Categories of Matroids

Vaia Patta

March 6, 2015
What is a matroid?
A matroid describes dependence.
What is a matroid?

A matroid describes *dependence*.
A matroid describes dependence.
A matroid describes *dependence*. 

\[
\begin{align*}
\mathbf{x} &= a_1 x_1 + a_2 x_2 + \ldots
\end{align*}
\]
What is a matroid?

A matroid describes *dependence*.

\[ x = ax_1 + bx_2 \]
What is a matroid?

A matroid describes *dependence*.

\[ x = ax_1 + bx_2 \quad \text{...etc} \]
Interesting classes of matroids

- **Pointed matroids**: distinguished element ○, where {○} is dependent (i.e. ○ is a loop)

- **Loopless matroids**: ○ is the only loop

- **Simple matroids**: loopless with no minimal dependent sets (circuits) of cardinality 2

- **Representable matroid**: those arising from linear dependence of vectors
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- *Pointed matroids*: distinguished element •, where \{•\} is dependent (i.e. • is a *loop*)

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Categories of matroids
What are the morphisms?

- **Rank of a set** $X$: Size of the largest independent set contained in $X$.
- **Closed sets** (or *flats*): Maximal sets of a fixed rank $r$.
- **Strong maps**: Functions between matroids for which the preimage of a closed set is closed.
What are the morphisms?

*Rank* of a set $X$: Size of the largest independent set contained in $X$
What are the morphisms?

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\[ \text{Rank of a set } X: \text{ Size of the largest independent set contained in } X \]

\[ \text{Closed sets (or flats): Maximal sets of a fixed rank } r \]

\[ \text{Strong maps: Functions between matroids for which the preimage of a closed set is closed} \]
Limits and Colimits

Matroids, Matroids Loopless, Simple Simple, Repr. over k Repr. over k
Limits and Colimits

Limits:

Matroids
Matroids Loopless
Simple
Simple
Repr. over k
Repr. over k
Limits and Colimits

Limits:

Products
Limits and Colimits

Limits:

Products
### Limits and Colimits

Matroids, Loopless, Simple, Simple, Repr. over k, Repr. over k

**Limits:**

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## Limits and Colimits

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Limits and Colimits

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### Limits:

- **Products**: 
  - Matroids: ✗
  - Matroids Loopless: ✗
  - Simple: ✗
  - Simple: ✗
  - Repr. over \( k \): ✗
  - Repr. over \( k \): ✗

- **Exponentials**: 
  - Matroids: ✗
  - Matroids Loopless: ✗
  - Simple: ✗
  - Simple: ✗
  - Repr. over \( k \): ✗
  - Repr. over \( k \): ✗

- **Pullbacks**: 
  - Matroids: ✗
  - Matroids Loopless: ✗
  - Simple: ✗
  - Simple: ✗
  - Repr. over \( k \): ✗
  - Repr. over \( k \): ✗

- **Equalisers**: 
  - Matroids: ✓
  - Matroids Loopless: ✓
  - Simple: ✓
  - Simple: ✓
  - Repr. over \( k \): ✓
  - Repr. over \( k \): ✓

### Colimits:

- **Coproducts**: 
  - Matroids: ✓
  - Matroids Loopless: ✓
  - Simple: ✓
  - Simple: ✓
  - Repr. over \( k \): ✓
  - Repr. over \( k \): ✓

- **Coequalisers**: 
  - Matroids: ✗
  - Matroids Loopless: ✗
  - Simple: ✗
  - Simple: ✗
  - Repr. over \( k \): ✗
  - Repr. over \( k \): ✗

- **Pushouts**: 
  - Matroids: ✗
  - Matroids Loopless: ✗
  - Simple: ✗
  - Simple: ✗
  - Repr. over \( k \): ✗
  - Repr. over \( k \): ✗
Functors and adjunctions

\[ \text{Matroids} \rightleftharpoons \text{Geometric Lattices} \]
Functors and adjunctions

Matroids (functor) ⊣ Geometric Lattices (adjunction) → Vect_k

Free matroids: Every set is independent.
Functors and adjunctions

\[ \text{Matroids} \Downarrow \text{Geometric Lattices} \Downarrow \text{Vect}_k \]

\textit{Free matroids}: Every set is independent.
Free matroids: Every set is independent.
Functors and adjunctions

\[
\begin{array}{c}
\text{Matroids} \leftarrow \text{Loopless Matroids} \\
\uparrow \quad \quad \downarrow \\
\text{Simple Matroids} \leftarrow \text{Free Matroids}
\end{array}
\]
Functors and adjunctions

\[
\begin{align*}
\text{Matroids} & \quad \text{\smallfrown} \quad \text{Loopless Matroids} \\
\text{Simple Matroids} & \quad \text{\smallfrown} \quad \text{Free Matroids}
\end{align*}
\]
Functors and adjunctions

Matroids as functors:

$$\text{Geometric Lattices} \longrightarrow \text{Sub}$$