Identifying All-vs-Nothing Arguments in Stabiliser Theory

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All-vs-Nothing arguments represent a powerful method for the detection of strong contextuality in quantum theory. The original formulation by Mermin [15] has been recently formalised and generalised using stabiliser quantum mechanics [2, 9]. In the present paper, we take advantage of this framework to introduce a computational technique to identify a large amount of strongly contextual empirical models admitting All-vs-Nothing arguments. We also present new insights into the stabiliser formalism and its connections with logic.

1 Introduction

Since the formulation of classic no-go theorems by Bell [6] and Kochen-Specker [14], contextuality has gained great relevance in the development of quantum information. This key characteristic feature of quantum mechanics represents one of the most valuable and fundamental resources at our disposal to break through the limits of classical computation [13, 20], with various concrete applications in quantum computing (e.g. in device-independent quantum security [12] and quantum speed-up [13]).

Among the most prominent contributors to the the study of contextuality is N. D. Mermin [15], whose 'All-vs-Nothing" argument for the proof of strong contextuality in GHZ states [10, 11, 16] is one of the most elegant and influential demonstrations of non-classicality in quantum theory. Recent work on the mathematical structure of contextuality [3] allowed a powerful formalisation and generalisation of Mermin's original proof to a large class of examples in quantum mechanics using stabiliser theory [2]. In the present paper, we take advantage of this formulation to introduce a computational method capable of producing examples of strongly contextual quantum-realisable models admitting generalised All-vs-Nothing arguments. We also illustrate new theoretical insights into the link between contextuality and logic.

We summarise the main results:

- We show the existence of a Galois correspondence between subgroups of the Pauli n-group $\mathscr{P}_n$ and their stabilisers in the Hilbert space of n-qubits $(\mathbb{C}^2)^{\otimes n}$, for any integer $n \geq 1$. It turns out that this correspondence is induced by a Galois connection between subgroups of $\mathscr{P}_n$ and vector subspaces of $(\mathbb{C}^2)^{\otimes n}$, which allows us to establish a relation with the Galois connection between syntax and semantics in logic [23].

Previous work has already established intriguing connections between logic and the study of contextuality. For instance, a direct link between the structure of quantum contextuality and classic semantic paradoxes is observed in [2], while [4] deals with logical proofs of contextuality. These findings suggest the existence of a more definite theoretical connection between the two domains. The Galois correspondence we illustrate is a first step towards a formal characterisation of said connection.

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- We formulate the "AvN triple conjecture" [1], which states that the presence of an AvN triple in a stabiliser group is not only a sufficient condition for the existence of an All-vs-Nothing argument (as shown in [2]), but it is also necessary.

This conjecture is based on the analysis of the All-vs-Nothing arguments which have appeared in the literature, which can always be seen to come down to exhibiting an AvN triple.

Finally, the central result of the paper

- We present a computational method to identify all AvN triples contained in the Pauli n-group $\mathscr{P}_n$.

Until now, we had a rather limited number of examples of quantum-realisable strongly contextual models giving rise to All-vs-Nothing arguments. The technique we introduce here gives us a large amount of instances of this specific type of models.

The paper is organised as follows. In Section 2, we recall the original All-vs-Nothing argument by Mermin, and generalise it to a class of possibilistic empirical models. Section 3 introduces the stabiliser formalism, the Galois connections and their relations with logic. Finally, in Section 4, we illustrate the method to identify AvN triples.

2 Mermin's original All-vs-Nothing argument

Quantum mechanics provides various examples of strong contextuality. Among the first to observe this highly non-classical behavior was Mermin [15], who showed that the GHZ state [10, 11, 16] is strongly contextual using what he defined as an "All-vs-Nothing argument". We summarise here the main ideas of his proof.

Recall the definition of **Pauli operators**, dichotomic observables corresponding to measuring spin in the x, y, z axis respectively

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\; Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;\; Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They have eigenvalues ± 1 and satisfy the following relations:

$$\begin{gathered} X^2 = Y^2 = Z^2 = I \\ XY = iZ, \;\; YZ = iX, \;\; ZX = iY, \\ YX = -iZ, \;\; ZY = -iX, \;\; XZ = -iY. \end{gathered}$$

The GHZ state is a tripartite state of qubits, defined as

$$\text{GHZ} := \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}.$$

We assume that each party $i = 1,2,3$ can perform a Pauli measurement in $\{X_i, Y_i\}$, obtaining, as a result, an eigenvalue in $\{\pm 1\}$[1]. By adopting the viewpoint of [3], this experiment can be seen as an empirical model whose support is partially described by Table 1.

[1] It is convenient to relabel $+1, -1, \times$ as $0, 1, \oplus$ respectively. The eigenvalues of a joint measurement $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ are the products of eigenvalues at each site, so they are also ± 1. Thus, joint measurements are still dichotomic and only distinguish joint outcomes up to parity.

1	2	3	$---$	$--+$	$-+-$	$-++$	$+--$	$+-+$	$++-$	$+++$
…	…	…	…	…	…	…	…	…	…	…
X_1	X_2	X_3	1	0	0	1	0	1	1	0
X_1	Y_2	Y_3	0	1	1	0	1	0	0	1
Y_1	X_2	Y_3	0	1	1	0	1	0	0	1
Y_1	Y_2	X_3	0	1	1	0	1	0	0	1
…	…	…	…	…	…	…	…	…	…	…

Table 1: A partial table for the GHZ possibilistic model.

By relabeling $+1,-1,\times$ as $0,1,\oplus$ respectively[1], these partial entries of the table can be characterised by the following equations in $\mathbb{Z}_2$

$$\begin{array}{ll} \bar{X}_1 \oplus \bar{X}_2 \oplus \bar{X}_3 = 1 & \qquad \bar{Y}_1 \oplus \bar{X}_2 \oplus \bar{Y}_3 = 0 \\ \bar{X}_1 \oplus \bar{Y}_2 \oplus \bar{Y}_3 = 0 & \qquad \bar{Y}_1 \oplus \bar{Y}_2 \oplus \bar{X}_3 = 0, \end{array}$$

where $\bar{P}_i \in \mathbb{Z}_2$ denotes the outcome of the measurement P_i, for all $P_i \in \{X_i, Y_i\}$. It is straightforward to see that this system is inconsistent. Indeed, if we sum all the equations, we obtain $0 = 1$, as each variable appears twice on the left hand side. This means that we cannot find a global assignment $\{X_1,Y_1,X_2,Y_2,X_3,Y_3\} \to \{0,1\}$ consistent with the model, showing that the GHZ state is strongly contextual.

2.1 General setting

The description of empirical models as **generalised probability tables** provided by [3] allows us to generalise Mermin's argument to a larger class of examples.

Let X be a finite set of measurement labels. A **measurement cover** is an antichain $\mathcal{M} \subseteq \mathcal{P}(X)$ that satisfies $\bigcup_{C\in\mathcal{M}} C = X$. This family contains the **measurement contexts**, i.e. the maximal sets of measurements that can be jointly performed. In this case, we assume that all the measurement are dichotomic and produce outcomes in $\mathbb{Z}_2$. This **measurement scenario** $\langle X, \mathcal{M}, \mathbb{Z}_2\rangle$ can be represented as an empty table featuring one row for each measurement context $C \in \mathcal{M}$ and one column for each possible joint outcome of measurements. An **empirical model** over the scenario $\langle X, \mathcal{M}, \mathbb{Z}_2\rangle$ is a family $\{e_C\}_{C\in\mathcal{M}}$ of probability distributions over each row of the table that satisfy the **compatibility condition** $e_C \mid_{C\cap C'} = e_{C'} \mid_{C\cap C'}$ for all $C, C' \in \mathcal{M}$, which is equivalent to no-signalling [3][2].

To an empirical model $e := \{e_C\}_{C\in\mathcal{M}}$ we can associate an **XOR theory** $\mathbb{T}_\oplus(e)$ in the following way. For each context $C \in \mathcal{M}$, $\mathbb{T}_\oplus(e)$ will have the assertion

$$\bigoplus_{x\in C} \bar{x} = 0,$$

(where $\bar{x} \in \mathbb{Z}_2$ is the outcome of the measurement x for all $x \in C$) whenever the support of e_C only contains joint outcomes of even parity (i.e. with an even number of 1s) and

$$\bigoplus_{x\in C} \bar{x} = 1$$

[2]Refer to [3] for the definition of the restriction of a probability distribution.

whenever it only contains joint outcomes of odd parity (i.e. with an odd number of 1s). We say that the model e is **AvN** if the theory $\mathbb{T}_{\oplus}(e)$ is inconsistent. Since an inconsistent theory implies the impossibility of defining a global assignment $X \to \mathbb{Z}_2$, we have the following result.

Proposition 2.1. *If an empirical model e is AvN, then it is strongly contextual.*

As an example, we consider the Popescu-Rohrlich (PR) Box model [18] given in Table 2.

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

Table 2: the PR-box model

The XOR theory of the PR box model consists of the following 4 equations

$$\begin{aligned} \bar{a}_1 \oplus \bar{b}_1 &= 0 & \qquad \bar{a}_2 \oplus \bar{b}_1 &= 0 \\ \bar{a}_1 \oplus \bar{b}_2 &= 0 & \qquad \bar{a}_2 \oplus \bar{b}_2 &= 1, \end{aligned}$$

which lead to $0 = 1$ when summed, showing the inconsistency of the theory and the strong contextuality of the model.

3 The stabiliser world

Stabiliser quantum mechanics [17, 8] is the natural setting for general All-vs-Nothing arguments, and allows us to see how AvN models can arise from quantum theory. In this section, we recall the main definitions and introduce a Galois correspondence between subgroups of the Pauli n-group $\mathscr{P}_n$ and their stabilisers.

Let $n \geq 1$ be an integer. The **Pauli** n**-group** $\mathscr{P}_n$ is the group whose elements are n-tuples $(P_i)_{i=1}^n$ of Pauli operators (i.e. $P_i \in \{X_i, Y_i, Z_i, I_i\}$), with global phase contained in $\{\pm 1, \pm i\}$. The multiplication is componentwise matrix multiplication and the unit is $(I_i)_{i=1}^n$. The group $\mathscr{P}_n$ acts on the Hilbert space of n-qubits $H_n := (\mathbb{C}^2)^{\otimes n}$ via the action

$$(P_i)_{i=1}^n \bullet \otimes_{i=1}^n |\psi_i\rangle := \otimes_{i=1}^n P_i \cdot |\psi_i\rangle \tag{1}$$

Given a subgroup $S \leq \mathscr{P}_n$, the **stabiliser** of S is the vector subspace

$$V_S := \{|\psi\rangle \in H_n \mid P \bullet |\psi\rangle = |\psi\rangle \ \forall P \in S\} \subseteq H_n.$$

Note that the subgroups of $\mathscr{P}_n$ which stabilise non-trivial subspaces must be commutative, and only contain elements with global phases ± 1.

3.1 Galois connections and relations with logic

We present a Galois correspondence between subgroups of the Pauli n-group and their stabilisers, which will enable us to establish a connection with logic.

Given two partially ordered sets A and B, an **(antitone) Galois connection** between A and B is a pair $\langle f, g\rangle$ of order-reversing maps $f : A \to B, g : B \to A$ such that $a \le g(f(a))$ for all $a \in A$, and $b \le f(g(b))$ for all $b \in B$. Such a connection is called an **(antitone) Galois correspondence** if f and g are inverses of each other.

Lemma 3.1. *A Galois connection $\langle f, g\rangle$ between A and B induces a Galois correspondence between* $\mathrm{im}(g)$ *and* $\mathrm{im}(f)$ *given by* $\langle f\,|_{\mathrm{im}(g)}, g\,|_{\mathrm{im}(f)}\rangle$.

Proof. Given an $a := g(b)$ for some $b \in B$, we have $a \le (g(f(a)))$ and $g(f(a)) = g(f(g(b))) \le g(b) = a$. Thus $a = g(f(a))$ for all $a \in \mathrm{im}(g)$ and, similarly, $b = f(g(b))$ for all $b \in \mathrm{im}(f)$. □

Let $(\mathscr{SG}(\mathscr{P}_n), \subseteq)$ denote the poset of subgroups of the Pauli n-group, ordered by inclusion. The bijection

$$F : (\mathscr{SG}(\mathscr{P}_n), \subseteq) \to (\{V_S \mid S \in \mathscr{SG}(\mathscr{P}_n)\}, \subseteq) :: S \mapsto V_S \tag{2}$$

together with its inverse $V_S \mapsto S$, forms an antitone Galois correspondence. Indeed, for all $S, T \in \mathscr{SG}(\mathscr{P}_n)$,

$$S \subseteq T \Rightarrow V_T \subseteq V_S.$$

Note that this correspondence is tight: a rank k subgroup determines a 2^{n-k} dimensional subspace [8]. Moreover, we can show that it is induced by a Galois connection between $(\mathscr{SG}(\mathscr{P}_n), \subseteq)$ and the poset of all vector subspaces of H_n, which we will denote by $(\mathscr{SS}(H_n), \subseteq)$. The easiest way to see this is to define a relation $R \subseteq \mathscr{P}_n \times (\mathbb{C}^2)^{\otimes n}$ by

$$gRv \Leftrightarrow g \bullet v = v,$$

which induces a Galois connection between the powersets $\mathscr{P}(\mathscr{P}_n)$ and $\mathscr{P}((\mathbb{C}^2)^{\otimes n})$, ordered by inclusion:

$$\begin{array}{ccc} \mathscr{P}(\mathscr{P}_n) & \longleftrightarrow & \mathscr{P}((\mathbb{C}^2)^{\otimes n}) \\ S & \longmapsto & S^\perp := \{v \mid \forall g(g \in S \Rightarrow gRv)\} \\ V^\perp := \{g \mid \forall v(v \in V \Rightarrow gRv)\} & \longleftarrow\!\shortmid & V. \end{array} \tag{3}$$

Note that **closed sets** $S^{\perp\perp}$ and $V^{\perp\perp}$ are subgroups of $\mathscr{P}_n$ and vector subspaces of $(\mathbb{C}^2)^{\otimes n}$ respectively. Therefore, by restricting (3) to closed sets, we obtain a Galois correspondence $\mathscr{SG}(\mathscr{P}_n) \leftrightarrow \mathscr{SS}(H_n)$ that induces the correspondence (2), since $S^\perp = V_S$ by definition. The following proposition is a more explicit version of the same result, which allows us to understand better the map $\mathscr{SS}(H_n) \to \mathscr{SG}(\mathscr{P}_n)$.

Proposition 3.2. *Let*

$$G : (\mathscr{SS}(H_n), \subseteq) \to (\mathscr{SG}(\mathscr{P}_n), \subseteq) :: V \mapsto \bigcap_{|\psi\rangle \in V} (\mathscr{P}_n)_{|\psi\rangle},$$

where $(\mathscr{P}_n)_{|\psi\rangle} := \{A \in \mathscr{P}_n \mid A \bullet |\psi\rangle = |\psi\rangle\}$ *denotes the isotropy group of* $|\psi\rangle$. *Then* $\langle F, G\rangle$ *is an antitone Galois connection.*[3] *Moreover, this connection induces the Galois correspondence* (2).

Proof. It is straightforward to show that G is order-reversing. Moreover, given $S \in \mathscr{SG}(\mathscr{P}_n)$, we have $G(F(S)) = \bigcap_{|\psi\rangle \in V_S} (\mathscr{P}_n)_{|\psi\rangle}$, and

$$P \in S \Rightarrow P \bullet |\psi\rangle = |\psi\rangle\ \forall |\psi\rangle \in V_S \Rightarrow P \in (\mathscr{P}_n)_{|\psi\rangle}\ \forall |\psi\rangle \in V_S \Rightarrow P \in \bigcap_{|\psi\rangle \in V_S} (\mathscr{P}_n)_{|\psi\rangle},$$

[3] Here, the codomain of F is extended to $\mathscr{SS}(H_n)$.

thus $S \subseteq G(F(S))$. On the other hand, given a subspace $W \subseteq H_n$,

$$F(G(W)) = F\left(\bigcap_{|\psi\rangle \in W} (\mathscr{P}_n)_{|\psi\rangle}\right) = V_{\left(\bigcap_{|\psi\rangle \in W} (\mathscr{P}_n)_{|\psi\rangle}\right)} = \left\{ |\psi\rangle \in H_n \;\middle|\; P \bullet |\psi\rangle = |\psi\rangle,\ \forall P \in \bigcap_{|\psi\rangle \in W} (\mathscr{P}_n)_{|\psi\rangle} \right\}$$
$$= \left\{ |\psi\rangle \in H_n \;\middle|\; P \bullet |\psi\rangle = |\psi\rangle,\ \forall P \text{ s.t. } \big(P \bullet |\psi\rangle = |\psi\rangle,\ \forall |\psi\rangle \in W\big) \right\},$$

and clearly W is a subset of this set. Thus F and G form indeed a Galois connection. By Lemma 3.1, there is a Galois correspondence $\text{im}(F) \leftrightarrow \text{im}(G)$. We already established a Galois correspondence $\mathscr{SG}(\mathscr{P}_n) \leftrightarrow \text{im}(F)$, thus, in particular, $\text{im}(G)$ is in 1-to-1 correspondence with $\mathscr{SG}(\mathscr{P}_n)$. Since $\mathscr{SG}(\mathscr{P}_n)$ is a finite set, and $\text{im}(G) \subseteq \mathscr{SG}(\mathscr{P}_n)$ we conclude that $\text{im}(g) = \mathscr{SG}(\mathscr{P}_n)$, i.e.

$$\left\{ \bigcap_{|\psi\rangle \in V} (\mathscr{P}_n)_{|\psi\rangle} \;\middle|\; V \in \mathscr{SS}(H_n) \right\} = \mathscr{SG}(\mathscr{P}_n)$$

This means that the Galois connection $\langle F, G\rangle$ induces the correspondence (2). It also shows that every subgroup of $\mathscr{P}_n$ can be written as an intersection of isotropy groups for the action (1). $\square$

This result suggests an intriguing relation with the Galois connection between syntax and semantics in logic [23]

$$\begin{array}{ccc} \mathscr{L}-\textbf{Theories} & \longleftrightarrow & \mathscr{P}(\mathscr{L}-\textbf{Structures}) \\ \Gamma & \longmapsto & \{\mathscr{M} \mid \forall \varphi(\varphi \in \Gamma \to \mathscr{M} \models \varphi)\} \\ \{\varphi \mid \forall \mathscr{M}(\mathscr{M} \in M \to \mathscr{M} \models \varphi)\} & \longleftarrow\!\shortmid & M, \end{array} \tag{4}$$

where $\mathscr{L}$ is a formal language. We investigate this aspect in the following section.

3.2 Stabiliser subgroups induce XOR theories

We formalise the connection with logic by showing that stabiliser subgroups give rise to XOR theories [2]. Given an observable P and a state v we have

$$\langle P\rangle_v := \langle v|P|v\rangle = 1 \Longleftrightarrow P|v\rangle = |v\rangle,$$

i.e. P stabilises v if and only if the expected value is 1. This means that if P is a dichotomic observable (e.g. $P \in \mathscr{P}_1$) and v is a state stabilised by P, the empirical model obtained by measuring P on v must contain only outcomes of even parity, while if $-P$ stabilises v, it must contain only outcomes of odd parity. Suppose $P = (P_i)_{i=1}^n \in \mathscr{P}_n$ and $v \in H_n$ a state stabilised by P. If the global phase of P is $+1$, we have the formula (see footnote 1)

$$\varphi_P := \left(\bigoplus_{i=1}^n \bar{P}_i = 0\right),$$

On the other hand, if the global phase of P is -1, we have the formula

$$\varphi_P := \left(\bigoplus_{i=1}^n \bar{P}_i = 1\right).$$

Therefore, to any subgroup $S \leq \mathscr{P}_n$ we can associate an XOR theory

$$\mathbb{T}_\oplus(S) := \{\varphi_P \mid P \in S\}.$$

Definition 3.3. Let S be a subgroup of $\mathscr{P}_n$. We say that S is **AvN** (AvN(S)) if $\mathbb{T}_\oplus(S)$ is inconsistent.

Given an AvN subgroup $S \leq \mathscr{P}_n$ and any state $|\psi\rangle \in V_S$, the n-partite empirical model realised by $|\psi\rangle$ under the Pauli measurements described by S is strongly contextual. Indeed, the inconsistency of $\mathbb{T}_\oplus(S)$ implies the impossibility of finding a global assignment compatible with the support of the empirical model:

Proposition 3.4. AvN *subgroups of* $\mathscr{P}_n$ *give rise to strongly contextual empirical models admitting All-vs-Nothing arguments.*

Now, let $\oplus$**-Th** denote the set of all XOR theories. The map

$$\mathbb{T}_\oplus : (\mathscr{SG}(\mathscr{P}_n), \subseteq) \to (\oplus\textbf{-Th}, \subseteq) :: S \mapsto \mathbb{T}_\oplus(S),$$

is order-preserving, and allows us to establish a link between the Galois connection $\mathscr{SG}(\mathscr{P}_n) \leftrightarrow \mathscr{SS}(H_n)$ and the one described in (4). In particular, we have the following commutative diagram

$$\begin{tikzcd}
\mathscr{SG}(\mathscr{P}_n) \arrow[r, bend left, "F"] \arrow[d, "\mathbb{T}_\oplus"'] & \mathscr{SS}(H_n) \arrow[l, bend left, "G"] \arrow[d, "\mathbb{M}_\oplus"] \\
\oplus\textbf{-Th} \arrow[r, bend left] & \mathscr{P}(\oplus\textbf{-Str}) \arrow[l, bend left]
\end{tikzcd}$$

where $\oplus$**-Str** is the set of XOR-structures, and the order preserving function $\mathbb{M}_\oplus$ maps a subspace V of H_n to the set

$$\mathbb{M}_\oplus := \left\{ \mathscr{M} \;\middle|\; \forall \varphi \left(\varphi \in \mathbb{T}_\oplus \left(\bigcap_{|\psi\rangle \in V} (\mathscr{P}_n)_{|\psi\rangle} \right) \to \mathscr{M} \models \varphi \right) \right\}.$$

3.3 AvN triples

Since AvN subgroups give rise to strongly contextual empirical models, we are naturally interested in characterising this property. In [2], this problem is addressed by introducing the notion of AvN triple. An **AvN triple** in $\mathscr{P}_n$ is a triple $\langle e, f, g\rangle$ (the order is important) with global phases $+1$, which pairwise commute, and which satisfy the following conditions:

1. For each $i = 1, \ldots, n$, at least two of e_i, f_i, g_i are equal.
2. The number of i such that $e_i = g_i \neq f_i$, all distinct from I, is odd.

A key result from [2] is that AvN triples provide a sufficient condition for All-vs-Nothing proofs of strong contextuality.

Theorem 3.5 (4.1 of [2])**.** *Any subgroup S of $\mathscr{P}_n$ generated by an AvN triple is AvN.*

Proof. Let $S \subseteq \mathscr{P}_n$ be generated by an AvN triple $\langle e, f, g\rangle$. Since e, f, g have global phase $+1$, $\mathbb{T}_\oplus(S)$ contains

$$\phi_e := \left(\bigoplus_{i=1}^n \bar{e}_i = 0 \right); \quad \phi_f := \left(\bigoplus_{i=1}^n \bar{f}_i = 0 \right); \quad \phi_g := \left(\bigoplus_{i=1}^n \bar{g}_i = 0 \right).$$

Condition 1 implies that, for all $1 \leq i \leq n$,

$$e_i \cdot f_i \cdot g_i = \begin{cases} e_i & \text{if } f_i = g_i, \\ g_i & \text{if } e_i = f_i, \\ -f_i & \text{if } e_i = g_i \neq f_i. \end{cases} \tag{5}$$

Therefore, the global phase of $e \cdot f \cdot g \in S$ is given by $(-1)^{|\{i : e_i = g_i \neq f_i\}|}$, which is equal to -1 by Condition 2. Hence $\mathbb{T}_{\oplus}(S)$ also contains

$$\phi_{efg} := \left(\bigoplus_{i=1}^{n} \overline{(efg)}_i = 1 \right)$$

Because of (5), we have

$$\overline{(efg)}_i = \overline{e_i f_i g_i} = \begin{cases} \bar{e}_i & \text{if } f_i = g_i, \\ \bar{g}_i & \text{if } e_i = f_i, \\ \bar{f}_i & \text{if } e_i = g_i \neq f_i, \end{cases}$$

since any phase at the i-th component becomes part of the global phase of egf. Therefore, if we add the i-th column of the system of equations composed by ϕ_e, ϕ_f, ϕ_g and ϕ_{efg} we always obtain 0. Indeed,

$$\begin{cases} \bar{e}_i \oplus \bar{f}_i \oplus \bar{g}_i \oplus \bar{e}_i = 2\bar{e}_i \oplus 2\bar{f}_i = 0 & \text{if } f_i = g_i, \\ \bar{e}_i \oplus \bar{f}_i \oplus \bar{g}_i \oplus \bar{g}_i = 2\bar{e}_i \oplus 2\bar{g}_i = 0 & \text{if } e_i = f_i, \\ \bar{e}_i \oplus \bar{f}_i \oplus \bar{g}_i \oplus \bar{f}_i = 2\bar{e}_i \oplus 2\bar{f}_i = 0 & \text{if } e_i = g_i \neq f_i, \end{cases}$$

Hence, if we sum all the equations of the system $\{\phi_e, \phi_f, \phi_g, \phi_{efg}\}$ we obtain 0 on the left hand side and, trivially, 1 on the right hand side. This shows the inconsistency of $\mathbb{T}_{\oplus}$.

□

Remarkably, any All-vs-Nothing argument which has appeared in the literature can be seen to come down to exhibiting AvN triples. For instance, Mermin's argument summarised in Section 2, is essentially based on the AvN triple

$$\langle (X_1, Y_2, Y_3), (Y_1, X_2, Y_3), (Y_1, Y_2, X_3) \rangle.$$

Another interesting example is the one of cluster states, a fundamental resource in measurement-based quantum computation [21, 19, 22]. The 4-qubit 1-dimensional cluster state is defined as the state stabilised by the subgroup S of $\mathscr{P}_4$ generated by

$$h := (X_1, Z_2, I_3, I_4), \;\; k := (Z_1, X_2, Z_3, I_4), \;\; l := (I_1, Z_2, X_3, Z_4), \;\; m := (I_1, I_2, Z_3, X_4).$$

The subgroup S contains the following AvN triple

$$\langle h \cdot l = (X_1, I_2, X_3, Z_4), \;\; k \cdot l = (Z_1, Y_2, Y_3, Z_4), \;\; h \cdot l \cdot m = (X_1, I_2, Y_3, Y_4) \rangle,$$

which can be used to prove the strong contextuality of the empirical model realised by applying measurements in S to the cluster states. Since AvN triples seem to be a key element of All-vs-Nothing arguments in quantum mechanics, we advance the following hypothesis:

Conjecture 3.6 (AvN Triple Conjecture [1])**.** *The presence of an AvN triple in a stabiliser subgroup S is a* ***necessary*** *as well as sufficient condition for the existence of an All-vs-Nothing proof of strong contextuality.*

4 Finding AvN triples

We devote this last section to the presentation of a computational method to identify all the AvN triples contained in $\mathscr{P}_n$. This technique allows us to obtain a large amount of quantum-realisable empirical models featuring All-vs-Nothing proofs of strong contextuality (until now, we only had a rather limited number of examples from the literature).

4.1 Check vector representation of an AvN triple

Check vectors [17] represent a useful way to represent elements of $\mathscr{P}_n$ in a more computation-friendly way. Given an element $P := (P_i)_{i=1}^n \in \mathscr{P}_n$, its check vector $r(P)$ is a $2n$-vector

$$r(P) = (x_1, x_2, \dots, x_n, z_1, z_2, \dots, z_n) \in \mathbb{Z}_2^{2n}$$

whose entries are defined as follows

$$(x_i, z_i) = \begin{cases} (0,0) & \text{if } P_i = I \\ (1,0) & \text{if } P_i = X \\ (1,1) & \text{if } P_i = Y \\ (0,1) & \text{if } P_i = Z. \end{cases}$$

Every check vector $r(P)$ completely determines P up to phase (i.e. $r(P) = r(\alpha P)$ for all $\alpha \in \{\pm 1, \pm i\}$). This representation can be used to characterise the conditions of an AvN triple in a computable way. We illustrate this procedure in the following paragraphs.

For a finitely generated subgroup $S := \langle g_1, \dots, g_l \rangle$ of $\mathscr{P}_n$, we say that its generators $g_1, \dots, g_l$ are independent if removing any generator g_i makes the group generated smaller. Therefore, it makes sense to impose the condition that AvN triples should indeed be constituted of independent elements of $\mathscr{P}_n$. This is also important because of the tightness of the Galois correspondence $S \leftrightarrow V_S$ of Section 3.1: if the elements e, f, g of an AvN triple have linearly independent check vectors, they generate a subgroup S such that V_S has dimension 2^{n-3} [8]. The following Lemma translates this notion in terms of check vectors.

Lemma 4.1. *Let $S = \langle g_1, \dots, g_l \rangle$ be a finitely generated subgroup of $\mathscr{P}_n$ such that V_S is not trivial. Then the generators $g_1, \dots, g_l$ are independent if and only if the matrix $C(S) \in M_{l \times 2n}(\mathbb{Z}_2)$, whose rows are constituted by the check vectors of $g_1, \dots g_l$, has rank l.*

Proof. Consider a general element $e \in \mathscr{P}_n$ and its check vector $r(e) = (x_1, \dots, x_n, z_1, \dots z_n)$. Notice that if we ignore phase, we can write $e_i = X^{x_i} Z^{z_i}$. Thus, since $X^2 = Z^2 = I$ and X, Z commute up to a phase factor, we can see that for each $e, f \in \mathscr{P}_n$ we have $r(ef) = r(e) \oplus r(f)$ (i.e. addition of check vectors corresponds to multiplication in S, up to phase). Suppose the rows of $C(S)$ are linearly dependent, then there exist $\{\lambda_1, \dots \lambda_l\}$ with at least one $\lambda_j \neq 0$, such that $\bigoplus_i \lambda_i r(g_i) = 0$. By the discussion above, this is true if and only if $\prod_i g_i^{\lambda_i} = I$ up to a phase still to determine. In [17], it is proven that $V_S \neq 0 \Leftrightarrow -I \notin S$. Thus by hypothesis we know that $-I \notin S$ and hence the phase must be 1. Thus the last condition corresponds to $g_j = g_j^{-1} = \prod_{i \neq j} g_i^{\lambda_i}$ and therefore $g_1, \dots, g_l$ are not independent. $\square$

Another condition we need to check is whether elements of an AvN triple pairwise commute. We can characterise this property in terms of check vectors as follows:

Lemma 4.2. *Let $g, g' \in \mathscr{P}_n$ with global phase* 1. *Then*

$$gg' = g'g \Leftrightarrow r(g)\Lambda r(g')^T = 0,$$

where $\Lambda = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Proof. For each i, g_i either commutes or anticommutes with g'_i. Let $m := |\{i : g_i g'_i = -g'_i g_i\}|$, then $gg' = (-1)^m g'g$, which implies that g and g' commute if and only if m is even. One can see that g_i and g'_i anticommute if and only if $x_i z'_i \oplus z_i x'_i = 1$, where $r(g) = (x_1, \dots, x_n, z_1, \dots, z_n)$ and $r(g') = (x'_1, \dots, x'_n, z'_1, \dots, z'_n)$. Therefore,

$$gg' = g'g \Leftrightarrow m \text{ is even } \Leftrightarrow \bigoplus_{i=1}^{n} x_i z'_i \oplus z_i x'_i = 0 \Leftrightarrow r(g)\Lambda r(g)^T = 0.$$

□

Finally, we need to characterise conditions 1 and 2. The former can be re-expressed as follows: for all $j = 1, \dots n$, the columns $C(S)_j$ and $C(S)_{n+j}$ must be equal for at least two row indices. More explicitly,

$$\forall 1 \leq j \leq n,\ \exists i, k \in \{1,2,3\} \text{ s.t. } \begin{cases} C(S)_{i,j} = C(S)_{k,j} \\ C(S)_{i,n+j} = C(S)_{k,n+j} \end{cases} \tag{6}$$

On the other hand, Condition 2 is equivalent to the following statement.

For all $j = 1, \dots, n$, the cardinality of

$$\begin{gathered} \{i \in \{1,2,3\} \,|(C_{1,j} = C_{3,j}) \wedge (C_{1,n+j} = C_{3,n+j}) \wedge ((C_{1,j} \neq C_{2,j}) \vee (C_{1,n+j} = C_{2,n+j})) \\ ((C_{1,j} \neq 0 \vee C_{1,n+j} \neq 0) \wedge (C_{2,j} \neq 0 \vee C_{2,n+j} \neq 0))\} \end{gathered} \tag{7}$$

is odd.

Now that we have characterised the conditions for an AvN triples in terms of check vectors, we can see that, in order to find all AvN triples in $\mathscr{P}_n$ we must solve the following problem:

$$\begin{array}{ll} \textbf{Find all} & M \in M_{3\times 2n}(\mathbb{Z}_2) \\ \textbf{such that:} & \text{rank}(M) = 3 \\ & M_i \cdot \Lambda \cdot M_i^T = 0\ \forall i \\ & M \text{ verifies (6)} \\ & M \text{ verifies (7)}, \end{array} \tag{8}$$

which is easily programmable.

An implementation of this method using Mathematica [24] can be found in [7], where we present the algorithm and the resulting list of AvN triples in $\mathscr{P}_n$, for $n = 3$ and $n = 4$. We obtain a considerable amount of new examples of strongly contextual models admitting All-vs-Nothing proofs of contextuality: 1296 for $n = 3$ and 114048 for $n = 4$ (although we should divide these number by two since the algorithm identifies AvN triples $\langle e, f, g\rangle$ and $\langle g, f, e\rangle$ as two separate entitites). To give an idea of the magnitude of these numbers, we can observe that the total number of triples composed of random elements of $\mathscr{P}_n$ is $2^{3 \cdot 2n}$. Thus, the precentage of AvN triples among random triples in $\mathscr{P}_3$ is $\sim 0.50\%$, while for $n = 4$ it is $\sim 0.68\%$.

Conclusions

The recent formalisation and generalisation of All-vs-Nothing arguments in stabiliser quantum mechanics [2] has allowed us to study their properties from a purely mathematical standpoint. The main result we have obtained is a computational method capable of producing examples of quantum-realisable strongly contextual empirical models admitting All-vs-Nothing arguments. This technique represents a major contribution to the rather limited amount of concrete examples in the literature. The new models found could potentially find relevant applications in quantum information and computation, as well as benefit the ongoing theoretical study of strong contextuality as a key feature of quantum mechanics [3, 2, 5]. The algorithm we have presented is fundamentally based on the notion of AvN triple, which gives rise to a sufficient condition for the existence of All-vs-Nothing arguments. In the course of our work, we have provided evidence to support the hypothesis that AvN triples actually characterise All-vs-Nothing proofs of contextuality. If this conjecture holds, the models we have obtained using the computational technique described in the paper are actually all the possible models admitting All-vs-Nothing arguments.

The abstract formulation of generalised AvN arguments has also allowed us to introduce new insights into the connections between logic and the study of contextuality. Recent work on logical Bell's inequalities [4] and the relation between contextuality and semantic paradoxes [2] suggests the existence of a definite interaction between these two domains. In this work, we have taken a first step towards a formal characterisation of this link by showing the existence of a Galois connection between subgroups of the Pauli n-group and subspaces of the Hilbert space of n-qubits, which can be seen as the stabiliser-theoretic counterpart of the Galois connection between syntax and semantics in logic.

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