Categorical perspectives on entropy

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Axiomatisation of Entropy

What is the minimal mathematical setup that gives rise to a notion of Entropy?[1]

- Thermodynamical states=Objects
- Adiabatic accessibility⇒Preorder
- Compound system⇒Monoidal product $\oplus$ (strict, symmetry strict).
- "Scaling" by $\lambda \in \mathbb{R}^+$ ⇒Monoidal endofunctors (symmetric, strict); composition is real multiplication.
- "Splitting and recombination"⇒ $X \cong (1-\lambda)X \oplus \lambda X$ for all $0 < \lambda < 1$.
- "Stability"⇒If $X \oplus \epsilon_i Z_0 \to Y \oplus \epsilon_i Z_1$ for a sequence $\epsilon_i \to 0$ then $X \to Y$.

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[1] Elliott H. Lieb and Jakob Yngvason, "The mathematical structure of the Second Law of Thermodynamics"

Generalisation

Goal: Adiabatic processes are different morphisms in the category; still minimal in terms of continuity.
Generalise in the following senses:

- Weaken monoidality and symmetry.
- Weaken scaling endofunctors λ.
- $\lambda \in R$ for a "well-behaved" topological semiring R.
- Allow more than one morphism in each homset.
    - The splitting and recombination isomorphism is natural.
    - Stability is a **function** and has the properties of convergence. It is a symmetric monoidal functor that commutes with scaling.

Application of weakening the endofunctors and changing the semiring: Kronecker product.
Functors between such categories: strict monoidal and commute with stability functor.

Topological sequential weak semivector space

- Topological category.
- Topologies are sequential.
- Weaker version of a 2-semivector space.
- The weak semivector space operations are continuous.

Functors between such categories: linear and continuous.

Characterisation

Lieb-Yngvason categories over semifields are precisely the "minimally-convergent" topological sequential weak semivector spaces.

$$\begin{tikzcd}
\mathrm{LY} \arrow[r, bend left, "\mathrm{Prim}"] \arrow[r, phantom, "\perp"] & \mathrm{TSWSVS} \arrow[l, bend left, "\mathrm{Seq}"] \arrow[loop right, "\mathrm{Prim}\circ\mathrm{Seq}\ \text{(idempotent monad)}"]
\end{tikzcd}$$

Traced monoidal categories

A Lieb-Yngvason category over a ring is tortile.

Hence *traced monoidal*.

Full submonoidal categories of tortile categories are also traced monoidal.[2]

This seems to be the case for Lieb-Yngvason categories over $\mathbb{R}^{\geq 0}$.

In a Lieb-Yngvason category over $\mathbb{R}^{\geq 0}$ there exists a Cancellation Law (*Lieb & Yngvason*), whose type matches the trace operation.

For which stability functions does the Lieb-Yngvason Cancellation Law coincide with the trace?

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[2] André Joyal, Ross Street, Dominic Verity, "Traced monoidal categories"