The category of matroids

Chris Heunen, Vaia Patta

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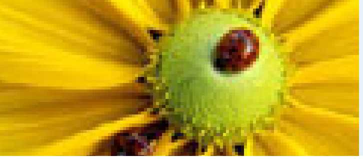
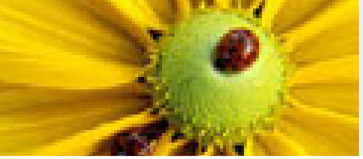
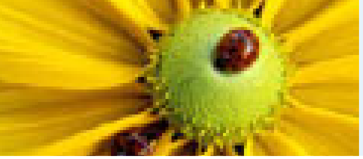


Objects



What’s a matroid?

A matroid describes a dependence relation on a *ground set* E.



What’s a matroid?

Objects
What’s a matroid?
No really, what’s a matroid?
Terminology
Morphisms
Basic properties
Functors
Limits and colimits
Adjunctions
Factorisation
Constructions
Greedy algorithm

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- Example: Linearly dependent columns of a matrix.

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- Example: Linearly dependent vectors in a vector space.

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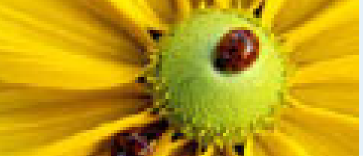
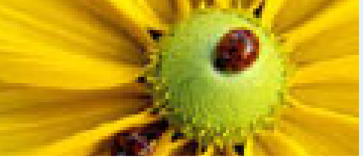
- Example: Cycles in a graph.



What’s a matroid?

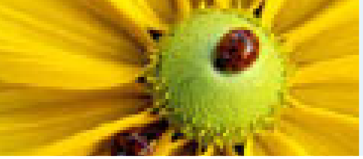
A matroid describes a dependence relation on a *ground set* E.

- Example: Linearly dependent columns of a matrix.
- Example: Linearly dependent vectors in a vector space.
- Example: Cycles in a graph.
- Example: Algebraic dependence.



No really, what's a matroid?

- Independent subsets $\mathcal{I}$ of E
  - downwards closed
  - nontrivial
  - independence augmentation
- Flats (closed sets) $\mathcal{F}$
  - nontrivial
  - closed under intersection
  - partitioning
- Rank function $r : 2^E \longrightarrow \mathbb{N}$
  - bounded
  - monotonic
  - valuation
- Closure $\mathrm{cl} : 2^E \longrightarrow 2^E$
- Bases $\mathcal{B}$

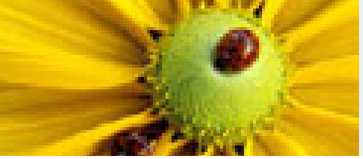


Terminology

loop

isthmus (coloop)

parallel elements



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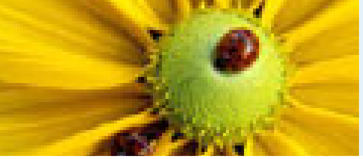
geometric lattice
lattice of flats



Terminology

loop
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parallel elements
geometric lattice
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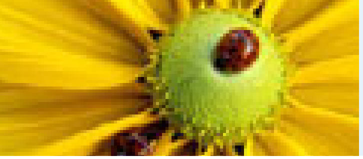
deletion
embedding
contraction
minor
“quotient”



Terminology

loop
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parallel elements
geometric lattice
lattice of flats
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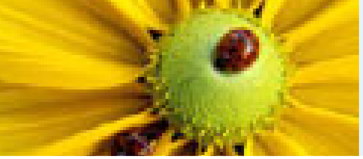
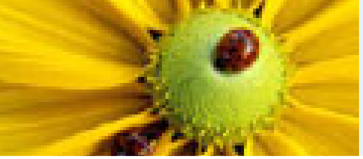
embedding
contraction
minor
"quotient"
loopless
simple
free
simplification
representable



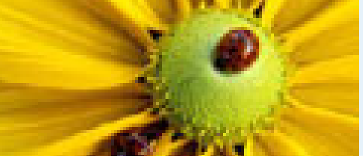
Morphisms



Strong maps

Strong map $f : M \to N$:

- preimage of a closed set is closed
- $r(f(Y)) - r(f(X)) \leq r(Y) - r(X)$ for all $X \subseteq Y \subseteq E(M)$
- $L(f) : L(M) \to L(N)$ preserves joins and sends atoms to atoms or bottom

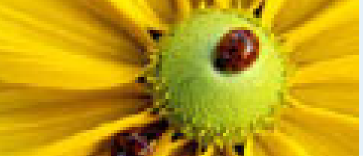


Subcategories

Pointed matroid:
Matroid with distinguished loop $\bullet$

Matroid categories	Pointed	Unpointed
All	$\mathsf{Matr}_\bullet$	Matr
Loopless	$\mathsf{LMatr}_\bullet$	LMatr
Simple	$\mathsf{SMatr}_\bullet$	SMatr
Free	$\mathsf{FMatr}_\bullet$	FMatr

Pointed categories have pointed maps as morphisms



Basic properties

Basic properties

$$\begin{tikzcd}
Matr_\bullet \arrow[r] \arrow[r, phantom, "\top"'] & Matr \arrow[l, bend left] \arrow[r] \arrow[r, phantom, "\perp"'] & Set \arrow[l, bend left]
\end{tikzcd}$$

Basic properties

$$\begin{tikzcd}
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- Monomorphism=injective

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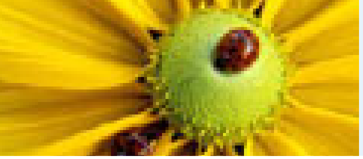
\end{tikzcd}$$

- Monomorphism=injective
- Epimorphism=surjective



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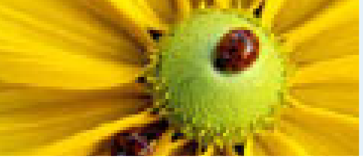
- Monomorphism=injective
- Epimorphism=surjective
- Isomorphism=bijective *and* flats map to flats



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- "quotient"=bijective

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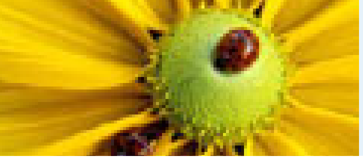
- Monomorphism=injective
- Epimorphism=surjective
- Isomorphism=bijective *and* flats map to flats
- “quotient”=bijective
- In pointed categories, contraction is a strong map.



Functors

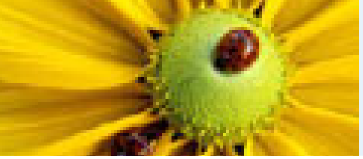
General notions

Functor $F : C \to D$

- "nearly full"=surjective on morphisms
- "nearly faithful": M monoid, C enriched in left M-actions, $F(g) = F(g) \Rightarrow f = m \cdot g$ for some $m \in M$

Geometric lattices

$$\begin{tikzcd}
Matr_\bullet \arrow[r, "L"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[loop below, "si_\bullet = S\circ L"'] & GLat \arrow[l, bend left, "S"]
\end{tikzcd}$$



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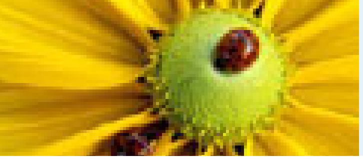
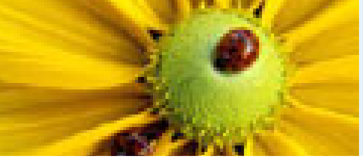
- L full (nearly full for unpointed)



Geometric lattices

$$\begin{tikzcd} Matr_{\bullet} \arrow[loop below, "si_{\bullet}=S\circ L"] \arrow[r, bend left=0, "L"] \arrow[r, phantom, shift right=2, "\perp"] & GLat \arrow[l, bend left, "S"] \end{tikzcd}$$

- L full (nearly full for unpointed)
- L maps “quotients” to quotients (and the associated restriction is nearly full)



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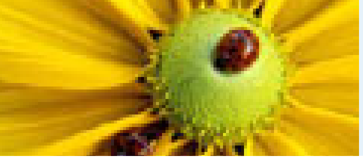
- Epimorphisms are surjections in $GLat$



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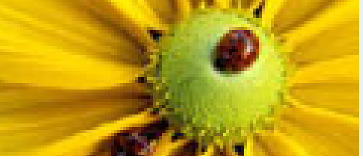

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- S embedding
- $SMatr_\bullet$ is the category of Eilenberg-Moore algebras of $si_\bullet$

Vector spaces

$$\begin{tikzcd}
FVect_k \arrow[r, "M", "\not\perp"'] \arrow[d, hook] & Matr_\bullet \\
MVect_k \arrow[ur, "\overline{M}"'] &
\end{tikzcd}$$

Vector spaces

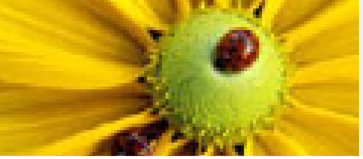
$$\begin{tikzcd}
FVect_k \arrow[r, "M"] \arrow[d, hook] \arrow[dr, phantom, "\not\perp" near start] & Matr_\bullet \\
MVect_k \arrow[ur, "\overline{M}"'] &
\end{tikzcd}$$

$\overline{M}$ left Kan extension of M

Vector Spaces

$$\begin{tikzcd}
 & Q \arrow[dr, "c"] & & \\
M \arrow[ur, "e"] \arrow[rr, "q"] \arrow[rrr, bend right, "f"'] & & N' \arrow[r, "\tau"] & N
\end{tikzcd}$$

For M and N representable over k, Q representable over extension of k



Vector Spaces

$$\begin{tikzcd}
& Q \arrow[dr, "c"] & & \\
M \arrow[ur, "e"] \arrow[rr, "q"] \arrow[rrr, bend right, two heads, "f"'] & & N' \arrow[r, two heads, "\tau"] & N
\end{tikzcd}$$

For M and N representable over k, Q representable over extension of k

$$\Downarrow$$

$\overline{M}$ nearly full for $k = \mathbb{Q}$

Vector spaces and geometric lattices

$$\begin{tikzcd}
FVect_k \arrow[r, "M"] \arrow[r, phantom, "\not\perp"'] \arrow[rr, bend left, "L\circ M"] \arrow[d, hook] & Matr_\bullet \arrow[r, "L"] \arrow[r, phantom, "\perp"'] \arrow[loop below, "si_\bullet=S\circ L"] & GLat \arrow[l, bend left, "S"] \\
MVect_k \arrow[ur, "\overline{M}"'] & &
\end{tikzcd}$$

$L \circ M$ nearly faithful

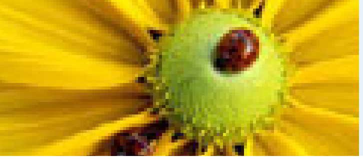
Graphs

Any functor $Graph \to Matr$ giving the cycle matroid on a graph cannot be surjective on objects, injective on objects, full or faithful.

Limits and colimits

Limits and colimits

- $FMatr$ is isomorphic to Set and $FMatr_\bullet$ is isomorphic to $Set_\bullet$.
- $SMatr$, $LMatr$, $Matr$, $SMatr_\bullet$, $LMatr_\bullet$ and $Matr_\bullet$ have all coproducts, all equalisers and do not generally have products, coequalisers, pullbacks, pushouts or exponentials.
- Every contraction is a coequaliser in $Matr_\bullet$.

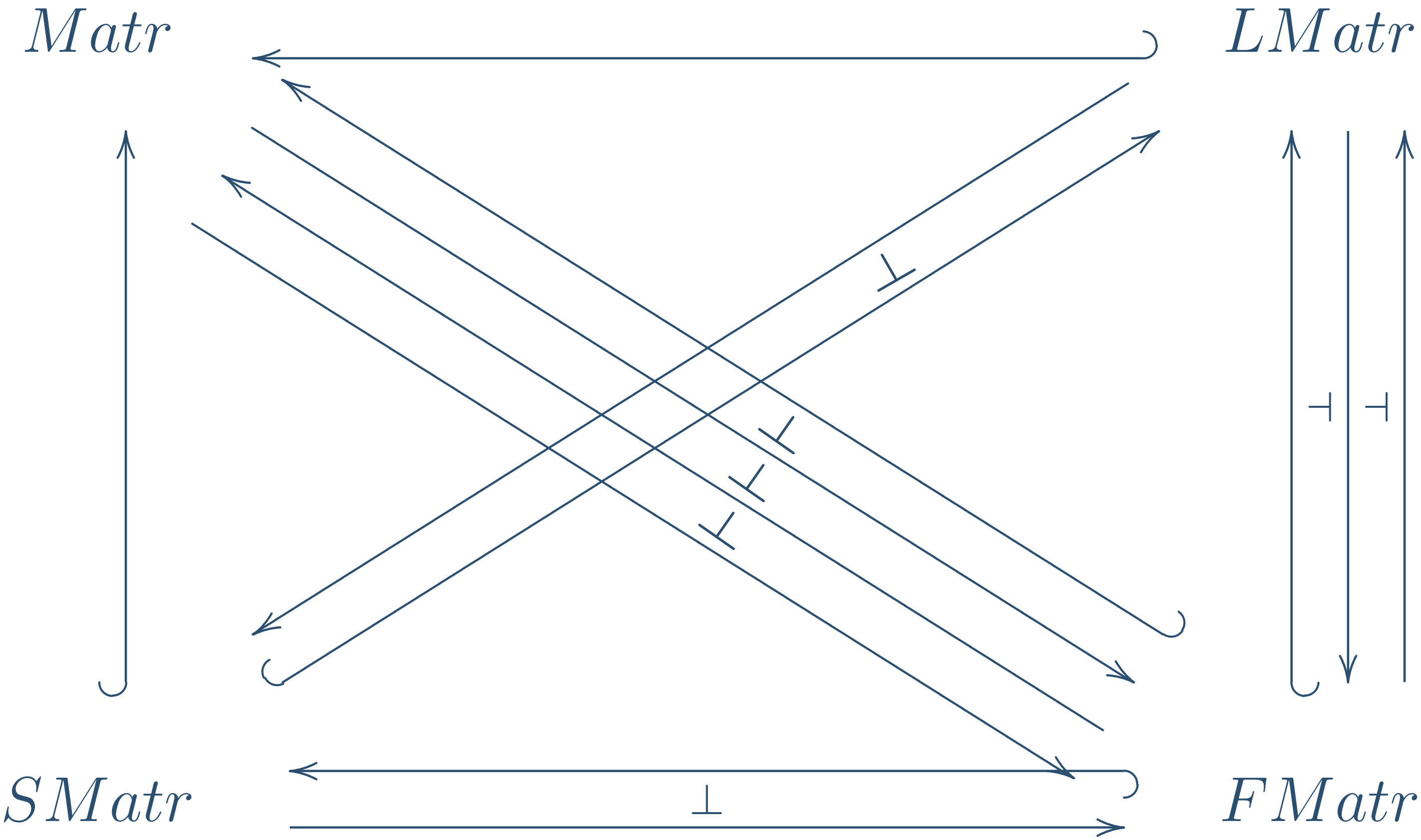


Adjunctions

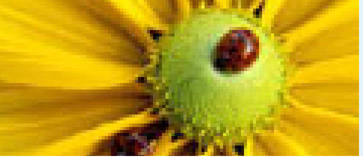
Pointed Categories

$$\begin{tikzcd}
Matr_\bullet \arrow[r, shift left=1.5] \arrow[r, phantom, "\perp"] \arrow[d, shift right=1.5] \arrow[d, phantom, "\dashv"] \arrow[dr, shift left=1.5] \arrow[dr, phantom, shift left=3, "\perp"] \arrow[dr, phantom, "\perp"] \arrow[dr, phantom, shift right=3, "\perp"] \arrow[dr, shift right=4.5] & LMatr_\bullet \arrow[l, hook, shift left=1.5] \arrow[d, shift left=1.5] \arrow[d, phantom, "\dashv"] \arrow[dl, shift right=1.5] \arrow[dl, phantom, "\perp"] \\
SMatr_\bullet \arrow[u, hook, shift right=1.5] \arrow[r, shift right=1.5] \arrow[r, phantom, "\perp"] \arrow[ur, hook, shift right=1.5] & FMatr_\bullet \arrow[l, hook, shift right=1.5] \arrow[u, hook, shift left=1.5] \arrow[ul, hook, shift right=4.5] \arrow[ul, shift left=1.5]
\end{tikzcd}$$

Unpointed Categories



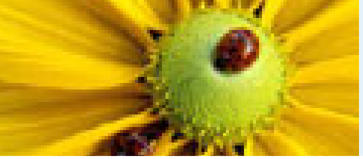
$$\begin{tikzcd}
Matr \arrow[rd, shift left=3] \arrow[rd, shift right=1] & LMatr \arrow[l, hook'] \arrow[ld, shift right] \arrow[d, shift left] \\
SMatr \arrow[u, hook] \arrow[ru, hook, shift right] \arrow[ru, phantom, "\perp"] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & FMatr \arrow[l, hook', shift right] \arrow[u, hook, shift left=3] \arrow[u, shift right=3] \arrow[u, phantom, shift left, "\dashv"] \arrow[u, phantom, shift right, "\dashv"] \arrow[lu, hook, shift right=3] \arrow[lu, shift left=1] \arrow[lu, phantom, shift left, "\perp"] \arrow[lu, phantom, "\perp"] \arrow[lu, phantom, shift right, "\perp"]
\end{tikzcd}$$



Factorisation

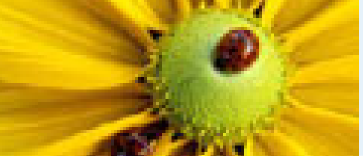
Factorisation systems

- $Matr_\bullet$
  - Orthogonal: (Epimorphisms, embeddings)
  - Orthogonal: (Lattice-preserving maps, maps injective on elements of each rank-1 flat)



Factorisation systems

- $Matr_{\bullet}$
  - Orthogonal: (Epimorphisms, embeddings)
  - Orthogonal: (Lattice-preserving maps, maps injective on elements of each rank-1 flat)
- $GLat$
  - Contraction in $GLat$
  - Embedding in $GLat$
  - Weak: (Embedding, contraction)



Factorisation systems

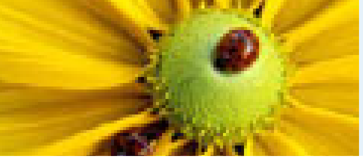
- $Matr$.
    - Orthogonal: (Epimorphisms, embeddings)
    - Orthogonal: (Lattice-preserving maps, maps injective on elements of each rank-1 flat)
- $GLat$
    - Contraction in $GLat$
    - Embedding in $GLat$
    - Weak: (Embedding, contraction)

Any orthogonal factorisation system $(\mathcal{L}, R)$ in $GLat$ induces an orthogonal factorisation system $(L^{-1}(\mathcal{L}), \mathcal{R}')$ in $Matr$.

Constructions

Functors

- Adding loops $Matr \longrightarrow Matr$
- Adding isthmuses $Matr \longrightarrow Matr$



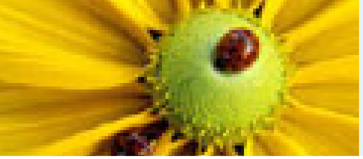
Functors

- Adding loops $Matr \to Matr$
- Adding isthmuses $Matr \to Matr$
- Category $Matr_*$
- Category $Matr_{*n}$
- Contraction $Matr_{*n+1} \to Matr_{*n}$
- Deletion $Matr_{*n+1} \to Matr_{*n}$ (right adjoint to inclusion)

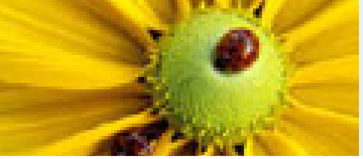
Taking minors is functorial.

Monoidal structure

$Matr_\times$ (matroids with a distinguished element) has parallel connection as coproduct and series connection, its dual operation, as an affine monoidal structure.



Greedy algorithm



Greedy algorithm

Every run of the greedy algorithm produces a maximal chain of epimorphisms in a subcategory of $Vect^b_{\mathbb{R}}$.The greedy algorithm solves the optimization problem if and only if the chains in $Vect^b_{\mathbb{R}}$ induced by all runs have the same limit.