Categorical perspectives on entropy

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Axiomatisation of Entropy

What is the minimal mathematical setup that gives rise to a notion of Entropy?¹

- Thermodynamical states=Objects
- Adiabatic accessibility⇒Preorder
- Compound system⇒Monoidal product ⊕ (strict, symmetry strict).
- “Scaling” by \( \lambda \in \mathbb{R}^+ \) ⇒ Monoidal endofunctors (symmetric, strict); composition is real multiplication.
- “Splitting and recombination” ⇒ \( X \cong (1 - \lambda)X \oplus \lambda X \) for all \( 0 < \lambda < 1 \).
- “Stability” ⇒ If \( X \oplus \epsilon_i Z_0 \to Y \oplus \epsilon_i Z_1 \) for a sequence \( \epsilon_i \to 0 \) then \( X \to Y \).

Goal: Adiabatic processes are different morphisms in the category; still minimal in terms of continuity.

Generalise in the following senses:

- Weaken monoidality and symmetry.
- Weaken scaling endofunctors \( \lambda \).
- \( \lambda \in R \) for a “well-behaved” topological semiring \( R \).
- Allow more than one morphism in each homset.
  - The splitting and recombination isomorphism is natural.
  - Stability is a function and has the properties of convergence. It is a symmetric monoidal functor that commutes with scaling.

**Application of weakening the endofunctors and changing the semiring:** Kronecker product.

Functors between such categories: strict monoidal and commute with stability functor.
Topological sequential weak semivector space

- Topological category.
- Topologies are sequential.
- Weaker version of a 2-semivector space.
- The weak semivector space operations are continuous.

Functors between such categories: linear and continuous.
Lieb-Yngvason categories over semifields are precisely the “minimally-convergent” topological sequential weak semivector spaces.

\[
\text{LY} \xleftarrow{\text{Prim}} \perp \xrightarrow{\text{Seq}} \text{TSWSVS} \xleftarrow{\text{Prim} \circ \text{Seq (idempotent monad)}}
\]
A Lieb-Yngvason category over a ring is tortile.

Hence \textit{traced monoidal}.

Full submonoidal categories of tortile categories are also traced monoidal.\(^2\)

This seems to be the case for Lieb-Yngvason categories over \(\mathbb{R} \geq 0\).

In a Lieb-Yngvason category over \(\mathbb{R} \geq 0\) there exists a Cancellation Law (\textit{Lieb & Yngvason}), whose type matches the trace operation.

\textit{For which stability functions does the Lieb-Yngvason Cancellation Law coincide with the trace?}

\(^2\)André Joyal, Ross Street, Dominic Verity, “Traced monoidal categories”