Preferences Single-Peaked on Nice Trees

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Abstract

Preference profiles that are single-peaked on trees enjoy desirable properties: they admit a Condorcet winner (Demange 1982), and there are hard voting problems that become tractable on this domain (Yu, Chan, and Elkind 2013). Trick (1989) proposed a polynomial-time algorithm that finds *some* tree with respect to which a given preference profile is single-peaked. However, some voting problems are only known to be easy for profiles that are single-peaked on "nice" trees, and Trick's algorithm provides no guarantees on the properties of the tree that it outputs. To overcome this issue, we build on the work of Trick and Yu et al. to develop a structural approach that enables us to compactly represent all trees with respect to which a given profile is single-peaked. We show how to use this representation to efficiently find the "best" tree for a given profile, according to a number of criteria; for other criteria, we obtain NP-hardness results. In particular, we show that it is NP-hard to decide whether an input profile is single-peaked with respect to a given tree. To demonstrate the applicability of our framework, we use it to identify a new class of profiles that admit an efficient algorithm for a popular variant of the Chamberlin–Courant (1983) rule.

1 Introduction

Preference aggregation is a difficult task when voters' preferences may be arbitrary: one has to deal with voting paradoxes (Arrow 1951) and computationally hard problems (Brandt, Conitzer, and Endriss 2013). This observation motivates the study of *domain restrictions*, i.e., special classes of voters' preferences that rule out paradoxical outcomes and/or allow one to circumvent computational hardness results. Perhaps the most-studied restricted domain is that of single-peaked preferences (Black 1948). This domain captures profiles where voters' preferences are determined by candidates' positions on a single issue, and has many desirable properties: for instance, single-peaked elections always have a Condorcet winner (a candidate that is preferred to every other candidate by a majority of voters), admit a non-manipulable voting rule (Moulin 1991), and allow for an efficient winner determination algorithm for a popular committee selection rule (Betzler, Slinko, and Uhlmann 2013).

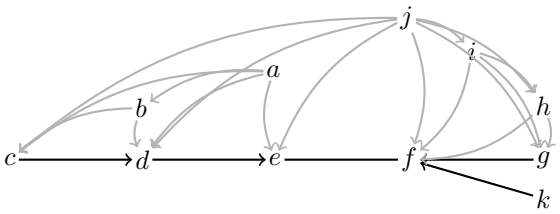


Figure 1: The 'attachment digraph' computed for the profile {*kfedghcijba, dcbeafghijk, gfhiedcbajk*} which is single-peaked on 336 different trees. All of them appear as subtrees of the above digraph, which can be computed in $O(|V| \cdot |C|^2)$ time. The black edges must appear in any tree; for each *free* vertex with gray outgoing arcs, we choose exactly one of them. The transitivity among the gray edges will be crucial to our approach.

Demange (1982) introduced a weaker domain restriction, namely, single-peakedness on a tree. Briefly, a profile is single-peaked on a tree if candidates can be mapped to the vertices of some tree so that the restriction of this profile to every path in this tree is single-peaked. This is a considerably broader domain than that of the single-peaked elections, which, nevertheless, retains some of the desirable properties of the latter: profiles that are single-peaked on a tree always have a Condorcet winner (Demange 1982), and there are committee selection problems that become easier when preferences are single-peaked on a tree (Yu, Chan, and Elkind 2013). However, some of the algorithms for this domain require the input profile to be single-peaked on a "nice" tree, such as a tree with a small number of leaves or a star (Yu, Chan, and Elkind 2013); indeed, positive results for single-peaked preferences can also be viewed in this light, as they require the preferences to be single-peaked on a specific tree, namely, a line (path).

Now, forcing Trick's algorithm to output a "nice" tree is not a trivial task: indeed, this algorithm may output a complex tree even when the input profile is single-peaked on a line. Fortunately, there are efficient algorithms for recognizing when a given profile is single-peaked on a line (Bartholdi and Trick 1986; Doignon and Falmagne 1994; Escoffier, Lang, and Öztürk 2008), and Yu, Chan, and Elkind (2013) explain how to modify Trick's algorithm so that it outputs a tree

with the minimum number of leaves and how to recognise when a profile is single-peaked on a star. However, prior to this work, no such algorithms were known for other types of "nice" trees, such as trees that have bounded diameter or a small number of internal nodes; in fact, it was an open question whether, given a profile V and a tree T, one can check in polynomial time whether V is single-peaked on T.

In this paper, we propose a general framework for answering such questions, and use it to obtain polynomial-time algorithms for identifying "nice" trees when they exist, for several appealing notions of "niceness". Specifically, we define a digraph that encodes, in a compact fashion, all trees with respect to which a given profile is single-peaked, an example is shown in Figure 1. This digraph enables us to count and/or enumerate all such trees. Moreover, we show that it has many useful structural properties, which can be exploited to efficiently find trees that have, e.g., the minimum degree, diameter, or number of internal nodes among all trees with respect to which a given profile is single-peaked, or to decide if a given profile is single-peaked on some specific type of tree, such as a caterpillar or a subdivision of a star (see Section 2 for definitions). However, there are limits to what we can accomplish in this way: we show that it is NP-hard to decide whether a given profile is single-peaked on a regular tree. Moreover, given a profile and a tree, it is NP-hard to decide if this profile is single-peaked on this specific tree.

Knowing whether a profile is single-peaked on a "nice" tree enables us to develop stronger intuition about the structure of voters' preferences. However, our recognition algorithms also have more tangible benefits: for at least one type of "nice" trees that can be identified by our algorithm (namely, trees with few internal vertices), we develop a winner determination algorithm for a variant of the Chamberlin–Courant committee selection rule (Chamberlin and Courant 1983) that, under plausible complexity assumptions, is more efficient than any algorithm for this rule under general preferences. Importantly, our winner determination algorithm works directly with the underlying tree, i.e., it relies on having an efficient procedure for constructing a tree with few internal vertices. We expect that similar results can be obtained for other types of trees that we can recognise; we leave this question for future work.

2 Preliminaries

For a finite set of *candidates* C, a *profile* V over C is a list of strict total orders over C; elements of V are called *votes*, or *preference orders*. We will view a vote $v \in V$ as a list of candidates, and use `Python`-like indices to refer to its entries: $v_{[1]}$, $v_{[2]}$, $v_{[-1]}$ are v's most, second-most, and least preferred candidates respectively, and $v_{[1:k]}$ is the set of v's k most preferred candidates. For readability, we write $c' \succ_v c$ to indicate that c' comes before c in the vote v. For a subset $C' \subseteq C$ of candidates, we denote by $v|_{C'}$ the preference order obtained from v by restricting it to $C' \times C'$.

A *tree* is a connected acyclic graph. A *leaf* of a tree is a vertex of degree 1. Vertices that are not leaves are *internal*. A *path* is a tree with exactly two leaves. The *diameter* of a tree T is the number of edges in a longest simple path in T. A *star* is a tree $K_{1,n}$ that has one internal vertex (the *center*) and n leaves. A *k-regular tree* is a tree in which every internal vertex has degree k. Note that paths are 2-regular (uniquely), and the star $K_{1,n}$ is n-regular. A *caterpillar* is a tree in which every vertex is within distance 1 of a central path; a *lobster* is a tree in which every vertex is within distance 2 of a central path. A *subdivision of a star* is a tree obtained from a star by replacing its edges by paths. The *path-width* of a tree T is the minimum width of a tree decomposition of T such that the underlying tree is a path (then called a *path decomposition*); see, e.g., (Bodlaender 1994) for a definition of a tree decomposition.

A profile V over C is *single-peaked (on a line)* if there is a linear order $\sqsubset$ on C such that for every $v \in V$, we have $x \succ_v y$ whenever $v_{[1]} \sqsubset x \sqsubset y$ or $y \sqsubset x \sqsubset v_{[1]}$. V is *single-peaked on* T, where T is a tree with vertex set C, if V is single-peaked when restricted to the vertex set of every path in T. Equivalently, V is single-peaked on T if for every $v \in V$ and each $k = 1, \dots, |C|$, the set $v_{[1:k]}$ induces a subtree of T. Given a profile V, we denote the set of all trees T such that V is single-peaked on T by $\mathcal{T}(V)$; we say that trees in $\mathcal{T}(V)$ are *suitable* for V. A profile V over C is *single-peaked on a tree* if $\mathcal{T}(V) \neq \varnothing$. In particular, V is single-peaked on a line if and only if it is single-peaked on a tree T that is a path.

A *digraph* $D = (\mathcal{V}, A)$ is a directed graph with no self-loops or multiple arcs. An arc $(u, v) \in A$ points from its *tail* u to its *head* v. We will write uv for (u, v). An *acyclic* digraph (a *dag*) is a digraph with no directed cycles. For a vertex v, its *out-degree* (resp., *in-degree*) $d^+(v)$ (resp., $d^-(v)$) is the number of arcs whose tail (resp., head) is v. A *sink* is a vertex v with $d^+(v) = 0$, a *source* is a vertex with $d^-(v) = 0$. Every dag has at least one sink and one source.

Given a digraph $D = (\mathcal{V}, A)$, we can *forget about its orientation* to obtain an underlying graph $G = (\mathcal{V}, E)$ where $\{u, v\} \in E$ if and only if $uv \in A$ or $vu \in A$.

3 The Attachment Digraph

We now introduce the essential tool in our study of the set $\mathcal{T}(V)$. Consider a profile V over a candidate set C that is single-peaked on *some* tree: $\mathcal{T}(V) \neq \varnothing$. We associate to this profile its *attachment digraph* D with vertex set C: this digraph is obtained by running Algorithm 1, which builds on the ideas of Trick (1989) and Yu, Chan, and Elkind (2013).

Algorithm 1 Build attachment digraph $D = (C, A)$ of V

```
D ← (C, A), A ← ∅        ▷ D is the empty digraph on C
C' ← C
while |C'| ⩾ 3 do
    L ← {(v|_{C'})_{[-1]} : v ∈ V}
    for each candidate c ∈ L do
        B_c = ⋂_{v∈V} B(v|_{C'}, c)
        if B_c = ∅ then
            return fail   ▷ V not single-peaked on any tree
        else
            add arcs cc' for each c' ∈ B_c to A
    C' ← C' \ L
return D
```

This algorithm runs in time $O(|V| \cdot |C|^2)$. It uses the operator $B(v, c)$ that takes as input a vote v and a candidate c, and returns a constraint on the candidates c' that c can be attached to in suitable trees as a leaf. It is defined as

$$B(v,c) = \begin{cases} \{c' : c' \succ_v c\} & \text{if } v_{[1]} \neq c, \\ \{v_{[2]}\} & \text{if } v_{[1]} = c. \end{cases}$$

This operator can be computed in time $O(|C|)$. To see that the definition makes sense, suppose that c is a leaf. If it is not the top candidate in vote v, then it must be attached to some c' that v prefers to c, since every path from $v_{[1]}$ to c goes through the parent of c. On the other hand, if c is the top candidate in v, then c must be attached to the candidate c' ranked immediately below c by v, as otherwise the path from c to $v_{[2]}$ violates the single-peakedness condition.

We start with a few easy properties of attachment digraphs.

Proposition 1. *Every attachment digraph (C, A) is acyclic and has at most two sinks. If it only has one sink t, then $d^-(t) \geqslant 2$.*

Proof. We follow the proof of Theorem 6.1 in the work of Yu et. al. (2013). Suppose that the while loop is executed $f - 1$ times, and denote the sets L found at each iteration by $L_1, \ldots, L_{f-1}$. Set $L_f := C \setminus (L_1 \cup \cdots \cup L_{f-1})$. Then $L_1, \ldots, L_f$ is a partition of C. Since for every $c \in C \setminus L_f$ we have $B_c \neq \varnothing$, at least one arc with tail c is added to A. Hence no $c \in C \setminus L_f$ can be a sink, so all sinks are in L_f. The condition of the while loop implies that $|L_f| \leqslant 2$, so there are at most two sinks. It also implies that $|L_{f-1} \cup L_f| \geqslant 3$, which gives $d^-(t) \geqslant 2$.

For acyclicity, note that since $B(v\restriction_{C'}, c) \subseteq C'$, we have $B_c \subseteq C'$ at the stage when we compute B_c. Thus if $c \in L_i$ then all arcs with tail c point into $L_{i+1} \cup \cdots \cup L_f$. So (C, A) has a topological order given by a linearisation of the partial order induced by the partition $L_1, \ldots, L_f$ of C. □

Often, it will be convenient to use the *pointed attachment digraph* D^+ of a profile, which is obtained from D by inserting a single arc (arbitrarily directed) between the two sinks in case there are two sinks. Thus, D^+ always has exactly one sink.

We call a subset $F \subseteq A$ of the arcs of a digraph an *arc-function* if every vertex that is not a sink has exactly one outgoing arc in F. The reason we are interested in attachment digraphs and their arc functions is given in the following theorem, which follows from results of Trick (1989).

Theorem 2. *For every profile V, the set $\mathcal{T}(V)$ is in bijection with the set of arc-functions for D^+, the pointed attachment digraph of V. Thus, every tree in $\mathcal{T}(V)$ appears as a subgraph of D^+, once we forget about its orientation.*

Indeed, given an arc-function F for the pointed attachment digraph, we can take the arcs in F, forget about their orientation, and obtain a suitable tree for V.

Corollary 3. *The number of suitable trees in $\mathcal{T}(V)$ is equal to the product of the out-degrees of the non-sink vertices of D^+. Hence we can compute $|\mathcal{T}(V)|$ in polynomial time.*

It turns out that attachment digraphs have a lot of structure beyond the results of Proposition 1. A key property, which will allow us to use essentially greedy algorithms, is what we call circumtransitivity.

Definition. *A dag $D = (C, A)$ is* circumtransitive *if its vertices can be partitioned into a set $C^{\rightarrow}$ of* forced *vertices and a set $C^{\rightrightarrows} = C \setminus C^{\rightarrow}$ of* free *vertices so that*

1. *every forced vertex has out-degree at most one, and all vertices reachable from it are also forced, and*
2. *every free vertex has out-degree at least two, and whenever x and y are free vertices with $xy, yz \in A$, then $xz \in A$.*

Note that in particular the sinks of D are forced. A circumtransitive dag thus consists of an inner part (the *forced part*) that even after forgetting about orientations is acyclic, and an outer part that is transitively attached to the inner part. Further, since every free vertex starts a path to a sink, by transitivity it must have an arc to some forced vertex.

Theorem 4. *Every attachment digraph (C, A) is circumtransitive.*

Proof. We will argue that Definition 3 is satisfied by taking the partition $C^{\rightarrow} = \{c : d^+(c) \leqslant 1\}, C^{\rightrightarrows} = \{c : d^+(c) \geqslant 2\}$.

Forced: Let $x \in C^{\rightarrow}$ be a forced vertex. If $d^+(x) = 0$, there is nothing to prove, so assume that $d^+(x) = 1$, i.e., $xy \in A$ for some $y \in C$. We will show that $d^+(y) \in \{0, 1\}$.

If y is a sink, we are done, so suppose $yz \in A$ for some $z \in C$. Then $xz \notin A$ and so $z \notin B_x$. This means that $z \notin B(v\restriction_{C'}, x)$ for some $v \in V$, where C' is the set of candidates remaining in the while-loop iteration in which x is attached. Note that $y \in C'$ and hence $z \in C'$ as well. Write $\bar{v} = v\restriction_{C'}$. We consider two cases:

(i) $x \neq \bar{v}_{[1]}$. Then $xy \in A$ implies $y \succ_v x$. Consider now the iteration in which y is attached. If y is ranked first in v at that point, then by construction $|B_y| = 1$, so we are done. Otherwise $yz \in A$ implies $z \succ_v y$. But then by transitivity $z \succ_v x$, so $z \in B(\bar{v}, x)$, a contradiction.

(ii) $x = \bar{v}_{[1]}$. Then $xy \in A$ implies that y is ranked second in $\bar{v}$. Hence, in the iteration in which y is attached, it must be the top candidate in v, and therefore $|B_y| = 1$.

Free: Consider vertices $x, y, z \in C$ with $x, y \in C^{\rightrightarrows}$ and $xy, yz \in A$. At x's attaching time, no vote can begin in x since $d^+(x) > 1$. Thus since $xy \in A$ we have $y \succ_v x$ for all $v \in V$. Similarly, since $yz \in A$ and $d^+(y) > 1$ we have $z \succ_v y$ for all $v \in V$. Hence, by transitivity, $z \succ_v x$ for all $v \in V$, so $z \in B_x$ and $xz \in A$. □

It follows that every tree in $\mathcal{T}(V)$ contains the forced part of D^+ as a subtree (after forgetting orientations).

Finally, let us study the free vertices $C^{\rightrightarrows}$ more closely.

Proposition 5. *Every free vertex of $D^+ = (C, A)$ has arcs to at least two forced vertices.*

Proof. We have observed that every free vertex has an arc to at least one forced vertex. Assume for the sake of contradiction that there is a free vertex x that only has a single arc to a forced vertex; let this forced vertex be z. Take a topological ordering of D^+; among all free vertices y such that $yz \in A$

but $yz' \notin A$ for every $z' \in C^{\rightarrow} \setminus \{z\}$, let v be the one that minimises the distance to z in the topological order. Since v is free, it also has an arc to some other vertex w, which is free by the choice of v. Now, w has an arc to some forced vertex z'. By transitivity we have $vz' \in A$, and hence $z' = z$. As w appears between v and z in the topological order, this contradicts minimality of the distance between v and z. □

Proposition 6. *For each free vertex $x \in C^{\rightrightarrows}$ of $D^+ = (C, A)$, the set $\{y \in C^{\rightarrow} : xy \in A\}$ induces a subtree in the undirected version of D^+.*

Proof. Suppose A contains arcs xy and xz where $y, z \in C^{\rightarrow}$. Let P be the unique y–z path in the undirected version of D^+ that is contained in $C^{\rightarrow}$, and let C_P be its vertex set. We will argue that $C_P \subseteq B_x$. Fix a tree $T \in \mathcal{T}(V)$; note that P is a path in T. Pick a vote $v \in V$. Since $|B_x| > 1$, we have $y \succ_v x, z \succ_v x$. Take a top segment of v that includes y and z, but not x. This set induces a subtree of T, and hence contains C_P. Thus, $w \succ_v x$ for each $w \in C_P$. As this holds for each $v \in V$, x can be attached to any vertex of C_P. □

Taken together, Propositions 5 and 6 imply that each free vertex can be attached to two forced vertices that are adjacent to each other in D^+.

4 Recognition Algorithms

Suppose we are given a profile V with $\mathcal{T}(V) \neq \varnothing$ and wish to find trees in $\mathcal{T}(V)$ that satisfy additional desiderata. As argued above, this can be done by computing the attachment digraph and then choosing the arc-function appropriately. Next, we give examples to that effect; the list is not meant to be exhaustive and, in addition to new results, includes already known results for paths, stars, and trees with few leaves.

Theorem 7. *Given a profile V that is single-peaked on a tree, we can find in polynomial time a suitable tree that among the trees in $\mathcal{T}(V)$ has a*

1. *minimum number of leaves,*
2. *minimum number of internal vertices,*
3. *minimum diameter,*
4. *minimum max-degree,*
5. *minimum path-width.*

Further, we can decide in polynomial time whether a given profile is single-peaked on a

6. *line,*
7. *star,*
8. *caterpillar,*
9. *lobster,*
10. *subdivision of a star.*

Proof sketches. Fix a profile V over a candidate set C, $|C| \geqslant 3$, and compute D, the attachment digraph of V, and D^+, the pointed attachment digraph.

1. This has already been done by Yu, Chan, and Elkind (2013). Their algorithm can be phrased in the language of attachment digraphs as follows: first we find a maximum partial arc-function such that no vertex has two incoming arcs, and then we extend it to a full arc-function arbitrarily. The first step can be achieved by matching techniques (as explained by Yu et al.) or by matroid intersection of two partition matroids (Gabow and Tarjan 1984).

2. We only need to decide how to attach free vertices. By Propositions 5 and 6, each $x \in C^{\rightrightarrows}$ can be attached to two forced vertices that are adjacent to each other. Thus, if $|C^{\rightarrow}| > 2$, we can attach x to a forced vertex that is internal in the forced part. Then every leaf of the forced part remains a leaf, and every free vertex becomes a leaf, which is clearly optimal. If $|C^{\rightarrow}| = 2$ (note that $|C^{\rightarrow}| \geqslant 2$ by Proposition 5), we can pick $y \in C^{\rightarrow}$ and attach all free vertices to y.

3. Run the algorithm of part 2; this produces a tree of the same diameter as the forced part if $|C^{\rightarrow}| = 2$ or else of diameter one larger than the forced part, which must be minimal.

4. We show how to find a suitable tree with max-degree at most k if there is one, for each fixed k. By repeatedly calling this algorithm with $k = 2, 3, \ldots, |C| - 1$, we find a tree of minimum max-degree. The algorithm is similar to part 1: we check whether there is an arc-function for D such that no vertex has more than $k - 1$ incoming arcs in the arc-function. Such an arc-function, if it exists, can be found using flow techniques or by matroid intersection of two partition matroids.

5. The path-width of any suitable tree is at least the path-width of the forced part. For trees, we can find a path decomposition of minimum width in linear time (Scheffler 1990). Run this algorithm on the forced part. Then attach free vertices to forced vertices, prioritising forced vertices that appear in a bag that is not of maximum cardinality among the bags in the path decomposition. If such a bag is found, duplicate it and add the free vertex to one of the copies. If some free vertex v is only attachable to forced vertices that only appear in maximum-cardinality bags, we need to fix things up. Take two different adjacent forced vertices w and x to which v is attachable (such vertices are guaranteed to exist). Take the left-most bag in which w and x appear together and duplicate it. Suppose that w does not appear to the left of the left copy. Then replace w by v in that copy and attach v to x. This yields a path-decomposition of width equal to the path-width of the forced part, which is optimal.

6. The fastest algorithms for the line are given by Doignon and Falmagne (1994) and Escoffier, Lang, and Öztürk (2008); they do not rely on the pre-computation of the attachment digraph and are thus faster than anything we can offer. That said, running algorithms for parts 1 or 4 above will return a path if possible, and if not, return something as close to a path as possible (according to the various senses).

7. As observed by Demange (1982) and Yu, Chan, and Elkind (2013), a profile is single-peaked on a star if and only if there exists a candidate that is ranked first or second in every vote. Hence the profiles single-peaked on a star form a regular language (under sensible encodings) and can be quickly recognised. In our framework we can look for a sink in D such that every non-sink vertex points to it.

8. A given profile can only be single-peaked on a caterpillar if the forced part of D^+ is a caterpillar. If so, use the algorithm from part 2 to make every free vertex a leaf. The

result is still a caterpillar. The lobster case (9) is similar.

10. The forced part is either a path or a subdivision of a star $K_{1,s}$ with $s > 2$. In the first case, it suffices to check whether all free vertices attach to a single vertex of this path (which then becomes the center). In the second case, first attach as many free vertices to the center as possible. By Proposition 6, the remaining free vertices can be attached to at most one of the leaves of the forced subdivision of a star. Then for each leaf separately, find a longest path in the sub-dag of free vertices attachable to it, and check that the union of these longest paths contains all the remaining free vertices. □

Applying these algorithms to the example in Figure 1, we see that suitable trees have between three and eight leaves, their diameter at least four, they can have max-degree three and path-width one, and include a caterpillar but not a subdivision of a star.

5 Hardness Results

The algorithms in the proof of Theorem 7 enable us to answer a wide range of questions about the set $\mathcal{T}(V)$. The NP-hardness results in this section, however, show that it is likely that not every such question can be answered efficiently.

Consider the following computational problem.

SINGLE-PEAKED TREE LABELLING
Instance: Profile V over C, unlabelled tree T on $|C|$ vertices
Question: Is there a labelling of the vertices of T with candidates in C s.t. V is single-peaked on that labelled tree?

Theorem 8. *Problem* SINGLE-PEAKED TREE LABELLING *is* NP*-complete even if the input trees are restricted to diameter at most four or to max-degree at most three.*

Proof. The problem is in NP since for a given labelling we can easily check whether it makes the profile single-peaked on T.

For the hardness proof, we reduce from X3C. Given an X3C-instance with objects $x_1, \dots, x_{3m}$ and sets $s_1, \dots, s_n$ we construct a tree T by taking a star $K_{1,n}$ and attaching three fresh leaves to exactly m of the leaves of the star. Then T has diameter four. We construct a profile over the candidate set $\{\star, x_1, \dots, x_{3m}, s_1, \dots, s_n\}$, with one vote for each object and for each set. In the following, all indifferences can be resolved arbitrarily.

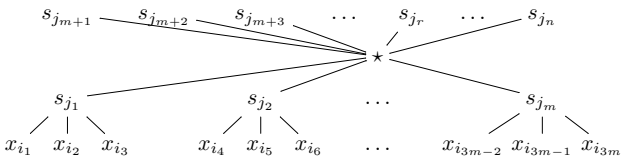
For each object x_i, vote to force x_i to be attached to $\star$ or to a set containing x_i:

$$\star, \{\text{sets containing } x_i\}, x_i, \{\text{other sets}\}, \{\text{other objects}\}.$$

For each set s_j, vote to force an edge from $\star$ to s_j:

$$\star, s_j, \{\text{sets other than } s_j\}, \{\text{objects}\}.$$

If there is a valid partition in the X3C-instance selecting precisely the sets $s_{j_1}, \dots, s_{j_m}$, then we can label T as follows: The center of the star is $\star$. Its n neighbours are the s_js. We assign the s_{j_i}s to the neighbours of $\star$ of degree four. Then we can assign each object to its set s_{j_i} in the X3C-solution. By considering top-initial segments of the votes given above, we see that this makes all votes single-peaked on this tree.



If there is a labelling of T making all the votes single-peaked, then there must be an X3C-solution. To see this, first note that the vertex labelled $\star$ must have degree at least n because of the set votes. There is only one such vertex in T, namely the center, which is thus labelled $\star$. It has exactly n neighbours, which then must all be labelled by some s_j. This leaves us to decide which vertices are labelled by which objects. Since we have already labelled all neighbours of the center, the objects must be attached to s_js. Hence by the constraints of the object votes, the labelling induces an X3C-solution. □

By copying the center vertex and adding some peripheral vertices, we can adjust this reduction so that T's maximum degree is three. Notice that the problem is (trivially) fixed-parameter tractable with parameter $k = |C|$ by just trying all $k!$ possible labellings of the input tree.

We can use a similar reduction to prove a hardness result complementing the easiness results of Theorem 7 (we omit the proof due to space constraints). Recall that a tree is k-regular if every non-leaf vertex has degree k.

Theorem 9. *Given a profile V, it is* NP*-complete to decide whether there exists a positive integer k such that V is single-peaked on a k-regular tree. The problem is also hard for each fixed $k \geqslant 4$.*

6 Application: Committee Selection

We will now demonstrate how to apply one of the recognition algorithms presented in the proof of Theorem 7 to obtain an algorithm for a committee selection problem that is known to be NP-hard for unrestricted preferences. Specifically, the computational problem we consider is winner determination under the Chamberlin–Courant rule with Borda misrepresentation function; our results extend to other misrepresentation functions that satisfy a mild condition. Our algorithm can be used for any profile that is single-peaked on a tree and is efficient for trees that have few internal vertices provided that the target committee size is not too large.

We start by defining the Chamberlin–Courant rule and providing a brief summary of complexity results for it, followed by the description of our algorithm and a proof of correctness.

Chamberlin and Courant (1983) propose a family of rules that take a candidate set C, a profile V over this set and a target committee size k as an input, and output a subset of candidates (committee) of size k. Given a candidate set C, $|C| = m$, every vector $\mathbf{s} = (s_1, \dots, s_m)$ of non-negative integers with $0 = s_1 \leqslant \dots \leqslant s_m$ defines a *positional misrepresentation function* $\mu_{\mathbf{s}} : V \times 2^C \to \mathbb{Z}$ as follows: $\mu_{\mathbf{s}}(v, C') = s_i$ if v ranks her most preferred candidate in C' in position i. The (utilitarian version of the) Chamberlin–Courant rule outputs some committee C' of size k that min-

imises the quantity $\sum_{v \in V} \mu_{\mathbf{s}}(v, C')$ (which we call the s-*score* of C') over all size-k subsets of C. The misrepresentation function associated with the vector $\mathbf{s} = (0, 1, \ldots, m-1)$ is known as the *Borda misrepresentation function*.

Finding a winning committee under the Chamberlin–Courant rule is known to be NP-hard, even for the Borda misrepresentation function (Lu and Boutilier 2011); however, this problem can be solved in polynomial time for an arbitrary misrepresentation function if the input profile is single-peaked (Betzler, Slinko, and Uhlmann 2013) or, more broadly, has bounded single-peaked width (Cornaz, Galand, and Spanjaard 2012), as well as for a large class of misrepresentation functions including the Borda misrepresentation function if the input profile is single-peaked on a star (Yu, Chan, and Elkind 2013). Yu et al. also provide an algorithm for profiles that are single-peaked on a tree, which works for arbitrary misrepresentation functions; its running time is polynomial in $|V|$ and the quantities $|C|^\lambda$ and k^λ, where λ is the number of leaves of a suitable tree (they also explain how to find a suitable tree with the minimum number of leaves).

While the latter algorithm is useful for profiles that are single-peaked on a tree with few *leaves*, we will now present an algorithm that is tailored for profiles that are single-peaked on trees with few *internal vertices*. It is inspired by Yu et al.'s algorithm for preferences that are single-peaked on a star.

Theorem 10. *Given a candidate set C, $|C| = m$, a profile V over C, $|V| = n$, a tree $T \in \mathcal{T}(V)$ with η internal vertices such that V is single-peaked on T, and a target committee size $k \geqslant 1$, we can find a winning committee of size k for (C, V) under the Chamberlin–Courant rule with the Borda misrepresentation function in time* $\mathrm{poly}(n, m, (k+1)^\eta)$.

Proof. Given a candidate $c \in C$, let $f(c)$ be the number of voters in V that rank c first, and let C° be the set of candidates that correspond to the internal vertices of T. For each candidate $c \in C^\circ$, let $\mathrm{ch}(c)$ denote the set of leaf candidates in $C \setminus C^\circ$ that are adjacent to c in T.

Our algorithm proceeds as follows. For each candidate $c \in C^\circ$ it guesses a pair $(b(c), \ell(c))$, where $b(c) \in \{0, 1\}$ and $0 \leqslant \ell(c) \leqslant k$: $b(c)$ indicates whether c itself is in the committee and $\ell(c)$ indicates how many candidates in $\mathrm{ch}(c)$ are in the committee. We require $\sum_{c \in C^\circ} (b(c) + \ell(c)) = k$. Next, it sets $C' = \{c \in C^\circ : b(c) = 1\}$, and then for each $c \in C^\circ$ it orders the candidates in $\mathrm{ch}(c)$ in non-increasing order of $f(c)$ (breaking ties according to a fixed ordering $\rhd$ over C), and adds the first $\ell(c)$ candidates in this order to C'.

Each guess corresponds to a committee of size k. Guessing can be implemented deterministically: consider all options for the collection $\{(b(c), \ell(c))\}_{c \in C^\circ}$ (there are at most $2^\eta \cdot (k+1)^\eta$ possibilities), compute the score of the resulting committee for each option, and output the best one.

It remains to argue that this algorithm finds a committee with the minimum Borda score. To see this, let $\mathcal{S}$ be the set of all size-k committees with the minimum Borda score, and pick a committee S^* from $\arg\max_{C' \in \mathcal{S}} |C' \cap C^\circ|$, breaking ties according to $\rhd$ (note that this means that there is no set $S \in \arg\max_{C' \in \mathcal{S}} |C' \cap C^\circ|$ such that $S^* \setminus S = \{c\}$, $S \setminus S^* = \{c'\}$ and $c' \rhd c$). For each $c \in C^\circ$, let $b^*(c) = 1$ if $c \in S^*$ and $b^*(c) = 0$ otherwise, and let $\ell^*(c) = |\mathrm{ch}(c) \cap S^*|$. Our algorithm will consider the collection $\{(b^*(c), \ell^*(c))\}_{c \in C^\circ}$ at some point, and output a committee S. We will now argue that $S = S^*$.

Indeed, we have $C^\circ \cap S = C^\circ \cap S^*$, so it remains to argue that $\mathrm{ch}(c) \cap S^* = \mathrm{ch}(c) \cap S$ for each $c \in C^\circ$. Suppose for the sake of contradiction that this is not the case, i.e., there exists a $c \in C^\circ$ and a pair of candidates $c', c'' \in \mathrm{ch}(c)$ with $c' \in S \setminus S^*$, $c'' \in S^* \setminus S$. If $c \in S^*$, consider the committee $S' = (S^* \setminus \{c''\}) \cup \{c'\}$. We claim that S' has the same Borda score as S^*. Indeed, the voters who do not rank c' or c'' first prefer c to either of these two candidates, so they are unaffected by the change, the misrepresentation of the $f(c'')$ voters who rank c'' first changes from 0 to 1, the misrepresentation of the $f(c')$ voters who rank c' first changes from 1 to 0, and we have $f(c') \geqslant f(c'')$ by construction of S. As we also have $c' \rhd c''$ by construction of S, this contradicts our choice of S^* from $\arg\max_{C' \in \mathcal{S}} |C' \cap C^\circ|$.

Now, suppose that $c \notin S^*$. Consider the committee $S' = (S^* \setminus \{c''\}) \cup \{c\}$. Again, we claim that S' has the same Borda score as S^*: we increase the misrepresentation of each of the $f(c'')$ voters who rank c'' first by 1 (as all of them rank c second), decrease the misrepresentation of each of the $f(c')$ voters who rank c' first by at least 1 (as all of them rank c second), and do not increase the misrepresentation of any other voter (as all of them prefer c to c''). Thus, the Borda score of S' does not exceed that of S^*, but $|S' \cap C^\circ| > |S^* \cap C^\circ|$, a contradiction with our choice of S^*. □

It is clear from our proof that Theorem 10 holds for every positional misrepresentation function whose score vector satisfies $s_1 = 0$, $s_2 = 1$, $s_3 \geqslant 2$. Observe also that our algorithm is in FPT with respect to the combined parameter (k, η); in contrast, for general preferences computing the Chamberlin–Courant winners is W[2]-hard with respect to k even under the Borda misrepresentation function (Betzler, Slinko, and Uhlmann 2013). Moreover, the algorithm of Yu, Chan, and Elkind (2013) for trees with few leaves is in XP with respect to the number of leaves λ, but is not in FPT with respect to λ or even (k, λ).

7 Conclusions and Future Work

We have designed polynomial-time algorithms for recognizing profiles that are single-peaked on special classes of trees, and demonstrated that such algorithms may be useful for efficient winner determination procedures under the Chamberlin–Courant rule. We believe that results similar to those of Section 6 can be obtained for other types of trees, and, more broadly, for other computational problems that are hard for general preferences, but easy for single-peaked preferences. Moreover, it seems plausible that such results can be extended to profiles that are "almost" single-peaked on a tree, for distance measures such as the ones proposed by Cornaz, Galand, and Spanjaard (2012), Faliszewski, Hemaspaandra, and Hemaspaandra (2014) or Erdélyi, Lackner, and Pfandler (2013). On the other hand, our analysis suggests new measures of closeness to single-peakedness (on the line) that are specialised to profiles that are single-peaked on a tree, namely, being single-peaked on a tree that is "almost" a line. Such measures may turn out to be easier to compute and exploit

than the ones for arbitrary profiles, which tend to be computationally demanding (Erdélyi, Lackner, and Pfandler 2013; Bredereck, Chen, and Woeginger 2013).

From a conceptual perspective, the notion of single-peakedness on a tree has been criticised for having less explanatory power than that of single-peakedness on a line. Indeed, profiles that are single-peaked on a star and ones that are single-peaked on a line have little in common, beyond the guaranteed existence of Condorcet winners, and arguably, a designation that lumps them together is of limited use. The tools developed in our work permit us to identify coherent subdomains of this broad domain, which, in turn, enables us to reason about structurally similar profiles (ones that are single-peaked on 'similar' trees) and their shared properties. We hope that this intuition will lead to new insights about real-life preference domains.

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