and finally the third to last, while on the right hand side we cancel the last element, then the penultimate and finally the third to last, obtaining the same result. Explicitly,

$$\begin{aligned}
\partial_{2q+1}\partial_{2q+2}\partial_{2q+2}\nu &= \partial_{2q+1}\partial_{2q+2}(\nu_0, \dots, \nu_{2q}, \nu_{2q+1}, \hat{\nu}_{2q+2}, \nu_{2q+3}) \\
&= \partial_{2q+1}(\nu_0, \dots, \nu_{2q}, \nu_{2q+1}, \hat{\nu}_{2q+2}, \hat{\nu}_{2q+3}) \\
&= \partial_{2q+1}\partial_{2q+2}(\nu_0, \dots, \nu_{2q}, \nu_{2q+1}, \nu_{2q+2}, \hat{\nu}_{2q+3}) \\
&= \partial_{2q+1}\partial_{2q+2}\partial_{2q+3}\nu
\end{aligned}$$

Thus we have

$$\begin{aligned}
\delta^{2q+2}(f(s))(\nu) &\overset{(4.9)(4.10)}{=} \sum_{k=0}^{2q+1} (-1)^k \rho_{|\nu|}^{|\partial_{2q+1}\partial_{2q+2}\partial_k \nu|}(s(\partial_{2q+1}\partial_{2q+2}\partial_k \nu)) \\
&= \sum_{k=0}^{2q+1} (-1)^k \rho_{|\nu|}^{|\partial_k\partial_{2q+1}\partial_{2q+2} \nu|}(s(\partial_k\partial_{2q+1}\partial_{2q+2} \nu)),
\end{aligned} \tag{4.11}$$

where the last equality is valid since now $0 \leqslant k \leqslant 2q+1$ and therefore it is unimportant whether we cancel the k-th term before or after having canceled the $(2q+2)$-th and the $(2q+1)$-th, explicitly

$$\begin{aligned}
\partial_{2q+1}\partial_{2q+2}\partial_k\nu &= \partial_{2q+1}\partial_{2q+2}(\nu_0, \dots, \hat{\nu}_k, \dots, \nu_{2q+1}, \nu_{2q+2}, \nu_{2q+3}) \\
&= (\nu_0, \dots, \hat{\nu}_k, \dots, \hat{\nu}_{2q+1}, \hat{\nu}_{2q+2}, \nu_{2q+3}) \\
&= \partial_k(\nu_0, \dots, \nu_k, \dots, \hat{\nu}_{2q+1}, \hat{\nu}_{2q+2}, \nu_{2q+3}) \\
&= \partial_k\partial_{2q+1}\partial_{2q+2}\nu.
\end{aligned}$$

We can now relabel

$$\partial_{2q+1}\partial_{2q+2}\nu := \tilde{\sigma} \in \mathcal{N}(\mathcal{M})^{2q+1}, \tag{4.12}$$

and obtain

$$\delta^{2q+2}(f(s))(\nu) \overset{(4.11)(4.12)}{=} \sum_{k=0}^{2q+1} (-1)^k \rho^{|\partial_k \tilde{\sigma}|}_{|\nu|}(s(\partial_k \tilde{\sigma})) = \delta^{2q}(s)(\tilde{\sigma}) = 0,$$

where the last equality is due to the fact that $s \in Z^{2q}(\mathcal{M}, \mathcal{F})$.

Now we need to check that $f(s)$ actually satisfies (4.7). Let $\sigma \in \mathcal{N}(\mathcal{M})^{2q+2}$. We have

$$\begin{aligned}
p^{C_0}_{|\sigma|}(f(s)(\sigma)) &= f(s)(\sigma) \mid_{|\sigma| \cap C_0} = \left(s(\partial_{2q+1}\partial_{2q+2}\sigma) \mid_{|\sigma|} \right) \mid_{|\sigma| \cap C_0} \\
&= s(\partial_{2q+1}\partial_{2q+2}\sigma) \mid_{|\sigma| \cap C_0} \overset{(4.12)}{=} s(\tilde{\sigma}) \mid_{|\sigma| \cap C_0} \\
&= s(\tilde{\sigma}) \mid_{\underbrace{|\tilde{\sigma}| \cap |\sigma|}_{=|\sigma|} \cap C_0} = \left(s(\tilde{\sigma}) \mid_{|\tilde{\sigma}| \cap C_0} \right) \mid_{|\sigma|} \\
&= \left(p^{C_0}_{|\tilde{\sigma}|}(s(\tilde{\sigma})) \right) \mid_{|\sigma|} \overset{(4.6)}{=} \left(c^{2q}_{s_0}(\tilde{\sigma}) \right) \mid_{|\sigma|} = \left(s_0 \mid_{|\tilde{\sigma}| \cap C_0} \right) \mid_{|\sigma|} \\
&= s_0 \mid_{|\tilde{\sigma}| \cap |\sigma| \cap C_0} = s_0 \mid_{|\sigma| \cap C_0} = c^{2q+2}_{s_0}(\sigma).
\end{aligned}$$

By proposition 4.1.4 this implies $\gamma^{q+1}(s_0) = 0$. We just proved

$$\gamma^q(s_0) = 0 \Rightarrow \gamma^{q+1}(s_0) = 0,$$

and, by recursion, we obtain the statement of the theorem.

□

This theorem suggests the existence of an infinite amount of types of contextuality organised in a strict hierarchy. Inspired by Definition 3.2.9, we are naturally driven to generalise it as follows.

Definition 4.2.2. Let $\mathcal{S}$ be an empirical model.

- If there exists a local section $s_0 \in \mathcal{S}(C_0)$ such that $\gamma^q_{F_R \mathcal{S}}(s_0) \neq 0$ we say that $\mathcal{S}$ is *cohomologically logically* (or *possibilistically*) q-*contextual at* s_0, or $\mathrm{CLC}^q_R(\mathcal{S}, s_0)$.
- We say that $\mathcal{S}$ is *cohomologically strongly* q-*contextual* (or $\mathrm{SC}^q_R(\mathcal{S})$) if $\mathrm{CLC}^q_R(\mathcal{S}, s)$ for all s.

The base case of Definition 3.2.9 then coincides with the case where $q = 0$.

By Proposition 4.1.4, a model $\mathcal{S}$ is cohomologically logically q-contextual at $s_0 \in \mathcal{S}(C_0)$ if and only if it is impossible to find a family $s \in Z^{2q}(\mathcal{M}, \mathcal{F})$ that restricts to s_0.

In the light of this definition, Theorem 4.2.1 would result in the following diagram of logical implications

$$\begin{tikzcd}
CSC_R(\mathcal{S}) \arrow[r, Rightarrow, "3.2.10"] \arrow[d, Rightarrow, "4.2.1"] & CLC_R(\mathcal{S}) \arrow[d, Rightarrow, "4.2.1"] \\
CSC^1_R(\mathcal{S}) \arrow[r, Rightarrow, "3.2.10"] \arrow[d, Rightarrow, "4.2.1"] & CLC^1_R(\mathcal{S}) \arrow[d, Rightarrow, "4.2.1"] \\
\vdots \arrow[d, Rightarrow, "4.2.1"] & \vdots \arrow[d, Rightarrow, "4.2.1"] \\
CSC^q_R(\mathcal{S}) \arrow[r, Rightarrow, "3.2.10"] \arrow[d, Rightarrow, "4.2.1"] & CLC^q_R(\mathcal{S}) \arrow[d, Rightarrow, "4.2.1"] \\
CSC^{q+1}_R(\mathcal{S}) \arrow[r, Rightarrow, "3.2.10"] \arrow[d, Rightarrow, "4.2.1"] & CLC^{q+1}_R(\mathcal{S}) \arrow[d, Rightarrow, "4.2.1"] \\
\vdots & \vdots
\end{tikzcd}$$

However, it turns out that this hierarchy is not applicable to further classify the contextuality of no-signaling empirical models. In fact, we will show in the following section that we cannot find any example of a no-signaling model which is cohomologically logically q-contextual for any $q > 0$.

4.3 NO-SIGNALING MODELS ARE q-NONCONTEXTUAL FOR $q \gneq 0$.

Consider a presheaf of abelian groups $\mathcal{F}$ relative to an empirical model $\mathcal{S}$ on a scenario $\langle X, \mathcal{M}, O\rangle$ (e.g. $\mathcal{F} := F_R\mathcal{S}$). To simplify notation we rewrite the cover as $\mathcal{M} := \{C_i\}_{i=0}^n$. Let us call $C_0 \in \mathcal{M}$ an arbitrary context, and let $s_{C_0} \in \mathcal{F}(C_0)$ be an arbitrary section. By no-signaling, there exists a family $\{s_{C_i} \in \mathcal{F}(C_i)\}_{i=0}^n$ such that

$$s_{C_i}|_{C_i \cap C_0} = s_{C_0}|_{C_i \cap C_0} \quad \forall 0 \leqslant i \leqslant n. \tag{4.13}$$

We define $z \in C^2(\mathcal{M}, \mathcal{F})$ by the expression

$$z(\omega) := s_{\partial_0 \partial_2 \omega}|_{|\omega|} \quad \in \mathcal{F}(|\omega|) \quad \forall \omega \in \mathcal{N}(\mathcal{M})^2$$

More explicitly, given an $\omega := (C_i, C_j, C_k) \in \mathcal{N}(\mathcal{M})^2$, we define

$$z(C_i, C_j, C_k) := s_{C_j}|_{C_i \cap C_j \cap C_k} \quad \in \mathcal{F}(C_i \cap C_j \cap C_k). \tag{4.14}$$

Given a general $\sigma := (C_i, C_j, C_k, C_l) \in \mathcal{N}(\mathcal{M})^3$, we have

$$\begin{aligned}
\delta^2(z)(\sigma) &= z(C_j, C_k, C_l)|_{|\sigma|} - z(C_i, C_k, C_l)|_{|\sigma|} \\
&\quad + z(C_i, C_j, C_l)|_{|\sigma|} - z(C_i, C_j, C_k)|_{|\sigma|} \\
&\overset{(4.14)}{=} s_{C_k}|_{|\sigma|} - s_{C_k}|_{|\sigma|} + s_{C_j}|_{|\sigma|} - s_{C_j}|_{|\sigma|} = 0,
\end{aligned}$$

thus $z \in Z^2(\mathcal{M}, \mathcal{F})$. Moreover, for any general $\omega = (C_i, C_j, C_k) \in \mathcal{N}(\mathcal{M})^2$ we have

$$\begin{aligned} p^{C_0}_{|\omega|}(z(\omega)) = z(\omega)\ |_{|\omega| \cap C_0} &\overset{(4.14)}{=} s_{C_j}\ |_{C_i \cap C_j \cap C_k \cap C_0} \\ &= \left(s_{C_j}\ |_{C_j \cap C_0}\right)\ |_{C_i \cap C_j \cap C_k \cap C_0} \\ &\overset{(4.13)}{=} \left(s_{C_0}\ |_{C_j \cap C_0}\right)\ |_{|\omega| \cap C_0} \\ &= s_{C_0}\ |_{|\omega| \cap C_0} =: c^2_{s_{C_0}}(\omega). \end{aligned}$$

By Proposition 4.1.4, this result implies $\gamma^1(s_{C_0}) = 0$, and by Theorem 4.2.1, we conclude

$$\gamma^q(s_{C_0}) = 0 \quad \forall q > 0.$$

Since s_{C_0} has been chosen arbitrarily, we conclude

$$\neg \mathsf{CLC}^q_R(\mathcal{S}) \quad \forall q > 0.$$

In other words, the image of ψ^{2q} is always included in $\mathsf{ker}(\gamma^q)$ in the case of no-signaling models, making the whole higher cohomology groups machinery effectively useless in the study of empirical models. This gives a definite answer to the question of whether higher cohomology groups can provide additional information on empirical models.

4.4 DISCUSSION

Motivated by Section 3.2.1 and particularly by Proposition 3.2.7, we successfully generalised the notion of cohomological obstruction to the context of higher cohomology groups. The proposed definition of higher obstruction is arguably the most natural and compact way to generalise the base case. The definition we gave ultimately coincides with the one given in [Ji13], but it is obtained using a completely different approach, which closely follows the outline of the base case and is independent of any ad hoc introduced definition (cf. joint/separate compatibility in [Ji13]). Thanks to the clear analogies with the 0-dimensional case, we were able to acquire a full understanding of the vanishing of the higher obstructions (cf. Proposition 4.1.4), and obtain a much stronger hierarchy result (cf. Theorem 4.2.1).

Nevertheless, our definition gives rise to some open questions that we aim to answer in future work.

- Is there a physical reason why the higher cohomological obstruction can be naturally generalised only to odd-dimensional cohomology groups?
- Is there an even-dimensional counterpart of the cohomological obstruction? Does it satisfy similar properties? What kind of information can it provide?

In Section 4.3, we showed that our natural generalisation cannot be used to further classify empirical models, giving a definite answer to previous attempts to generalise the cohomological obstruction to higher cohomology groups. However, we still believe this machinery is interesting in its own right, as a mathematical entity. We aim to contemplate the possibility of applying it to different contexts, possibly to explain the relation between multiple models on the same scenario. We also have a number of interesting questions that remain unanswered and that we hope to solve in future work.

- Does the failure of the higher obstruction method in the scope of empirical models implies that higher odd-dimensional cohomology groups cannot help us refining our classification of contextuality?
- What information can even-dimensional cohomology groups provide?

5 COHOMOLOGY AND AVN ARGUMENTS

In this final chapter we will describe the relation between the two methods of detecting contextuality studied so far: cohomological obstructions and All vs Nothing arguments. Although the two approaches seem to be completely unrelated, they can actually be organised in a precise sequence of logical implications. In particular, the main result of this chapter tells us that if an empirical model admits an AvN argument, then the cohomological obstructions witness its contextuality [ABK+15].

5.1 A GENERALISATION OF AVN ARGUMENTS

In conventional proofs of contextuality involving AvN arguments like the one we proposed in Chapter 2, we typically associate to every context of the empirical model a $\mathbb{Z}_2$-linear equation. We then proceed to prove that such a system is inconsistent. A natural generalisation of this idea is to use R-linear equations, where R is a general ring. We will here implement this viewpoint. It will be convenient to consider scenarios of the form $\langle X, \mathcal{M}, R\rangle$ for an arbitrary ring R.

Definition 5.1.1. Let R be a ring and $\mathcal{S}$ an empirical model on a scenario $\langle X, \mathcal{M}, R\rangle$. An R-*linear equation* is a triple $\phi := \langle C := V_\phi, a, b\rangle$, where $C \in \mathcal{M}$, $a : C \to R$ and $b \in R$. We say that an event $s \in \mathcal{E}(C)$ *satisfies* ϕ, or $s \models \phi$, if

$$\sum_{m \in C} a(m)s(m) = b.$$

We can consider an R-linear equation associated to a context C as a logic formula concerning the outcomes of the measurements in C (exactly like in the usual case, where each equation can be seen as a statement about the parity of the outcomes). This allows us to consider the model from a model-theoretical point of view.

Definition 5.1.2. Consider a scenario $\langle X, \mathcal{M}, R\rangle$.

- To every system of R-linear equations Γ associated to a context $C \in \mathcal{M}$ we can associate its *model*

  $$\mathbb{M}(\Gamma) := \{s \in \mathcal{E}(C) \mid s \models \phi \ \forall \phi \in \Gamma\},$$

  which is the set of all the events that satisfy all the equations in the system.

- To every set of events $S \subseteq \mathcal{E}(C)$ we can associate its R-*linear theory*

  $$\mathbb{T}_R(S) := \{\phi \mid s \models \phi \ \forall s \in S\},$$

  which is the set of all the equations satisfied by all its events.

Notice that Definition 5.1.2 does not involve empirical models.

Definition 5.1.3. Let $\mathcal{S}$ be an empirical model. We define its R*-linear theory* to be

$$\mathbb{T}_R(\mathcal{S}) := \bigcup_{C \in \mathcal{M}} \mathbb{T}_R(\mathcal{S}(C)) = \{\phi \mid s \models \phi, \ \forall s \in \mathcal{S}(V_\phi)\}.$$

We say that $\mathcal{S}$ is AvN_R (denoted by $AvN_R(\mathcal{S})$, if $\mathbb{T}_R(\mathcal{S})$ is inconsistent, i.e. there is no global assignment $g : X \to R$ such that $g\,|_{V_\phi} \models \phi \ \forall \phi \in \mathbb{T}_R(\mathcal{S})$.

Proposition 5.1.4. *Let $\mathcal{S}$ be an empirical model. We have*

$$AvN_R(\mathcal{S}) \Rightarrow SC(\mathcal{S}).$$

Proof. We will show the opposite statement. Suppose $\neg SC(\mathcal{S})$. Then there exists a $g \in \mathcal{S}(X)$. Then $g\,|_{V_\phi} \in \mathcal{S}(V_\phi)$ for each $\phi \in \mathbb{T}_R(\mathcal{S})$, thus $g\,|_{V_\phi} \models \phi$. We conclude that $\mathbb{T}_R(\mathcal{S})$ is consistent, hence the model is not AvN_R. □

5.2 AFFINE CLOSURES

Consider a subset $U \subseteq X$ (where X can be seen as the usual set of measurements). We denote by **Theories** the poset containing all logical theories, ordered by inclusion. Consider the power set of events $\mathcal{P}\mathcal{E}(U)$, seen as a poset, also ordered by inclusion. We have two order reversing poset-maps (i.e. controvariant functors between poset categories)

$$\begin{array}{rccc} \mathbb{T}_R : & \mathcal{P}\mathcal{E}(U) & \longrightarrow & \textbf{Theories} \\ & S \subseteq \mathcal{E}^U = R^U & \longmapsto & \mathbb{T}_R(S), \end{array}$$

and

$$\begin{array}{rccc} \mathbb{M} : & \textbf{Theories} & \longrightarrow & \mathcal{P}\mathcal{E}(U) \\ & \Gamma & \longmapsto & \mathbb{M}(\Gamma), \end{array}$$

Proposition 5.2.1. *The maps $\mathbb{T}_R$ and $\mathbb{M}$ form an antitone Galois connection (cf. Definition 2.1.4).*

Proof. For each theory Γ, we have $\Gamma \subseteq \mathbb{T}_R(\mathbb{M}(\Gamma))$. In fact

$$\begin{aligned} \mathbb{T}_R(\mathbb{M}(\Gamma)) &= \mathbb{T}_R(\{s \in \mathcal{E}(U) \mid s \models \phi, \ \forall \phi \in \Gamma\}) \\ &= \{\phi \mid s \models \phi, \ \forall s \in \{s \in \mathcal{E}(U) \mid s \models \phi, \ \forall \phi \in \Gamma\}\} \\ &= \{\phi \mid s \models \phi, \forall s \in \mathcal{E}(U) : (s \models \phi, \ \forall \phi \in \Gamma)\}, \end{aligned}$$

and Γ is clearly included in this set. On the other hand we have $S \subseteq \mathbb{M}(\mathbb{T}_R(S))$ for each $S \subseteq \mathcal{E}(U)$, in fact

$$\begin{aligned} \mathbb{M}(\mathbb{T}_R(S)) &= \mathbb{M}(\{\phi \mid s \models \phi, \forall s \in S\}) \\ &= \{s \in \mathcal{E}(U) \mid s \models \phi, \ \forall \phi \in \{\phi \mid s \models \phi, \forall s \in S\}\} \\ &= \{s \in \mathcal{E}(U) \mid s \models \phi, \ \forall \phi : (s \models \phi, \forall s \in S)\}, \end{aligned}$$

and clearly S is contained in this set.

□

We consider the closure operator $\mathbb{M} \circ \mathbb{T}_R$. When applied to a set of events, this operator gives us the largest set of events whose theory is still the same, in fact, we already showed that $S \subseteq \mathbb{M} \circ \mathbb{T}_R(S)$ and we also have $\mathbb{T}_R\mathbb{M} \circ \mathbb{T}_R(S) = \mathbb{T}_R(S)$. To see this, notice that, since $\mathbb{T}_R$ is order reversing, we get $\mathbb{T}_R\mathbb{M} \circ \mathbb{T}_R(S) \subseteq \mathbb{T}_R(S)$. Moreover, let $\phi \in \mathbb{T}_R(S)$. Then $s \models \phi$ for each $s \in S$. Let $t \in \mathcal{E}(U)$ be such that $t \models \psi$ for each ψ satisfied by any $s \in S$, then in particular $t \models \phi$. Thus $\phi \in \mathbb{T}_R\mathbb{M} \circ \mathbb{T}_R(S)$.

Note that $\mathcal{E}(U) = R^U$ is a (free) R-module (it is even an R-vector space if R is a field), it thus make sense to take R-linear combinations of events. Now, suppose we have solutions $s_1, \dots, s_n$ to an R-linear equation $\phi = \langle C, a, b \rangle$. We claim that any affine linear combination of them, i.e. any

$$\sum_{i=1}^{n} r_i s_i \text{ s.t. } \sum_{i=1}^{n} r_i = 1,$$

is still a solution of the linear equation. In fact

$$\begin{aligned} s_i \models \phi,\ \forall 1 \leqslant i \leqslant n \Rightarrow & \sum_{m \in C} a(m) s_i(m) = b,\ \forall \leqslant i \leqslant n \\ & \Rightarrow \sum_{m \in C} a(m) \sum_{i=1}^{n} r_i s_i(m) = \underbrace{\sum_{i=1}^{n} r_i}_{=1} \underbrace{\sum_{m \in C} a(m) s_i(m)}_{=b} \\ & = b \Rightarrow \sum_{i=1}^{n} r_i s_i \models \phi. \end{aligned}$$

This means that the set of solutions $\mathbb{M}(\Gamma)$ to a system Γ is an affine submodule of $\mathcal{E}(U)$, which implies that

$$\mathrm{aff} \leqslant \mathbb{M} \circ \mathbb{T}_R,$$

where aff is defined as follows.

Definition 5.2.2. Let $S \subseteq \mathcal{E}(U)$. We define the *affine closure* of S as the set

$$\mathrm{aff}_U\, S := \left\{ \sum_{i=1}^{n} r_i s_i \;\middle|\; s_i \in S, r_i \in R, \sum_{i=1}^{n} r_i = 1 \right\}.$$

Lemma 5.2.3. *The* affine operator $\mathrm{aff} : \mathcal{P}\mathcal{E} \Rightarrow \mathcal{P}\mathcal{E}$ *is natural.*

Proof. Given two subsets $U \subseteq U' \subseteq X$, we denote by $S|_U$ the set $S|_U := \{ s|_U \mid s \in S \}$, and by $\hat{\rho}^{U'}_U := \mathcal{P}\mathcal{E}(U \subseteq U') :: S \mapsto S|_U$. Naturality for aff corresponds to the commutativity of the following diagram.

$$\begin{tikzcd}
 & \mathcal{P}\mathcal{E}(U) \arrow[rr, "\mathrm{aff}_U"] \arrow[dl, dashed, no head] & & \mathcal{P}\mathcal{E}(U) \arrow[dl, dashed, no head] \\
 S|_U \arrow[rr, mapsto] & & \mathrm{aff}_U S|_U & \\
 & \mathcal{P}\mathcal{E}(U') \arrow[uu, "\hat{\rho}^{U'}_U"'] \arrow[rr, "\mathrm{aff}_{U'}"] \arrow[dl, dashed, no head] & & \mathcal{P}\mathcal{E}(U') \arrow[uu, "\hat{\rho}^{U'}_U"'] \arrow[dl, dashed, no head] \\
 S \arrow[uu, mapsto] \arrow[rr, mapsto] & & \mathrm{aff}_{U'} S \arrow[uu, mapsto] &
\end{tikzcd}$$

We can prove that this diagram commutes by starting from the bottom left corner with a subset $S \subseteq \mathcal{E}(U')$. A the top right corner we obtain $\mathrm{aff}_U S|_U$

coming from the left and $(\mathrm{aff}_{U'}S)|_U$ coming from the bottom right. These two results coincide, in fact

$$
\begin{aligned}
\mathrm{aff}_U S|_U &= \left\{ \sum_{i=0}^{n} r_i s_i \;\middle|\; s_i \in S|_U, \sum_{i=0}^{n} r_i = 1 \right\} \\
&= \left\{ \sum_{i=0}^{n} r_i s_i|_U \;\middle|\; s_i \in S, \sum_{i=0}^{n} r_i = 1 \right\} \\
&= \left\{ \left(\sum_{i=0}^{n} r_i s_i \right)|_U \;\middle|\; s_i \in S, \sum_{i=0}^{n} r_i = 1 \right\} \\
&= \left\{ s|_U \;\middle|\; s \in \left\{ \sum_{i=0}^{n} r_i s_i \;\middle|\; s_i \in S, \sum_{i=0}^{n} r_i = 1 \right\} \right\} \\
&= \left\{ s|_U \;\middle|\; s \in \mathrm{aff}_{U'}(S) \right\} = (\mathrm{aff}_{U'}S)|_U\,.
\end{aligned}
$$

□

We use the affine operator to define the affine closure of an empirical model.

Definition 5.2.4. Let $\mathcal{S}$ be an empirical model on a scenario $\langle X, \mathcal{M}, R\rangle$. We define its *affine closure* $\mathrm{Aff}\,\mathcal{S}$, as the empirical model given, at each $C \in \mathcal{M}$ by

$$(\mathrm{Aff}\,\mathcal{S})(C) := \mathrm{aff}\,\mathcal{S}(C).$$

Note that, since $\mathbb{T}_R(\mathcal{S})$ is defined as the union of all the theories at each context, the Galois connection of Proposition 5.2.1 can be lifted to the level of empirical models. We also clearly have $\mathrm{Aff} \leqslant \mathbb{M} \circ \mathbb{T}_R$ (with equality if R is a field).

Proposition 5.2.5. *Let $\mathcal{S}$ be an empirical model on a $\langle X, \mathcal{M}, R\rangle$ scenario. We have*

$$\mathsf{AvN}_R(\mathcal{S}) \Rightarrow \mathsf{SC}(\mathrm{Aff}\,\mathcal{S}).$$

If R is a field, then the converse is also true.

Proof. Since $\mathrm{Aff} \leqslant \mathbb{M} \circ \mathbb{T}_R$, which is the closure operator, the R-linear theory of $\mathcal{S}$ coincides with the R-linear theory of $\mathrm{Aff}\,\mathcal{S}$. Therefore, if $\mathcal{S}$ is AvN_R, then $\mathrm{Aff}\,\mathcal{S}$ is also AvN_R, thus strongly contextual by Proposition 5.1.4. Now, suppose R is a field and that $\mathbb{T}_R(\mathcal{S})$ is consistent. Then there is a global event $g : X \to R$ such that it satisfies all the equations in $\mathbb{T}_R(\mathcal{S})$. But $g \in (\mathrm{Aff}\,\mathcal{S})(X)$, since $\mathbb{M} \circ \mathbb{T}_R(\mathcal{S}) = \mathrm{Aff}\,\mathcal{S}$ (since R is a field). Therefore, $\mathrm{Aff}\,\mathcal{S}$ is not strongly contextual. □

5.3 COHOMOLOGY DETECTS THE CONTEXTUALIY OF AvN MODELS

Consider again the functor (3.4) F_R that constructs the free R-module on a given set X previously used in practice to define the presheaf of abelian groups associated to an empirical model. Since $F_R X$ is the free R-module generated by X, we know that the functor F_R is the left adjoint of the forgetful functor $\mathcal{U} : R-\mathbf{Mod} \to \mathbf{Set}$. The adjunction is defined by the unit $\eta : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathcal{U} \circ F_R$ given by

$$\eta_X : X \to \mathcal{U}(F_R X) :: x \to 1 \cdot x,$$

and by the counit $\epsilon : F_R \circ \mathcal{U} \Rightarrow \mathrm{Id}_{R-\mathbf{Mod}}$ defined as

$$\epsilon_M : F_R\mathcal{U}(M) \to M :: r \mapsto \sum_{x \in M} r(x) \cdot x. \tag{5.1}$$

Now, let M be an R-module and let $S \subseteq \mathcal{U}(M)$. Notice that $F_R S$, being the set of all formal linear combinations of elements in S, can be seen as the linear span of the set $\mathtt{im}(\eta_S) = \{1 \cdot s \mid s \in S\}$ inside $F_R\mathcal{U}(M)$, denoted by $\mathtt{span}_{F_R\mathcal{U}(M)}\mathtt{im}(\eta_S)$. Since ϵ is a homomorphism of R-modules, it maps $\mathtt{span}_{F_R\mathcal{U}(M)}\mathtt{im}(\eta S)$ to $\mathtt{span}_M S$, i.e. the linear span of S in M. Moreover, it also maps formal affine combinations $F_R^{\mathtt{aff}}(S) = \mathtt{aff}_{F_R\mathcal{U}(M)}\mathtt{im}(\eta_S)$ to the affine closure $\mathtt{aff}_M S$.

The specific scenarios we are studying have R as an outcome set. Therefore, all the events in each $\mathcal{E}(U)$ are in fact R-modules. This allows us to rewrite the sheaf of events as $\mathcal{E} : \mathcal{P}(X)^{\mathrm{op}} \to R - \mathbf{Mod}$. Therefore, we can construct a map of presheaves $\mathrm{Id}_{\mathcal{E}} * \epsilon : F_R \circ \mathcal{U} \circ \mathcal{E} \Rightarrow \mathcal{E}$, given at each context by the counit (5.1) $\epsilon_{\mathcal{E}(U)}$. If we have an empirical model $\mathcal{S}$, we exploit this fact to subsets of the module at every context. Note that if U is beneath the cover, then $\mathtt{aff}_{\mathcal{E}(U)}\mathcal{S}(U) = (\mathtt{Aff}\ \mathcal{S})(U)$ by definition, thus summarizing the whole discussion, we conclude that the map $\mathrm{Id}_{\mathcal{E}} * \epsilon$ restricts as shown in the following diagram.

$$\begin{tikzcd}
F_R^{\mathtt{aff}}\mathcal{U}\mathcal{S} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F_R\mathcal{U}\mathcal{S} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F_R\mathcal{U}\mathcal{E} \arrow[d, Rightarrow, "\epsilon"] \\
\mathtt{Aff}\ \mathcal{S} \arrow[r, Rightarrow] & \mathtt{Span}\ \mathcal{S} \arrow[r, Rightarrow] & \mathcal{E}
\end{tikzcd}$$

We can finally formulate that cohomological obstructions witness the contextuality of AvN models by providing a chain of logical implications that bring together these two methods.

Theorem 5.3.1. *Let $\mathcal{S}$ be an empirical model on a scenario $\langle X, \mathcal{M}, R\rangle$. Then*

$$\mathrm{AvN}_R(\mathcal{S}) \Rightarrow \mathrm{AC}(\mathtt{Aff}\ \mathcal{S}) \Rightarrow \mathrm{CSC}_R(\mathcal{S}) \Rightarrow \mathrm{SC}(\mathcal{S}).$$

Proof. The first implication has already been proved in Proposition 5.2.5, and the third in Proposition 3.2.10. We will prove the second implication by showing the converse. If $\mathcal{S}$ is not CSC_R, then there exists a context C_0 and a section $s_0 \in F_R\mathcal{S}(C_0)$ such that $\gamma_{F_R\mathcal{S})}(s_0) = 0$. By Proposition 3.2.7, this means that there exists a compatible family $\{s_C \in F_R\mathcal{S}(C)\}_{C \in \mathcal{M}}$ extending s_0 (i.e. $s_{C_0} = s_0$). Each s_C is a formal linear combination of elements in $\mathcal{S}(C)$, thus the presheaf map $F_R^{\mathtt{aff}}\mathcal{U}\mathcal{S} \Rightarrow \mathtt{Aff}\ \mathcal{S}$ of the diagram above maps this compatible family to a compatible family of $\mathtt{Aff}\ \mathcal{S}$, showing that $\mathtt{Aff}\ \mathcal{S}$ cannot be strongly contextual. $\square$

6 APPENDIX

6.1 SOME MORE DETAILS ON THE GALOIS CORRESPONDENCE $S \leftrightarrow V_S$

We have seen in Section 2 that there is an antitone Galois correspondence between $(\mathcal{SG}(\mathcal{P}), \subseteq)$ and $(\{V_S \mid S \in \mathcal{SG}(\mathcal{P}_n)\}, \subseteq)$. We will show here that this correspondence is in fact induced by an antitone Galois connection between $(\mathcal{SG}(\mathcal{P}), \subseteq)$ and $(\mathcal{SS}(\mathbb{C}^n), \subseteq)$, i.e. the set of all sub-vector spaces of $\mathbb{C}^n$.

Proposition 6.1.1. *Let*

$$\begin{array}{rccccccc} F: & \mathcal{SG}(\mathcal{P}) & \longrightarrow & \mathcal{SS}(\mathbb{C}^n) & :: & S & \longmapsto & V_S \\ G: & \mathcal{SS}(\mathbb{C}^n) & \longrightarrow & \mathcal{SG}(\mathcal{P}) & :: & V & \longmapsto & \bigcap_{|\psi\rangle \in V} (\mathcal{P}_n)_{|\psi\rangle}, \end{array}$$

where $(\mathcal{P}_n)_{|\psi\rangle} := \{A \in \mathcal{P}_n \mid A \bullet |\psi\rangle = |\psi\rangle\}$ *denotes the isotropy group of* $|\psi\rangle$. *Then* F *and* G *form a* Galois connection *between* $(\mathcal{SG}(\mathcal{P}), \subseteq)$ *and* $(\mathcal{SS}(\mathbb{C}^n), \subseteq)$. *Explicitly, this means that* F *and* G *are order-reversing and satisfy*

$$S \subseteq G(F(S)) \quad W \subseteq F(G(W)),$$

for all $S \in \mathcal{SG}(\mathcal{P})$ *and* $W \in \mathcal{SS}(\mathbb{C}^n)$.

Proof. We have already shown that F is order reversing. Let $V \subseteq W$ be two subspaces of $\mathbb{C}^n$. We have

$$G(W) := \bigcap_{|\psi\rangle \in W} (\mathcal{P}_n)_{|\psi\rangle} \subseteq \bigcap_{|\psi\rangle \in V} (\mathcal{P}_n)_{|\psi\rangle} =: G(V),$$

thus, G is also order-reversing. Now, let $S \in \mathcal{SG}(\mathcal{P})$, we have

$$G(F(S)) = G(V_S) = \bigcap_{|\psi\rangle \in V_S} (\mathcal{P}_n)_{|\psi\rangle},$$

and

$$\begin{aligned} A \in S \Rightarrow A \bullet |\psi\rangle = |\psi\rangle,\ \forall |\psi\rangle \in V_S \Rightarrow A \in (\mathcal{P}_n)_{|\psi\rangle},\ \forall |\psi\rangle \in V_S \\ \Rightarrow A \in \bigcap_{|\psi\rangle \in V_S} (\mathcal{P}_n)_{|\psi\rangle}, \end{aligned} \tag{6.1}$$

thus $S \subseteq G(F(S))$. On the other hand, let $W \subseteq \mathbb{C}^n$ be a subspace. We have

$$\begin{aligned} F(G(W)) &= F\left(\bigcap_{|\psi\rangle \in W} (\mathcal{P}_n)_{|\psi\rangle}\right) = V_{\left(\bigcap_{|\psi\rangle \in W} (\mathcal{P}_n)_{|\psi\rangle}\right)} \\ &= \left\{ |\psi\rangle \in \mathbb{C}^n \;\middle|\; A \bullet |\psi\rangle = |\psi\rangle,\ \forall A \in \bigcap_{|\psi\rangle \in W} (\mathcal{P}_n)_{|\psi\rangle} \right\} \\ &= \left\{ |\psi\rangle \in \mathbb{C}^n \;\middle|\; A \bullet |\psi\rangle = |\psi\rangle,\ \forall A : \left(A \in (\mathcal{P}_n)_{|\psi\rangle},\ \forall |\psi\rangle \in W\right) \right\} \\ &= \left\{ |\psi\rangle \in \mathbb{C}^n \;\middle|\; A \bullet |\psi\rangle = |\psi\rangle,\ \forall A : (A \bullet |\psi\rangle = |\psi\rangle,\ \forall |\psi\rangle \in W) \right\}, \end{aligned}$$

and clearly W is a subset of this set. Thus F and G form indeed a Galois connection.

□

Remark 6.1.2. Notice that this connection is not a correspondence, in fact (for instance) we don't necessarily have $S = G(F(S))$ for every S. To see this we can observe that, although every other logical implication of (6.1) can be reversed, the first one is strict in general:

$$A \bullet |\psi\rangle = |\psi\rangle,\ \forall |\psi\rangle \in V_S \not\Rightarrow A \in S$$

We will use the following general result concerning Galois connections (the proof is quite simple and left in exercise, cf. [EKMS92] for deeper insights)

Proposition 6.1.3. *Any antitone Galois connection* $f : A \to B$, $g : B \to A$ *induces an antitone Galois correspondence between* $\mathrm{im}(f)$ *and* $\mathrm{im}(g)$.

Thus, the Galois connection of Proposition 6.1.1 induces a Galois correspondence

$$\mathrm{im}(F) = \{V_S \mid S \in \mathcal{SG}(\mathcal{P}_n)\} \longleftrightarrow \left\{ \bigcap_{|\psi\rangle \in V} (\mathcal{P}_n)_{|\psi\rangle} \,\middle|\, V \in \mathcal{SS}(\mathbb{C})^n \right\} = \mathrm{im}(G).$$

Since we have proved in the project that we have an antitone Galois correspondence $\mathcal{SG}(\mathcal{P}_n) \sim \mathrm{im}(F)$ and we have just shown that $\mathrm{im}(F) \sim \mathrm{im}(G)$, we infer, in particular, that $\mathrm{im}(G)$ is in bijection with $\mathcal{SG}(\mathcal{P}_n)$. Since $\mathcal{SG}(\mathcal{P}_n)$ is a finite set and $\mathrm{im}(G) \subseteq \mathcal{SG}(\mathcal{P}_n)$, we conclude that $\mathrm{im}(G) = \mathcal{SG}(\mathcal{P}_n)$, i.e.

$$\left\{ \bigcap_{|\psi\rangle \in V} (\mathcal{P}_n)_{|\psi\rangle} \,\middle|\, V \in \mathcal{SS}(\mathbb{C})^n \right\} = \mathcal{SG}(\mathcal{P}_n)$$

This is a interesting result: it tells us that every subgroup of $\mathcal{P}_n$ can be written as an intersection of isotropy groups for the action on $\mathbb{C}^n$. It also implies that the Galois connection of Proposition 6.1.1 induces the correspondence of Proposition 2.1.5.

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