Programming Research Group

THE LOGIC OF ENTANGLEMENT. AN INVITATION.

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Abstract

SHORT: We expose the information flow capabilities of quantum entanglement.

LONG: This paper contains several components:

- We prove a general characterization theorem on *information flow through bipartite entanglement*. This theorem will enable us to provide a unified view on protocols such as *quantum teleportation* [9], *quantum logic gate teleportation* [33] and *entanglement swapping* [61].
- We accomplish the extension of the above to *multipartite entanglement* which exposes the necessity of logical tools such as *typing* [22, 53]. Also the need for *linear logic connectives* [29] and *polarities* [45] arises naturally.
- We expose a methodology emerging from our *information flow based reasoning about entanglement* which yields a *two-way compilation scheme* enabling design of computational and communicational protocols. This tool allows evident reconstruction of the above mentioned protocols of quantum information processing and also the design of new ones in terms of a classical *travelling token-interpretation*. We use this methodology to realize a *passage from sequential to parallel composition for quantum logic gates*. This mechanism also yields a *fault-tolerant methodology to prepare multipartite entangled states*.
- At a more advanced level this methodology allows to accommodate classical functional programming features such as *Currying* [12], *λ-calculi* [7], *geometry of interaction* in the sense of [1, 3] and other *high-level specification logics*.
- Finally, the information flow capabilities of entanglement exposed in this paper yield a *canonical family of entanglement measures for multipartite systems*. They also provide an interpretation in terms of information flow capabilities for *non-local untary operations*.

The basic idea goes as follows. Consider a network containing specification of bipartite entanglement in terms of bipartite projectors on one-dimensional subspaces. We show that we can apply classical functional reasoning on a *virtual information flow*:

- The virtual information flow is manifestly *acausal*. It seems to flow from one Hilbert space to the other via EPR-bridges, however, it does that **as if**:

  "Time goes backward at the other side of the EPR-bridge".

- The functions involved are not the projectors themselves. When considering the isomorphism between the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ and the vector space of linear maps $\mathcal{H}_1^* \to \mathcal{H}_2$, the function involved will be in the isomorphic image in $\mathcal{H}_1^* \to \mathcal{H}_2$ of the projector's range in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

The passage from bipartite entanglement to multipartite entanglement goes with the passage of a network with a virtual flow to one with virtual function boxes. We show that in this way we can realize any network with (linear) functional actions of any order. The *compilation scheme* then translates such entanglement specification networks in an ordinary quantum computational setting of measurements, unitary transformations and classical communication.

Contents

Some conventions

We will be liberal about the use of the words *map*, *function*, *operator* and *transformation* when talking about (anti-)linear and (anti-)unitary maps. We will use the words *projector* and *projection* as equivalent when referring to linear maps.

We will use *entangled state* as a generic term for any element of the tensor product of two Hilbert spaces. If we want to make clear that such an element is a pure tensor we will say so or refer to it as being *disentangled*.

We completely ignore *normalization constants* for quantum states. Writing those redundancies might have added some ten extra pages to this paper. We preferred to use that space for some beautiful clarifying pictures.

Acknowledgements

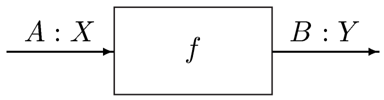
We are unequivocally indebted to Samson Abramsky, Howard Barnum, Sam Braunstein, Ross Duncan, Peter Hines, Radha Jagadeesan, Elham Kashefi, Keye Martin, Prakash Panangaden, Mehrnoosh Sadrzadeh and Vlatko Vedral for useful discussions, feedback, suggestions and indication of references. The main inspiration for the results of this paper is the "Physical traces" paper by S. Abramsky and the author [3]. All 160 pictures in this paper are made with Aldus Freehand™ v 3.0 (1991). The 6 diagrams are made with Paul Taylor's package v 3.88 (2000). These results were presented for the first time on the 16th of June 2003 at the Fields Institute Summer School on Logic and Foundations of Computation, Workshop on Quantum Programming Languages, Ottawa, Canada.

1 Introduction

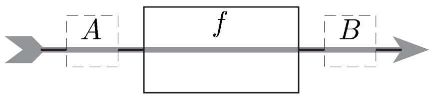
In classical functional programming, pre- and post- specification refer to properties imposed on the input and the output of a function:

- *Pre-specification* of a property A stands for the assertion that the input should satisfy A. *Post-specification* of a property B stands for the assertion that the output should satisfy B.

In pictures, given a functional *action* $f : X \to Y$ (which we represent as a box) we can put specifications at the input and the output of the box



where $A : X$ and $B : Y$ express the fact that the input is of *type* X and the output is of *type* Y. In physical terms X is the state space describing the possible input states accepted by f and Y is the state space in which the output states live. It is then reasonable to assume that we can express each such property as the subset of the state space consisting of those states which satisfy that property i.e. $A \subseteq X$ and $B \subseteq Y$. Taking the perspective of a *token carrier of state* traveling through the network one can think of the information which it carries as first being subjected to pre-specification A, then to the functional action f, and finally being subjected to the post-specification B:



Specification, of course, is non-physical, it is merely the statement that a certain property should be satisfied (= *assertion*). It does have an operational content in the sense that one can only know for sure that specifications are satisfied if one verifies them i.e. read the state of the system before and after effectuating f.

Quantum information processing is to be conceived as a *physical carrier of state* traveling through a network such that the interaction between this carrier and the network obeys the laws of quantum mechanics. In practice it might be the case that the carrier of state is stationary and subjected to operations. In fact this is the case for most current attempts to implement quantum computation. But it requires little imagination to pass from such a passive perspective to an active one in which one considers the physical carrier of state as moving and the network as stationary.

The implicit passive nature of the verificational feature of pre- and post-specificational assertions obstructs straightforward translation of *classical process semantics* and its *specification logics* to quantum information processing. Indeed, in quantum theory verification changes the state of the system due to von *Neumann's projection postulate* [Postulate B.2]. This means that specification of a quantum property is itself some kind of function box. A language designed for quantum information processing should take this feature into account. In this paper we will develop a high-level functional perspective on quantum information processing which takes this feature into account. And we do even more. In **physical terms**:

[1] We provide an interpretation of the behavior of entanglement both in the bipartite and the multipartite cases. In particular is the behavioral structure of multipartite entanglement an essentially open problem. We claim to provide an account of it in classical logical terms.

In **computational terms**:

[2] We show that *entanglement specification* only (by "only" we mean without the presence of any other kind of function boxes) suffices for the design of arbitrary chains of concatenated linear functions, tensoring, currying and so on. Conversely, distributed interacting entanglement specifications can at there turn be interpreted in terms of arbitrary chains of concatenated linear functions.

[3] We provide a two-way compilation scheme for arbitrary networks of (i) non-local quantum measurements [Definition B.3], (ii) local unitary operations and (iii) classical communication as networks in which only entanglement specification appears.

[2+3] Combining the above results provides a powerful tool for network and protocol design. We will demonstrate how this procedure simplifies re-designing the quantum teleportation protocol [9], the quantum logic gate teleportation protocol [33] and the entanglement swapping protocol [61]. And we produce some new ones including the passage from sequential to parallel composition for quantum logic gates.

Thus we encourage both computer scientists and physicists to read this paper:

- For the *practicing physicists* it will provide insight in why protocols like teleportation and entanglement swapping actually work the way they do. It will moreover enable them to produce much more sophisticated ones.

- For *computer scientists* it provides a way to do quantum information processing in terms of their usual functional way of reasoning and writing using their high-level specification languages without having to think all the time in terms of Hilbert spaces, self-adjoint operators and other beasts of that kind.

- For the *conceptually minded physicist*, the *philosopher* and the *logician* it will provide insight in how information (seems to) flow(s) through entanglement. Hence we expose a new insight in quantum behavior, namely the existence of a logic governing its information flow capabilities.

We will make some vague references to category theory [5, 22, 26] but the subject is not at all a prerequisite for understanding this paper. The categorical development of the ideas exposed in this paper is a non-trivial research project in itself which is currently in full development [4].

2 Entanglement specification

Before we introduce the notion of entanglement specification we will clarify how specification occurs naturally within quantum mechanics and in particular how it relates to quantum measurements. Next we introduce entanglement specification as functionally labeled operations on a system. Then we show how this notion of entanglement specification naturally yields the *teleportation protocol*. This example is a paradigmatic case for the rest of the paper.

2.1 Quantum measurements as specifications

Appendix B provides a minimal overview of elementary quantum theory. Appendix A does the same on Hilbert spaces and projectors. Although in quantum theory states correspond to one-dimensional subspaces of a Hilbert space $\mathcal{H}$ we will represent a state by any non-zero vector $\psi \in \mathcal{H}$ contained in the corresponding subspace $ray(\psi)$.

i. *Projectors from measurements.* In terms of its spectral decomposition [Theorem A.8] a general finitary non-degenerated quantum measurement [Definition B.1] looks like

$$M = 1 \cdot \mathrm{P}_1 + 2 \cdot \mathrm{P}_2 + 3 \cdot \mathrm{P}_3 + \ldots + n \cdot \mathrm{P}_n$$

where each $\mathrm{P}_i : \mathcal{H} \to \mathcal{H}$ is a projector [Definition A.6] which is orthogonal [Definition A.7] to all the other ones appearing in the sum, and where, in benefit of transparency, we assumed that the spectrum $\sigma(M)$ of this measurement is an integer enumeration. The actual measurement process constitutes two correlated events:

1. A value i is delivered to the observer;
2. The state of the system undergoes a transition $\mathrm{P}_i : \mathcal{H} \to \mathcal{H} :: \psi \mapsto \mathrm{P}_i(\psi)$.

Hence the state of the system changes from its *initial state* ψ to a *terminal state* $\mathrm{P}_i(\psi)$ while the observer gets informed about this change by receiving the value i. The actual values of the probabilities with which these transitions take place are of no crucial importance in this paper. Also of no importance is the actual value of i itself since in the eye of the observer the delivered value is merely a token witness to the change the system has undergone. Thus, more abstractly, one could conceive a finitary non-degenerated quantum measurement M as a list

$$(\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \ldots, \mathrm{P}_n) \tag{1}$$

of mutually orthogonal projectors (with n the dimension of the Hilbert space) where the actual process then consists of one of the transitions $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3 \ldots, \mathrm{P}_n$ taking place and the observer getting informed on which one happened. Every projector on $\mathcal{H}$ with a one-dimensional space of fixed points arises as an element in such a list. It will be these *one-dimensional projectors* that play a central role in this paper.

ii. *Specification as projectors.* On the other hand any projector P on $\mathcal{H}$ (and thus also any one-dimensional one) defines itself a unique dichotomic measurement since

$$\mathrm{P} = 1 \cdot \mathrm{P} + 0 \cdot \mathrm{P}^{\perp}$$

where $\mathrm{P}^{\perp}$ is P's orthocomplement [Definition A.7]. A projector "seen as a measurement" is the closest one gets to a verificational process in quantum theory. After the action P the system is in a state contained in the set of P's fixed points

$$A_{\mathrm{P}} := \{\psi \in \mathcal{H} \mid \mathrm{P}(\psi) = \psi\}$$

and this *linear subspace* of $\mathcal{H}$ represents the projector faithfully. We moreover have the following behavior of a system subjected to consecutive measurements.

1. If we perform measurement $1 \cdot \mathrm{P} + 0 \cdot \mathrm{P}^{\perp}$ immediately after the system has been subjected to the action P then we will obtain outcome 1 in that measurement and consequently the system will be again subjected to the action P [Postulate B.2] .

2. Every projector P is idempotent, that is, $\mathrm{P} \circ \mathrm{P} = \mathrm{P}$ [Definition A.6] .

Hence it is fair to say that after the system has been subjected to the action P the system satisfies a "property" A_{P}, explicitly, $A_{\mathrm{P}} :=$

> "If we perform $1 \cdot \mathrm{P} + 0 \cdot \mathrm{P}^{\perp}$ immediately after the system has been subjected to the action P then we will obtain outcome 1."

This motivates the following definition.

Definition 2.1 We refer to the process of imposing a property A_{P} on a system by subjecting it to the corresponding action P as *specification of* A_{P}.

However, imposing P on a system by means of a measurement is a probabilistic process. Thus a particular specification might fail. But we can read these specifications conditionally, assuming that the system has been subjected to it. An example of this kind of conditioning is the process of *preparing a system in a certain state.* This view will be very helpful when reasoning about networks involving entanglement specification.

iii. *Non-probabilistic implementation of specifications.* The probabilistic feature of "applying a projector" will not obstruct us to design protocols such as the teleportation protocol which are globally non-probabilistic. Assume that we want to specify the state of a system in terms of a one-dimensional projector P. We can *complete* such a projector into a non-degenerated measurement by adjoining it with $n-1$ other one-dimensional projectors such that we obtain a list of type (1) of n mutually orthogonal projectors. In terms of specification this means that whenever we effectuate that measurement and receive outcome i we have specified that the system is in state $\psi_i \in A_i := \mathrm{P}_i[A_i]$. We don't know in advance which ψ_i it will be but it will always be one in the list $(\psi_1, \ldots, \psi_n)$. *A strategy to design networks* should then be such that when effectuating a measurement, whatever state gets specified in the list $(\psi_1, \ldots, \psi_n)$, ultimately, we always end up with the desired result by making later actions depend on the measurement outcome. In Subsection 2.3 we illustrate how this strategy enables to design the teleportation protocol and in Subsection 3.3 how it enables the design of the logic gate teleportation protocol. In Subsection 6.2 we do the same for entanglement swapping. In Subsection 3.4 we realize the passage from sequential to parallel composition for quantum logic gates.

2.2 Functional labeling of bipartite projectors

Bellow all Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ will be finite dimensional. Let $\{e^{(i)}_{\alpha_i}\}_{\alpha_i}$ be a base of the Hilbert space $\mathcal{H}_i$.

Definition 2.2 *Entanglement specification* consists of specifying the state

$$\Psi := \sum_{\alpha_1 \ldots \alpha_m} f_{\alpha_1 \ldots \alpha_m} \cdot e^{(1)}_{\alpha_1} \otimes \ldots \otimes e^{(m)}_{\alpha_m} \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_m$$

of a compound quantum system by means of the projector

$$\mathrm{P}_\Psi : \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_m \to \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_m :: \Phi \mapsto \langle \Psi \mid \Phi \rangle \cdot \Psi$$

where we assume $|\Psi| = 1$.

In the case $m = 2$ we obtain *bipartite entanglement* specification

$$\mathrm{P}_\Psi : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 :: \Phi \mapsto \langle \Psi \mid \Phi \rangle \cdot \Psi$$

with

$$\Psi := \sum_{\alpha\beta} f_{\alpha\beta} \cdot e^{(1)}_\alpha \otimes e^{(2)}_\beta \in \mathcal{H}_1 \otimes \mathcal{H}_2 .$$

By $\mathcal{H}_1 \to \mathcal{H}_2$ we denote the set of linear functions between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. This set $\mathcal{H}_1 \to \mathcal{H}_2$ is itself a vector space over $\mathbb{C}$. Thus we can read

$$\mathrm{P}_\Psi : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$$

as a representation of the states in $\mathcal{H}_1 \otimes \mathcal{H}_2$ within a Hilbert space of linear maps, explicitly

$$\mathrm{P}_{-} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to (\mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2) :: \Psi \mapsto \mathrm{P}_\Psi .$$

This correspondence is itself not linear. But we can represent the bipartite states of $\mathcal{H}_1 \otimes \mathcal{H}_2$ in a vector space of linear functions of a smaller dimension than the above one via a correspondence which is linear. Let $\{e^{(1)}_\alpha\}_\alpha$ and $\{e^{(2)}_\beta\}_\beta$ be bases respectively of $\mathcal{H}_1$ and $\mathcal{H}_2$. By identifying the bases

$$\{e^{(1)}_\alpha \otimes e^{(2)}_\beta\}_{\alpha\beta} \quad \stackrel{\simeq}{\longleftrightarrow} \quad \{\langle e^{(1)}_\alpha \mid - \rangle \cdot e^{(2)}_\beta\}_{\alpha\beta}$$

respectively of

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \Big\{ \sum_{\alpha\beta} f_{\alpha\beta} \, \cdot e_\alpha^{(1)} \otimes e_\beta^{(2)} \;\Big|\; (f_{\alpha\beta})_{\alpha\beta} \in \mathbb{C}^{n\times n} \Big\}$$

and

$$\mathcal{H}_1 \to \mathcal{H}_2 = \Big\{ \sum_{\alpha\beta} f_{\alpha\beta} \, \langle e_\alpha^{(1)} \mid - \rangle \cdot e_\beta^{(2)} \;\Big|\; (f_{\alpha\beta})_{\alpha\beta} \in \mathbb{C}^{n\times n} \Big\}$$

it follows that these vector spaces have the same dimension and thus that there exists a linear isomorphism

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \;\simeq\; \mathcal{H}_1 \to \mathcal{H}_2 .$$

An explicit correspondence is

$$\sum_{\alpha\beta} f_{\alpha\beta} \cdot e_\alpha^{(1)} \otimes e_\beta^{(2)} \quad \stackrel{\simeq}{\longleftrightarrow} \quad \sum_{\alpha\beta} f_{\alpha\beta} \, \langle e_\alpha^{(1)} \mid - \rangle \cdot e_\beta^{(2)} \,.$$

It however depends on the choice of $\{e_\alpha^{(1)}\}_\alpha$ e.g. passage to another base $\{c \cdot e_\alpha^{(1)}\}_\alpha$ yields

$$(c \cdot e_\alpha^{(1)}) \otimes e_\beta^{(2)} = c \cdot (e_\alpha^{(1)} \otimes e_\beta^{(2)}) \quad \stackrel{\not\simeq}{\longleftrightarrow} \quad \bar{c} \cdot (\langle e_\alpha^{(1)} \mid - \rangle \cdot e_\beta^{(2)}) = \langle c \cdot e_\alpha^{(1)} \mid - \rangle \cdot e_\beta^{(2)} \,.$$

There is no unique base independent canonical isomorphism between $\mathcal{H}_1 \to \mathcal{H}_2$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$. This can easily be seen as follows. The dual of $\mathcal{H}$, that is, the Hilbert space

$$\mathcal{H}^* := \mathcal{H} \to \mathbb{C}$$

of linear functionals is presentable as

$$\{ \langle \psi \mid - \rangle : \mathcal{H} \to \mathbb{C} \mid \psi \in \mathcal{H} \} \,.$$

The canonical correspondence between the Hilbert spaces $\mathcal{H}^*$ and $\mathcal{H}$ is not linear but *anti-linear* [Definition A.9] since

$$\sum_\alpha \psi_\alpha \cdot e_\alpha = \psi \;\stackrel{\simeq}{\longleftrightarrow}\; \langle \psi \mid - \rangle = \Big\langle \sum_\alpha \psi_\alpha \cdot e_\alpha \;\Big|\; - \Big\rangle = \sum_\alpha \bar{\psi}_\alpha \cdot \langle e_\alpha \mid - \rangle \,.$$

This forces any canonical correspondence between $\mathcal{H}_1 \to \mathcal{H}_2$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$ to behave linear with respect to $\mathcal{H}_2$ and anti-linear with respect to $\mathcal{H}_1$, which is impossible since for the tensor product we have

$$(c \cdot \psi) \otimes \phi = c \cdot (\psi \otimes \phi) = \psi \otimes (c \cdot \phi) \,.$$

But this does indicate that there exists a canonical isomorphism

$$\mathcal{H}_1^* \otimes \mathcal{H}_2 \;\simeq\; \mathcal{H}_1 \to \mathcal{H}_2$$

explicitly given by

$$\sum_{\alpha\beta} f_{\alpha\beta} \cdot \langle e_\alpha^{(1)} \mid - \rangle \otimes e_\beta^{(2)} \quad \stackrel{\simeq}{\longleftrightarrow} \quad \sum_{\alpha\beta} f_{\alpha\beta} \, \langle e_\alpha^{(1)} \mid - \rangle \cdot e_\beta^{(2)}$$

where $\{\langle e^{(1)}_\alpha \mid -\rangle\}_\alpha$ is now a base for $\mathcal{H}_1^*$. This isomorphism is *natural* in the strict categorical sense [26]. Since $\mathcal{H}_1^{**} \simeq \mathcal{H}_1$ via the (categorical) natural correspondence

$$\Big\langle \langle\psi \mid -\rangle_{\mathcal{H}_1} \Big| - \Big\rangle_{\mathcal{H}_1^*} \;\overset{\simeq}{\longleftrightarrow}\; \psi$$

we also have a natural isomorphism

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \;\simeq\; (\mathcal{H}_1^*)^* \otimes \mathcal{H}_2 \;\simeq\; \mathcal{H}_1^* \to \mathcal{H}_2$$

explicitly given by

$$\sum_{\alpha\beta} f_{\alpha\beta} \cdot e^{(1)}_\alpha \otimes e^{(2)}_\beta \;\overset{\simeq}{\longleftrightarrow}\; \sum_{\alpha\beta} f_{\alpha\beta} \Big\langle \langle e^{(1)}_\alpha \mid -\rangle_{\mathcal{H}_1} \Big| - \Big\rangle_{\mathcal{H}_1^*} \cdot e^{(2)}_\beta \,.$$

We will use both the natural and the base-dependent isomorphisms in this paper. For computer scientists a natural correspondence is to be preferred due to the important role which category theory plays in the semantics of programing languages [5]. Physicists might rather choose the base dependent one since in quantum information theory they are used to work with fixed bases anyway e.g. $\{|0\rangle, |1\rangle\}$ in the case of so-called *qubits*. In this two-dimensional case they would then set

$$\begin{pmatrix} f_{00} & f_{10} \\ f_{01} & f_{11} \end{pmatrix} \;\overset{\simeq}{\longleftrightarrow}\; f_{00}|00\rangle + f_{01}|01\rangle + f_{10}|10\rangle + f_{11}|11\rangle \,.$$

Another kind of isomorphism that will be of major interest is

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \;\simeq\; \mathcal{H}_1 \looparrowright \mathcal{H}_2$$

where

$$\mathcal{H}_1 \looparrowright \mathcal{H}_2 = \Big\{ \sum_{\alpha\beta} f_{\alpha\beta} \, \langle - \mid e^{(1)}_\alpha \rangle \cdot e^{(2)}_\beta \;\Big|\; (f_{\alpha\beta})_{\alpha\beta} \in \mathbb{C}^{n\times n} \Big\}$$

is the vector space of anti-linear maps. This correspondence is also canonical via the linear correspondence

$$\sum_{\alpha\beta} f_{\alpha\beta} \cdot e^{(1)}_\alpha \otimes e^{(2)}_\beta \;\overset{\simeq}{\longleftrightarrow}\; \sum_{\alpha\beta} f_{\alpha\beta} \, \langle - \mid e^{(1)}_\alpha \rangle \cdot e^{(2)}_\beta \,.$$

Unfortunately, the passage from linear to anti-linear functions blurs any straightforward categorical status of the above kind of canonicity. Also the strict correspondence between composition of functions and multiplication of corresponding matrices diminishes [Lemmas 5.4, 5.5 and 5.7 in Subsection 5.1]. Nonetheless $\mathcal{H}_1 \looparrowright \mathcal{H}_2$ will prove to play a crucial role for the developments in this paper and for the understanding of information flow through entanglement in general [Subsection 4.1].

Definition 2.3 By a *functionally labeled* (bipartite) *state* $\Psi_f \in \mathcal{H}_1 \otimes \mathcal{H}_2$ we mean the isomorphic image of a function f via one of the above linear isomorphisms. By a *functionally labeled* (bipartite) *projector* P_f we mean the projector

$$\mathrm{P}_f \equiv \mathrm{P}_{\Psi_f} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 :: \Phi \mapsto \langle \Psi_f \mid \Phi \rangle \cdot \Psi_f \,.$$

This functional labeling provides a bridge between applying quantum measurements and classical functional reasoning. It also indicates what's so great about entanglement. In terms of types, taking a system of type $\mathcal{H}_1$ together with one of type $\mathcal{H}_2$ provides one that is in bijective correspondence with one of type $\mathcal{H}_1 \to \mathcal{H}_2$. This is in sharp contrast to the classical situation where in case of *pairing types* we would just obtain $\mathcal{H}_1 \times \mathcal{H}_2$ as global type. A physical discussion on how entanglement lies at the core of the speedup of quantum algorithms as compared to classical ones can be found in [27].

In fact, what we study in this paper is to which extend we can replace in the sentence above "one that is in bijective correspondence with one of type $\mathcal{H}_1 \to \mathcal{H}_2$" by "one of type $\mathcal{H}_1 \to \mathcal{H}_2$". This mainly involves proving that *compositionality of functions*, that is, given $f : A \to B$ and $g : B \to C$ we can define a composite

$$g \circ f : A \to C :: \psi \mapsto g(f(\psi)) ,$$

carries over to the world of entanglement of physical states. Moreover, we will show that also the *tensor structure*, that is, given $f : A \to C$ and $g : B \to D$ we can define a product

$$f \times g : A \times B \to C \times D ,$$

and *currying of arguments* [12], that is, passage from a two argument function to a one argument function via the isomorphism

$$(A \times B) \to C \quad \simeq \quad A \to (B \to C) ,$$

will carry over to our setting.

Our notion of entanglement specification was used by S. Abramsky and the author in [3] to realize the traced monoidal category of vector spaces in terms of quantum system. Therefore [3] is the main predecessor to this paper. On the other hand, the results of this paper explain why the constructions in [3] are truly natural — we will elaborate a bit more on the connection between the results in this paper and those presented in [3] in Subsection 6.6 and Appendix C.

2.3 Example: recovering the teleportation protocol

We will not start by describing the standard quantum teleportation protocol [9]. We will reproduce it from a much simpler (probabilistic) version introduced in [3]. Although at first sight the steps we have to take to accomplish this might seem somewhat arbitrary, they incarnate a generally applicable canonical procedure.

In this subsection all Hilbert spaces are two-dimensional. We refer to each component of a compound system as a material/physical *carrier of state*. Thus $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_m$ describes the state jointly carried by m carriers of state. In [3] we observed the following. Consider three carriers of state jointly described in

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \quad \text{with} \quad \mathcal{H}_1 \simeq \mathcal{H}_2 \simeq \mathcal{H}_3$$

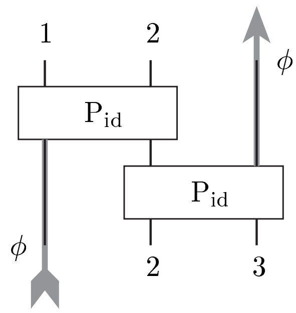
via (pseudo-) *identities*

$$\mathrm{id} : \mathcal{H}_1 \to \mathcal{H}_2 \qquad \text{and} \qquad \mathrm{id} : \mathcal{H}_2 \to \mathcal{H}_3 \,.$$

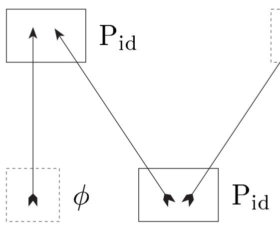
Thus if $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ have qubit bases we have

$$\mathrm{id} : \mathcal{H}_{1/2} \to \mathcal{H}_{2/3} :: \phi_0|0\rangle + \phi_1|1\rangle \;\mapsto\; \phi_0|0\rangle + \phi_1|1\rangle \,.$$

If we first subject carrier 2 and 3 to the base-dependent functionally labeled projector $\mathrm{P}_{\mathrm{id}} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ and then subject carrier 1 and 2 to $\mathrm{P}_{\mathrm{id}} : \mathcal{H}_2 \otimes \mathcal{H}_3 \to \mathcal{H}_2 \otimes \mathcal{H}_3$, then, if the input state of carrier 1 is not entangled to the others and is in state ϕ then the output state of carrier 3 will also not be entangled to the others carriers and will also be in state ϕ. The input state of carrier 2 and 3 plays no role at all (except when the state of carrier 2 and 3 is badly chosen and cannot pass through the first projector). In a picture this yields:



Thus we are able to teleport the state ϕ (probabilistically) from carrier 1 to carrier 3. To see that this truly embodies teleporting a state from one region in space to another one it suffices to tilt the trajectories of the carriers as in this picture:



We now verify the above claim for the particular case of qubits — this is the case for which the original teleportation protocol was formulated in [9].

Convention 2.4 *We ignore normalization constants and will continue to do so throughout the paper in order not to cloud it with those meaningless constants.*

Tensor symbols are omitted as it is usual in the quantum information literature. By

$$\mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;\stackrel{\simeq}{\longleftrightarrow}\; \Psi_{\mathrm{id}} = |00\rangle + |11\rangle$$

we have, assuming the input at carrier/qubit 2 and 3 is not orthogonal to $|00\rangle + |11\rangle$,

$$\mathrm{P}_{\mathrm{id}} = \underbrace{\langle 00 + 11 \mid -\rangle}_{\text{constant}}(|00\rangle + |11\rangle)\,.$$

For input $|\phi\rangle = \phi_0|0\rangle + \phi_1|1\rangle$ at carrier/qubit 1, after applying the projector P_{id} to carrier/qubit 2 and 3 the state of the system is:

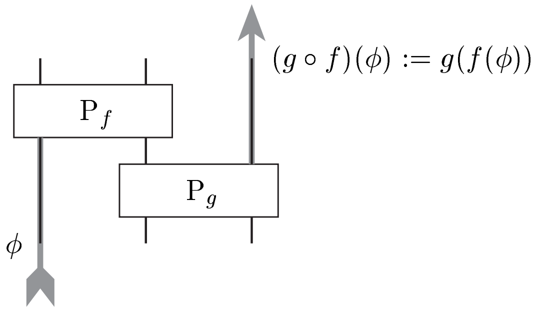
$$|\phi\rangle(|00\rangle + |11\rangle) = \phi_0|000\rangle + \phi_0|011\rangle + \phi_1|100\rangle + \phi_1|111\rangle\,.$$

Next, applying P_{id} to the first two carriers/qubits yields

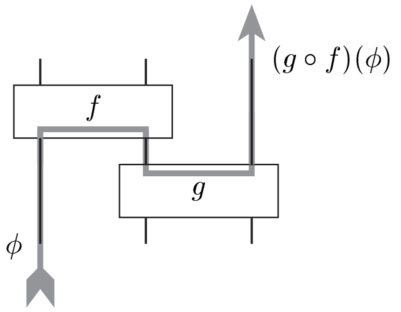
$$\phi_0(|00\rangle + |11\rangle)|0\rangle + \phi_1(|00\rangle + |11\rangle)|1\rangle = (|00\rangle + |11\rangle)|\phi\rangle$$

what proves the claim for the qubit case.

This result actually generalizes in a remarkable manner to projectors labeled by arbitrary functions between Hilbert spaces of arbitrary (finite) dimension [Section 3.1]:

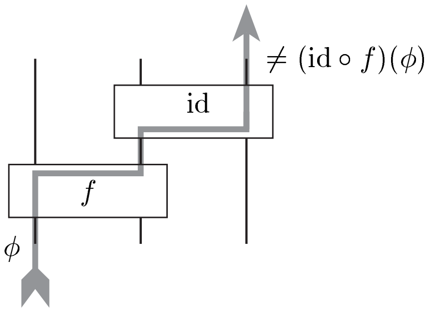


It then follows that the teleportation in the first example is merely a consequence of the fact that $(\mathrm{id} \circ \mathrm{id})(\phi) = \phi$. It seems that the information flows as follows:



That is, it flows **as if** not the projectors but the functions that label them act on the state in a simular way as the boxes do in classical functional models. However, for carrier 2 it is **as if** the information flows backward in time. This seemingly backward in time flow has been pointed at in [57] for the usual teleportation protocol but the additional functional behavior we expose here makes it far more striking. This more general case is at its turn only one particular incarnation of the general theorem which we will prove on compositional behavior (in the above sense) of the labels of functionally labeled projectors which specify entanglement.

We will have to play this game carefully, as is shown in the following counter example:



The third carrier will be entangled to the second one because they constitute the entangled state Ψ_{id} and there will be no spoor of the function f in the final state of the third carrier. Thus compositionality fails. The crucial question to answer in order to establish a true functional paradigm for analyzing and designing *entanglement specification networks* will consists of nailing down how information propagates through projectors that specify entanglement — this of course in the **as if** sense.

We will now pass from a probabilistic to a non-probabilistic protocol. We can conceive the first projector (in physical time) as specifying an in advance prepared state. This eliminates the uncertainty which goes with that projector. And this is indeed what one does in most of the standard protocols. One assumes the existence of some entangled state which is for most protocols the *EPR-state* Ψ_{id} — sometimes also referred to as an *EPR-pair*. As we will show in Subsection 3.2 one can eliminate the uncertainty which goes with the first projector without having to rely on prepared states.

Next consider the following alternative labels for projectors:

$$\pi := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \stackrel{\simeq}{\longleftrightarrow} \quad \Psi_\pi = |01\rangle + |10\rangle$$

$$\mathrm{id}^* := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \stackrel{\simeq}{\longleftrightarrow} \quad \Psi_{\mathrm{id}^*} = |00\rangle - |11\rangle$$

$$\pi^* := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \stackrel{\simeq}{\longleftrightarrow} \quad \Psi_{\pi^*} = |01\rangle - |10\rangle$$

It is well-known that the set of functions $\{\mathrm{id}\,,\,\pi\,,\,\mathrm{id}^*\,,\,\pi^*\}$ constitute a base for $\mathcal{H}_1 \otimes \mathcal{H}_2$, namely the so-called *Bell-basis*. The labeling functions themselves "almost" constitute the *Pauli matrices*

$$\sigma_x \equiv X := \pi \qquad\qquad \sigma_y \equiv Y := i\pi^* \qquad\qquad \sigma_z \equiv Z := \mathrm{id}^*$$

up to multiplication of π^* by the imaginary unit i. We stick the functions $\{\mathrm{id}\,,\,\pi\,,\,\mathrm{id}^*\,,\,\pi^*\}$ rather than to the Pauli matrices because of their agreement with the Bell-basis.

With each $f \in \{\mathrm{id}\,,\,\pi\,,\,\mathrm{id}^*\,,\,\pi^*\}$ we now associate a distinct number a_f and define a self-adjoint operator

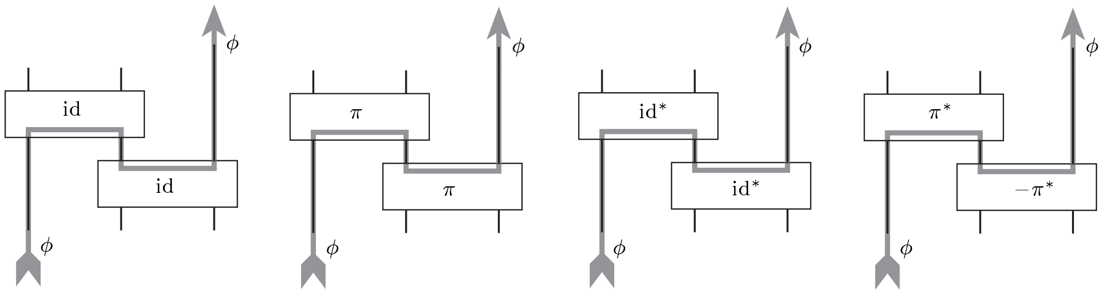
$$M(\mathrm{id}, \pi, \mathrm{id}^*, \pi^*) := a_{\mathrm{id}} \cdot \mathrm{P}_{\mathrm{id}} + a_\pi \cdot \mathrm{P}_\pi + a_{\mathrm{id}^*} \cdot \mathrm{P}_{\mathrm{id}^*} + a_{\pi^*} \cdot \mathrm{P}_{\pi^*}\,,$$

that is, a four-outcome measurement in quantum theory of which (as discussed above) the crucial part is the list

$$(\mathrm{P}_{\mathrm{id}}, \mathrm{P}_{\pi}, \mathrm{P}_{\mathrm{id}^*}, \mathrm{P}_{\pi^*})\,.$$

Our teleportation protocol can be conceived such that the second projector which we apply is actually the measurement $M(\mathrm{id}, \pi, \mathrm{id}^*, \pi^*)$ where we condition on the fact that the outcome when we effectuate it has to be a_{id}. What should we do if the measurement outcome ends up not being a_{id}?

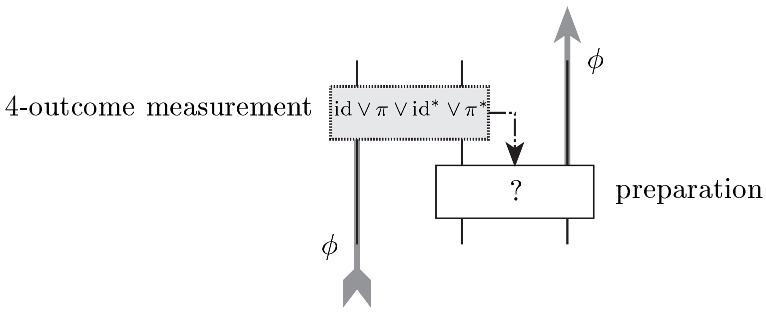
Consider the following four variations on the same theme:



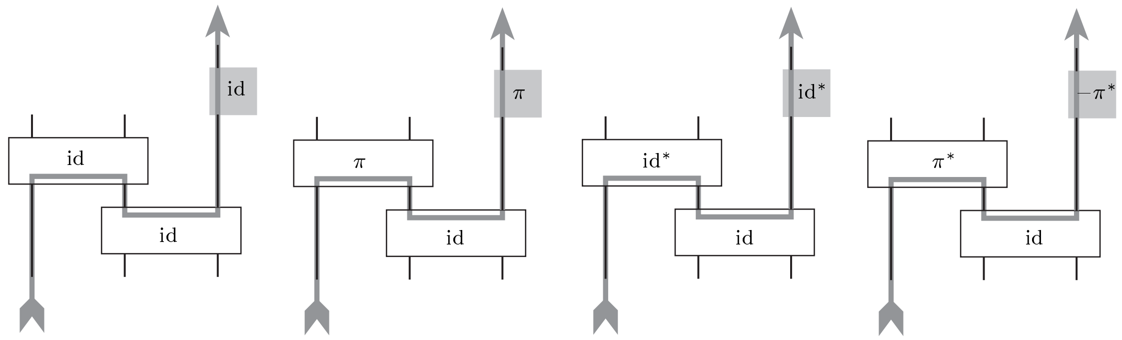
These variations indeed all teleport since

$$\mathrm{id} \circ \mathrm{id} = \pi \circ \pi = \mathrm{id}^* \circ \mathrm{id}^* = -\pi^* \circ \pi^* = \mathrm{id}\,.$$

Thus we have four distinct (probabilistic) teleportation protocols, where the four appearing second projectors are those that constitute the spectral decomposition of the self-adjoint operator $M(\mathrm{id}, \pi, \mathrm{id}^*, \pi^*)$. In these variations the first projector (i.e. the preparation of an entangled state) which we (have to) apply has to be such that the composition of its label with the label of the second projector yields the identity. Could we choose the first projector as a function of the obtained result in the measurement $M(\mathrm{id}, \pi, \mathrm{id}^*, \pi^*)$ hence eliminating the probabilistic feature of our protocol?



Of course we cannot do that since we would violate physical causality. But we can correct the error of applying the wrong projector at a later stage (after the measurement) as a unitary transformation on carrier 3. For each of the possible outcomes that is:

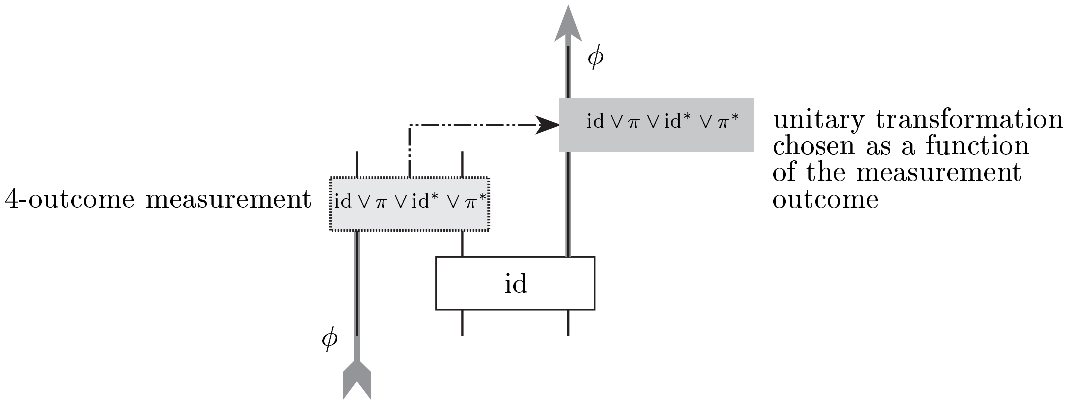


since

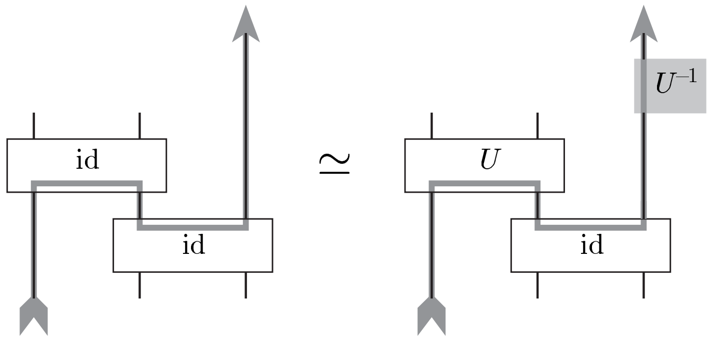
$$\pi \circ \mathrm{id} \circ \pi = \mathrm{id} \circ \mathrm{id} \circ \mathrm{id} = \mathrm{id}^* \circ \mathrm{id} \circ \mathrm{id}^* = -\pi^* \circ \mathrm{id} \circ \pi^* = \mathrm{id}\,.$$

Here of course the symbols id , π , id* , π^* in the grey boxes are not anymore the labels of projectors but actual unitary maps — all the maps id , π , id* , π^* are indeed unitary. We thus produced a non-probabilistic teleportation protocol and the reader might realize that it is the usual one.

The classical information that has to be transmitted in this protocol consists of the outcome of the measurement $M(\mathrm{id}, \pi, \mathrm{id}^*, \pi^*)$ being send to track 3 such that the appropriate unitary transformation can be applied. All this results in the following picture:



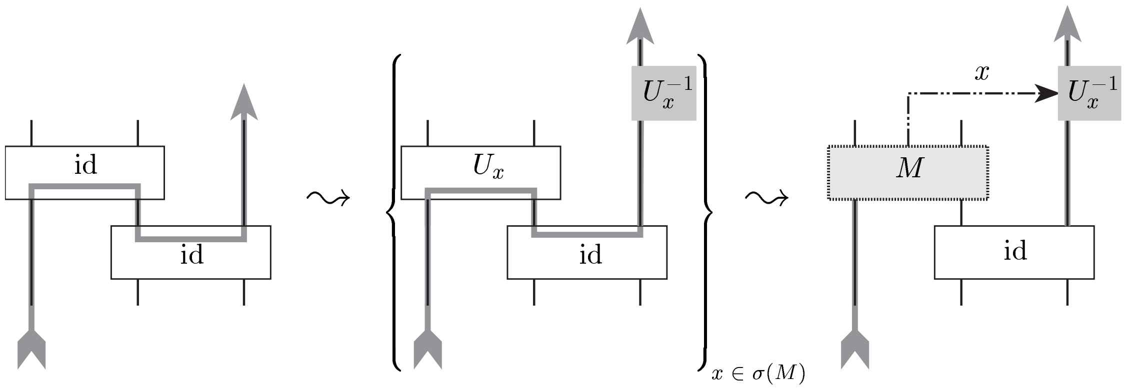
with the doted line being the transmitted classical information. The crucial step in all this is that compositionality provides us with the following substitution:



relying on the arithmetic

$$\begin{aligned}
\mathrm{id}^{(2,3)} \circ \mathrm{id}^{(1,2)} &= \mathrm{id}^{(2,3)} \circ (U^{-1} \circ U)^{(1,2)} \\
&= (\mathrm{id} \circ U^{-1})^{(2,3)} \circ U^{(1,2)} \\
&= (U^{-1} \circ \mathrm{id})^{(2,3)} \circ U^{(1,2)} \\
&= (U^{-1})^{(3)} \circ \mathrm{id}^{(2,3)} \circ U^{(1,2)}
\end{aligned}$$

which uses commutation of id and U^{-1}. As we will see in Subsection 3.3 commutation is not a necessity to perform this trick. In that subsection we will also elaborate a bit more on the teleportation protocol shaping it as one usually finds it in text books. This substitution allows to pass from probabilistic teleportation in terms of *entanglement specification* via a family of supplementary probabilistic teleportation protocols to the usual teleportation protocol in terms of *preparation*, *measurement*, *classical communication* and *local unitary transformations.* Setting $M := M(\mathrm{id}, \pi, \mathrm{id}^*, \pi^*)$, denoting by $x \in \sigma(M)$ that x is a possible outcome of the measurement M and denoting by $U_x \in \{\mathrm{id}\,,\, \pi\,,\, \mathrm{id}^*\,,\, \pi^*\}$ the label of the corresponding projector we obtain:



Conversely we can interpret teleportation in terms of entanglement specification by conditioning the measurement outcome e.g. $U_x := \mathrm{id}$. This two-way principle generalizes into a two-way *compilation tool* between:

- General entanglement specification networks ;
- Networks of prepared states, non-local measurements, local unitary operations and exchange of classical information between components.

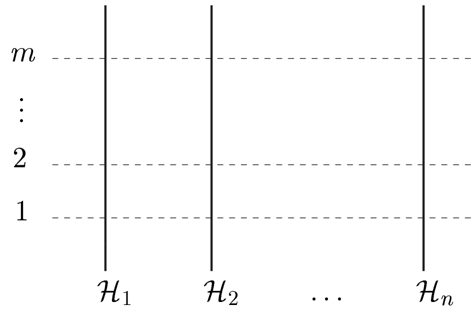
This procedure allows us to (i) design communication and computation protocols in terms of functional considerations, (ii) using high-level formal tools for example from process calculi, (iii) express this then as a family of entanglement specification networks and (iv) compile it then to (prepared states,) measurements, unitary transformations and classical communication. We now start the full formal development of these ideas beginning with the study of the compositional behavior of bipartite entanglement specification.

3 Compositionality for bipartite entanglement specification

Consider n (material) carriers of state, in parallel, respectively described in a Hilbert space $\mathcal{H}_i$, such that the whole set is described in

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n \,.$$

Let $1, \ldots, m$ be discrete consecutive instances of time.

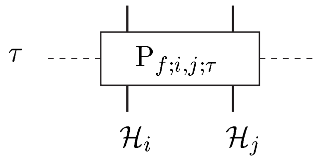


We refer to the vertical lines as *tracks*. At several locations this network has entanglement specification ($\sim$ **eP**)

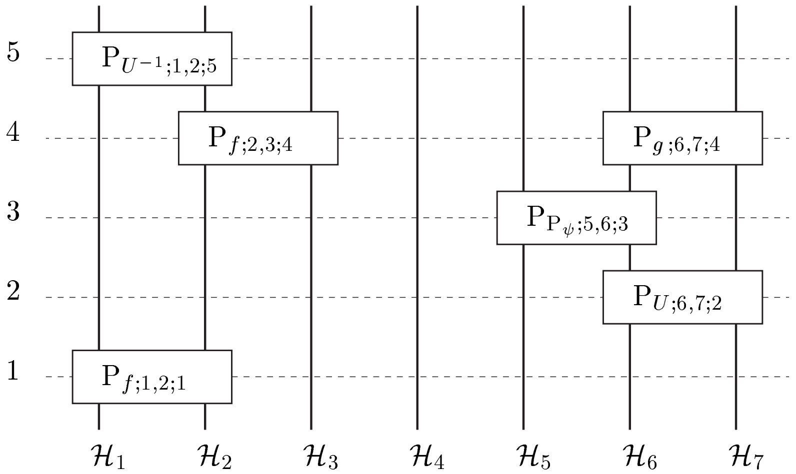
$$\mathrm{P}_{f;i,j;\tau} : \mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_i \otimes \mathcal{H}_j$$

the indices respectively pointing at:

- The labeling function f ;
- The indices of material carriers $\{i, j\}$ to which the **eP** is applied ;
- The time τ at which it is applied.



Of course i and j are implicit in $f : \mathcal{H}_i \to \mathcal{H}_j$ as the indices of the domain and codomain, but it will be useful to have them explicitly. Further, although i and j are arbitrary non-equal indices in $\{1, \ldots, n\}$, in our graphical representation we will conveniently represent them as adjacent. The general situation then looks for example as follows:



Let

$$\Psi^{\mathbf{in}} := \sum_{\alpha_1 \ldots \alpha_n} \Psi^{\mathbf{in}}_{\alpha_1 \ldots \alpha_n} \cdot e^{(1)}_{\alpha_1} \otimes \ldots \otimes e^{(n)}_{\alpha_n} \;\in\; \bigotimes_{\nu=1}^{\nu=n} \mathcal{H}_\nu$$

be the initial state at time 0. Let $P_{f;i,j;1}$ be the only **eP** at time 1. The state at time $1+\epsilon < 2$ is then

$$\Psi^1 := P_{f;i,j;1}(\Psi^{\mathbf{in}})$$

where slightly abusively by $P_{f;i,j;1}$ we actually mean

$$P_{f;i,j;1} \otimes id_{\neg\{i,j\}} : \bigotimes_{\nu=1}^{\nu=n} \mathcal{H}_\nu \to \bigotimes_{\nu=1}^{\nu=n} \mathcal{H}_\nu$$

with $id_{\neg\{i,j\}}$ being the identity on

$$\bigotimes \{\mathcal{H}_\nu \mid \nu \in \{1, \ldots, n\} \setminus \{i,j\}\}\,.$$

Let $P_{g;k,l;\tau}$ be the only **eP** at time τ. Analogously set

$$\Psi^\tau := P_{g;k,l;\tau}(\Psi^{\tau-1})$$

for all $\tau \in \{1, \ldots, n\}$ and set $\Psi^{\mathbf{out}} := \Psi^n$.

Definition 3.1 An *entanglement specification network* is one of the above kind. We call $\mathcal{H}_i$ the *type* of the i-th carrier and $\bigotimes_{\nu=1}^{\nu=n} \mathcal{H}_\nu$ the *horizontal type* of the network.

Given is an entanglement specification network Ξ. We will now define a notion of *path* within such a network. Any pair (ν, τ) with $\nu \in \{1, \ldots, n\}$ and $\tau \in \{1, \ldots, m\}$ can be conceived as a *coordinate* in an entanglement specification network:

$$
\begin{array}{ccccc}
(1,\mathbf{out}) & (2,\mathbf{out}) & & (n,\mathbf{out}) & \\
| & | & & | & \\
\text{-- } (1,m) \text{ ---} & (2,m) & \text{--------} & (n,m) \text{ --} & \\
\Big| & \Big| & & \Big| & \\
\text{-- } (1,2) \text{ ---} & (2,2) & \text{--------} & (n,2) \text{ --} & \\
| & | & & | & \\
\text{-- } (1,1) \text{ ---} & (2,1) & \text{--------} & (n,1) \text{ --} & \\
| & | & & | & \\
(1,\mathbf{in}) & (2,\mathbf{in}) & \ldots & (n,\mathbf{in}) &
\end{array}
$$

where we have added input coordinates

$$\mathbb{I}(\Xi) := \{(\nu, \mathbf{in}) \mid \nu \in \{1, \ldots, n\}\}$$

and output coordinates

$$\mathbb{O}(\Xi) := \{(\nu, \mathbf{out}) \mid \nu \in \{1, \ldots, n\}\} .$$

The space of coordinates is

$$\mathbb{S}(\Xi) := \{1, \ldots, n\} \times \{\mathbf{in}, 1, \ldots, m, \mathbf{out}\} .$$

Let $\mathbf{P}(\Xi)$ be the set of **eP**'s occurring in Ξ. Define

$$\mathbb{P}(\Xi) := \Big\{(\nu, \tau) \in \mathbb{S}(\Xi) \,\Big|\, \exists\, i, f : \mathrm{P}_{f;\nu,i;\tau} \in \mathbf{P}(\Xi) \;\text{ or }\; \mathrm{P}_{f;i,\nu;\tau} \in \mathbf{P}(\Xi)\Big\} .$$

Thus $\mathbb{P}(\Xi)$ is the set of all coordinates where we have an **eP**. Define

$$](\nu, \tau_1), (\nu, \tau_2)[\;:= ((\nu, \tau_1 + 1), \ldots, (\nu, \tau_2 - 1))$$

$$](\nu, \tau_2), (\nu, \tau_1)[\;:= ((\nu, \tau_2 - 1), \ldots, (\nu, \tau_1 + 1))$$

for $1 \leq \tau_1 < \tau_2 \leq m$. Note that $](\nu, \tau_i), (\nu, \tau_j)[$ can be empty. Further set

$$[(\nu, \mathbf{in}), (\nu, \tau)[\;:= ((\nu, \mathbf{in}), (\nu, 1) \ldots, (\nu, \tau - 1))$$

$$](\nu, \tau), (\nu, \mathbf{out})] \;:= ((\nu, \tau + 1), \ldots, (\nu, m), (\nu, \mathbf{out})) .$$

By a path in Ξ we mean a list $\Gamma := (\Gamma_1, \ldots, \Gamma_{|\Gamma|})$ of coordinates in $\mathbb{S}(\Xi)$ of some specific shape (to be defined below).

3.1 Compositionality for forward paths

i. *Full forward paths.* In this section we first consider paths of the following shape:

$$[(\nu_1,), (\nu_1, \tau_1)[\cdot ((\nu_1, \tau_1), (\nu_2, \tau_1)) \cdot](\nu_2, \tau_1), (\nu_2, \tau_2)[\cdot ((\nu_2, \tau_2), (\nu_3, \tau_2)) \cdot](\nu_3, \tau_2), (\nu_3, \tau_3)[\cdot \ldots$$

$$\ldots \cdot](\nu_{||\Gamma||}, \tau_{||\Gamma||-1}), (\nu_{||\Gamma||}, \tau_{||\Gamma||})[\cdot ((\nu_{||\Gamma||}, \tau_{||\Gamma||}), (\nu_{||\Gamma||+1}, \tau_{||\Gamma||})) \cdot](\nu_{||\Gamma||+1}, \tau_{||\Gamma||}), (\nu_{||\Gamma||+1},)]$$

implying $\Gamma_1 \in \mathbb{I}(\Xi)$ and $\Gamma_{|\Gamma|} \in \mathbb{O}(\Xi)$, and where we additionally require

$$](\nu_{\gamma+1}, \tau_\gamma), (\nu_{\gamma+1}, \tau_{\gamma+1})[\;\cap\, \mathbb{P}(\Xi) = \emptyset$$

for all $\gamma \in \{1, \ldots, ||\Gamma|| - 1\}$,

$$[(\nu_1,), (\nu_1, \tau_1)[\ \cap \mathbb{P}(\Xi) = \emptyset\,, \qquad](\nu_{||\Gamma||+1}, \tau_{||\Gamma||}), (\nu_{||\Gamma||+1},)]\ \cap\ \mathbb{P}(\Xi) = \emptyset$$

and

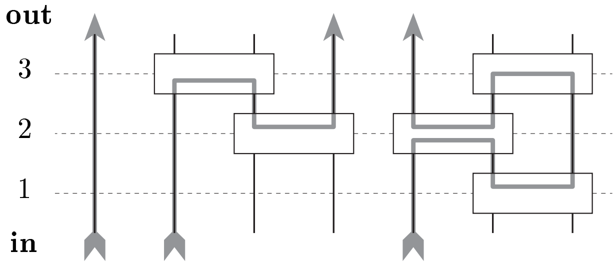
$$\mathrm{P}_{f_\gamma;\nu_\gamma,\nu_{\gamma+1};\tau_\gamma} \in (\Xi)$$

for all $\gamma \in \{1, \ldots, ||\Gamma||\}$. For convenience we set:

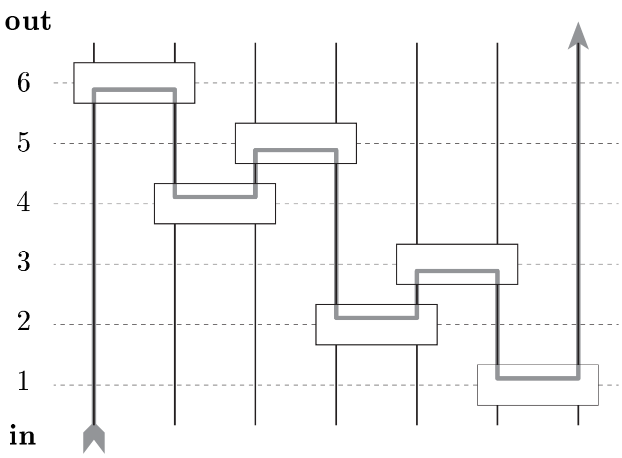
$$\nu_{in} := \nu_1 \qquad \mathcal{H}_{in} := \mathcal{H}_{\nu_1}$$

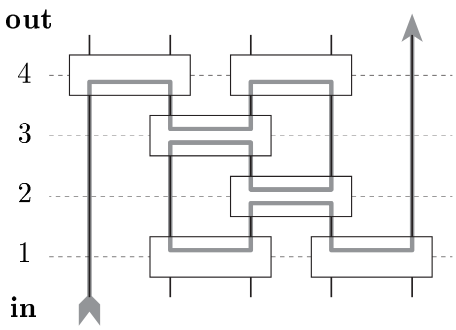
$$\nu_{out} := \nu_{||\Gamma||+1} \qquad \mathcal{H}_{out} := \mathcal{H}_{\nu_{||\Gamma||+1}}$$

Note that $||\Gamma||$ is the number of **eP**'s through which Γ passes. Some basic examples are:



Here are some more sophisticated ones:





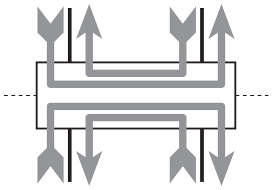
In words and referring to the pictures, such a path

- starts at time **in** and ends at time **out** ;
- proceeds along the tracks ;
- whenever it encounters an **eP** it:

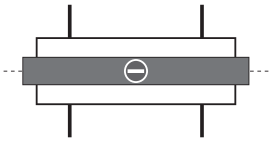
  1. jumps from track ν_γ to another one, namely track $\nu_{\gamma+1}$;

2. reverses its direction (with respect to the direction of the actual physical time).

It follows that paths are manifestly *acausal* with respect to the physical flow of time. In this definition it is also implicitly encoded that an **eP** allows paths to pass in four different ways:



No vertical passage is however possible:



Definition 3.2 By a *full forward path* we mean one of the above kind.

We introduce the notation $x_1 \ldots \hat{x}_i \ldots x_n$ for $x_1 \ldots x_{i-1}x_{i+1} \ldots x_n$. Assume that the initial state can be written as

$$\Psi := \phi_{in} \otimes \sum_{\alpha_1 \ldots \hat{\alpha}_{in} \ldots \alpha_n} \Phi_{\alpha_1 \ldots \hat{\alpha}_{in} \ldots \alpha_n} \cdot e^{(1)}_{\alpha_1} \otimes \ldots \otimes \hat{e}^{(\nu_{in})}_{\alpha_{in}} \otimes \ldots \otimes e^{(n)}_{\alpha_n}$$

with $\phi_{in} \in \mathcal{H}_{in}$, that is, the ν_{in}-th carrier of state is initially not entangled to any other one. We will express this fact by saying that *ϕ_{in} is free in Ψ*. Analogously we conceive *ϕ_{out} being free in Ψ* whenever

$$\Psi := \phi_{out} \otimes \sum_{\alpha_1 \ldots \hat{\alpha}_{out} \ldots \alpha_n} \Phi_{\alpha_1 \ldots \hat{\alpha}_{out} \ldots \alpha_n} \cdot e^{(1)}_{\alpha_1} \otimes \ldots \otimes \hat{e}^{(\nu_{out})}_{\alpha_{out}} \otimes \ldots \otimes e^{(n)}_{\alpha_n} .$$

for $\phi_{out} \in \mathcal{H}_{out}$.

Definition 3.3 We call the pair (Ξ, Ψ) *regular* if Ψ is not equal to $\mho \in \bigotimes_{\nu=1}^{\nu=n} \mathcal{H}_\nu$.

As we will discus in Subsection 4.4 Ψ will only be $\mathbf{0}$ in very singular cases and they can be avoided either by a different choice of Ψ while retaining ϕ_{in} or by a slight modification of the network that doesn't alter its compositionality and the set of paths it admits.

Theorem 3.4 (Full forward path compositionality) *Given are:*

- *An entanglement specification network Ξ of horizontal type $\bigotimes_{\nu=1}^{\nu=n} \mathcal{H}_\nu$;*
- *A full forward path Γ passing through $||\Gamma||$* **eP***'s respectively labeled $\mathrm{P}_{f_\gamma;\nu_\gamma,\nu_{\gamma+1};\tau_\gamma}$;*

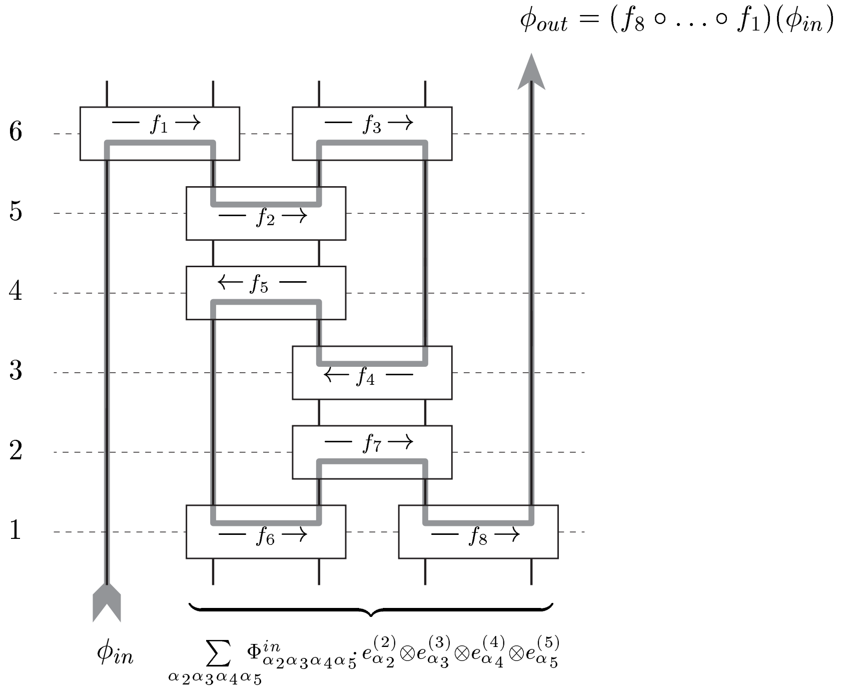
- *An input state* $\Psi \in \bigotimes_{i=1}^{i=n} \mathcal{H}_i$ *with* (Ξ, Ψ) *regular.*

If ϕ_{in} *is free in* Ψ *then* ϕ_{out} *is free in* Ψ *and we have*

$$\phi_{out} = (f_{||\Gamma||} \circ \cdots \circ f_{\gamma+1} \circ f_\gamma \circ f_{\gamma-1} \circ \cdots \circ f_1)(\phi_{in}) .$$

Proof. See Section 5. $\square$

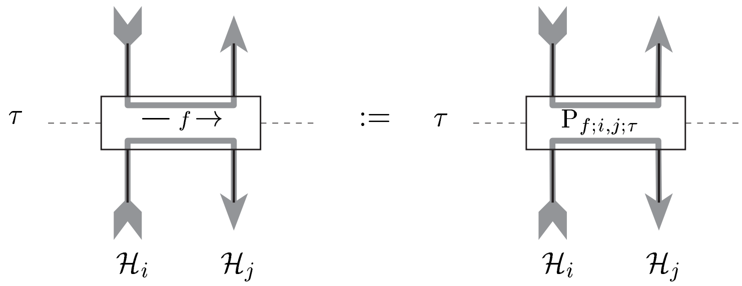
In a picture this means that for the following network:



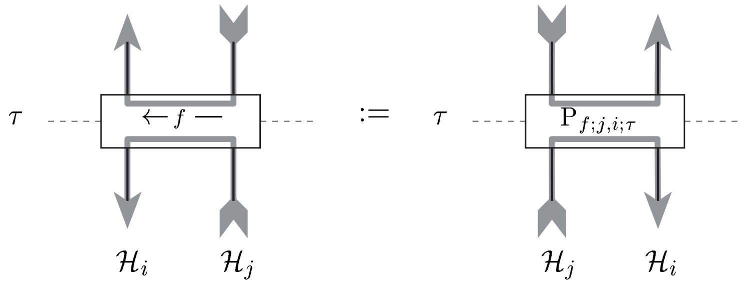
we have

$$\phi_{out} = (f_8 \circ f_7 \circ f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(\phi_{in})$$

where we simplified denotation by:



and



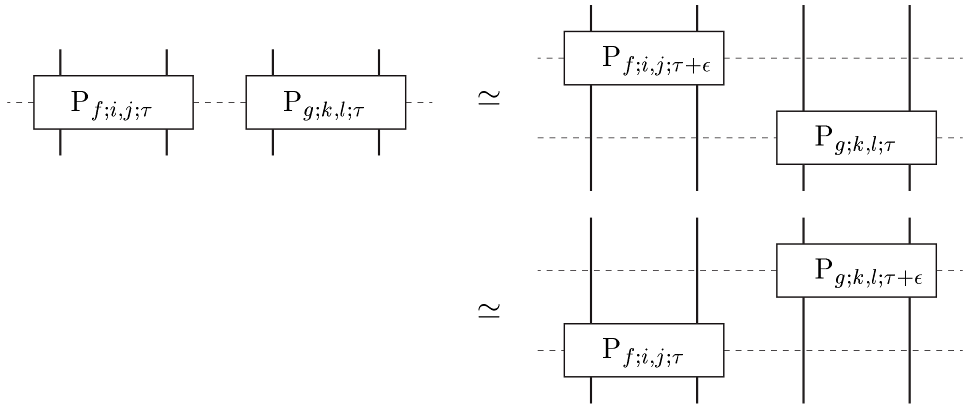
Thus this theorem teaches us that information flows in an entanglement specification network **as if** it flows along full forward paths in a compositional manner, the functions applied to it being the labeling function of the corresponding **eP**. Of course, the acausal nature of such path forces us to stress that we have no actual physical flow but only an **as if** flow through full forward paths. Note in particular the sharp contrast with the actual physical causal order when applying projections (we omit track specification):

$$\Psi = (\underbrace{\mathrm{P}_{f_3;6} \circ \mathrm{P}_{f_1;6}}_{\mathrm{P}_{f_1;6} \circ \mathrm{P}_{f_3;6}} \circ \mathrm{P}_{f_2;5} \circ \mathrm{P}_{f_5;4} \circ \mathrm{P}_{f_4;3} \circ \mathrm{P}_{f_7;2} \circ \underbrace{\mathrm{P}_{f_8;1} \circ \mathrm{P}_{f_6;1}}_{\mathrm{P}_{f_6;1} \circ \mathrm{P}_{f_8;1}})(\Psi)$$

We can indeed freely commute those **eP**'s that happen at the same time, although obviously in general we don't have $f_3 \circ f_1 = f_1 \circ f_3$ and $f_8 \circ f_6 = f_6 \circ f_8$ at all. For $\{i,j\} \cap \{k,l\} = \emptyset$, $i \neq j$ and $k \neq l$ we have

$$\begin{aligned}
\mathrm{P}_{f;i,j;\tau} \otimes \mathrm{P}_{g;k,l;\tau} \otimes \mathrm{id}_{\neg\{i,j,k,l\}} &= \Big(\mathrm{P}_{f;i,j;\tau+\epsilon} \otimes id_{\neg\{i,j\}}\Big) \circ \Big(\mathrm{P}_{g;k,l;\tau} \otimes id_{\neg\{k,l\}}\Big) \\
&= \Big(\mathrm{P}_{g;k,l;\tau+\epsilon} \otimes id_{\neg\{k,l\}}\Big) \circ \Big(\mathrm{P}_{f;i,j;\tau} \otimes id_{\neg\{i,j\}}\Big) ,
\end{aligned}$$

for ϵ sufficiently small. Thus, in pictures:



It then also follows that considering only one **eP** at each time is no actual restriction. Remarkable is the independence of ϕ_{out} on

$$\sum_{\alpha_2\alpha_3\alpha_4\alpha_5} \Phi_{\alpha_2\alpha_3\alpha_4\alpha_5} \cdot e^{(2)}_{\alpha_2} \otimes e^{(3)}_{\alpha_3} \otimes e^{(4)}_{\alpha_4} \otimes e^{(5)}_{\alpha_5}$$

in the above example or on

$$\sum_{\alpha_1\ldots\hat{\alpha}_{in}\ldots\alpha_n} \Phi_{\alpha_1\ldots\hat{\alpha}_{in}\ldots\alpha_n} \cdot e^{(1)}_{\alpha_1} \otimes \ldots \otimes \hat{e}^{(i_{in})}_{\alpha_{in}} \otimes \ldots \otimes e^{(n)}_{\alpha_n}$$

in general — except for the few singular cases where Ψ becomes $\mho$ [Subsection 4.4].

The cautious reader might have some concerns with respect to typing and interchangeability of the different labelings discussed in Subsection 2.2. And he is very right to do so. We postpone the discussion of these matters to Proposition 4.4 in Subsection 4.1.

ii. *Partial forward paths.* Theorem 3.4 on compositionality extends beyond the case of full forward paths. Consider paths of the shape:

$$[(\nu_1, \tau_{in}), (\nu_1, \tau_1)[\cdot((\nu_1, \tau_1), (\nu_2, \tau_1))\cdot](\nu_2, \tau_1), (\nu_2, \tau_2)[\cdot\ldots$$
$$\ldots\cdot](\nu_{||\Gamma||}, \tau_{||\Gamma||-1}), (\nu_{||\Gamma||}, \tau_{||\Gamma||})[\cdot((\nu_{||\Gamma||}, \tau_{||\Gamma||}), (\nu_{||\Gamma||+1}, \tau_{||\Gamma||}))\cdot](\nu_{||\Gamma||+1}, \tau_{||\Gamma||}), (\nu_{||\Gamma||+1}, \tau_{out})]$$

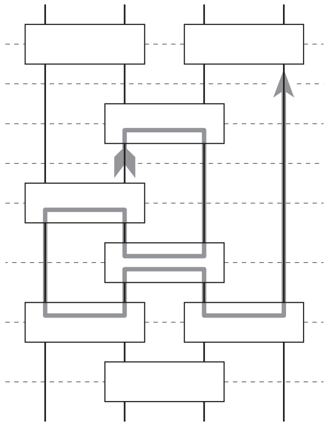
provided that at times τ_{in} and τ_{out} no **eP**'s occur. So we drop $\Gamma_1 \in \mathbb{I}(\Xi)$ and $\Gamma_{|\Gamma|} \in \mathbb{O}(\Xi)$ but replace these by:

- $\tau_{in} < \tau_1$ and $\tau_{||\Gamma||} < \tau_{out}$ assuring the path is forward;
- $\forall\gamma \in \{1, \ldots, ||\Gamma||\} : \tau_\gamma < \tau_{out}$.

The condition

$$[(\nu_1, \tau_{in}), (\nu_1, \tau_1)[\ \cap \mathbb{P}(\Xi) = \emptyset$$
$$](\nu_{||\Gamma||+1}, \tau_{||\Gamma||}), (\nu_{||\Gamma||+1}, \tau_{out})]\ \cap\ \mathbb{P}(\Xi) = \emptyset$$

replaces $[(\nu_1,), (\nu_1, \tau_1)[\ \cap \mathbb{P}(\Xi) = \emptyset$ and $](\nu_{||\Gamma||+1}, \tau_{||\Gamma||}), (\nu_{||\Gamma||+1},)]\ \cap\ \mathbb{P}(\Xi) = \emptyset$ as compared to full forward paths. Thus, the generalization is such that the input can be at any time (before the output) while the output should be after (in physical time) all other **eP**'s that take part in the path:



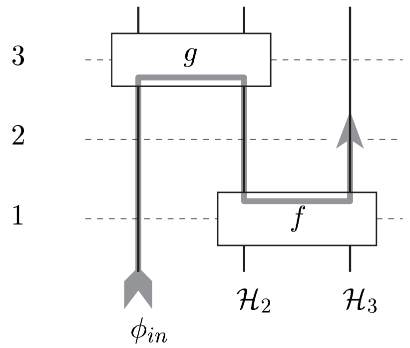
The time at which the path starts and ends doesn't coincide anymore with **in** and **out**. We indicate this by referring to the *initial* and *terminal path time* in *italic* contra **boldface** which we use for the **initial** and **terminal physical time** — as we have done so far and will continue to do so.

Definition 3.5 By a *partial forward path* we mean one of the above kind.

Corollary 3.6 *Theorem 3.4 extends to partial forward paths.*

There is a clear asymmetry in the above result with respect to the flow of physical time. We can have $\tau_\lambda < \tau_0$ but we cannot have $\tau_\lambda > \tau_{out}$. It is easy to find counter examples which establish this.

Counter example 3.7 Consider the following path.



At time 2 the third carrier of state is in general entangled to the second one since they are in the entangled state labeled by an arbitrary function f.

There also seems to be an obvious *physical argument* purely relying on causality. If there is an **eP** through which the path passes at a time later than the time at which we evaluate ϕ_{out}, we could still decide not to effectuate the **eP** or change it. Having a functional correspondence between ϕ_{in} and ϕ_{out} which depends on this "still changeable" function g would be in conflict with causality. However!

Riddle 3.8 (Cancelability on the left for atomically singular maps) If for partial forward paths we drop the condition $\tau_\lambda < \tau_{out}$, whenever in addition to ϕ_{in} being free in Ψ it is also given that ϕ_{out} is free in Ψ, then we can actually prove that compositionality as it is expressed in Corollary 3.6 still holds. How can this be possible in view of the above outlined argument on causality with respect to the actual physical time?

Solution. See Subsection 5.5. □

The title of this riddle refers to a notion which we will introduce in Subsection 5.3.

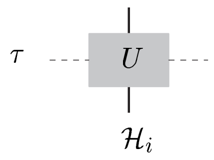
Many more surprising facts of this kind will be discussed in Subsection 4.1 and Subsection 4.3. One crucial lesson which follows from the fact that only the partial forward paths defined above assure that the output is free is that for arbitrary intermediate points of the path no statement can be made on compositional dependence on the input. We can deduce from this that the compositionality result is a *global* one and not a *local* one. That is, the compositionality theorem only makes statements on how the input and the output relate for a path of a particular shape with restrictions both on

1. how it passes through **eP**'s, and,
2. where the input and output are located relative to the rest of the network;

it makes no statements on the state at intermediate points of a path.

3.2 Local unitary actions

We will now allow our networks to contain local unitary transformations. This means that at coordinates (i, τ) where no is located we might encounter:



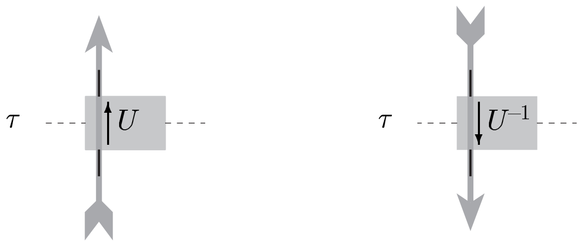
The definition of a path stays the same:

- Paths pass unitary transformations as if they would not be there.

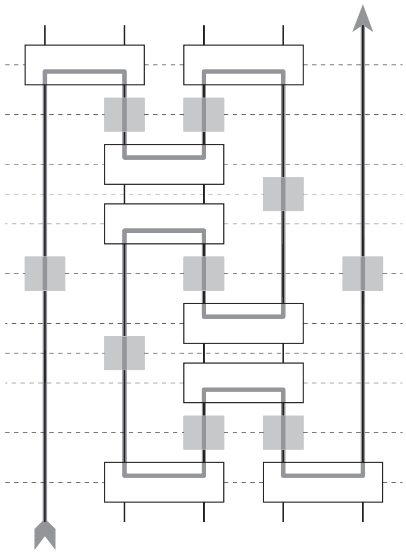
We introduce a labeling convention.

Convention 3.9 *If a path goes forward (in physical time) through the unitary transformation we label it as U. If the path however goes backward through the unitary transformation we label it by U^{-1}, that is, by U's inverse.*

This "reversal" of the label should be interpreted as representing the unitary transformation "as seen by the path", and not by physical time.



Extending our example from the previous section this could yield:



Theorem 3.10 (Extended compositionality) *With the assumptions of Theorem 3.4 when* Γ *additionally passes through unitary transformation*

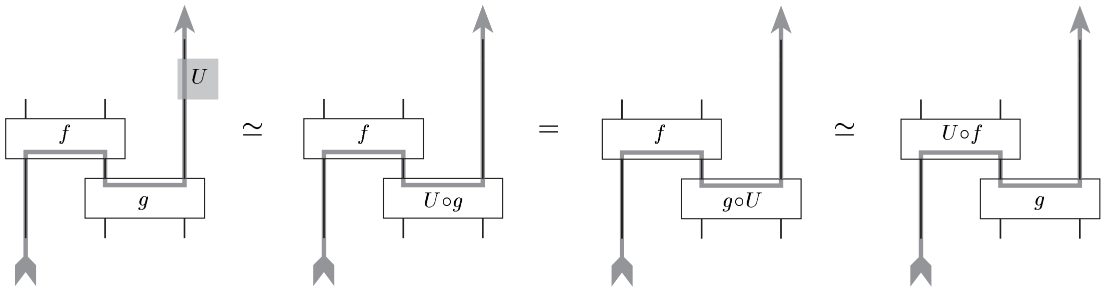
$$U_\gamma : \mathcal{H}_{\nu_{\gamma+1}} \to \mathcal{H}_{\nu_{\gamma+1}}$$

either or both after $\mathrm{P}_{f_{\gamma-1};\nu_{\gamma-1},\nu_\gamma;\tau_{\gamma-1}}$ *and before* $\mathrm{P}_{f_\gamma;\nu_\gamma,\nu_{\gamma+1};\tau_\gamma}$ *then the functionality theorem still hold with as modified compositional expression:*

$$\phi_{out} = (U_{||\Gamma||+1} \circ f_{||\Gamma||} \circ U_{||\Gamma||} \circ \cdots \circ f_{\gamma+1} \circ U_{\gamma+1} \circ f_\gamma \circ U_\gamma \circ f_{\gamma-1} \circ \cdots \circ U_2 \circ f_1 \circ U_1)(\phi_{in})\,.$$

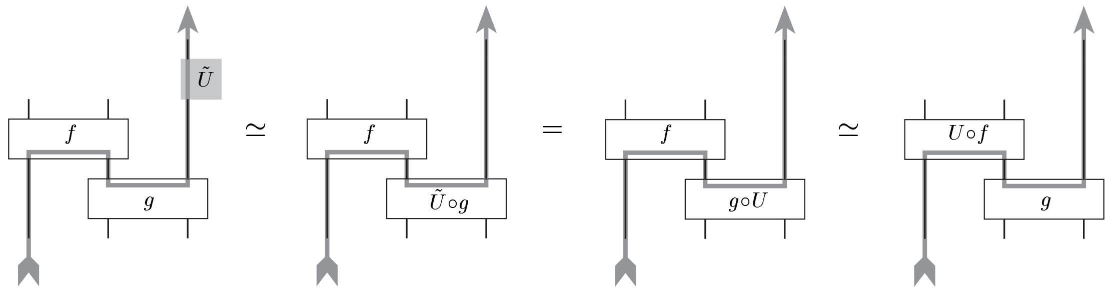
Thus, in terms of the labels, there is no difference between the path passing through an **eP** or through a unitary transformation. Except then for the fact that in the case of unitary transformations it are these transformations themselves which appear as functions while for **eP**'s it are the functional labels. It then follows that we can factor unitary components out as we did in Subsection 2.3.

Corollary 3.11 (Unitary correction 1) *If* g *and* U *commute then we have:*



Hence effectuating the unitary action U "at the end of the path" is equivalent to having implemented $U \circ f$. Thus we can use U to correct f into the desired result $U \circ f$. We now weaken the commutation requirement.

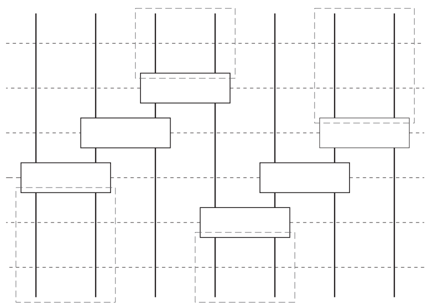
Corollary 3.12 (Unitary correction 2) *If $g \circ U = \tilde{U} \circ g$ then we have:*



In this more general case we can use $\tilde{U}$ to correct f into the desired result $U \circ f$ since effectuating the unitary action $\tilde{U}$ at the end of the path is equivalent to having implemented $U \circ f$. This non-commutative version has important applications as it is shown in Subsections 3.3 and 3.4.

Definition 3.13 An **eP** is *positive* for a path Γ iff Γ enters from above; an **eP** is *negative* for a path Γ iff Γ enters from below. In pictures:

Definition 3.14 An **eP** $\mathrm{P}_{f;i,j;\kappa} \in (\Xi)$ *is free from below* iff for each $\tau \in \{1, \ldots, \kappa - 1\}$ there exists neither $\mathrm{P}_{-;k,j;\tau} \in (\Xi)$ nor $\mathrm{P}_{-;i,l;\tau} \in (\Xi)$. Analogously we define *being free from above in* Ξ for an **eP**. In pictures:



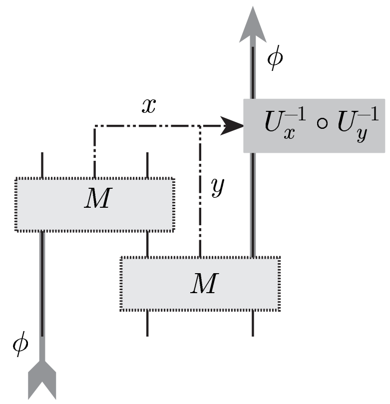
We conclude by mentioning some "causality respecting" strategies one can use to compile entanglement specification networks into networks of prepared states, non-local measurements, local unitary transformations and classical communication.

(1a) A negative and free from below appearing **eP** can be conceived as a prepared state.

(1b) A negative appearing **eP** can be corrected by a unitary transformation.

(2) A positive appearing **eP** can be corrected either via Corollary 3.11 or Corollary 3.12 depending on its commutation properties with the negative appearing **eP** which it precedes (along the path).

Note that combining (1b) and (2) allows to produce a non-probabilistic teleportation protocol without relying on prepared states.



We will mostly rely on (1a) for eliminating uncertainties for negative appearing **eP**'s. In Subsections 3.3 and 3.4 we present some more refined thoughts on correction of unwanted measurement outcomes.

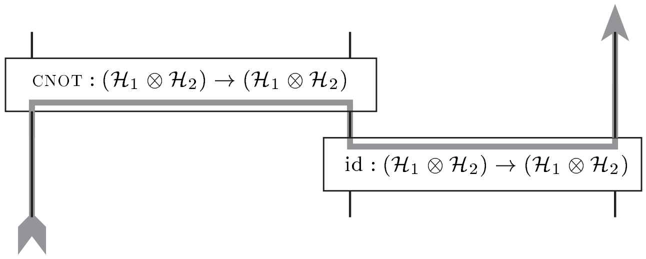
3.3 Example: logic gate teleportation

We now reproduce the quantum logic gate teleportation protocol of [33]. Quantum gate teleportation is an ingenious variant of the teleportation protocol. One does not just teleport a *pair of* states but at the same time reproduces the pair as if a CNOT-gate has acted on it. The protocol only uses measurements and locally unitary operations [Definition B.3] while a CNOT-gate itself is a non-local unitary operation. The action of a CNOT-gate on the standard qubit base of pure tensors is

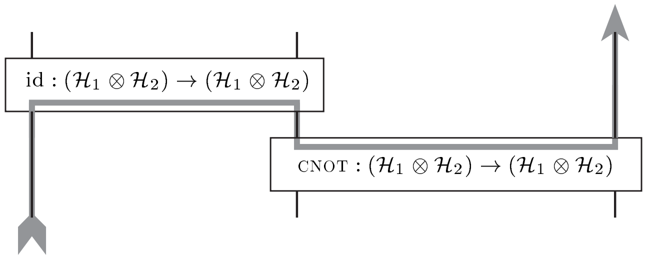
$$\left|\begin{matrix}0\\0\end{matrix}\right\rangle \mapsto \left|\begin{matrix}0\\0\end{matrix}\right\rangle \qquad \left|\begin{matrix}0\\1\end{matrix}\right\rangle \mapsto \left|\begin{matrix}0\\1\end{matrix}\right\rangle \qquad \left|\begin{matrix}1\\0\end{matrix}\right\rangle \mapsto \left|\begin{matrix}1\\1\end{matrix}\right\rangle \qquad \left|\begin{matrix}1\\1\end{matrix}\right\rangle \mapsto \left|\begin{matrix}1\\0\end{matrix}\right\rangle$$

Thus, the action on the first qubit is id but on the second qubit is either id or π depending on the value of the first qubit. The action is clearly unitary since it merely swaps the base vectors $|10\rangle$ and $|11\rangle$. It is obvious that such a unitary action does not factor into two local unitary operators $U_1 \otimes U_2$ each acting on one qubit since U_1 would have to be id what makes U_2 undefinable.

In view of the results obtained above there are two immediate candidates to do such a thing "at least probabilistically", namely implementation of $\mathrm{id} \circ \mathrm{CNOT}$



and implementation of CNOT $\circ$ id



The problem is now reduced to eliminating the probabilistic nature as we did it for ordinary teleportation in Subsection 2.3. In the case of id$\circ$CNOT we prepare the $\mathrm{id}_{\mathcal{H}_1\otimes\mathcal{H}_2}$-labeled state and in the case of CNOT$\circ$id we prepare the CNOT-labeled state, eliminating the uncertainty due to the first (in time) **eP**. In addition to this we have to perform a unitary correction in order to eliminate any "unwanted measurement outcome" in the measurement which we perform to implement the second **eP**.

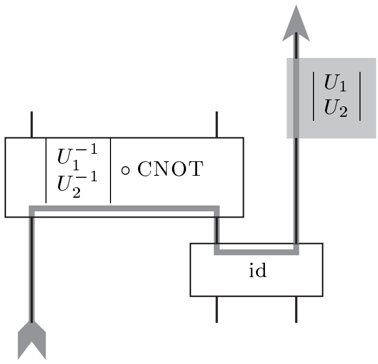
i. *Implementing* id $\circ$ CNOT. We can apply Corollary 3.11 since id commutes with everything. Since we want the unitary correction to consist of local unitary operation we have to complete

$$\Psi_{\mathrm{CNOT}} = \left|\begin{matrix}0\\0\end{matrix}\right\rangle \otimes \left|\begin{matrix}0\\0\end{matrix}\right\rangle + \left|\begin{matrix}0\\1\end{matrix}\right\rangle \otimes \left|\begin{matrix}0\\1\end{matrix}\right\rangle + \left|\begin{matrix}1\\0\end{matrix}\right\rangle \otimes \left|\begin{matrix}1\\1\end{matrix}\right\rangle + \left|\begin{matrix}1\\1\end{matrix}\right\rangle \otimes \left|\begin{matrix}1\\0\end{matrix}\right\rangle$$

into a measurement base $\{\Psi_{\mathrm{CNOT}}, \Phi, \ldots\}$ such that the labeling function f_Φ of each base vector Φ is unitary and in particular that CNOT$\circ f_\Phi^{-1}$ (which is the corresponding required unitary correction) is local. Hence for some unitary transformations $U_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $U_2 : \mathcal{H}_2 \to \mathcal{H}_2$ we have

$$\mathrm{CNOT} \circ f_\Phi^{-1} = \left|\begin{matrix}U_1\\U_2\end{matrix}\right| \qquad \text{that is} \qquad f_\Phi = \left|\begin{matrix}U_1^{-1}\\U_2^{-1}\end{matrix}\right| \circ \mathrm{CNOT}\,,$$

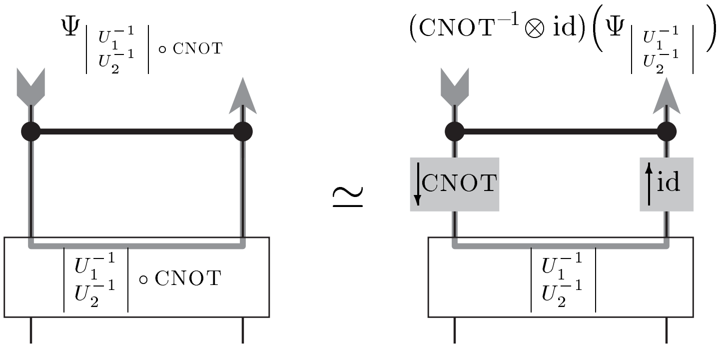
where we used the vertical representation of the respective tensors $U_1\otimes U_2$ and $U_1^{-1}\otimes U_2^{-1}$ to make types graphically match with the above representation of CNOT and Ψ_{CNOT}. This yields the following situation for correcting the obtained measurement outcome.



For $U_1 = U_2 = \mathrm{id}$ we have $\Phi = \Psi_{\mathrm{CNOT}}$. As alternative choices for U_1 and U_2 let $U_1, U_2 \in \{\mathrm{id}\,,\, \pi\,,\, \mathrm{id}^*\,,\, \pi^*\}$. We show orthogonality of the resulting base vectors. Observe that

$$\Psi_{\left|\begin{smallmatrix} U_1^{-1} \\ U_2^{-1}\end{smallmatrix}\right| \circ \mathrm{CNOT}} = (\mathrm{CNOT}^{-1} \otimes \mathrm{id})\Big(\Psi_{\left|\begin{smallmatrix} U_1^{-1} \\ U_2^{-1}\end{smallmatrix}\right|}\Big)\,.$$

If this doesn't seem obvious at first, just look at the following pictures having our compositionality results in mind.



Recalling Convention 3.9, the physical outputs in the two pictures give exactly the lefthandside and righthandside in the equation. Strictly spoken this doesn't follow from Theorem 3.10 but it does have the same flavor. It does follow by Lemma 5.8 in Section 5 where we prove Theorem 3.10 — this graphical justification of the above equation nicely illustrates how one can use compositionality in quantum arithmetic.

Proposition 3.15 *Functional labeling of states commutes with the tensor:*

1. $\Psi_f \otimes \Psi_g = \Psi_{f\otimes g}$;

2. $\Psi_{f_1\otimes g_1} \perp \Psi_{f_2\otimes g_2} \iff \Psi_{f_1} \perp \Psi_{f_2}$ or $\Psi_{f_1} \perp \Psi_{f_2}$.

Proof. Using the construction of the tensor product presented in Appendix A we have

$$f \otimes g \;=\; \Big(\sum_{\alpha\beta} f_{\alpha\beta}\langle e^{(1)}_\alpha \mid -\rangle \cdot e^{(3)}_\beta\Big) \otimes \Big(\sum_{\gamma\delta} g_{\gamma\delta}\langle e^{(2)}_\gamma \mid -\rangle \cdot e^{(4)}_\delta\Big)$$

$$
\begin{aligned}
&= \sum_{\alpha\beta\gamma\delta} f_{\alpha\beta} g_{\gamma\delta} \langle e^{(1)}_\alpha \mid -\rangle \langle e^{(2)}_\gamma \mid -\rangle \cdot e^{(3)}_\beta \otimes e^{(4)}_\delta \\
&= \sum_{\alpha\beta\gamma\delta} f_{\alpha\beta} g_{\gamma\delta} \langle e^{(1)}_\alpha \otimes e^{(2)}_\gamma \mid -\rangle \cdot e^{(3)}_\beta \otimes e^{(4)}_\delta
\end{aligned}
$$

such that

$$
\begin{aligned}
\Psi_f \otimes \Psi_g &= \Big(\sum_{\alpha\beta} f_{\alpha\beta} \cdot e^{(1)}_\alpha \otimes e^{(3)}_\beta\Big) \otimes \Big(\sum_{\gamma\delta} g_{\gamma\delta} \cdot e^{(2)}_\gamma \otimes e^{(4)}_\delta\Big) \\
&= \sum_{\alpha\beta\gamma\delta} f_{\alpha\beta} g_{\gamma\delta} \cdot e^{(1)}_\alpha \otimes e^{(2)}_\gamma \otimes e^{(3)}_\beta \otimes e^{(4)}_\delta \\
&= \Psi_{f\otimes g}\,.
\end{aligned}
$$

It is further obvious that the above proof does not depend on the chosen labeling. The second claim follows by Proposition A.11 in Appendix A. □

By this proposition we have

$$
\Psi_{\left|\begin{smallmatrix} U_1^{-1} \\ U_2^{-1} \end{smallmatrix}\right|} = \left|\begin{matrix} \Psi_{U_1^{-1}} \\ \Psi_{U_2^{-1}} \end{matrix}\right\rangle .
$$

Since $\mathrm{CNOT}^{-1} \otimes \mathrm{id}$ is a unitary transformation which preserves orthogonality Proposition 3.16 reduces the required orthogonality to the Bell-base being orthogonal.

So we now have a protocol which does what we required it to do. *This is however not the protocol introduced in* [33]. A problem with this protocol might be the actual physical realizability of the measurement which is highly non-local and requires itself effectuation of a CNOT-gate [Appendix B]. A crucial feature considered by the authors to justify their protocol [33] is *fault tolerance* [58]. We will slightly elaborate on features of that kind below but our main goal here is to illustrate a methodology which uses the compositionality theorems.

ii. *Implementing* $\mathrm{CNOT} \circ \mathrm{id}$*.* The protocol proposed in [33] will be obtained when we implement $\mathrm{CNOT}\circ\mathrm{id}$. Since CNOT does not commute with arbitrary unitary transformations this requires the use of Corollary 3.12. As measurement base we take

$$
\left|\begin{matrix} \Psi_{U_1} \\ \Psi_{U_2} \end{matrix}\right\rangle \qquad \text{with} \qquad U_1, U_2 \in \{\mathrm{id}\,,\, \pi\,,\, \mathrm{id}^*\,,\, \pi^*\}\,,
$$

that is, we take the product of the respective Bell-bases of $\mathcal{H}_1 \otimes \mathcal{H}_1$ and $\mathcal{H}_2 \otimes \mathcal{H}_2$. By Proposition 3.16 we know that the corresponding labeling functions are

$$
\left|\begin{matrix} U_1 \\ U_2 \end{matrix}\right| \qquad \text{with} \qquad U_1, U_2 \in \{\mathrm{id}\,,\, \pi\,,\, \mathrm{id}^*\,,\, \pi^*\}\,.
$$

One verifies that

$$
\mathrm{CNOT}\circ\left|\begin{matrix} \pi \\ \mathrm{id} \end{matrix}\right| = \left|\begin{matrix} \pi \\ \pi \end{matrix}\right|\circ\mathrm{CNOT} \qquad \mathrm{CNOT}\circ\left|\begin{matrix} \mathrm{id}^* \\ \mathrm{id} \end{matrix}\right| = \left|\begin{matrix} \mathrm{id}^* \\ \mathrm{id} \end{matrix}\right|\circ\mathrm{CNOT} \qquad \mathrm{CNOT}\circ\left|\begin{matrix} \pi^* \\ \mathrm{id} \end{matrix}\right| = \left|\begin{matrix} \pi \\ \pi \end{matrix}\right|\circ\left|\begin{matrix} \mathrm{id}^* \\ \mathrm{id} \end{matrix}\right|\circ\mathrm{CNOT}
$$

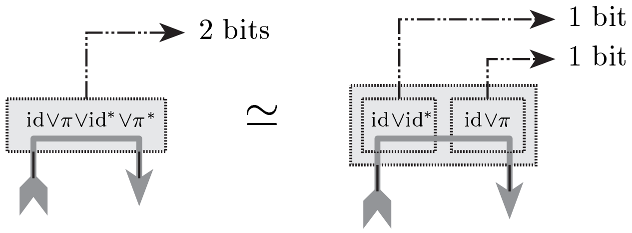
$$\text{CNOT}\circ\left|\begin{matrix}\text{id}\\ \pi\end{matrix}\right| = \left|\begin{matrix}\text{id}\\ \pi\end{matrix}\right|\circ\text{CNOT} \qquad \text{CNOT}\circ\left|\begin{matrix}\text{id}\\ \text{id}^*\end{matrix}\right| = \left|\begin{matrix}\text{id}^*\\ \text{id}^*\end{matrix}\right|\circ\text{CNOT} \qquad \text{CNOT}\circ\left|\begin{matrix}\text{id}\\ \pi^*\end{matrix}\right| = \left|\begin{matrix}\text{id}\\ \pi\end{matrix}\right|\circ\left|\begin{matrix}\text{id}^*\\ \text{id}^*\end{matrix}\right|\circ\text{CNOT}$$

the latter column since $\pi^* = \pi \circ \text{id}^*$. Thus by

$$\left|\begin{matrix}U_1\\ U_2\end{matrix}\right| = \left|\begin{matrix}U_1\\ \text{id}\end{matrix}\right| \circ \left|\begin{matrix}\text{id}\\ U_2\end{matrix}\right|$$

it follows that all necessary unitary corrections factor in a pair of local unitary corrections. In principle this completes the design of the protocol.

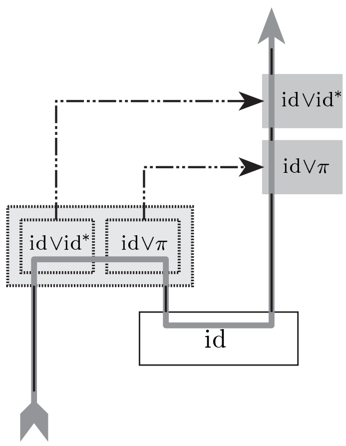
iii. *Using the Pauli group structure.* The protocol designed above is the one to be found in [33] but still it might appear not that familiar to the reader. We will reshape both the teleportation and the logic gate teleportation protocol. To do this it suffices to notice that by $\pi^* = \pi \circ \text{id}^*$ we can think of a measurement in the Bell-base as consisting of two sub-measurements



This operationally incarnates the by Booleans x and z parametrized representation

$$\psi_{xz} := |0x\rangle + (-1)^z|1(1-x)\rangle$$

of the Bell-base as can be found in the standard literature. This observation transforms the above teleportation protocol into:



In case of logic gate teleportation we can additionally use the following.

Proposition 3.16 *Functional labeling of* **eP***'s commutes with the tensor:*

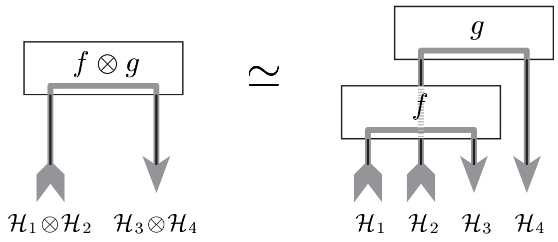
$$\mathrm{P}_f \otimes \mathrm{P}_g = \mathrm{P}_{f\otimes g}\,.$$

Proof. We have

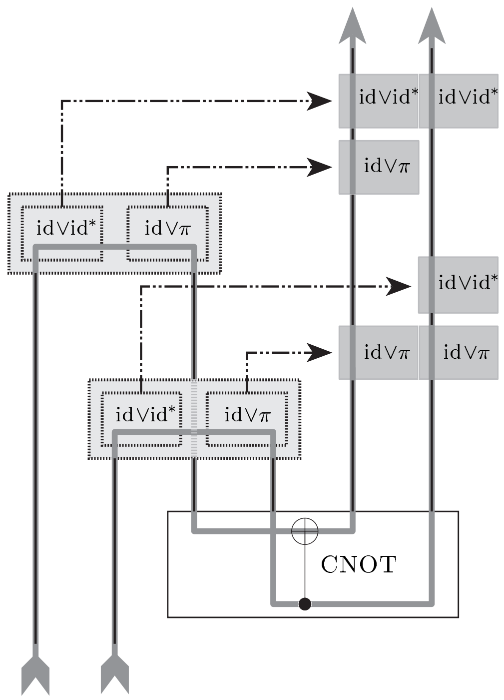
$$\mathrm{P}_f \otimes \mathrm{P}_g = \mathrm{P}_{\Psi_f} \otimes \mathrm{P}_{\Psi_g} = \mathrm{P}_{\Psi_f \otimes \Psi_g} = \mathrm{P}_{\Psi_{f\otimes g}} = \mathrm{P}_{f\otimes g}$$

by Proposition 3.16 and Proposition B.4. □

Thus we can think of an **eP** labeled by the tensor of a pair of functions as a pair of **eP**'s:



This transforms the above protocol into:



If in this picture we substitute "id∨id*" by "Z" and "id∨π" by "X" and permute the carriers a bit then we exactly obtain FIG 2 in [33]. We represented the "virtual" action of the CNOT-label on the path by the usual representation of the CNOT-gate in quantum circuits. This clearly exposes where effectuation of the CNOT-gate is "hidden" in the logic gate teleportation protocol.

iv. *Locally correctable gate teleportation via Clifford groups.* We reproduced logic gate teleportation for the particular case of a CNOT-gate. As shown in [33] this example extends to all unitary transformations in the *Clifford group*, that is, the group generated by the CNOT-gate, the *Hadamard gate* and the *phase gate* via composition and tensors, given a fixed number of qubits ($\sim$ size of the space) on which they act. This can easily be seen since the Clifford group can be alternatively defined as

$$\mathcal{CG}_n := \{V \in \mathcal{U}_n \mid V \circ [\mathcal{G}_n] \circ V^{-1} \subseteq \mathcal{G}_n\}$$

where the *Pauli group* is defined as

$$\mathcal{G}_n := \{U_1 \otimes \ldots \otimes U_n \mid U_1, \ldots, U_n \in \mathcal{G}_1\}$$

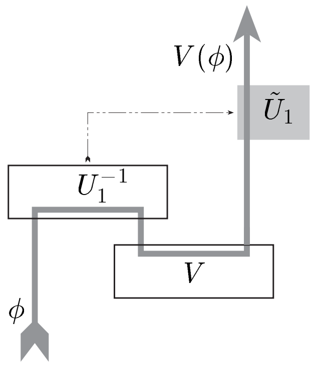
with

$$\mathcal{G}_1 := \{\alpha \cdot \mathrm{id}\,, \alpha \cdot \pi,\, \alpha \cdot \mathrm{id}^*,\, \alpha \cdot \pi^* \mid \alpha \in \{1, i, -1, -i\}\}$$

and $\mathcal{U}_n$ being all unitary transformations acting on n qubits. Thus each $V \in \mathcal{CG}_n$ is such that for each $U_1 \in \mathcal{G}_n$ there is a $\tilde{U}_1 \in \mathcal{G}_n$ such that

$$V \circ U_1 = \tilde{U}_1 \circ V\,.$$

This brings us within the scope of Corollary 3.12 while assuring that the necessary correction $\tilde{U}_1$ is a tensor of local unitary operations. We obtain



since

$$V = V \circ U_1 \circ U_1^{-1} = \tilde{U}_1 \circ V \circ U_1^{-1}.$$

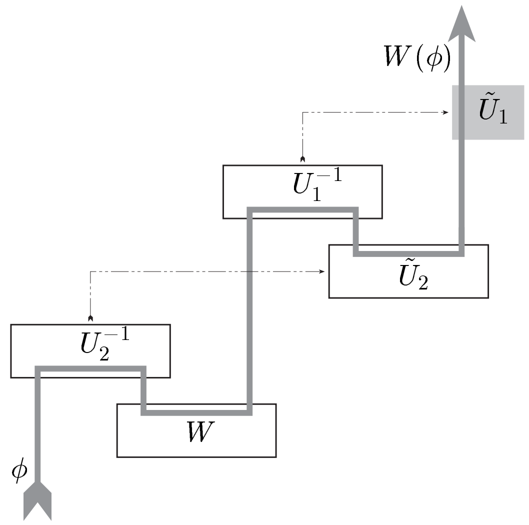
As also indicated in [33] we can push this line of reasoning further when considering

$$\mathcal{CG}_n^2 := \{W \in \mathcal{U}_n \mid W \circ [\mathcal{G}_n] \circ W^{-1} \subseteq \mathcal{CG}_n\}$$

which contains some unitary transformations not in $\mathcal{CG}_n$ such as the *Toffoli gate*, the $\pi/8$ *gate* and the *controlled phase gate*. Each $W \in \mathcal{CG}_n^2$ is such that for each $U_2 \in \mathcal{G}_n$ there is a $\tilde{U}_2 \in \mathcal{CG}_n$ such that

$$W \circ U_2 = \tilde{U}_2 \circ W\,.$$

Of course $\tilde{U}_2 \in \mathcal{CG}_n$ is itself not necessarily local but we can implement it exactly as we implemented $V \in \mathcal{CG}_n$ above. We obtain



since

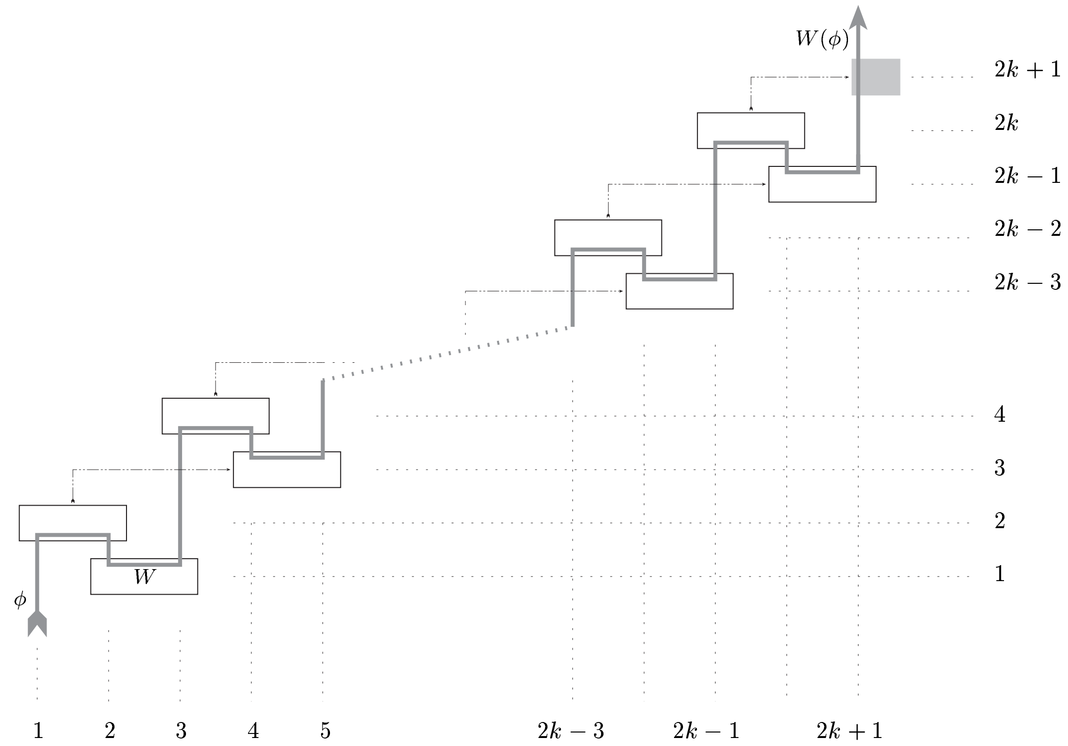
$$W = W \circ U_2 \circ U_2^{-1} = \tilde{U}_2 \circ W \circ U_2^{-1} = \tilde{U}_2 \circ U_1 \circ U_1^{-1} \circ W \circ U_2^{-1} = \tilde{U}_1 \circ \tilde{U}_2 \circ U_1^{-1} \circ W \circ U_2^{-1}.$$

In this setting $\tilde{U}_1$ is the only unitary transformation which we have to apply and it is a tensor of local ones. In terms of measurements and classical communication, the "outcome" U_2^{-1} of the first measurements conditions the second prepared state as being $\tilde{U}_2$ while the outcome U_1^{-1} imposes effectuation of a unitary transformation $\tilde{U}_1$. Note here also that the temporal order of the **eP**'s is fully determined by the causal flow of classical information.

This procedure clearly extends inductively for $k \geq 2$ to all unitary transformations in

$$\mathcal{CG}_n^{k+1} := \{W \in \mathcal{U}_n \mid W \circ [\mathcal{G}_n] \circ W^{-1} \subseteq \mathcal{CG}_n^k\}\,.$$

Causality of the flow of classical information again fully fixes the **eP**'s temporal order.



Hence we possess a method for implementing all unitary transformations in

$$\mathcal{CG}_n^\omega := \bigcup_{k\in\mathbb{N}} \mathcal{CG}_n^k$$

while restricting to

- Bell-base measurements, and
- Unitary transformations of the Pauli group (and hence local).

On the other hand the prepared states can be any

$$\Psi_f \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n \quad \text{with} \quad f \in \mathcal{CG}_n^\omega .$$

Definition 3.17 The *height* of an **eP** network is the minimal required number of distinct instances of time at which there is an **eP** or a unitary transformation. The *depth* of an **eP** network is the number of **eP**'s and unitary transformations acting on each carrier. The *width* of an **eP** network is the number of participating carriers.

The height of the above network is $2k+1$ given that $W \in \mathcal{CG}_n^k$. We need k measurements, $k-1$ corrections in terms of a prepared state, 1 correction in terms of a local unitary operator and of course also the initially prepared state encoding W itself. The causal dependencies in terms of classical information obstruct any of these operations to be performed at the same time. The depth is only 2. Each qubit in its prepared state will only be subjected to one additional operation, either a Bell-base measurement or a unitary transformation. The width is also $2k+1$ since we need k **eP**'s representing prepared entangled states and to this we have to add the initial state.

3.4 Example: parallel composition via Bell-base measurements

As an exercise we will produce an *algorithm* which, for unitary operations, substitutes *sequential* by *parallel* composition and hence reduces the depth of a network. We will team this line of thought up with the imposed constraints and corresponding developments of the previous subsection. In particular does the passage from sequential to parallel composition allow to implement the so-called first law of *fault-tolerant computation* put forward in [55] : "Don't use the same bit twice". As mentioned above fault-tolerance was also the main motivation behind the logic gate construction [33]. Of course, with this prophecy of a reality with quantum computers in mind, one has to admit that currently it is still sort of a bit early to foresee how the ultimate architecture of those machines will be. Form a computational point of view the operational compilation tool needed to pass from an entanglement specification network to a setting of (prepared states,) measurements, unitary transformations and classical communication has to take this architecture into account. But it seems fair to say that restricting the necessary required operations to a limited number of already realized ones would increase the global realizability of any proposed model for computation. Thus we will do so. We do repeat that our main goal is exposing a methodology emerging from information flow based reasoning about entanglement rather than proposing a particular model for quantum computation.

i. *Parallel composition via Bell-base measurements.* Assume that we have a variety of available entangled states but that we can only perform Bell-base measurements. We want to implement a composition of unitary transformations

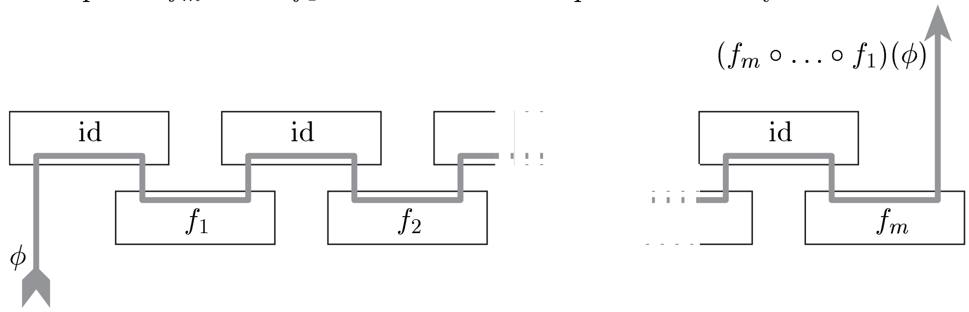
$$f_m \circ \ldots \circ f_1$$

where we assume that all involved unitary transformations and thus also $f_m \circ \ldots \circ f_1$ have the same type namely

$$f_1\,,\ \ldots\ ,f_m\,,(f_m \circ \ldots \circ f_1) : \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n \to \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n\,.$$

A motivation for performing such a composition could be that one possesses a primitive set of gates from which by composition and tensoring one can accomplish the action of a much larger set. As an example, given the CNOT-gate, the Hadamard gate and the phase gate, by composition and tensor one can produce arbitrary unitary operations in $\mathcal{CG}_n$.

The composite $f_m \circ \ldots \circ f_1$ would be realized "probabilistically" when effectuating:



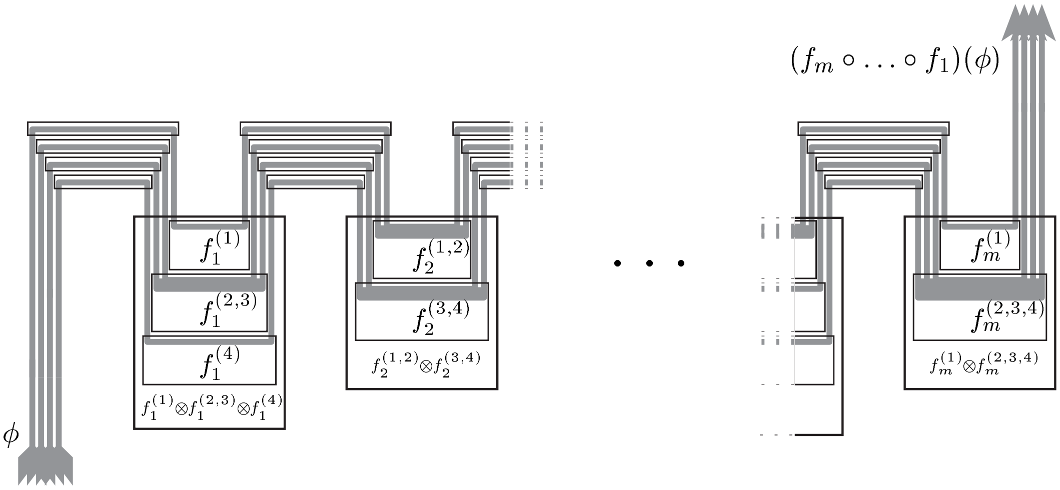
Each

$$(\mathrm{id} : \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n \to \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n)\text{-labeled}$$

is of course at each turn a tensor of n

$$(\mathrm{id} : \mathcal{H}_\nu \to \mathcal{H}_\nu)\text{-labeled 's}\,.$$

We can think of the "global path" as consisting of n *subpaths* which interact in the f_i-labeled **eP**'s. In general each f_i is itself also a tensor of several functions.



In benefit of clarity of the picture we re-indexed the types of the **eP** a bit such that now

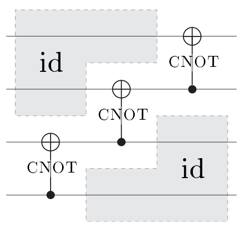
$$\mathrm{P}_{f_1}\,, \ldots, \mathrm{P}_{f_m} : \mathcal{H}_n \otimes \ldots \otimes \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n \to \mathcal{H}_n \otimes \ldots \otimes \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n\,.$$

We have the following quantitative facts:

- Both the height and the depth of this network are 2;
- The width of the network is $(2m + 1) \times n$;
- The required number of Bell-base measurements is $m \times n$.

The above mentioned limited variety of available gates which motivates performing composition now translates in a limited variety of available prepared states. However, preparing these states in general involves application of the corresponding gate.

Some of the $f_i^{(\alpha,\ldots,\zeta)}$'s might be identities and in that case there is a redundancy in the number of Bell-base measurements since we will have occurrences of $\mathrm{id} \circ \ldots \circ \mathrm{id} = \mathrm{id}$ encoded as several **eP**'s. As an example consider the case of



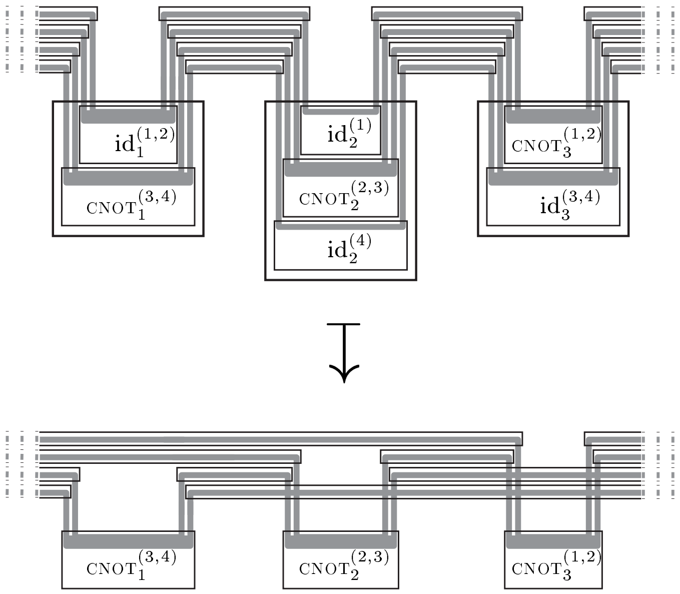
that is

$$\left(\mathrm{CNOT}_3^{(1,2)} \otimes \mathrm{id}_3^{(3,4)}\right) \circ \left(\mathrm{id}_2^{(1)} \otimes \mathrm{CNOT}_2^{(2,3)} \otimes \mathrm{id}_2^{(4)}\right) \circ \left(\mathrm{id}_1^{(3,4)} \otimes \mathrm{CNOT}_1^{(1,2)}\right).$$

We can reduce the number of measurements by substituting "triples"

$$(\text{measurement}\,;\ \text{id-prepared state}\,;\ \text{measurement})$$

by only one measurement.



In the case of this example this passage reduces the number of measurements to $(m \times n)/2$. If more than 4 initial states are involved while applying one two-qubit gate at the time the reduction factor will be $1/(n-2)$ in stead of $1/2$. Note also that after this reduction the network obtained when starting from a composition which has two or more non-trivial gates at the same time is equivalent ($\simeq$ the same up to permutation of the carriers) as the one obtained when starting from the modified composition which one gets after substituting each occurrence of

$$\ldots \circ (f^{(\alpha_1,\ldots,\zeta_1)} \otimes g^{(\alpha_2,\ldots,\zeta_2)} \otimes \ldots) \circ \ldots$$

by either

$$\ldots \circ (\mathrm{id}^{(\alpha_1,\ldots,\zeta_1)} \otimes g^{(\alpha_2,\ldots,\zeta_2)} \otimes \ldots) \circ (f^{(\alpha_1,\ldots,\zeta_1)} \otimes \mathrm{id}^{(\alpha_2,\ldots,\zeta_2)} \otimes \ldots) \circ \ldots$$

or

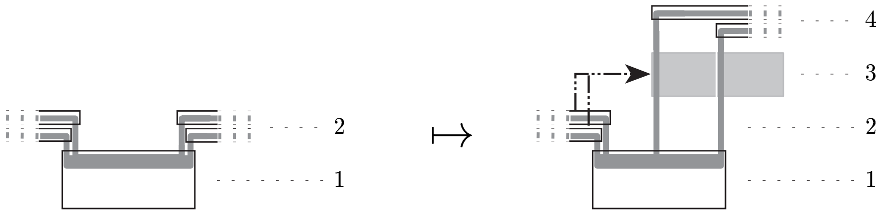
$$\ldots \circ (f^{(\alpha_1,\ldots,\zeta_1)} \otimes \mathrm{id}^{(\alpha_2,\ldots,\zeta_2)} \otimes \ldots) \circ (\mathrm{id}^{(\alpha_1,\ldots,\zeta_1)} \otimes g^{(\alpha_2,\ldots,\zeta_2)} \otimes \ldots) \circ \ldots$$

(and hence increasing m). Below we put this reduction in an algorithmic shape.

ii. *Computation via a generating set of gates.* Of course one desires to pass to a deterministic protocol via some correction taking into account "undesired measurement outcomes". Let us for the sake of the argument assume that

$$f_1, \ldots, f_m \in \mathcal{CG}_n \,.$$

As demonstrated in Subsection 3.3 we can correct undesired measurement outcomes immediately "after" (with respect to the path direction) the **eP** which follows the measurement via unitary transformations effectuated strictly "later" (with respect to the physical time) than the measurement itself.



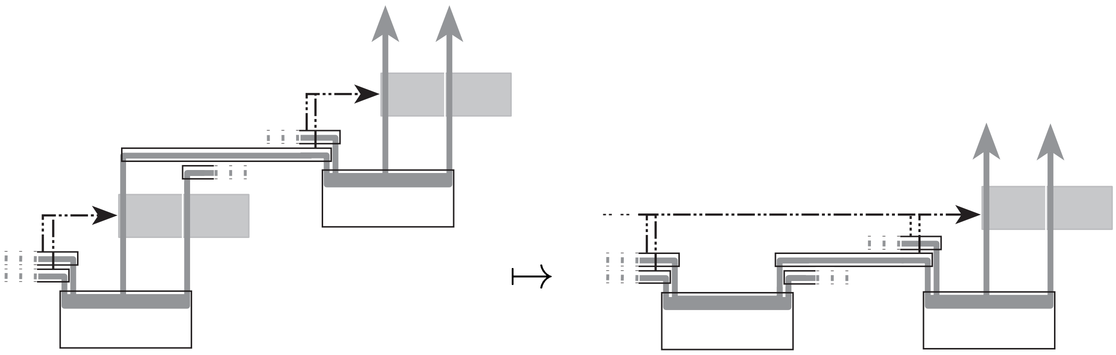
These corrections are *local unitary transformations* in $\mathcal{G}_n$ since $f_1, \ldots, f_m \in \mathcal{CG}_n$. Roughly spoken, the required number of unitary corrections as well as the height of the resulting network is proportional to the number m of gates one applies. However, one can reduce the height and the required number of unitary corrections by further exploiting the group structure of $\mathcal{G}_n$ and for each $\nu \in \{1, \ldots, n\}$ only applying one unitary correction at the end of each subpath. That is, each occurrence of

(measurement ; prepared state ; correction ; measurement ; prepared state ; correction)

along a subpath can be replaced by

(measurement ; prepared state ; measurement ; prepared state ; correction)

Below we make this reduction of unitary transformations precise.



In terms of m both the height and the number of required unitary corrections are now constants. Most importantly, by reducing the number of unitary transformations that we have to perform we reduce the errors that they might cause. The depth is now 2 and the height is 3. It is easy to see that we can extend this procedure to

$$f_1, \ldots, f_m \in \mathcal{CG}_n^k$$

following Subsection 3.3 paragraph (iv) but this will increase the height to $2k+1$. Also the width will increase.

iii. *A compilation algorithm.* We assume that at each instance of time we only apply one gate — it should be obvious that this does not impose any restriction. For convenience we will represent the gates themselves as f_i, that is, we trow away the identities in the tensor. Given are:

- Two numbers n and m respectively being the number of qubits involved and the number of gates one desires to apply consecutively.
- The respective initial states ϕ_ν of the qubits.
- The kinds of gates f_i we apply where the index $i \in \{1, \ldots, m\}$ specifies the order of application; specification of the list $\mathrm{Tracks}(f_i)$ being the qubits to which we apply f_i; below we will sometimes conceive $\mathrm{Tracks}(f_i)$ as a set rather than as a list.

Desired is the state of the qubits after being subjected to the gates. Available are

- Local unitary operations in $\mathcal{G}_n$;
- Bell-base measurements ;
- Prepared entangled states of which the functional labels include $f_1, \ldots, f_m$.

Assuming that all $f_i \in \mathcal{CG}_n$ we now define the network which produces this state.

- Denote by $|\mathrm{Tracks}(f_i)|$ the cardinality of $\mathrm{Tracks}(f_i)$. Consider

  $$n + \left(2 \cdot \sum_{i=1}^{i=m} |\mathrm{Tracks}(f_i)|\right)$$

  carriers of state.
- The initial state of these is

  $$\left(\bigotimes_{\nu=1}^{\nu=n} \phi_\nu\right) \otimes \left(\bigotimes_{i=1}^{i=m} \Psi_{f_i}\right).$$

- Set

  $$\mathrm{Gates}(\nu) := \left\{ i \in \{1, \ldots, m\} \;\middle|\; \nu \in \mathrm{Tracks}(f_i) \right\}.$$

  For each $\nu \in \{1, \ldots, m\}$ and each $i \in \mathrm{Gates}(\nu)$ set

  $$\underline{\mathrm{I}}(\nu, i) := \mathrm{Sup}(\{j \in \mathrm{Gates}(\nu) \mid j < i\} \cup \{0\})$$

and let $\mathrm{Order}(\nu, i)$ be the position of ν in the list $\mathrm{Tracks}(f_i)$. Now define for each such pair (ν, i) the numbers

$$\begin{aligned}
\mathrm{In}_i^{(\nu)} &:= \nu \quad \underline{\text{iff}} \quad \underline{\mathrm{I}}(\nu,i) = 0 \\
&:= n + \left(2 \cdot \sum_{j=1}^{j=\underline{\mathrm{I}}(\nu,i)-1} |\mathrm{Tracks}(f_j)|\right) + |\mathrm{Tracks}(f_{\underline{\mathrm{I}}(\nu,i)})| + \mathrm{Order}(\nu, \underline{\mathrm{I}}(\nu,i)) \quad \underline{\text{otherwise}} \\
\mathrm{Out}_i^{(\nu)} &:= n + \left(2 \cdot \sum_{j=1}^{j=i-1} |\mathrm{Tracks}(f_j)|\right) + \mathrm{Order}(\nu, i)
\end{aligned}$$

and apply a Bell-base measurement $M_i^{(\nu)}$ to the carriers $\mathrm{In}_i^{(\nu)}$ and $\mathrm{Out}_i^{(\nu)}$.

- Let $\left(U_i^{(\nu)}\right)^{-1}$ be the function labeling the outcome state of the measurement $M_i^{(\nu)}$. Then solve the equations

  $$\begin{cases} \forall i: \ \bigotimes \left\{ \tilde{U}_i^{(\nu)} \,\middle|\, \nu \in \mathrm{Tracks}(f_i) \right\} \circ f_i = f_i \circ \bigotimes \left\{ \tilde{U}_{\underline{\mathrm{I}}(\nu,i)}^{(\nu)} \circ U_i^{(\nu)} \,\middle|\, \nu \in \mathrm{Tracks}(f_i) \right\} \\ \forall \nu: \ \tilde{U}_0^{(\nu)} = \mathrm{id} \end{cases}$$

  using the Clifford group structure. For each $\nu \in \{1, \ldots, n\}$ define

  $$\overline{\mathrm{I}}(\nu) := \mathrm{Sup}(\mathrm{Gates}(\nu))$$

  and apply the unitary transformation $\tilde{U}_m^{(\nu)}$ to carrier

  $$n + \left(2 \cdot \sum_{j=1}^{j=\overline{\mathrm{I}}(\nu)-1} |\mathrm{Tracks}(f_j)|\right) + |\mathrm{Tracks}(f_{\overline{\mathrm{I}}(\nu)})| + \mathrm{Order}(\nu, \overline{\mathrm{I}}(\nu)) \,.$$

Using all the above one straightforwardly verifies that this indeed produces the desired state as output. Some quantitative facts.

- The height of this network is 3;
- The depth of this network is 2;
- The width of the network is

  $$n + \left(2 \cdot \sum_{i=1}^{i=m} |\mathrm{Tracks}(f_i)|\right) \,.$$

  This number lies between $2m + n$ and $2(m \times n) + n$.
- The required number of Bell-base measurements is

  $$\sum_{i=1}^{i=m} |\mathrm{Tracks}(f_i)| = \sum_{\nu=1}^{\nu=n} |\mathrm{Gates}(\nu)| \,.$$

  This number lies between m and $m \times n$.

- The required number of one-qubit unitary transformations is n.

One can extend this algorithm for arbitrary unitary transformations in $\mathcal{CG}_n^\omega$ using the method exposed in 3.3. But this of course changes the numbers.

- The height of the network is now $2k+1$ with
$$k := \mathrm{Inf}\{j \in \mathbb{N} \mid f_1, \ldots, f_m \in \mathcal{CG}_n^j\}\,.$$
- The depth of this network is still 2;
- The width of the network is now
$$n + \left(2 \cdot \sum_{i=1}^{i=m} k(i) \cdot |\mathrm{Tracks}(f_i)|\right)$$
with
$$k(i) := \mathrm{Inf}\{j \in \mathbb{N} \mid f_i \in \mathcal{CG}_n^j\}\,.$$
This number lies between $2m+n$ and $2(k \times m \times n) + n$.
- The required number of Bell-base measurements is now
$$\sum_{i=1}^{i=m} k(i) \cdot |\mathrm{Tracks}(f_i)|\,.$$
This number lies between m and $k \times m \times n$.
- The required number of one-qubit unitary transformations is still n.

3.5 Example: specificational quantum logic

We started this paper of by discussing the notions of pre- and post-specification in orthodox functional computing. This involved asserting that the input and the output of functional actions satisfy certain properties. We can do the same for our virtual paths. So far we only made assertions about the state of the input and the output of a path. We can extend this to asserting that at the input and the output the corresponding carrier satisfies certain properties [Subsection 2.1]. From the discussion in Subsection 2.1 it follows that one can put forward the following (at least conceptually) "operational" definition of a physical property [40].

Definition 3.18 A physical system *possesses a property* A_{P} iff in case we would perform the measurement $1 \cdot \mathrm{P} + 0 \cdot \mathrm{P}^{\perp}$ we obtain outcome 1 with certainty.

The properties which a quantum systems can possess are in bijective correspondence with the *complete lattice* of closed subspaces $\mathbb{L}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ [13]. An axiomatic characterization of this lattice can be found in [54].

Question 3.19 *Given is a path in an entanglement specification network. Is there a weakest pre-specification which guarantees a given post-specification?*

For a path Γ we can write $A \stackrel{\Gamma}{\Rightarrow} B$ whenever pre-specification A guarantees post-specification B and the *weakest pre-specification* for B, provided it exists, is then

$$\bigvee\{A \in \mathbb{L}(\mathcal{H}) \mid A \stackrel{\Gamma}{\Rightarrow} B\}\,.$$

Weakest pre-specifications are well-known in orthodox functional computing [23, 36] — also referred to as *weakest preconditions*. They extend to quantum systems and even more so to general property lattices [18, 20, 28]. This generalization has the classical case as its limit. Briefly, if we consider the supremum preserving pointwise extension $f : \mathbb{L}_1 \to \mathbb{L}_2$ of the functional action to the lattices of properties then the weakest pre-specification for B is $f^*(B)$ where $f^* : \mathbb{L}_2 \to \mathbb{L}_1$ is the infimum preserving Galois adjoint to f [20]. The orthodox case arises when both lattices are complete atomistic Boolean algebras. A *specificational quantum logic* would be the corresponding analogue to Hoare logic [35]. We think that a fruitful collaboration could emerge if some members of the quantum logic community and some of the theoretical computer science community would join forces on issues like these.

4 Requirements and time witnesses for compositionality

The passage from a *causal* reality to an *acausal* formal tool to reason about it obviously cannot exist without certain limitations. We expose them in this section.

4.1 Time reversal and complex phase

Let us indeed be courageous and expose some edges to the above.

Deceit 4.1 Above we boldly cheated. In Section 2.2 we discussed the existence of a canonical functional labeling of bipartite states and projectors via a linear isomorphism

$$\eta : (\mathcal{H}_1^* \otimes \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_2) :: \bar{\psi} \otimes \phi \mapsto \bar{\psi} \cdot \phi\,.$$

where

$$\bar{\psi} := \langle\psi \mid -\rangle_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathbb{C}\,.$$

Using $\mathcal{H} \simeq \mathcal{H}^{**}$ we obtain a complementary functional labeling via the linear isomorphism

$$\epsilon : (\mathcal{H}_1 \otimes \mathcal{H}_2) \to (\mathcal{H}_1^* \to \mathcal{H}_2) :: \psi \otimes \phi \mapsto \bar{\bar{\psi}} \cdot \phi$$

where

$$\bar{\bar{\psi}} := \langle\bar{\psi} \mid -\rangle_{\mathcal{H}_1^*} : \mathcal{H}_1^* \to \mathbb{C}\,.$$

Other linear isomorphisms which also allow functional labeling of bipartite states in terms of linear functions are

$$\epsilon^{\{e_i\}} : (\mathcal{H}_1 \otimes \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_2) :: e_i \otimes \phi \mapsto \bar{e}_i \cdot \phi\,,$$

where $\{e_i\}_i$ is a chosen base of $\mathcal{H}_1$, but these linear isomorphisms depend on the chosen base $\{e_i\}_i$. There are two obvious concerns which arise with respect to the results claimed in Subsection 3.1 and Subsection 3.2:

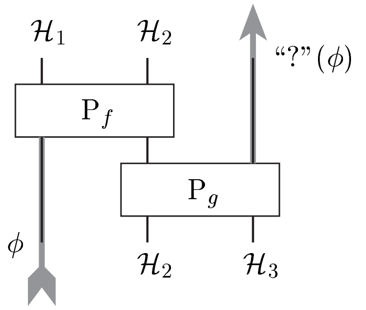
(1) First with respect to the ϵ-labeling. If we have two $\mathbf{eP}$'s in a teleportation-style setting projecting on $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\Phi \in \mathcal{H}_2 \otimes \mathcal{H}_3$, then the canonical labeling functions

$$f = \epsilon(\Psi) : \mathcal{H}_1^* \to \mathcal{H}_2 \qquad \text{and} \qquad g = \epsilon(\Phi) : \mathcal{H}_2^* \to \mathcal{H}_3$$

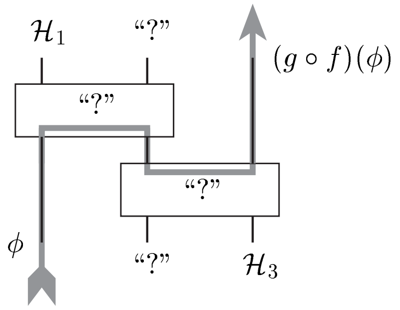
don't have matching types which allow composition since

$$\mathrm{codom}(f) = \mathcal{H}_2 \neq \mathcal{H}_2^* = \mathrm{dom}(g)\,.$$

Hence the composite "$g \circ f$" is ill-defined.



Dually, when we want to implement the composite of functions with matching types e.g. $f : \mathcal{H}_1 \to \mathcal{H}_2$ and $g : \mathcal{H}_2 \to \mathcal{H}_3$, then the projectors on the states $\eta^{-1}(f) \in \mathcal{H}_1^* \otimes \mathcal{H}_2$ and $\eta^{-1}(g) \in \mathcal{H}_2^* \otimes \mathcal{H}_3$ don't match for a "teleportation-style" implementation.



(2) Secondly with respect to an $\epsilon^{\{e_i\}}$-labeling. For a teleportation-style setting with projectors on $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\Phi \in \mathcal{H}_2 \otimes \mathcal{H}_3$ the functions

$$\epsilon^{\{e_i^{(1)}\}}(\Psi) : \mathcal{H}_1 \to \mathcal{H}_2 \qquad \text{and} \qquad \epsilon^{\{e_i^{(2)}\}}(\Phi) : \mathcal{H}_2 \to \mathcal{H}_3$$

depend on the choices of $\{e_i^{(1)}\}_i$ and $\{e_i^{(2)}\}_i$ so how can the compositionality result still hold for all possible choices? Dually, when implementing $f \circ g$ via a teleportation-style network different choices of bases yield different networks and hence different dependencies of the input $\phi_{in} \in \mathcal{H}_1$ on the output $\phi_{out} \in \mathcal{H}_3$.

However, the example of teleportation which we explicitly worked out in Subsection 2.3 using the base-dependent labeling seems to confirm that the compositionality theorems are indeed correct and so do all the investigations we did around the logic gate teleportation protocol. Further there is the explicitly proven compositionality in [3]. What is the catch to all this?

Recalling the in Section 2.2 introduced notation $\mathcal{H}_1 \looparrowright \mathcal{H}_2$ to denote the anti-linear functions from $\mathcal{H}_1$ to $\mathcal{H}_2$, the labeling

$$\epsilon^* : (\mathcal{H}_1 \otimes \mathcal{H}_2) \to (\mathcal{H}_1 \looparrowright \mathcal{H}_2) :: \psi \otimes \phi \mapsto \langle - \mid \psi \rangle \cdot \phi$$

seems to escape both of the above mentioned concerns since it is both canonical and provides matching types, and indeed, with respect to this labeling the compositionality theorems perfectly hold. Below we will derive from this fact how to make the ϵ-labeling. The adjustment which we will have to make has the following "feel":

> "When reading the time of a transparent clock from its back we have to conjugate the angles which the hands make relative to the noon position."

We have to do a similar thing when "reading" our paths whenever they undergo time-reversal due to an **eP**. This "conjugation of angles" produces an additional component to the composite which will make the types match. Although one could argue that this makes matters slightly more complicated it also provides a syntactical merit by providing a witness for reversal of the temporal direction of the path. In particular will the reader who has some experience with *-autonomous categories [8] and classical linear logic [29] welcome these witnesses.

This also affects the implementation of the $\epsilon^{\{e_i\}}$-labeling. As the compositionality theorem currently stands, in general it does not hold for the $\epsilon^{\{e_i\}}$-labeling. However, it does hold in many cases, as for example for teleportation, logic gate teleportation, parallel implementation of composition and its use in [3]. The additional component encoding time reversal mentioned above requires only a minor adjustment to Theorem 3.4 for the case of an $\epsilon^{\{e_i\}}$-labeling and the differences due to different choices of bases will then cancel each other out. More precisely, in order to make the compositionality theorems work we have to conjugate all elements in the matrix of the functions labeling negative appearing **eP**'s [Definition 3.13]. It then follows that if these matrices only have real coefficients then the result will remain invariant under this conjugation. And this was exactly the case for the maps id and π, id^*, π^* we needed for implementing teleportation, logic gate teleportation and parallel composition. All this becomes evident after doing the appropriate bit of mathematics.

i. *Interchangeability of different labelings.* First we introduce three bijections and their respective inverses which will be useful for the rest of the discussion. These are the *anti-linear* correspondences

$$r^* : \mathcal{H} \looparrowright \mathcal{H}^* :: \psi \mapsto \bar{\psi} \qquad \text{and} \qquad r_* : \mathcal{H}^* \looparrowright \mathcal{H} :: \bar{\psi} \mapsto \psi\,,$$

the *base-dependent* linear correspondences

$$s^{\{e_i\}} : \mathcal{H} \to \mathcal{H}^* :: e_i \mapsto \bar{e}_i \qquad \text{and} \qquad s_{\{e_i\}} : \mathcal{H}^* \to \mathcal{H} :: \bar{e}_i \mapsto e_i$$

and the involutive *base-dependent* and *anti-linear* conjugation of coefficients

$$c = c^{-1} : \mathcal{H} \nrightarrow \mathcal{H} :: \sum_i \psi_i \cdot e_i \mapsto \sum_i \bar{\psi}_i \cdot e_i$$

where we dropped explicit reference to the chosen base of $\mathcal{H}$ to simplify notation. The commutative diagram below expresses how these maps compose.

$$\begin{tikzcd}
& \mathcal{H} \arrow[dl, "r^*"'] \arrow[dd, leftrightarrow, "c"] \\
\mathcal{H}^* \arrow[ur, "r_*"'] \arrow[dr, "s_{\{e_i\}}"] & \\
& \mathcal{H} \arrow[ul, "s^{\{e_i\}}"]
\end{tikzcd}$$

For $\mathcal{H} \simeq \mathcal{H}^{**}$ there exists a correspondence $\psi \mapsto \bar{\bar{\psi}}$ which is natural in the categorical sense [26] when considering the category $\mathbb{C}$ of finite dimensional complex vector spaces with linear functions as morphisms. There exists no such thing with respect to $\mathcal{H} \simeq \mathcal{H}^*$. However, the non-linear correspondence $r^* : \mathcal{H} \nrightarrow \mathcal{H}^*$ is canonical in the sense that it does not depend on the choice of base.

We recall the general forms discussed in Subsection 2.2

$$\begin{aligned}
f \quad &:= \sum_{\alpha\beta} f_{\alpha\beta} \left\langle \langle e^{(1)}_\alpha \mid - \rangle_{\mathcal{H}_1} \middle| - \right\rangle_{\mathcal{H}_1^*} \cdot e^{(2)}_\beta \\
f^* \quad &:= \sum_{\alpha\beta} f_{\alpha\beta} \, \langle - \mid e^{(1)}_\alpha \rangle \cdot e^{(2)}_\beta \\
f^{\{e_i\}} &:= \sum_{\alpha\beta} f_{\alpha\beta} \, \langle e^{(1)}_\alpha \mid - \rangle \cdot e^{(2)}_\beta
\end{aligned}$$

of the maps respectively in $\mathcal{H}_1^* \to \mathcal{H}_2$, $\mathcal{H}_1 \nrightarrow \mathcal{H}_2$ and $\mathcal{H}_1 \to \mathcal{H}_2$. One verifies that these maps satisfy

$$\begin{array}{lll}
f^* = f \circ r^* & \qquad f^{\{e_i\}} = f \circ s^{\{e_i\}} & \qquad f^* = f^{\{e_i\}} \circ c \\
f = f^* \circ r_* & \qquad f = f^{\{e_i\}} \circ s_{\{e_i\}} & \qquad f^{\{e_i\}} = f^* \circ c \,.
\end{array}$$

Notice that the equalities at the right already show that in order to cancel the base-dependency of $f^{\{e_i\}}$ out it suffices to conjugate the coefficients of the input vector in the same base. They also guide us towards defining the actions corresponding to the above introduced maps r^*, $s^{\{e_i\}}$ and c, these respectively being

$$\tilde{r}^* : (\mathcal{H}_1^* \to \mathcal{H}_2) \to (\mathcal{H}_1 \nrightarrow \mathcal{H}_2) :: f \mapsto f^* = f \circ r^*$$

$$\tilde{s}^{\{e_i\}} : (\mathcal{H}_1^* \to \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_2) :: f \mapsto f^{\{e_i\}} = f \circ s^{\{e_i\}}$$

$$\tilde{c} : (\mathcal{H}_1 \to \mathcal{H}_2) \to (\mathcal{H}_1 \looparrowright \mathcal{H}_2) :: f^{\{e_i\}} \mapsto f^* = f^{\{e_i\}} \circ c$$

while their respective inverses are

$$\tilde{r}_*(f^*) := f^* \circ r_* \qquad \tilde{c}(f^*) := f^* \circ c \qquad \tilde{s}_{\{e_i\}}(f^{\{e_i\}}) := f^{\{e_i\}} \circ s_{\{e_i\}}$$

— we indeed use notation $\tilde{c}$ both for $\tilde{c}$ itself and for its inverse. The commutative diagram below concisely expresses how these actions compose.

$$\begin{tikzcd}
 & \mathcal{H}_1 \looparrowright \mathcal{H}_2 \arrow[dl, shift right, "\tilde{r}^*"'] \arrow[dd, leftrightarrow, "\tilde{c}"] \\
\mathcal{H}_1^* \to \mathcal{H}_2 \arrow[ur, shift right, "\tilde{r}_*"'] \arrow[dr, shift left, "\tilde{s}_{\{e_i\}}"] & \\
 & \mathcal{H}_1 \to \mathcal{H}_2 \arrow[ul, shift left, "\tilde{s}^{\{e_i\}}"]
\end{tikzcd}$$

Recalling that also $\mathcal{H}_1 \looparrowright \mathcal{H}_2$ is a vector space these maps $\tilde{r}^*$, $\tilde{r}_*$, $\tilde{s}^{\{e_i\}}$, $\tilde{s}_{\{e_i\}}$ and $\tilde{c}$ are isomorphisms between three distinct isomorphic copies of the vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$. When regarding

$$\Psi_f \;:=\; \sum_{\alpha\beta} f_{\alpha\beta}\, e_\alpha^{(1)} \otimes e_\beta^{(2)}$$

as the general form of the vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$ itself it easily follows that it are these three isomorphisms which express how the different labelings interrelate.

Proposition 4.2 (Labeling interchangeability) *The commuting diagram*

$$\begin{tikzcd}
 & & \mathcal{H}_1 \looparrowright \mathcal{H}_2 \arrow[dd, leftrightarrow, "\tilde{c}"] \\
\mathcal{H}_1 \otimes \mathcal{H}_2 \arrow[urr, "\epsilon^*"] \arrow[r, "\epsilon"] \arrow[drr, "\epsilon^{\{e_i\}}"'] & \mathcal{H}_1^* \to \mathcal{H}_2 \arrow[ur, leftrightarrow, "\tilde{r}"] \arrow[dr, leftrightarrow, "\tilde{s}"] & \\
 & & \mathcal{H}_1 \to \mathcal{H}_2
\end{tikzcd}$$

expresses how the different labelings interrelate.

Unfortunately the compositionality result cannot hold for all labelings.

Proposition 4.3 *In its current form Theorem 3.4 only holds for the ϵ^*-labeling.*

Proof. In Section 5 we prove that compositionality holds for the ϵ^*-labeling. It then follows by the discussion below that Theorem 3.4 could not generally hold for any arbitrary $\epsilon^{\{e_i\}}$-labeling as well. The compositionality theorem is ill-defined for the ϵ-labeling since types don't match. $\square$

Hence interchangeability of labelings should be expressed with the ϵ^*-labeling as reference.

Proposition 4.4 (Labeling interchangeability bis) *For* $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ *let*

$$\begin{array}{lll} f & := \epsilon(\Psi) & \in \mathcal{H}_1^* \to \mathcal{H}_2 \\ f^* & := \epsilon^*(\Psi) & \in \mathcal{H}_1 \looparrowright \mathcal{H}_2 \\ f^{\{e_i\}} & := \epsilon^{\{e_i\}}(\Psi) & \in \mathcal{H}_1 \to \mathcal{H}_2 \,. \end{array}$$

The commuting diagram

$$\begin{tikzcd}
 & \mathcal{H}_1^* \arrow[dr, "f"] & \\
\mathcal{H}_1 \arrow[ur, "r^*"] \arrow[rr, "f^*"] \arrow[dr, "c"'] & & \mathcal{H}_2 \\
 & \mathcal{H}_1 \arrow[ur, "f^{\{e_i\}}"'] &
\end{tikzcd}$$

expresses, in view of Proposition 4.3, the interchangeability of the ϵ*-,* ϵ^**- and* $\epsilon^{\{e_i\}}$*-labelings with respect to the compositionality theorems. Thus*

$$f = \tilde{r}_*(f^*) \qquad \text{and} \qquad f^{\{e_i\}} = \tilde{c}(f^*)\,.$$

This results in the following:

- In case of an ϵ-labeling the composite $f_{||\Gamma||} \circ \cdots \circ f_1$ should be replaced by

  $$\tilde{r}^*(f_{||\Gamma||}) \circ \cdots \circ \tilde{r}^*(f_1)$$

  what results in $||\Gamma||$ additional identical components

  $$r^* : \mathcal{H} \looparrowright \mathcal{H}^* :: \psi \mapsto \bar{\psi}$$

  which act each time Γ passes an **eP**. As mentioned above, this conjugation can be seen as an explicit witness of the time-reversal and an adequate heuristics is provided by the "reading a transparent clock from the back" metaphor.

- In case of an $\epsilon^{\{e_i\}}$-labeling the composite $f_{||\Gamma||} \circ \cdots \circ f_1$ should be replaced by

  $$\tilde{c}(f^{\{e_i\}}_{||\Gamma||}) \circ \cdots \circ \tilde{c}(f^{\{e_i\}}_1)$$

  where we dropped labels which distinguish between the bases of different Hilbert spaces to avoid notational overkill. For a forward path which always passes through an even number of **eP**'s this can be rewritten as

  $$f^{\{e_i\}}_{||\Gamma||} \circ (c \circ f^{\{e_i\}}_{||\Gamma||-1} \circ c) \circ \ldots \circ f^{\{e_i\}}_{2k} \circ (c \circ f^{\{e_i\}}_{2k-1} \circ c) \circ \ldots \circ f^{\{e_i\}}_2 \circ (c \circ f^{\{e_i\}}_1 \circ c)\,.$$

The functions appearing as
$$c \circ f_{2k-1}^{\{e_k\}} \circ c$$
are exactly those that label negative appearing **eP**'s. For their matrices we have
$$\begin{aligned}
(c \circ f_{2k-1}^{\{e_i\}} \circ c)\Big(\sum_\alpha \psi_\alpha \cdot e_\alpha\Big) &= (c \circ f_{2k-1}^{\{e_i\}})\Big(\sum_\alpha \bar{\psi}_\alpha \cdot e_\alpha\Big) \\
&= c\Big(\sum_{\alpha\beta} \bar{\psi}_\alpha \, (f_{2k-1}^{\{e_i\}})_{\alpha\beta} \cdot e_\beta\Big) \\
&= \sum_{\alpha\beta} \psi_\alpha \, (\bar{f}_{2k-1}^{\{e_i\}})_{\alpha\beta} \cdot e_\beta
\end{aligned}$$
while
$$f_{2k-1}^{\{e_i\}}\Big(\sum_\alpha \psi_\alpha \cdot e_\alpha\Big) = \sum_{\alpha\beta} \psi_\alpha \, (f_{2k-1}^{\{e_i\}})_{\alpha\beta} \cdot e_\beta \,.$$
Hence the net effect of all this comprises:

- The coefficients in the matrix of the functions which label positive appearing **eP**'s remain unaffected.
- The coefficients in the matrix of the functions which label negative appearing **eP**'s have to be conjugated.

Explicitly, simplifying notation by writing $(f_{2k-1}^{\{e_i\}})_{\alpha\beta}$ as $f_{\alpha\beta}^{2k-1}$, we have
$$\phi_{out} = \sum_{\alpha i_1 \ldots i_{||\Gamma||-1}\beta} \phi_\alpha^{in} \bar{f}_{\alpha i_1}^{1} f_{i_1 i_2}^{2} \ldots \bar{f}_{i_{2k-2} i_{2k-1}}^{2k-1} f_{i_{2k-1} i_{2k}}^{2k} \ldots \bar{f}_{i_{||\Gamma||-2} i_{||\Gamma||-1}}^{||\Gamma||-1} f_{i_{||\Gamma||-1}\beta}^{||\Gamma||} \cdot e_\beta^{(\nu_{out})}$$
for $\phi_{in} = \sum_\alpha \phi_\alpha^{in} \cdot e_\alpha^{(\nu_{in})}$ and $\phi_{out} = \sum_\beta \phi_\beta^{out} \cdot e_\beta^{(\nu_{out})}$.

This confirms our claims made at the beginning of this subsection.

The necessity to conjugate the coefficients in negatively appearing **eP**'s is not just a calculative accident, but can also be exposed in a canonical fashion. An analogous construction as the one above for $\epsilon^{\{e_i\}}$ can be done for ϵ too. We rewrite
$$\tilde{r}^*(f_{||\Gamma||}) \circ \cdots \circ \tilde{r}^*(f_1)$$
as
$$f_{||\Gamma||} \circ (r^* \circ f^{||\Gamma||-1} \circ r^*) \circ \ldots \circ f_{2k} \circ (r_* \circ f_{2k-1} \circ r^*) \circ \ldots \circ f_2^{\{e_i\}} \circ (r^* \circ f_1 \circ r^*) \,.$$
Note that for $f = \epsilon(\Psi)$ the anti-linear bijection
$$\bar{\epsilon} : (\mathcal{H}_1 \otimes \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_2^*) :: \Psi \mapsto r^* \circ \epsilon(\Psi) \circ r^*$$
is canonical. This yields an alternative formulation of the compositionality theorem where positive appearing **eP**'s should be labeled by ϵ while negative appearing **eP**'s should be

labeled by $\bar{\epsilon}$. This provides matching types for the composite which are alternatingly the duals of the types in the network:

$$\mathcal{H}_1 \stackrel{\bar{\epsilon}(\Psi_1)}{\longrightarrow} \mathcal{H}_2^* \stackrel{\epsilon(\Psi_2)}{\longrightarrow} \mathcal{H}_3 \ \cdots \ \mathcal{H}_{||\Gamma||-1} \stackrel{\bar{\epsilon}(\Psi_{||\Gamma||-1})}{\longrightarrow} \mathcal{H}_{||\Gamma||}^* \stackrel{\epsilon(\Psi_{||\Gamma||})}{\longrightarrow} \mathcal{H}_{||\Gamma||+1} \, .$$

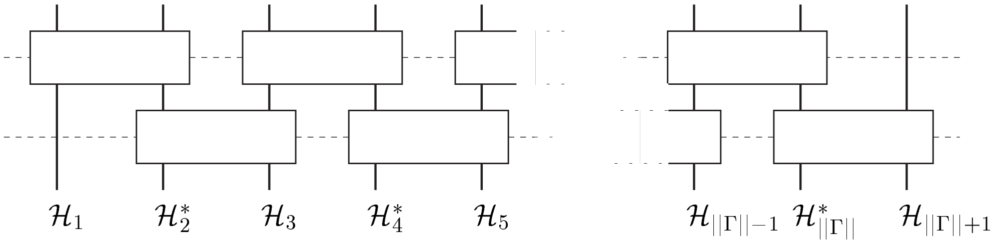
ii. *Implementing composition of linear functions.* Let us now address the question on how to implement a composition of linear functions. This is indeed a different problem than the one addressed above which only enables us to implement composition of anti-linear functions. Given is

$$\mathcal{H}_1 \stackrel{f_1}{\longrightarrow} \mathcal{H}_2 \stackrel{f_2}{\longrightarrow} \mathcal{H}_3 \ \cdots \ \mathcal{H}_{||\Gamma||-1} \stackrel{f_{||\Gamma||-1}}{\longrightarrow} \mathcal{H}_{||\Gamma||} \stackrel{f_{||\Gamma||}}{\longrightarrow} \mathcal{H}_{||\Gamma||+1}$$

with $||\Gamma||$ even. We consider carriers in

$$\mathcal{H}_1 \otimes \mathcal{H}_2^* \otimes \mathcal{H}_3 \otimes \ldots \otimes \mathcal{H}_{||\Gamma||-1} \otimes \mathcal{H}_{||\Gamma||}^* \otimes \mathcal{H}_{||\Gamma||+1} \, .$$

We arrange the **eP**'s as follows:



It remains to be specified on which states they project.

- If an **eP** appears positive we attribute $\eta^{-1}(f_{2k})$.
- If an **eP** appears negative we attribute $(r \circ \eta^{-1})(f_{2k-1})$ where we introduced a canonical (= base-independent) anti-linear mapping

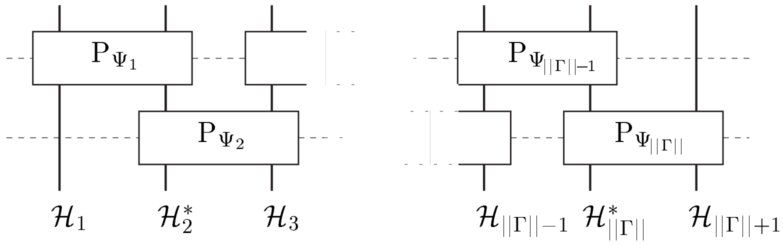
  $$r : (\mathcal{H}_{2k-1}^* \otimes \mathcal{H}_{2k}) \to (\mathcal{H}_{2k-1} \otimes \mathcal{H}_{2k}^*) :: \Psi \mapsto \bar{\Psi}$$

  with

  $$\bar{\Psi} = \sum_{ij} \bar{f}_{ij} \cdot e_i \otimes \bar{e}_j \qquad \text{for} \qquad \Psi = \sum_{ij} f_{ij} \cdot \bar{e}_i \otimes e_j \, .$$

  Note that $r \circ \eta^{-1}$ involves conjugation of the coefficients in the matrix $(f_{ij})_{ij}$.

Since both r and η^{-1} are canonical so is their composite. We claim that this is an implementation of the composition $f_{||\Gamma||} \circ \ldots \circ f_1$. To show this we apply Theorem 3.4 to the proposed network



with

$$\Psi_{2k-1} = (r \circ \eta^{-1})(f_{2k-1}) \qquad \text{and} \qquad \Psi_{2k} = \eta^{-1}(f_{2k})$$

being aware that that theorem only holds for the ϵ^*-labeling. We obtain

$$\mathcal{H}_1 \xrightarrow{g_1} \mathcal{H}_2^* \xrightarrow{g_2} \mathcal{H}_3 \ \cdots \ \mathcal{H}_{||\Gamma||-1} \xrightarrow{g_{||\Gamma||-1}} \mathcal{H}_{||\Gamma||}^* \xrightarrow{g_{||\Gamma||}} \mathcal{H}_{||\Gamma||+1}$$

with

$$g_{2k-1} = (\epsilon^* \circ r \circ \eta^{-1})(f_{2k-1}) \qquad \text{and} \qquad g_{2k} = (\epsilon^* \circ \eta^{-1})(f_{2k})$$

as the composite which relates ϕ^{out} to ϕ^{in}. Moreover,

$$\begin{aligned}
(\epsilon^* \circ r \circ \eta^{-1})(f_{2k-1})(\psi) &= \epsilon^*\Big(\sum_{ij} \bar{f}_{ij}^{2k-1} e_i \otimes \bar{e}_j\Big)(\psi) \\
&= \sum_{ij} \bar{f}_{ij}^{2k-1} \langle \psi \mid e_i \rangle \cdot \bar{e}_j \\
&= \sum_{ij} \bar{\psi}_i \bar{f}_{ij}^{2k-1} \cdot \bar{e}_j \\
&= (r^* \circ f_{2k-1})(\psi)
\end{aligned}$$

so

$$(\epsilon^* \circ r \circ \eta^{-1})(f_{2k-1}) = r^* \circ f_{2k-1}$$

and analogously

$$(\epsilon^* \circ \eta^{-1})(f_{2k}) = f_{2k} \circ r_* \,.$$

Hence

$$\begin{aligned}
g_{||\Gamma||} \circ \ldots \circ g_1 &= f_{||\Gamma||} \circ r_* \circ r^* \circ f_{||\Gamma||-1} \circ \ldots \circ f_2 \circ r_* \circ r^* \circ f_1 \\
&= f_{||\Gamma||} \circ \ldots \circ f_1
\end{aligned}$$

since $r_* \circ r^* = id$. This concludes the proof of our claim. Note that it is again the combination of base-independence and matching types which yields a correct (alternative) implementation of the compositionality theorem.

iii. *Corrections to unitary actions.* We now look at the implications of the above for Theorem 3.10 which extends compositionality to local unitary actions.

Proposition 4.5 *In its current form Theorem 3.10 only holds for the ϵ^*-labeling.*

Proof. See the proof of Proposition 4.3. $\square$

This implies the following for the other labelings.

- In case of an ϵ-labeling the composite

$$U_{||\Gamma||+1} \circ f_{||\Gamma||} \circ U_{||\Gamma||} \circ \cdots \circ U_2 \circ f_1 \circ U_1$$

should be replaced by

$$U_{||\Gamma||+1} \circ \tilde{r}^*(f_{||\Gamma||}) \circ U_{||\Gamma||} \circ \cdots \circ U_2 \circ \tilde{r}^*(f_1) \circ U_1$$

with identical interpretation as discussed above.

- In case of an $\epsilon^{\{e_i\}}$-labeling the composite should be replaced by

$$U_{||\Gamma||+1} \circ \cdots \circ \tilde{c}(f_{2k}^{\{e_i\}}) \circ U_{2k} \circ \tilde{c}(f_{2k-1}^{\{e_i\}}) \circ U_{2k-1} \circ \cdots \circ U_1\,,$$

that is,

$$\begin{aligned} U_{||\Gamma||+1} \circ \cdots \circ f_{2k}^{\{e_i\}} \circ c \circ U_{2k} \circ f_{2k-1}^{\{e_i\}} \circ c \circ U_{2k-1} \circ \cdots \circ U_1 \\ = U_{||\Gamma||+1} \circ \cdots \circ f_{2k}^{\{e_i\}} \circ \bar{U}_{2k}^{\{e_i\}} \circ (c \circ f_{2k-1}^{\{e_i\}} \circ c) \circ U_{2k-1} \circ \cdots \circ U_1 \end{aligned}$$

where

$$\bar{U}_{2k}^{\{e_i\}} := c \circ U_{2k} \circ c : \mathcal{H}_{2k} \to \mathcal{H}_{2k}$$

is the unitary transformation obtained by conjugating the matrix of U_{2k} in the base $\{e_i^{(2k)}\}_i$. The net effect of this is that in addition to the elements of the matrix of the functions that label negative appearing **eP**'s we additionally have to do the following:

  - When the path goes backward through a local unitary action then the elements of the matrices (in the chosen base for the Hilbert space on which it acts) of the unitary transformations which label it have to be conjugated. Since they are labeled by the inverse of the local unitary action itself this results in transposing the original matrix in the base $\{e_i^{(2k)}\}_i$.

This necessity to conjugate matrix of the unitary transformations that label local unitary actions through which the path goes backward again can be derived from reasoning via forcing matching types and canonicity. When using the ϵ-labeling for positively appearing **eP**'s and the $\bar{\epsilon}$-labeling for the negatively appearing ones the alternatingly dualized types requires substitution of $U_{2k} : \mathcal{H}_{2k} \to \mathcal{H}_{2k}$ by

$$\bar{U}_{2k} := r^* \circ U_{2k} \circ r_* : \mathcal{H}_{2k}^* \to \mathcal{H}_{2k}^*$$

such that we get

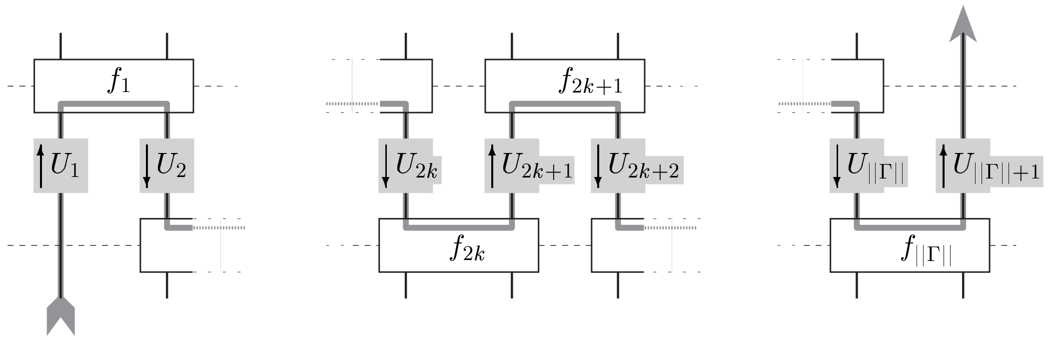
$$\mathcal{H}_1 \cdots \mathcal{H}_{2k-1} \stackrel{\bar{\epsilon}(\Psi_{2k-1})}{\longrightarrow} \mathcal{H}_{2k}^* \stackrel{\bar{U}_{2k}}{\longrightarrow} \mathcal{H}_{2k}^* \stackrel{\epsilon(\Psi_{2k})}{\longrightarrow} \mathcal{H}_{2k+1} \cdots \mathcal{H}_{||\Gamma||+1}\,.$$

We conclude by summarizing the formal results for the $\epsilon^{\{e_i\}}$-labeling.

Lemma 4.6 *Let*

$$g := U_{||\Gamma||+1} \circ f_{||\Gamma||} \circ U_{||\Gamma||} \circ \cdots \circ U_{2k+2} \circ f_{2k+1} \circ U_{2k+1} \circ f_{2k} \circ U_{2k} \circ \cdots \circ U_2 \circ f_1 \circ U_1$$

be a composite of $\bar{\epsilon}$-labels for **eP***'s in a configuration such that the path passes through them in the same order and the same temporal direction as for the network below.*



Then we have

$$g(\phi_{in}) = \phi_{out}$$

iff for the matrices of the corresponding $\epsilon^{\{e_i\}}$-labels we have

$$\phi^{out}_\beta = \sum_{\alpha i_1 \ldots i_{||\Gamma||+1} j_1 \ldots j_{||\Gamma||}} \phi^{in}_\alpha U^1_{\alpha i_1} \bar{f}^1_{i_1 j_1} \bar{U}^2_{j_1 i_2} \cdots$$

$$\cdots \bar{U}^{2k}_{j_{2k-1} i_{2k}} f^{2k}_{i_{2k} j_{2k}} U^{2k+1}_{j_{2k} i_{2k+1}} \bar{f}^{2k+1}_{i_{2k+1} j_{2k+1}} \bar{U}^{2k+2}_{j_{2k+1} i_{2k+2}} \cdots$$

$$\cdots \bar{U}^{||\Gamma||}_{j_{||\Gamma||-1} i_{||\Gamma||}} f^{||\Gamma||}_{i_{||\Gamma||} j_{||\Gamma||}} U^{||\Gamma||+1}_{j_{||\Gamma||} i_{||\Gamma||+1}}$$

with $\phi_{in} = \sum_\alpha \phi^{in}_\alpha \cdot e^{(\nu_{in})}_\alpha$ and $\phi_{out} = \sum_\beta \phi^{out}_\beta \cdot e^{(\nu_{out})}_\beta$.

iv. *Using matrices to represent anti-linear maps.* The above also motivates a definition for the matrix of an anti-linear map.

Definition 4.7 The *matrix of an anti-linear map* $f^* : \mathcal{H}_1 \looparrowright \mathcal{H}_2$ for the bases $\{e^{(1)}_\alpha\}_\alpha$ of $\mathcal{H}_1$ and $\{e^{(2)}_\beta\}_\beta$ of $\mathcal{H}_2$ is the matrix of the linear map $f^* \circ r_* : \mathcal{H}_1^* \to \mathcal{H}_2$ for the bases $\{\langle e^{(1)}_\alpha \mid -\rangle\}_\alpha$ of $\mathcal{H}_1^*$ and $\{e^{(2)}_\beta\}_\beta$ of $\mathcal{H}_2$.

It follows that the matrix of an anti-linear function f are exactly the coefficients $(f_{\alpha\beta})_{\alpha\beta}$ to be found in the generic form

$$f = \sum_{\alpha\beta} f_{\alpha\beta} \, \langle - \mid e^{(1)}_\alpha \rangle \cdot e^{(2)}_\beta \, .$$

Since this is also the case for both linear labelings the following is then obvious.

Proposition 4.8 *Let $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$. The following matrices are the same.*

- *The matrix $(f_{\alpha\beta})_{\alpha\beta}$ of coefficients when Ψ is expressed in the base $\{e_\alpha^{(1)} \otimes e_\beta^{(2)}\}_{\alpha\beta}$.*
- *The matrix of $\epsilon(\Psi) : \mathcal{H}_1^* \to \mathcal{H}_2$ in the bases $\{\langle e_\alpha^{(1)} \mid -\rangle\}_\alpha$ of $\mathcal{H}_1^*$ and $\{e_\beta^{(2)}\}_\beta$ of $\mathcal{H}_2$.*
- *The matrix of $\epsilon^*(\Psi) : \mathcal{H}_1 \looparrowright \mathcal{H}_2$ in the bases $\{e_\alpha^{(1)}\}_\alpha$ of $\mathcal{H}_1$ and $\{e_\beta^{(2)}\}_\beta$ of $\mathcal{H}_2$.*
- *The matrix of $\epsilon^{\{e_\alpha^{(1)}\}}(\Psi) : \mathcal{H}_1 \to \mathcal{H}_2$ in the bases $\{e_\alpha^{(1)}\}_\alpha$ of $\mathcal{H}_1$ and $\{e_\beta^{(2)}\}_\beta$ of $\mathcal{H}_2$.*

In particular do we have that the labeling

$$\epsilon^* : (\mathcal{H}_1 \otimes \mathcal{H}_2) \looparrowright (\mathcal{H}_1 \to \mathcal{H}_2)$$

as well as the labeling transformations

$$\tilde{c} : (\mathcal{H}_1 \looparrowright \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_2) \qquad \text{and} \qquad r_* : (\mathcal{H}_1 \looparrowright \mathcal{H}_2) \to (\mathcal{H}_1^* \to \mathcal{H}_2)$$

preserve matrices.

Remark 4.9 The base $\{e_\alpha^{(1)}\}_\alpha$ of $\mathcal{H}_1$ used to express the matrix and the one used to define the labeling $\epsilon^{\{e_\alpha^{(1)}\}} : (\mathcal{H}_1 \otimes \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_2)$ have to coincide.

Since matrix calculus is a product of "strictly linear" algebra something will change. What changes is that composition of anti-linear functions does not coincide anymore with multiplication of matrices. Fortunately, the required correction only involves complex conjugation of certain coefficients. Lemma 4.6 exposes an example of an expression involving matrices of anti-linear maps (and of linear maps) which express composition of functions. The basic rule is that every time one multiplies with a matrix of an anti-linear function one has to conjugate all coefficients on the left of that matrix. Lemmas 5.4, 5.5 and 5.7 in Subsection 5.1 are explicitly proven examples of this rule.

4.2 Reversal of path direction

The $\bar{\epsilon}$-labeling we needed as supplement to the ϵ-labeling in order to make types match can be conceived as providing an answer to the following question:

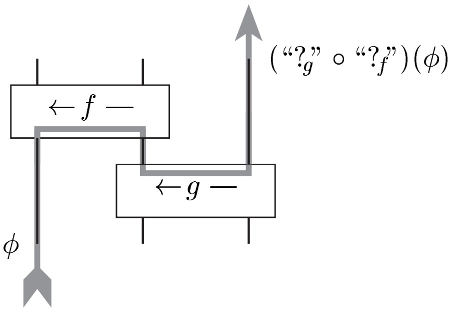
> "How does an **eP** act if it acts negative on a path (as compared to positive)?"

Indeed, we could consider the action of **eP**'s appearing positive with respect to a path as a reference and conceive the action of those appearing negative as a variation on this.

Deceit 4.10 We arranged the network such that the functions labeling the **eP**'s acted in the same direction as in which the path passed though it. This raises the following question:

> "How does an **eP** contribute to the action on the input if a path Γ path passes through it in the opposite direction of its labeling function?"

E.g. which functions will appear at "$?_g$" and "$?_f$" in



Obviously this cannot be f itself because in general the types won't match.

We refer to Appendix A for the definition of the *adjoint of a linear map*. The adjoint of a linear map would be the obvious candidate for reversing the labeling. We extend the definition of the adjoint of a linear map by setting

$$(r^*)^\dagger = r_* \,.$$

Since every anti-linear map $f : \mathcal{H}_1 \nrightarrow \mathcal{H}_2$ arises as $g \circ r^* : \mathcal{H}_1 \to \mathcal{H}_2$ for some linear map $g := \tilde{r}_*(f) : \mathcal{H}_1^* \to \mathcal{H}_2$ we then have

$$f^\dagger = (g \circ r^*)^\dagger = (r^*)^\dagger \circ g^\dagger = r_* \circ g^\dagger$$

by the usual rules of adjoints [Proposition A.3]. This insinuates the following.

Definition 4.11 We define the *anti-adjoint of an anti-linear map* $f : \mathcal{H}_1 \nrightarrow \mathcal{H}_2$ as the anti-linear map

$$f^\dagger = r_* \circ (f \circ r_*)^\dagger : \mathcal{H}_2 \nrightarrow \mathcal{H}_1 \,.$$

We can now formulate the answer to the above question.

Proposition 4.12 *With respect to compositionality theorems we have*

$$\mathrm{P}_{f;i,j;\tau} \simeq \mathrm{P}_{f^\dagger;j,i;\tau}$$

where $f^\dagger$ is the anti-adjoint to f.

Proof. See Section 5. □

Again, of course, with the following restriction to the labelings.

Proposition 4.13 *Proposition 4.12 only holds for the ϵ^*-labeling.*

In the case of the ϵ-labeling again the types don't match and in case of a $\epsilon^{\{e_i\}}$-labeling the result needs an adjustment. An again we rely on Proposition 4.4 and its notations.

- Given is a linear function $f : \mathcal{H}_1^* \to \mathcal{H}_2$ which is the ϵ-labeling of some **eP**. Then the corresponding ϵ-labeling $f^\ddagger$ with respect to a path passing in the opposite direction should be of type $\mathcal{H}_2^* \to \mathcal{H}_1$ and thus cannot be f's adjoint. We have

$$f \circ r^* = f^* \quad \text{and} \quad f^\ddagger \circ r^* = (f^*)^\dagger \quad \text{with} \quad (f^*)^\dagger = r_* \circ f^\dagger\,.$$

  Thus

$$f^\ddagger = f^\ddagger \circ r^* \circ r_* = (f^*)^\dagger \circ r_* = r_* \circ f^\dagger \circ r_*\,.$$

- Given is a linear function $f^{\{e_i\}} : \mathcal{H}_1 \to \mathcal{H}_2$ which is the $\epsilon^{\{e_i\}}$-labeling of some **eP**. Then the corresponding $\epsilon^{\{e_i\}}$-labeling $(f^{\{e_i\}})^\ddagger$ with respect to a path passing in the opposite direction should be of type $\mathcal{H}_2^* \to \mathcal{H}_1$. We have

$$f^{\{e_i\}} \circ c = f^* \qquad \text{and} \qquad (f^{\{e_i\}})^\ddagger \circ c = (f^*)^\dagger\,.$$

  Thus

$$(f^{\{e_i\}})^\ddagger = (f^{\{e_i\}})^\ddagger \circ c \circ c = (f^*)^\dagger \circ c = r_* \circ f^\dagger \circ c\,.$$

  It then easily follows that the matrix of $(f^{\{e_i\}})^\ddagger$ will be the transposed matrix of $f^{\{e_i\}}$, that is, the adjoint of $f^{\{e_i\}}$ with conjugated elements, explicitly

$$(f^{\{e_i\}})^\ddagger_{\alpha\beta} = f^{\{e_i\}}_{\beta\alpha}\,.$$

By Proposition 4.8 on the matrices for the different labelings we also have the following.

Proposition 4.14 *The matrix of the anti-adjoint of an anti-linear map $f : \mathcal{H}_1 \looparrowright \mathcal{H}_2$ is the transposed of the matrix of f, that is,*

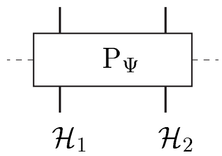
$$(f^\dagger_{\alpha\beta})_{\alpha\beta} = (f_{\beta\alpha})_{\alpha\beta}\,.$$

We leave the implementation of all this for the $\bar{\epsilon}$-labeling supplemented ϵ-labeling to the interested reader. We summarize part of the above in a lemma.

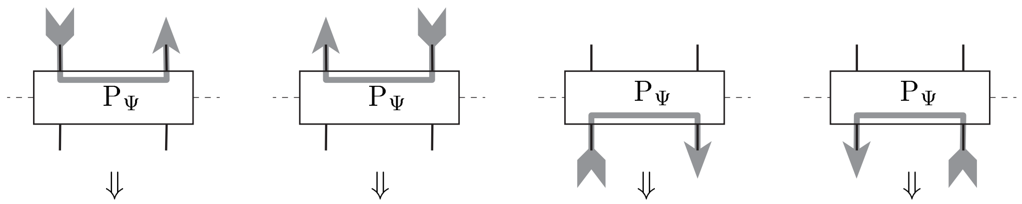
Lemma 4.15 *Let*

$$f = \epsilon^*(\Psi) : \mathcal{H}_1 \looparrowright \mathcal{H}_2$$

be the ϵ^*-labeling of an **eP** *which projects on $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$.*



The following matricial shapes appear in the matrix composition of Lemma 4.6



$$\ldots(f_{ij}^{\{e_i\}})_{ij}\ldots \qquad \ldots(f_{ji}^{\{e_i\}})_{ij}\ldots \qquad \ldots(\bar{f}_{ij}^{\{e_i\}})_{ij}\ldots \qquad \ldots(\bar{f}_{ji}^{\{e_i\}})_{ij}\ldots$$

as a function of how the path passes through the **eP**.

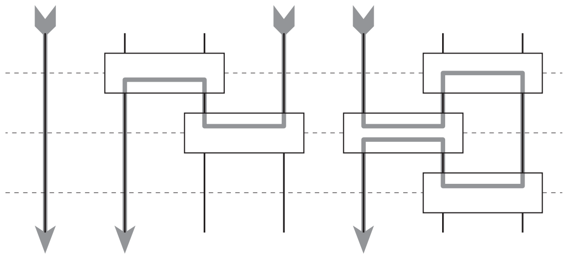
The adjoint of a linear map is in general not its inverse, and neither is the anti-adjoint of an anti-linear map its inverse, even if the map admits an inverse. This will impose limitations on the compositional interpretation of information flow through entanglement when we want to go beyond the type of paths discussed in the previous subsection. This will be the scope of the following subsection.

4.3 Temporal location of inputs and outputs

We investigate if the compositionality theorem still holds when generalizing the notion of path by varying the temporal direction of the path's input and output.

i. *Backward paths.* Since the virtual flow of information as exposed in Theorem 3.4 doesn't seem to care about the actual physical direction of time one could wonder whether the theorem extends to the case where we reverse the direction of a path.

Definition 4.16 By a *full backward path* we mean one which becomes a forward path after reversing the list's order.



Definition 4.17 An anti-linear map $f : \mathcal{H}_1 \nleftrightarrow \mathcal{H}_2$ is called *anti-unitary* if

$$f = U \circ r^* \quad \text{where} \quad U : \mathcal{H}_1^* \to \mathcal{H}_2 \quad \text{is} \quad \text{unitary}\,.$$

Proposition 4.18 *If $f : \mathcal{H}_1 \to \mathcal{H}_2$ is anti-unitary then $f^\dagger = f^{-1}$.*

Proof. We have

$$f^\dagger = r_* \circ (f \circ r_*)^\dagger = r_* \circ (U)^\dagger = r_* \circ (U)^{-1} = (U \circ r_*)^{-1} = f^{-1}$$

by Proposition A.5 in Appendix A. □

In benefit of notational simplicity we formulate the next corollary for the ϵ^*-labeling. The version for the ϵ-labeling is analogous except for replacing anti-unitary by unitary and adding the necessary maps r_* and r^* in order to make types match.

Corollary 4.19 (Backward path compositionality) *Given are:*

- *An entanglement specification network* Ξ *of horizontal type* $\bigotimes_{i=1}^{i=n}\mathcal{H}_i$;
- *A backward path* Γ *passing through* $||\Gamma||$ **eP***'s respectively labeled* $\mathrm{P}_{f_\gamma;\nu_\gamma,\nu_{\gamma+1};\tau_\gamma}$;
- *An input state* $\Psi\in\bigotimes_{i=1}^{i=n}\mathcal{H}_i$ *with* (Ξ,Ψ) *regular.*

If we both have

- ϕ_{in} *is free in* Ψ ;
- ϕ_{out} *is free in* Ψ ;

then

$$(f_1^\dagger\circ\cdots\circ f_{\gamma-1}^\dagger\circ f_\gamma^\dagger\circ f_{\gamma+1}^\dagger\circ\cdots\circ f_{||\Gamma||}^\dagger)(\phi_{out})=\phi_{in}\,.$$

If in addition all maps $f_{||\Gamma||},\cdots,f_1$ *are anti-unitary we have*

$$\phi_{out}=(f_{||\Gamma||}\circ\cdots\circ f_{\gamma+1}\circ f_\gamma\circ f_{\gamma-1}\circ\cdots\circ f_1)(\phi_{in})\,.$$

Proof. Given that ϕ_{out} is free in Ψ we can apply Theorem 3.4 to the path obtained by reversing the list order. Then by Proposition 4.12 the result follows. If all maps $f_1,\cdots,f_{||\Gamma||}$ are anti-unitary there inverses coincide with their adjoints [Proposition 4.18] so we obtain

$$(f_1^{-1}\circ\cdots\circ f_{\gamma-1}^{-1}\circ f_\gamma^{-1}\circ f_{\gamma+1}^{-1}\circ\cdots\circ f_{||\Gamma||}^{-1})(\phi_{out})=\phi_{in}$$

from which the result follows. □

Of course this is a much weaker statement then the compositionality claim made in Theorem 3.4 since we require ϕ_{out} to be free in Ψ and also that all maps $f_{||\Gamma||},\cdots,f_1$ are anti-unitary. This is unfortunately the strongest statement we can make. Let us look at some counter examples which expose this failure of compositionality. We will only consider matrices with real coefficients such that the $\epsilon^{\{e_i\}}$-labeling applies without the necessity of conjugation.

Counter example 4.20 Consider the linear function

$$\mathrm{P}_\psi:\mathcal{H}\to\mathcal{H}::\phi\mapsto\langle\psi\mid\phi\rangle\cdot\psi\,.$$

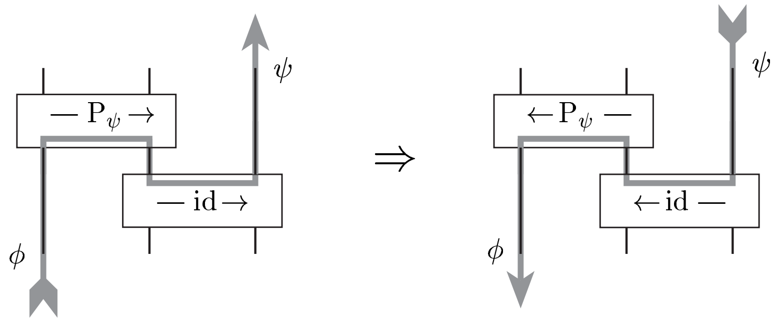
Note that in the base $\{\psi,e_2\}$ with $\psi\perp e_2$ this function has

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

as matrix. Of course this is a projector but we use it as a labeling function and not as a specification. The actual specification it defines is the **eP**

$$\mathrm{P}_{\mathrm{P}_\psi}:\mathcal{H}\otimes\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}\,.$$

Since P_ψ is a projector it is self-adjoint and thus $\mathrm{P}_\psi^\dagger = \mathrm{P}_\psi$. By applying the forward path compositionality theorem, Proposition 4.12 and the fact that $\mathrm{P}_\psi(\phi) = \psi$ we obtain:



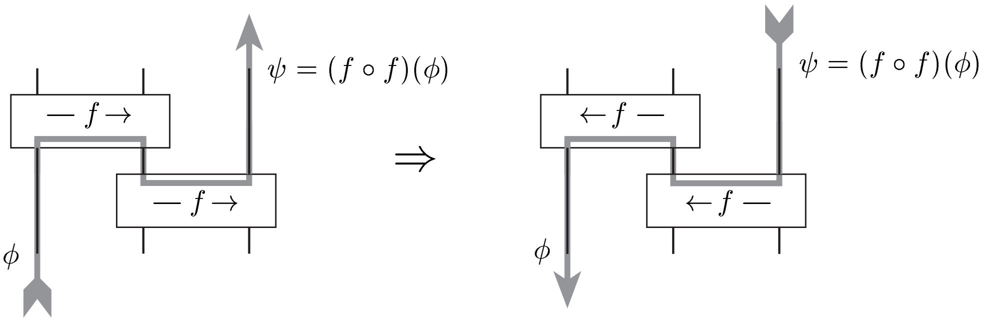
what would indicate that $\mathrm{P}_\psi(\psi) = \phi$. But this is only the case if $\phi = \psi$.

The cautious reader might notice that it is the irreversibility of P_ψ which obstructs extending compositionality to backward paths. This is however not the only reason as shown in the counter example bellow which only involves reversible labeling functions.

Counter example 4.21 Consider the self-adjoint endomap $f : \mathcal{H} \to \mathcal{H}$ with matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^\dagger = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

within



By forward compositionality we have $\psi_0 = 4\phi_0$ and $\psi_1 = \phi_1$ while backward path compositionality would indicate that $\phi_0 = 4\psi_0$ and $\phi_1 = \psi_1$.

Hence in this case backward path compositionality doesn't hold as a consequence of the fact that reversing passage through an **eP** does not correspond with inverting the labeling function but by taking its adjoint [Proposition 4.12].

Interlude 4.22 Now for something merely of recreational interest. It contributes to the game on exposing so-called *quantum weirdness*. In certain situations we can make statements for backward paths on freeness of ϕ_{out} given ϕ_{in} is free. If the path involves carriers $\nu_1, \ldots \nu_k$ and not carriers $\nu_{k+1}, \ldots \nu_n$, then, provided that all the maps $f_1, \cdots, f_{||\Gamma||}$ are surjective, the output of the path will not be entangled to any of the carriers $\nu_{k+1}, \ldots \nu_n$.

A similar thing holds for forward paths. Given that the output is free in Ψ, if again the path only involves carriers $\nu_1, \ldots \nu_k$ and not carriers $\nu_{k+1}, \ldots \nu_n$, then, provided that all the maps $f_1, \cdots, f_{||\Gamma||}$ are now injective, the input of the path will again not be entangled to any of the carriers $\nu_{k+1}, \ldots \nu_n$. Note here that forward flow injectivity assures reversibility in terms of a partial map. However, we are not talking about the **eP**'s themselves being reversible because they aren't at all. Actually they are all equally "maximally irreversible" [Section 5.3]. What we are talking about here is reversibility of the labeling functions. A proof of this useless but entertaining observation can be found in Section 5. Nothing however can be said on the fact whether the output of the path will be entangled to the other carriers $\nu_{k+1}, \ldots \nu_n$. This will also be shown in Section 5.

ii. *Output only paths.* The above might insinuate that compositional behavior is restricted to forward paths only. This is not the case. Consider paths of the following shape:

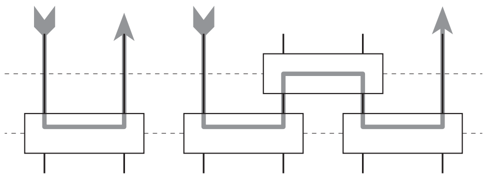
$$[(\nu_1,), (\nu_1, \tau_1)[\cdot((\nu_1, \tau_1), (\nu_2, \tau_1))\cdot](\nu_2, \tau_1), (\nu_2, \tau_2)[\cdot \ldots$$

$$\ldots \cdot](\nu_{||\Gamma||}, \tau_{||\Gamma||-1}), (\nu_{||\Gamma||}, \tau_{||\Gamma||})[\cdot((\nu_{||\Gamma||}, \tau_{||\Gamma||}), (\nu_{||\Gamma||+1}, \tau_{||\Gamma||}))\cdot](\nu_{||\Gamma||+1}, \tau_{||\Gamma||}), (\nu_{||\Gamma||+1},)]$$

implying both $\Gamma_1, \Gamma_{|\Gamma|} \in \mathbb{O}(\Xi)$ with the condition

$$[(\nu_1,), (\nu_1, \tau_1)[\ \cap \mathbb{P}(\Xi) = \emptyset$$

replacing $[(\nu_1,), (\nu_1, \tau_1)[\ \cap \mathbb{P}(\Xi) = \emptyset$ as compared to full forward paths. Both ϕ_{in} and ϕ_{out} are now to be identified at time τ_{out}. Thus, the main feature of such a path is that it both starts and ends in the physical output:



Definition 4.23 By an *output only path* we mean one of the above kind.

Theorem 4.24 (Output only path compositionality) *Given are:*

- *An entanglement specification network* Ξ *of horizontal type* $\bigotimes_{i=1}^{i=n} \mathcal{H}_i$;
- *An output only path* Γ *passing through* $||\Gamma||$ **eP***'s respectively labeled* $\mathrm{P}_{f_\gamma;\nu_\gamma,\nu_{\gamma+1};\tau_\gamma}$;
- *An input state* $\Psi \in \bigotimes_{i=1}^{i=n} \mathcal{H}_i$ *with* (Ξ, Ψ) *regular.*

If ϕ_{in} *is free in* Ψ *then* ϕ_{out} *is free in* Ψ *and we have*

$$\phi_{out} = (f_{||\Gamma||} \circ \cdots \circ f_{\gamma+1} \circ f_\gamma \circ f_{\gamma-1} \circ \cdots \circ f_1)(\phi_{in})\,.$$

Proof. See Section 5. □

This theorem is an exact copy of the full forward path theorem except for the fact that 'full forward path' is replaced by 'output only path' and Ψ by Ψ. It's significance is however manifestly different. To start with, we have no control at all on whether ϕ_{in} is free since it is part of the physical output-state and not of the input state. This actually makes for the following weird observations.

Riddle 4.25 (Pathology of output only paths with free outputs) If we reverse a forward path then we obtain a backward path for which compositionality fails. This is in particular due to the fact that reversal of the action of an **eP** along a path encodes as taking the adjoint of the labeling function. If we however reverse an output only path we again obtain an output only path. Thus when considering compositionality for these two paths at the same time we both have

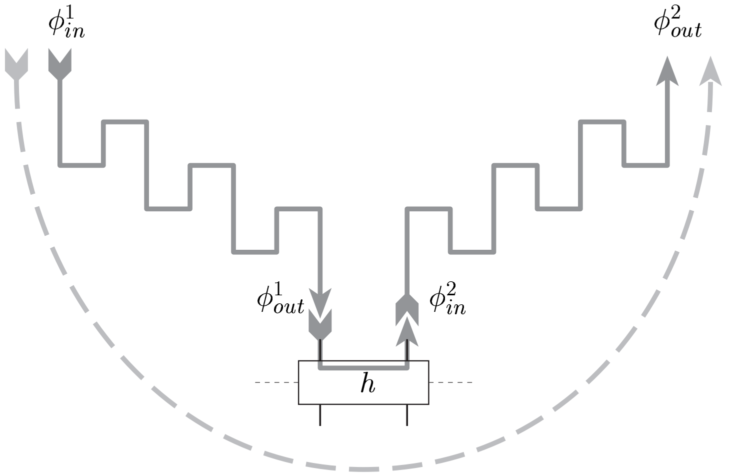
$$\phi_{out} = (f_{||\Gamma||} \circ \cdots \circ f_1)(\phi_{in}) \qquad \text{and} \qquad (f_1^\dagger \circ \cdots \circ f_{||\Gamma||}^\dagger)(\phi_{out}) = \phi_{in} \,.$$

where ϕ_{in} and ϕ_{out} refer to the initial path direction of the output only path. In general $f_1, \ldots, f_{||\Gamma||}$ are not unitary, that is, the adjoints do not coincide with the inverses. This smells like a contradiction. How can this be the case?

Solution. See Subsection 5.5. □

The title of the next riddle refers again to the notion of atomically singular map which we will introduce in Subsection 5.3.

Riddle 4.26 (Pathology of atomically singular maps) Now we consider a situation in which we connect up a full backward path and a full forward path by one additional **eP** as in the picture below.



We obtain an output only path. We know that compositionality holds for the whole path while it fails for the backward subpath. How can this be the case?

Solution. See Subsection 5.5. □

As it was the case for forward paths we can weaken the restrictions on the location of the input and output. Whenever freeness of the output is guaranteed we can again prove claims on compositionality for output only paths analogous to the ones made in Riddle 3.8 for forward paths. They are however somewhat pathological as will be discussed in the solution to Riddle 3.8 and Riddle 4.25 in Subsection 5.5.

There is a slight but truly qualitative change with respect to the ϵ- and the $\epsilon^{\{e_i\}}$-labelings. The respective additional components r^* and c are now odd in number. This yields a conjugation of the coefficients of the input as an additional net effect:

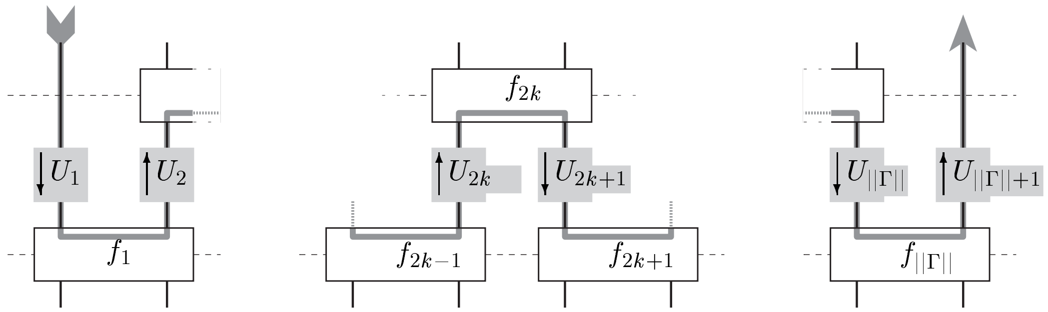
$$\phi_{out} = \sum_{\alpha i_1 \ldots i_{||\Gamma||-1}\beta} \bar{\phi}^{in}_{\alpha} f^1_{\alpha i_1} \bar{f}^2_{i_1 i_2} \ldots f^{2k-1}_{i_{2k-2} i_{2k-1}} \bar{f}^{2k}_{i_{2k-1} i_{2k}} f^{2k+1}_{i_{2k} i_{2k+1}} \ldots \bar{f}^{||\Gamma||-1}_{i_{||\Gamma||-2} i_{||\Gamma||-1}} f^{||\Gamma||}_{i_{||\Gamma||-1}\beta} \cdot e^{(\nu_{out})}_{\beta} .$$

It should be clear that this conjugation exposes the fact that the path "as a whole" produces a time reversal. We conclude with a lemma which states the matricial shape of output only path compositionality.

Lemma 4.27 *Let*

$$g := U_{||\Gamma||+1} \circ f_{||\Gamma||} \circ U_{||\Gamma||} \circ \cdots \circ f_{2k+1} \circ U_{2k+1} \circ f_{2k} \circ U_{2k} \circ f_{2k-1} \circ \cdots \circ U_2 \circ f_1 \circ U_1$$

be a composite of $\bar{\epsilon}$-labels for **eP***'s in a configuration such that the path passes through them in the same order and the same temporal direction as for the network below.*



Then we have

$$g(\phi_{in}) = \phi_{out}$$

iff for the matrices of the corresponding $\epsilon^{\{e_i\}}$-labels we have

$$\phi^{out}_{\beta} = \sum_{\alpha i_1 \ldots i_{||\Gamma||+1} j_1 \ldots j_{||\Gamma||}} \bar{\phi}^{in}_{\alpha} \bar{U}^1_{\alpha i_1} f^1_{i_1 j_1} U^2_{j_1 i_2} \cdots$$
$$\cdots f^{2k-1}_{i_{2k-1} j_{2k-1}} U^{2k}_{j_{2k-1} i_{2k}} \bar{f}^{2k}_{i_{2k} j_{2k}} \bar{U}^{2k+1}_{j_{2k} i_{2k+1}} f^{2k+1}_{i_{2k+1} j_{2k+1}} \cdots$$
$$\cdots \bar{U}^{||\Gamma||}_{j_{||\Gamma||-1} i_{||\Gamma||}} f^{||\Gamma||}_{i_{||\Gamma||} j_{||\Gamma||}} U^{||\Gamma||+1}_{j_{||\Gamma||} i_{||\Gamma||+1}}$$

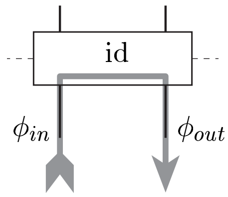
with $\phi_{in} = \sum_{\alpha} \phi^{in}_{\alpha} \cdot e^{(\nu_{in})}_{\alpha}$ and $\phi_{out} = \sum_{\beta} \phi^{out}_{\beta} \cdot e^{(\nu_{out})}_{\beta}$.

iii. *Input only paths.* To close the circle we take a brief look at the remaining case of the four possible alternatives with respect to the direction of the path at its input and its output.

Statement 4.28 *There is no analogous result for input only paths.*

We substantiate this statement by a counter example.

Counter example 4.29 Consider the following path:



where ϕ_{in} and ϕ_{out} are completely arbitrary and thus $\mathrm{id}(\phi_{in}) \neq \phi_{out}$.

There is of course also the purely physical argument that it would be absurd to have a statement on relational dependence of current events depending on things that *might* happen in the future. It is always the experimentalist's option to choose to cancel an intended act at any time.

iv. *General paths.* All the above illustrates the subtile interplay between the virtual path's time (e.g. *in* and *out*) and real physical time (e.g. **in** and **out**). We will need the proofs of the above results in order to obtain insight in the exact combinatorics of a general statement on compositionality for general paths and why it arises the way it does. Corollary 6.2 contains elements of such a thing.

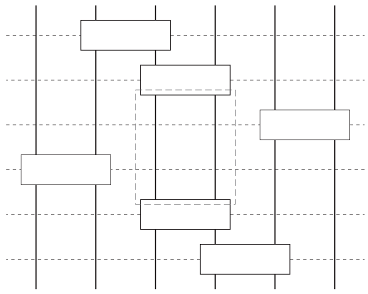
4.4 Amplitude and regularity

The compositionality theorem interpretation involves the condition that there is a non-zero chance of "passage" of the actual physical state Ψ through all **eP**'s. The actual values of the non-zero probability do not play any role as we for example illustrated for teleportation in Subsection 2.3 — we can eliminate probabilities by introducing an additional classical information flow which allows unitary correction.

Question 4.30 *When does the output state* Ψ *become* $\mathbf{0}$*?*

One verifies that there are only two cases in which this can happen namely in the case of a badly chosen input or in the case of a badly constructed network. Both can easily be overcome.

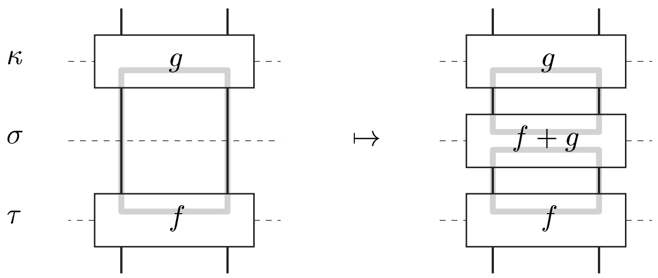
Definition 4.31 Let $\kappa < \tau$. An **eP** $\mathrm{P}_{g;i,j;\tau} \in (\Xi)$ *covers* an **eP** $\mathrm{P}_{f;i,j;\kappa} \in (\Xi)$ if for each $\sigma \in \{\kappa + 1, \ldots, \tau - 1\}$ there exists neither $\mathrm{P}_{-;k,j;\sigma} \in (\Xi)$ nor $\mathrm{P}_{-;i,l;\sigma} \in (\Xi)$.



The two cases in which Ψ becomes $\mathbf{0}$ are:

1. When $\mathrm{P}_{f;i,j;\kappa} \in (\Xi)$ is free from below [Definition 3.14] and
$$\Psi = \sum_{\alpha_i\alpha_j} g_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j} \otimes \sum_{\alpha_1\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\alpha_n} \Phi^{in}_{\alpha_1\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\alpha_n} \cdot e^{(1)}_{\alpha_1} \otimes \ldots \otimes \hat{e}^{(i)}_{\alpha_i} \otimes \ldots \otimes \hat{e}^{(j)}_{\alpha_j} \otimes \ldots \otimes e^{(n)}_{\alpha_n}$$
with
$$\sum_{\alpha_i\alpha_j} f_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j} \perp \sum_{\alpha_i\alpha_j} g_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j} .$$
This can easily be avoided by choosing the component $\sum_{\alpha_i\alpha_j} g_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j}$ differently within Ψ. We recall again that the choice of input doesn't affect compositionality in any way.

2. When $\mathrm{P}_{g;i,j;\tau} \in (\Xi)$ covers $\mathrm{P}_{f;i,j;\kappa} \in (\Xi)$ and
$$\sum_{\alpha_i\alpha_j} f_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j} \perp \sum_{\alpha_i\alpha_j} g_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j} .$$
The magic of quantum mechanics allows to eliminate this case by introducing one additional **eP** which projects on the state
$$\sum_{\alpha_i\alpha_j} f_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j} + \sum_{\alpha_i\alpha_j} g_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j} = \sum_{\alpha_i\alpha_j} (f_{\alpha_i\alpha_j} + g_{\alpha_i\alpha_j}) \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j}$$
at time σ with $\tau < \sigma < \kappa$. Note that no path could pass through this newly introduced **eP** anyway. Indeed, the region between an **eP** and one that covers it only allows loops in view of how a full or partial forward or backward path travels through an **eP**.



Therefore this additional **eP** will not influence the information flow in any way.

Thus somewhat surprisingly, the measurement used to implement it will not require any correction of "unwanted outcomes".

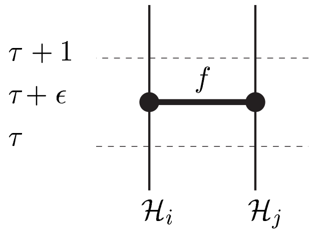
We do not prove the above claims on regularity. They follow straightforwardly from the proofs which follow below in Section 5.

5 Proofs and solutions

In this section we collected most of the proofs of the previous sections. Although they involve a lot of algebraic manipulation they provide structural insights and indicate a graphical representation of those manipulations.

5.1 Propagation of entangled states

Let $i \neq j$. We introduce a graphical representation



to indicates that at time $\tau + \epsilon < \tau + 1$ the state Ψ^τ has shape

$$\Big(\sum_{\alpha_i\alpha_j} f_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j}\Big) \otimes \Big(\sum_{\alpha_1\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\alpha_n} \Phi^\tau_{\alpha_1\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\alpha_n} \cdot e_{\alpha_1\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\alpha_n}\Big)$$

where

$$e_{\alpha_1\ldots\hat{\alpha}_i\ldots\hat{\alpha}_j\ldots\alpha_n} := e^{(1)}_{\alpha_1} \otimes \ldots \otimes \hat{e}^{(i)}_{\alpha_i} \otimes \ldots \otimes \hat{e}^{(j)}_{\alpha_j} \otimes \ldots \otimes e^{(n)}_{\alpha_n} .$$

We fix the choice of a labeling.

Convention 5.1 *In this section we conceive all labeling functions $f, g, h, \ldots$ as being an anti-linear, that is, they are in the image of the ϵ^*-labeling. E.g.*

$$f := \epsilon^*\Big(\sum_{\alpha_i\alpha_j} f_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j}\Big) : \mathcal{H}_i \nrightarrow \mathcal{H}_j .$$

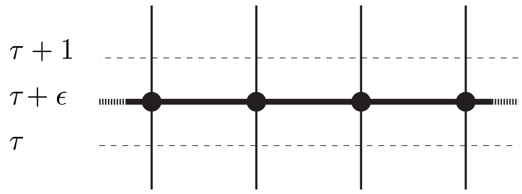
As mentioned above we ignore global normalization [Convention 2.4] what results in

$$\mathrm{P}_g \simeq \Big\langle \sum_{\alpha\beta} g_{\alpha\beta} \cdot e^{(i)}_\alpha \otimes e^{(j)}_\beta \,\Big|\, - \Big\rangle \cdot \sum_{\alpha\beta} g_{\alpha\beta} \cdot e^{(i)}_\alpha \otimes e^{(j)}_\beta .$$

Recall that the passage from an ϵ^*-labeling to matrices in a chosen base requires some conjugations of coefficients [Lemmas 4.6, 4.15 and 4.27]. An arbitrary entangled state

$$\Psi^\tau = \sum_{\alpha_1\ldots\alpha_n} \Psi^\tau_{\alpha_1\ldots\alpha_n} \cdot e^{(1)}_{\alpha_1} \otimes \ldots \otimes e^{(n)}_{\alpha_n}$$

will be graphically represented by



where the thick lines represent the entanglement between the individual carriers of state.

First we verify how a bipartite projector acts on an arbitrary multipartite entangled state Ψ^τ. We have

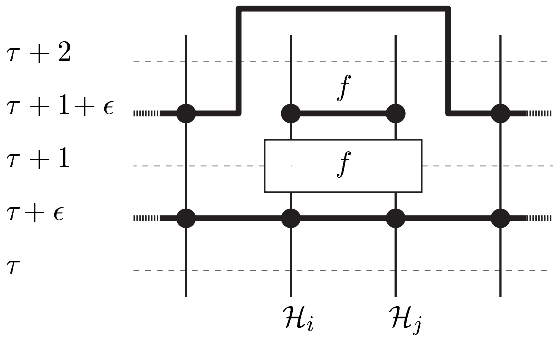
$$
\begin{aligned}
&(\mathrm{P}_{f;i,j;\tau+1} \otimes \mathrm{id}_{\neg\{i,j\}})(\Psi^\tau) \\
&\ = \sum_{\alpha_1\ldots\alpha_n} \Psi^\tau_{\alpha_1\ldots\alpha_n} \Big\langle \sum_{\alpha\beta} f_{\alpha\beta} \cdot e^{(i)}_\alpha \otimes e^{(j)}_\beta \;\Big|\; e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j} \Big\rangle \cdot \Big(\sum_{\alpha\beta} f_{\alpha\beta} \cdot e^{(i)}_\alpha \otimes e^{(j)}_\beta \Big) \otimes e_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_n} \\
&\ = \sum_{\alpha_1\ldots\alpha_n} \Psi^\tau_{\alpha_1\ldots\alpha_n} \Big(\sum_{\alpha\beta} \bar f_{\alpha\beta} \delta_{\alpha\alpha_i} \delta_{\beta\beta_j} \Big) \cdot \Big(\sum_{\alpha\beta} f_{\alpha\beta} \cdot e^{(i)}_\alpha \otimes e^{(j)}_\beta \Big) \otimes e_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_n} \\
&\ = \Big(\sum_{\alpha\beta} f_{\alpha\beta} \cdot e^{(i)}_\alpha \otimes e^{(j)}_\beta \Big) \otimes \Big(\sum_{\alpha_1\ldots\alpha_n} \Psi^\tau_{\alpha_1\ldots\alpha_n} \, \bar f_{\alpha_i\alpha_j} \cdot e_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_n} \Big) \\
&\ = \Big(\sum_{\alpha\beta} f_{\alpha\beta} \cdot e^{(i)}_\alpha \otimes e^{(j)}_\beta \Big) \otimes \Big(\sum_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_n} \Phi^{\tau+1}_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_n} \cdot e_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_n} \Big)
\end{aligned}
$$

where

$$\Phi^{\tau+1}_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_n} := \sum_{\alpha_i\alpha_j} \bar f_{\alpha_i\alpha_j} \; \Psi^\tau_{\alpha_1\ldots\alpha_n} \,.$$

We express this graphically as a lemma.

Lemma 5.2 (State preparation)



The above calculation teaches us something else too. If Ψ^τ is itself of the shape

$$\mathcal{X}^\tau \otimes \mathcal{Y}^\tau = \Big(\sum_{\alpha_1\ldots\alpha_k} \mathcal{X}^\tau_{\alpha_1\ldots\alpha_k} \cdot e^{(1)}_{\alpha_1} \otimes \ldots \otimes e^{(k)}_{\alpha_k} \Big) \otimes \Big(\sum_{\alpha_{k+1}\ldots\alpha_n} \mathcal{Y}^\tau_{\alpha_{k+1}\ldots\alpha_n} \cdot e^{(k+1)}_{\alpha_{k+1}} \otimes \ldots \otimes e^{(n)}_{\alpha_n} \Big)$$

with $i,j<k<n$ then we have

$$
\begin{aligned}
\Phi^{\tau+1}_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_n} &= \sum_{\alpha_i\alpha_j} \bar f_{\alpha_i\alpha_j}\, \Psi^{\tau}_{\alpha_1\ldots\alpha_n}\\
&= \sum_{\alpha_i\alpha_j} \bar f_{\alpha_i\alpha_j}\, \mathcal{X}^{\tau}_{\alpha_1\ldots\alpha_k}\, \mathcal{Y}^{\tau}_{\alpha_{k+1}\ldots\alpha_n}\\
&= \mathcal{Y}^{\tau}_{\alpha_{k+1}\ldots\alpha_n} \sum_{\alpha_i\alpha_j} \bar f_{\alpha_i\alpha_j}\, \mathcal{X}^{\tau}_{\alpha_1\ldots\alpha_k}
\end{aligned}
$$

and thus

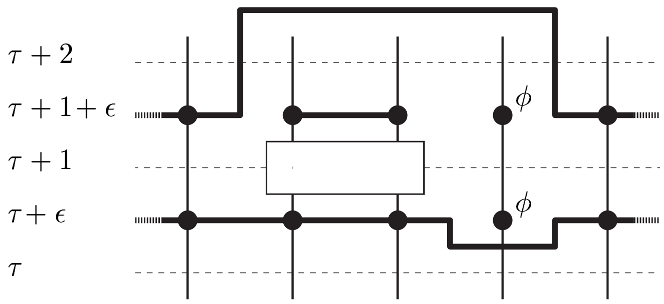
$$
\begin{aligned}
\Psi^{\tau+1} &= \Big(\sum_{\alpha\beta} f_{\alpha\beta}\cdot e^{(i)}_{\alpha}\otimes e^{(j)}_{\beta}\Big)\otimes\Bigg(\sum_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_k}\Big(\sum_{\alpha_i\alpha_j}\bar f_{\alpha_i\alpha_j}\,\mathcal{X}^{\tau}_{\alpha_1\ldots\alpha_k}\Big)\cdot e_{\alpha_1\ldots\hat\alpha_i\ldots\hat\alpha_j\ldots\alpha_k}\Bigg)\otimes\mathcal{Y}^{\tau}\\
&= \mathcal{X}^{\tau+1}\otimes\mathcal{Y}^{\tau}\,.
\end{aligned}
$$

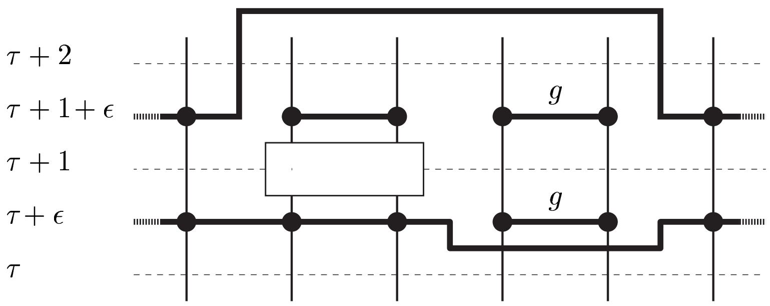
Of specific interest to us will be the cases $\mathcal{Y}^{\tau}=\sum_{\alpha}\phi_{\alpha}\cdot e_{\alpha}$ and $\mathcal{Y}^{\tau}=\sum_{\alpha\beta}g_{\alpha\beta}\cdot e_{\alpha}\otimes e_{\beta}$.

Lemma 5.3 (Factor independence) *Let $1\le i<j<k<n$. If $\Psi^{\tau}=\mathcal{X}^{\tau}\otimes\mathcal{Y}^{\tau}$ with*

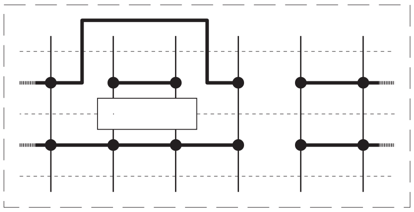
$$
\mathcal{X}^{\tau}\in\bigotimes_{\nu=1}^{\nu=k}\mathcal{H}_{\nu}\qquad\text{and}\qquad\mathcal{Y}^{\tau}\in\bigotimes_{\nu=k+1}^{\nu=n}\mathcal{H}_{\nu}
$$

then the action of $\mathrm{P}_{f;i,j;\tau+1}$ does not alter the factor $\mathcal{Y}^{\tau}$. In particular do we have

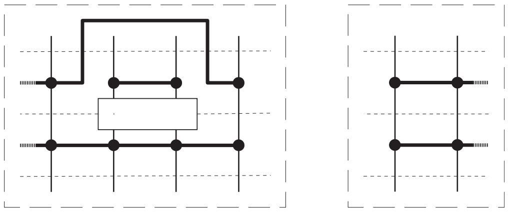




Nor does the presence of the factor $\mathcal{Y}^{\tau}$ alters the action of $\mathrm{P}_{f;i,j;\tau+1}$ on $\mathcal{X}^{\tau}$.



$$\simeq$$



Next consider

$$\Psi := \Big(\sum_{ij} f_{ij} \cdot e_i^{(1)} \otimes e_j^{(2)}\Big) \otimes \Big(\sum_{kl} h_{kl} \cdot e_k^{(3)} \otimes e_l^{(4)}\Big)$$

in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$. We have

$$\begin{aligned}
&(id_{\mathcal{H}_1} \otimes \mathrm{P}_{g;2,3} \otimes \mathrm{id}_{\mathcal{H}_4})\,(\Psi) \\
&\quad = \sum_{ijkl} f_{ij} h_{kl} \cdot e_i^{(1)} \otimes \Big\langle \sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(2)} \otimes e_\beta^{(3)} \;\Big|\; e_j^{(2)} \otimes e_k^{(3)} \Big\rangle \cdot \Big(\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(2)} \otimes e_\beta^{(3)}\Big) \otimes e_l^{(4)} \\
&\quad = \sum_{ijkl} f_{ij} h_{kl} \Big(\sum_{\alpha\beta} \bar{g}_{\alpha\beta} \delta_{\alpha j} \delta_{\beta k}\Big) \cdot e_i^{(1)} \otimes \Big(\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(2)} \otimes e_\beta^{(3)}\Big) \otimes e_l^{(4)} \\
&\quad = \sum_{ijkl} f_{ij} \bar{g}_{jk} h_{kl} \cdot e_i^{(1)} \otimes \Big(\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(2)} \otimes e_\beta^{(3)}\Big) \otimes e_l^{(4)} \\
&\quad = \Big(\sum_{ijkl} f_{ij} \bar{g}_{jk} h_{kl} \cdot e_i^{(1)} \otimes e_l^{(4)}\Big) \otimes \Big(\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(2)} \otimes e_\beta^{(3)}\Big)\,.
\end{aligned}$$

Lemma 5.4 (Anti-linear composition) *Let $f : \mathcal{H}_1 \nleftrightarrow \mathcal{H}_2$, $g : \mathcal{H}_2 \nleftrightarrow \mathcal{H}_3$ and $h : \mathcal{H}_3 \nleftrightarrow \mathcal{H}_4$ be anti-linear maps defined by*

$$f := \sum_{ij} f_{ij} \langle - \mid e_i^{(1)} \rangle \cdot e_j^{(2)} \qquad g := \sum_{jk} g_{jk} \langle - \mid e_j^{(2)} \rangle \cdot e_k^{(3)} \qquad h := \sum_{kl} h_{kl} \langle - \mid e_k^{(3)} \rangle \cdot e_l^{(4)}\,.$$

Then we have

$$h \circ g \circ f := \sum_{ijkl} f_{ij} \bar{g}_{jk} h_{kl} \langle - \mid e_i^{(1)} \rangle \cdot e_l^{(4)}.$$

Proof. We have

$$\begin{aligned}
h \circ g \circ f &= \sum_{k_2 l} h_{k_2 l} \Bigg\langle \sum_{j_2 k_1} g_{j_2 k_1} \Big\langle \sum_{i j_1} f_{i j_1} \langle - \mid e_i^{(1)} \rangle \cdot e_{j_1}^{(2)} \;\Big|\; e_{j_2}^{(2)} \Big\rangle \cdot e_{k_1}^{(3)} \;\Bigg|\; e_{k_2}^{(3)} \Bigg\rangle \cdot e_l^{(4)} \\
&= \sum_{i j_1 j_2 k_1 k_2 l} \bar{\bar{f}}_{i j_1} \bar{g}_{j_2 k_1} h_{k_2 l} \langle - \mid e_i^{(1)} \rangle \langle e_{j_1}^{(2)} \mid e_{j_2}^{(2)} \rangle \langle e_{k_1}^{(3)} \mid e_{k_2}^{(3)} \rangle \cdot e_l^{(4)} \\
&= \sum_{ijkl} f_{ij} \bar{g}_{jk} h_{kl} \langle - \mid e_i^{(1)} \rangle \cdot e_l^{(4)}
\end{aligned}$$

what completes the proof. □

We prove a slight generalization of this lemma which we will need later in the text.

Lemma 5.5 (Anti-linear composition bis) *Let*

$$f_1 : \mathcal{H}_1 \looparrowright \mathcal{H}_2 \quad \ldots \quad f_i : \mathcal{H}_i \looparrowright \mathcal{H}_{i+1} \quad \ldots \quad f_{2m+1} : \mathcal{H}_{2m+1} \looparrowright \mathcal{H}_{2m+2}$$

be anti-linear maps defined by

$$f_i := \sum_{\alpha_i \beta_i} f^i_{\alpha_i\beta_i} \langle - \mid e^{(i)}_{\alpha_i}\rangle \cdot e^{(i+1)}_{\beta_i} .$$

for all $i \in \{1 \ldots, 2m+1\}$. Then we have

$$f_{2k} \circ \ldots \circ f_1 := \sum_{\alpha\alpha_1 \ldots \alpha_{2m+1}\beta} f^1_{\alpha\alpha_1} \bar{f}^2_{\alpha_1\alpha_2} \cdots$$

$$\cdots f^{2k-1}_{\alpha_{2k-2}\alpha_{2k-1}} \bar{f}^{2k}_{\alpha_{2k-1}\alpha_{2k}} f^{2k+1}_{\alpha_{2k}\alpha_{2k+1}} \cdots$$

$$\cdots \bar{f}^{2m}_{\alpha_{2m-1}\alpha_{2m}} f^{2m+1}_{\alpha_{2m+1}\beta} \langle - \mid e^{(1)}_{\alpha}\rangle \cdot e^{(2m+2)}_{\beta} .$$

Proof. Follows from Lemma 5.4 by induction. □

Applying Lemma 5.4 to the above yields

$$(id_{\mathcal{H}_1} \otimes \mathrm{P}_{g;2,3} \otimes \mathrm{id}_{\mathcal{H}_4})\,(\Psi) = \Big(\sum_{il} (h \circ g \circ f)_{il} \cdot e^{(1)}_i \otimes e^{(4)}_l\Big) \otimes \Big(\sum_{\alpha\beta} g_{\alpha\beta} \cdot e^{(2)}_\alpha \otimes e^{(3)}_\beta\Big) .$$

Following Lemma 5.3 on factor independence we can introduce an extra factor

$$\Phi^\tau \in \bigotimes \{\mathcal{H}_\xi \mid \xi \in \{1, \ldots, n\} \setminus \{i, j, k, l\}\}$$

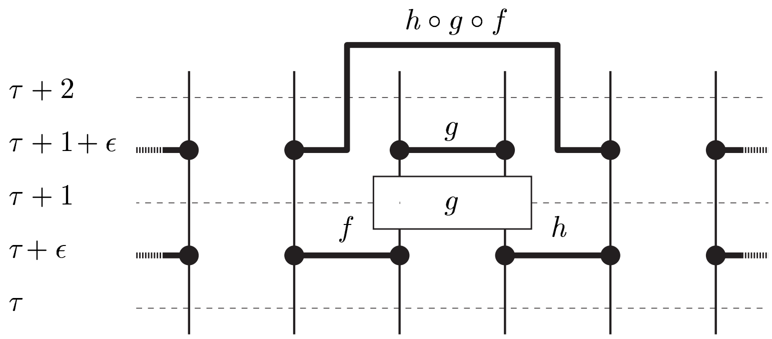
such that

$$\Psi^\tau = \Big(\sum_{\alpha_i\alpha_j} f_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j}\Big) \otimes \Big(\sum_{\alpha_k\alpha_l} h_{\alpha_k\alpha_l} \cdot e^{(k)}_{\alpha_k} \otimes e^{(l)}_{\alpha_l}\Big) \otimes \Phi^\tau .$$

After re-indexing we obtain

$$(\mathrm{P}_{f;j,k;\tau+1} \otimes \mathrm{id}_{\neg\{j,k\}})(\Psi^\tau) = \Big(\sum_{\alpha_i\alpha_l} (h \circ g \circ f)_{\alpha_i\alpha_l} \cdot e^{(i)}_{\alpha_i} \otimes e^{(l)}_{\alpha_l}\Big) \otimes \Big(\sum_{\alpha_j\alpha_k} g_{\alpha_j\alpha_k} \cdot e^{(j)}_{\alpha_j} \otimes e^{(k)}_{\alpha_k}\Big) \otimes \Phi^\tau .$$

Lemma 5.6 (Compositionality)



For $U : \mathcal{H}_2 \to \mathcal{H}_2$ unitary we have

$$(\mathrm{id}_{\mathcal{H}_1} \otimes U)_{ik;jl} = \delta_{ij} U_{kl} \, ,$$

hence

$$\begin{aligned} (id_{\mathcal{H}_1} \otimes U)\Big(\sum_{ik} f_{ik} \cdot e_i^{(1)} \otimes e_k^{(2)}\Big) &= \sum_{ik;jl} f_{ik} \delta_{ij} U_{kl} \cdot e_j^{(1)} \otimes e_l^{(2)} \\ &= \sum_{jkl} f_{jk} U_{kl} \cdot e_j^{(1)} \otimes e_l^{(2)} \, . \end{aligned}$$

For $V : \mathcal{H}_1 \to \mathcal{H}_1$ unitary we have

$$(V \otimes \mathrm{id}_{\mathcal{H}_2})_{ik;jl} = U_{ij} \delta_{kl}$$

and by Propositions A.3 and A.5 we have

$$V_{ij} = \overline{(V^{-1})}_{ji} \, ,$$

hence

$$\begin{aligned} (V \otimes \mathrm{id}_{\mathcal{H}_2})\Big(\sum_{ik} f_{ik} \cdot e_i^{(1)} \otimes e_k^{(2)}\Big) &= \sum_{ik;jl} f_{ik} V_{ij} \delta_{kl} \cdot e_j^{(1)} \otimes e_l^{(2)} \\ &= \sum_{ijk} f_{ik} V_{ij} \cdot e_j^{(1)} \otimes e_k^{(2)} \\ &= \sum_{ijk} \overline{(V^{-1})}_{ji} f_{ik} \cdot e_j^{(1)} \otimes e_k^{(2)} \\ &= \sum_{ijk} \overline{(V^{-1})}_{ij} f_{jk} \cdot e_i^{(1)} \otimes e_k^{(2)} \, . \end{aligned}$$

Lemma 5.7 (Anti-linear composition tris) *Let $f : \mathcal{H}_1 \nrightarrow \mathcal{H}_2$ be an anti-linear map defined by*

$$f := \sum_{jk} f_{jk} \langle - \mid e_j^{(1)} \rangle \cdot e_k^{(2)}$$

and let $U : \mathcal{H}_2 \to \mathcal{H}_2$ and $V : \mathcal{H}_1 \to \mathcal{H}_1$ be unitary transformations defined by

$$U := \sum_{kl} U_{kl} \langle e_k^{(2)} \mid - \rangle \cdot e_l^{(2)} \qquad V := \sum_{ij} V_{ij} \langle e_i^{(1)} \mid - \rangle \cdot e_j^{(1)} \, .$$

Then we have

$$U \circ f = \sum_{jkl} f_{jk} U_{kl} \langle - \mid e_j^{(1)} \rangle \cdot e_l^{(2)} \qquad f \circ V = \sum_{ijk} \bar{V}_{ij} f_{jk} \langle - \mid e_i^{(1)} \rangle \cdot e_k^{(2)} \, .$$

Proof. We have

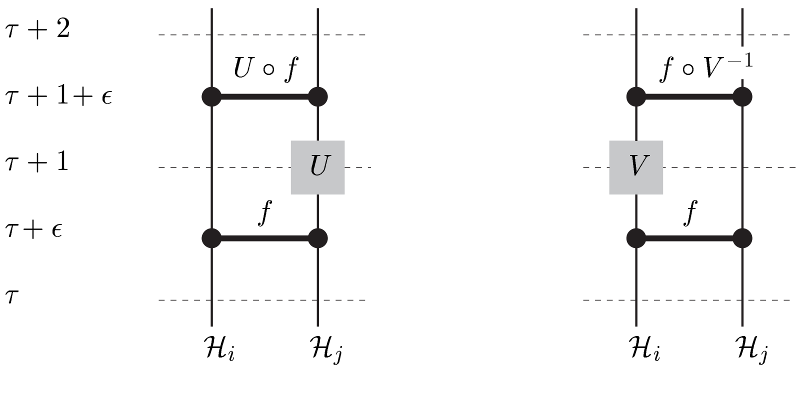
$$\begin{aligned}
U\circ f &= \sum_{k_2 l} U_{k_2 l}\Big\langle e^{(2)}_{k_2}\;\Big|\;\sum_{jk_1} f_{jk_1}\langle - \mid e^{(1)}_j\rangle\cdot e^{(2)}_{k_1}\Big\rangle\cdot e^{(2)}_l\\
&= \sum_{jk_1k_2l} f_{jk_1}U_{k_2l}\langle - \mid e^{(1)}_j\rangle\langle e^{(2)}_{k_2}\mid e^{(2)}_{k_1}\rangle\cdot e^{(2)}_l\\
&= \sum_{jkl} f_{jk}U_{kl}\langle - \mid e^{(1)}_j\rangle\cdot e^{(2)}_l\\
f\circ V &= \sum_{j_2k} f_{j_2k}\Big\langle\sum_{ij_1} V_{ij_1}\langle e^{(1)}_i \mid -\rangle\cdot e^{(1)}_{j_1}\;\Big|\;e^{(1)}_{j_2}\Big\rangle\cdot e^{(2)}_k\\
&= \sum_{ij_1j_2k} f_{j_2k}\bar{V}_{ij_1}\overline{\langle e^{(1)}_i \mid -\rangle}\langle e^{(1)}_{j_1}\mid e^{(1)}_{j_2}\rangle\cdot e^{(2)}_k\\
&= \sum_{ijk}\bar{V}_{ij}f_{jk}\langle - \mid e^{(1)}_i\rangle\cdot e^{(2)}_k
\end{aligned}$$

what completes the proof. □

Applying Lemma 5.4 to the above then yields

$$\begin{aligned}
(id_{\mathcal{H}_1}\otimes U)\Big(\sum_{ik} f_{ik}\cdot e^{(1)}_i\otimes e^{(2)}_k\Big) &= \sum_{jl}(U\circ f)_{jl}\cdot e^{(1)}_j\otimes e^{(2)}_l\\
(V\otimes \mathrm{id}_{\mathcal{H}_2})\Big(\sum_{ij} f_{ij}\cdot e^{(1)}_i\otimes e^{(2)}_j\Big) &= \sum_{kl}(f\circ V^{-1})_{kl}\cdot e^{(1)}_k\otimes e^{(2)}_l\,.
\end{aligned}$$

Lemma 5.8 (Unitary action)



We will now extend the compositionality lemma by allowing unitary operators to act on the incoming entangled state before the projector does. Again setting

$$\Psi := \Big(\sum_{ij} f_{ij}\cdot e^{(1)}_i\otimes e^{(2)}_j\Big)\otimes\Big(\sum_{kl} h_{kl}\cdot e^{(3)}_k\otimes e^{(4)}_l\Big)$$

considering Ψ as above by Lemma 5.8 we have

$$(id_{\mathcal{H}_1} \otimes U \otimes V \otimes \mathrm{id}_{\mathcal{H}_4}) = \Big(\sum_{ij} (U \circ f)_{ij} \cdot e_i^{(1)} \otimes e_j^{(2)}\Big) \otimes \Big(\sum_{kl} (h \circ V^{-1})_{kl} \cdot e_k^{(3)} \otimes e_l^{(4)}\Big)\,.$$

By Lemma 5.6 it then follows that

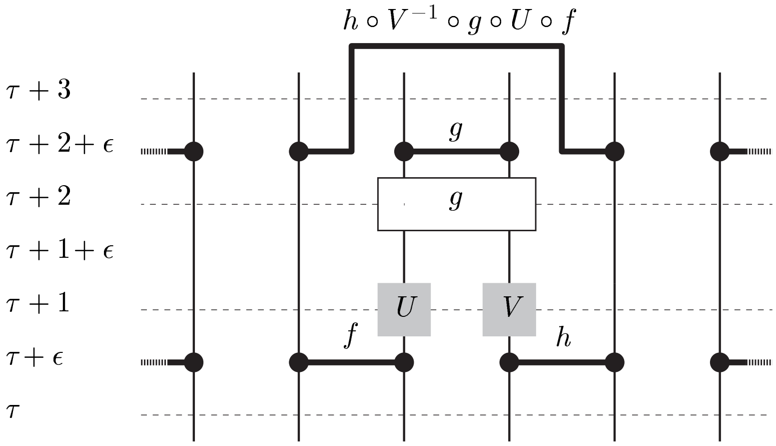
$$\begin{aligned}&\Big((id_{\mathcal{H}_1} \otimes \mathrm{P}_{g;2,3} \otimes \mathrm{id}_{\mathcal{H}_4}) \circ (id_{\mathcal{H}_1} \otimes U \otimes V \otimes \mathrm{id}_{\mathcal{H}_4})\Big)(\Psi) \\ &\qquad = \Big(\sum_{il} (h \circ V^{-1} \circ g \circ U \circ f)_{il} \cdot e_i^{(1)} \otimes e_l^{(4)}\Big) \otimes \Big(\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(2)} \otimes e_\beta^{(3)}\Big)\,.\end{aligned}$$

For $U : \mathcal{H}_j \to \mathcal{H}_j$ and $V : \mathcal{H}_k \to \mathcal{H}_k$ we have

$$\begin{aligned}&\Big((\mathrm{P}_{f;j,k;\tau+1} \otimes \mathrm{id}_{\neg\{j,k\}}) \circ (U \otimes V \otimes \mathrm{id}_{\neg\{j,k\}})\Big)(\Psi^\tau) \\ &\qquad = \Big(\sum_{\alpha_i\alpha_l} (h \circ V^{-1} \circ g \circ U \circ f)_{\alpha_i\alpha_l} \cdot e_{\alpha_i}^{(i)} \otimes e_{\alpha_l}^{(l)}\Big) \otimes \Big(\sum_{\alpha_j\alpha_k} g_{\alpha_j\alpha_k} \cdot e_{\alpha_j}^{(j)} \otimes e_{\alpha_k}^{(k)}\Big) \otimes \Phi^\tau\end{aligned}$$

with Ψ^τ and Φ^τ as defined above.

Lemma 5.9 (Extended compositionality)



5.2 Core of compositionality proof

Consider now an entanglement specification network Ξ and an *output only path* Γ. Assume, as we did in the assumptions of Theorem 3.4 and Theorem 4.24, that the index $\gamma \in \{1, \ldots, ||\Gamma||\}$ in f_γ stands for the order in which Γ passes through the **eP**'s. We denote by $\mathbf{Pos}(\Gamma)$ all **eP**'s in (Ξ) which are positive for Γ and by $\mathbf{Neg}(\Gamma)$ all **eP**'s in (Ξ) which are negative for Γ [Definition 3.13]. Recall that an **eP** can be both positive and negative for some Γ. The *disjoint* union $\mathbf{Neg}(\Gamma) + \mathbf{Pos}(\Gamma)$ then contains $||\Gamma||$ **eP**'s. Clearly Γ passes alternatingly through $\mathbf{Pos}(\Gamma)$ and $\mathbf{Neg}(\Gamma)$. We also know that $||\Gamma||$ is odd. Therefore we have the following cardinalities for the sets of **eP**'s of each kind:

$$|(\Gamma)| = \frac{||\Gamma|| + 1}{2} \qquad \text{and} \qquad |(\Gamma)| = \frac{||\Gamma|| - 1}{2}\,.$$

Let $T_\Gamma \subseteq \{1,\ldots,m\}$ be the $|(\Gamma)|$-element ordered set of time instances at which an **eP** contained in $\mathbf{Neg}(\Gamma)$ is specified. Note that it is no restriction to assume that no two **eP**'s contained in $\mathbf{Neg}(\Gamma)$ coincide timewise. We will refer to an **eP** in $\mathbf{Neg}(\Gamma)$ by labeling it by $\tau \in T_\Gamma$. We fix some more notations by stipulating

$$\mathrm{P}_\tau := \mathrm{P}_{f_{\gamma(\tau)};\nu_\tau^\lambda,\nu_\tau^\rho;\tau} \in (\Gamma)\,,$$

that is,

- $\gamma(\tau)$ is the order of P_τ along Γ ;
- ν_τ^λ is the track where Γ enters P_τ ;
- ν_τ^ρ is the track where Γ leaves P_τ .

For each $\mathrm{P}_\tau \in (\Gamma)$ we define subpaths of Γ:

- Let Γ_τ^{in} be the list obtained by first removing all elements from the list Γ that come after (ν_τ^λ, τ) and then, starting from the first of the remaining list-elements, removing all elements until all the remaining have time before τ.
- Let Γ_τ^{out} be the list obtained by first removing all elements from the list Γ that come before (ν_τ^ρ, τ) and then, starting from the last of the remaining list-elements, removing all elements until the remaining all have time before τ.

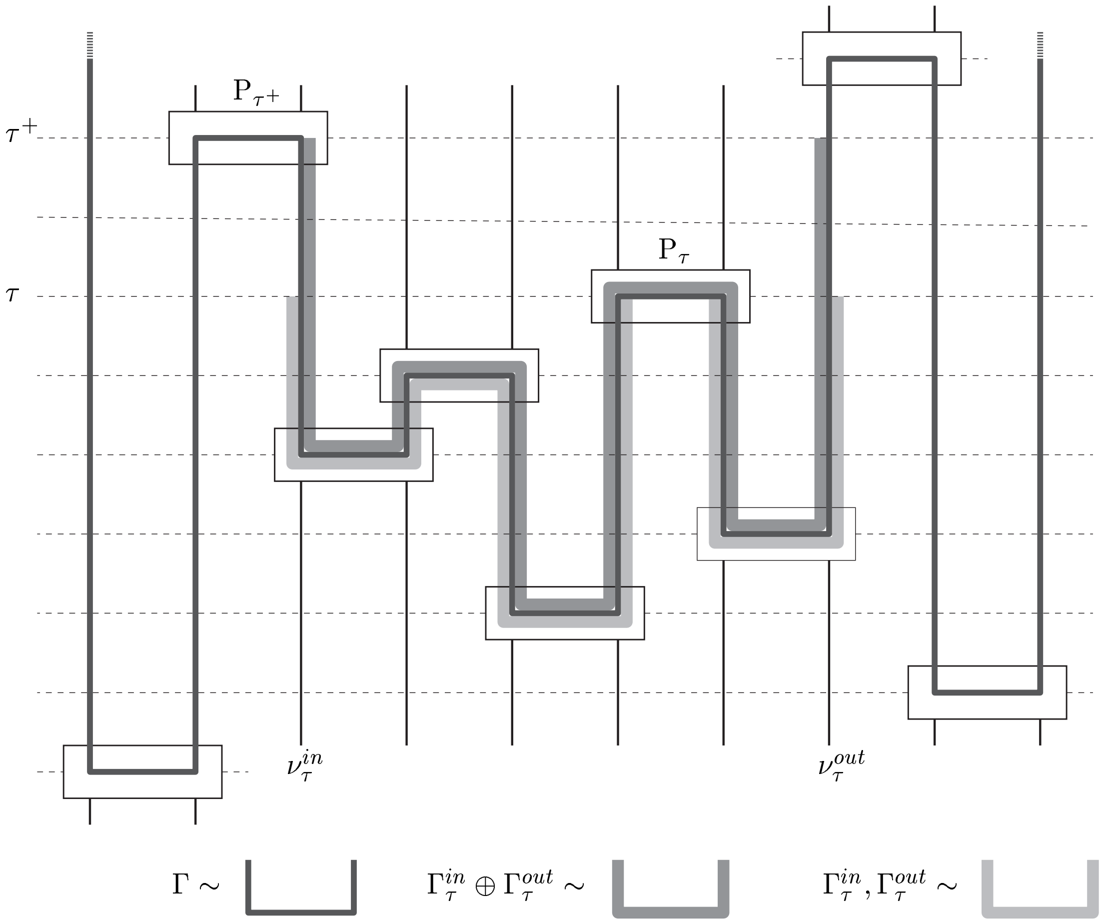
Let (ν_τ^{in}, τ) be the first element of the list Γ_τ^{in} and let (ν_τ^{out}, τ) be the last element of the list Γ_τ^{out}. We define yet another sublist of Γ:

$$\Gamma_\tau^{in} \oplus \Gamma_\tau^{out} := [(\nu_\tau^{in}, \tau^+), (\nu_\tau^{in}, \tau)[\,\cdot\, \Gamma_\tau^{in}\ \cdot\ \Gamma_\tau^{out}\,\cdot\,](\nu_\tau^{out}, \tau), (\nu_\tau^{out}, \tau^+)]$$

where:

- τ^+ is the smallest number in $T_\Gamma \cap \{\tau+1,\ldots,m\}$ such that either $(\nu_\tau^{in}, \tau^+) \in \mathbb{P}(\Xi) \cap \Gamma$ or $(\nu_\tau^{out}, \tau^+) \in \mathbb{P}(\Xi) \cap \Gamma$;
- If no such $\tau^+ \in T_\Gamma \cap \{\tau+1,\ldots,m\}$ exists we set $\tau^+ :=$.

All this becomes much clearer with a picture.



We have the following.

Lemma 5.10 *For all $\tau \in T_\Gamma$ the states $\Psi^{\tau-1} \in \bigotimes_{\nu=1}^{\nu=n} \mathcal{H}_\nu$ have the shape*

$$\Psi^{\tau-1} = \Big(\sum_{\alpha\beta} g^\lambda_{\alpha\beta} \cdot e_\alpha^{(\nu_\tau^{in})} \otimes e_\beta^{(\nu_\tau^\lambda)}\Big) \otimes \Big(\sum_{\alpha\beta} g^\rho_{\alpha\beta} \cdot e_\alpha^{(\nu_\tau^\rho)} \otimes e_\beta^{(\nu_\tau^{out})}\Big) \otimes \Phi^{\tau-1}$$

with

$$g^\lambda := f_{\gamma(\tau)-1} \circ \ldots \circ f_{\gamma(\tau)-||\Gamma_\tau^{in}||} \qquad g^\rho := f_{\gamma(\tau)+||\Gamma_\tau^{out}||} \circ \ldots \circ f_{\gamma(\tau)+1}$$

and the states $\Psi^\tau \in \bigotimes_{\nu=1}^{\nu=n} \mathcal{H}_\nu$ have the shape

$$\Psi^\tau = \Big(\sum_{\alpha\beta} f_{\alpha\beta} \cdot e_\alpha^{(\nu_\tau^\lambda)} \otimes e_\beta^{(\nu_\tau^\rho)}\Big) \otimes \Big(\sum_{\alpha\beta} h_{\alpha\beta} \cdot e_\alpha^{(\nu_\tau^{in})} \otimes e_\beta^{(\nu_\tau^{out})}\Big) \otimes \Phi^\tau$$

with

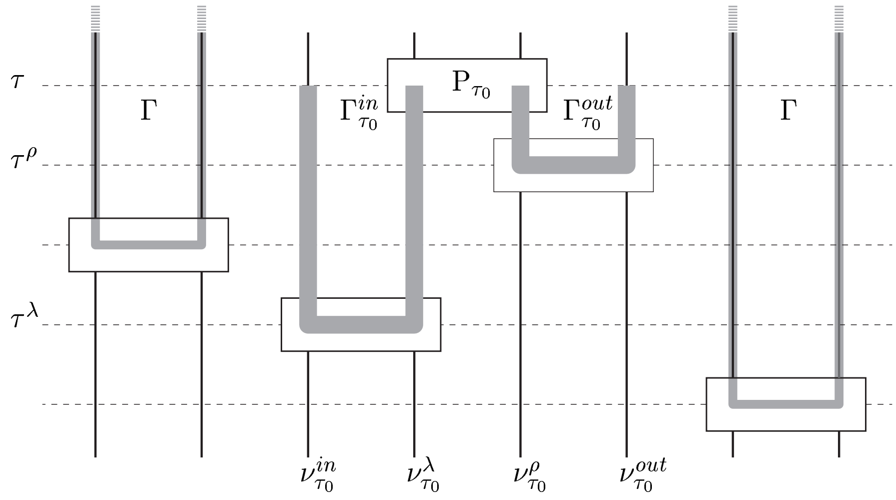
$$f := f_{\gamma(\tau)} \qquad h := f_{\gamma(\tau)+||\Gamma_\tau^{in}||} \circ \ldots \circ f_{\gamma(\tau)-||\Gamma_\tau^{in}||}\,.$$

Proof. Observe that an output only path alternatively goes through negative and positive **eP**'s and that the first and last **eP** through which it passes are both positive. We now proceed by induction on the value of τ.

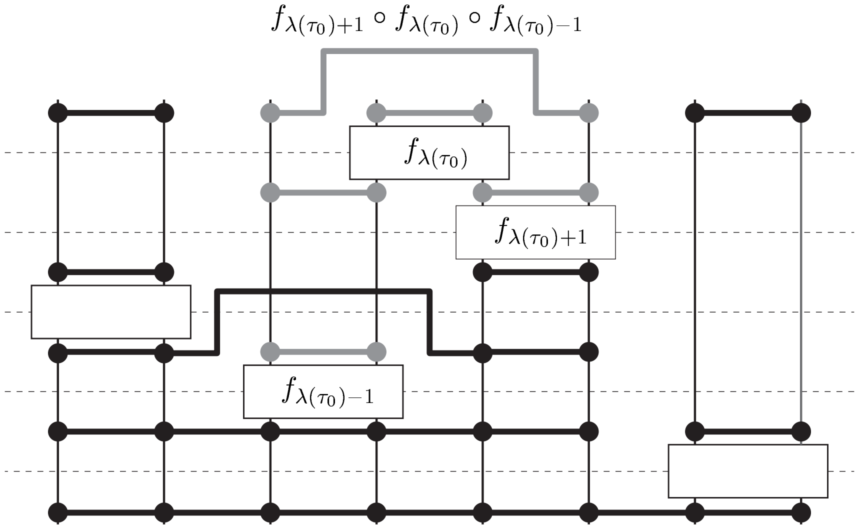
(i) *Base case.* Denote by τ_0 the infimum of T_Γ. Since P_{τ_0} is the first **eP** (in physical time) of the network which is negative for Γ it follows that $\Gamma^{in}_{\tau_0}$ and $\Gamma^{out}_{\tau_0}$ can only pass through positive **eP**'s so they each only can pass through one other **eP**, that is, $||\Gamma^{in}_{\tau_0}|| = ||\Gamma^{out}_{\tau_0}|| = 1$. Refer to the corresponding times at which these two positive **eP**'s act as τ^λ and τ^ρ. Hence the claim made in this lemma for this case is

$$g^\lambda := f_{\gamma(\tau_0)-1} \qquad g^\rho := f_{\gamma(\tau_0)+1} \qquad h := f_{\gamma(\tau_0)+1} \circ f_{\gamma(\tau_0)} \circ f_{\gamma(\tau_0)-1}\,.$$

Observing the geometry of an entanglement specification network and a path:



in view of the factor independence lemma, the above claim is true for Ψ^{τ_0-1} by Lemma 5.2 on state preparation and it is true for Ψ^{τ_0} by Lemma 5.6 on compositionality.



(ii) *Inductive step.* For τ fixed define τ^λ as the supremum in (Γ_τ^{in}) and τ^ρ as the supremum in (Γ_τ^{out}). It then follows by the geometry of an entanglement specification network and a path that

$$\Gamma_\tau^{in} = \Gamma_{\tau^\lambda}^{in} \oplus \Gamma_{\tau^\lambda}^{out} \qquad\qquad \Gamma_\tau^{out} = \Gamma_{\tau^\rho}^{in} \oplus \Gamma_{\tau^\rho}^{out}\,.$$

Recall also that by the definition of a path we have that neither of

$$](\nu_\tau^{in},\tau),(\nu_\tau^{in},\tau^\lambda)[\quad](\nu_\tau^{\lambda},\tau^\lambda),(\nu_\tau^{\lambda},\tau)[\quad](\nu_\tau^{\rho},\tau),(\nu_\tau^{\rho},\tau^\rho)[\quad](\nu_\tau^{out},\tau^\rho),(\nu_\tau^{out},\tau)[$$

intersects with an **eP**. Applying the induction hypothesis to τ^λ and τ^ρ then results in the fact that Ψ^{τ^λ} and Ψ^{τ^ρ} respectively contain a factor

$$\sum_{\alpha\beta} h^\lambda_{\alpha\beta}\cdot e_\alpha^{(\nu^{in}_{\tau^\lambda})}\otimes e_\beta^{(\nu^{out}_{\tau^\lambda})} \qquad\qquad \sum_{\alpha\beta} h^\rho_{\alpha\beta}\cdot e_\alpha^{(\nu^{in}_{\tau^\rho})}\otimes e_\beta^{(\nu^{out}_{\tau^\rho})}$$

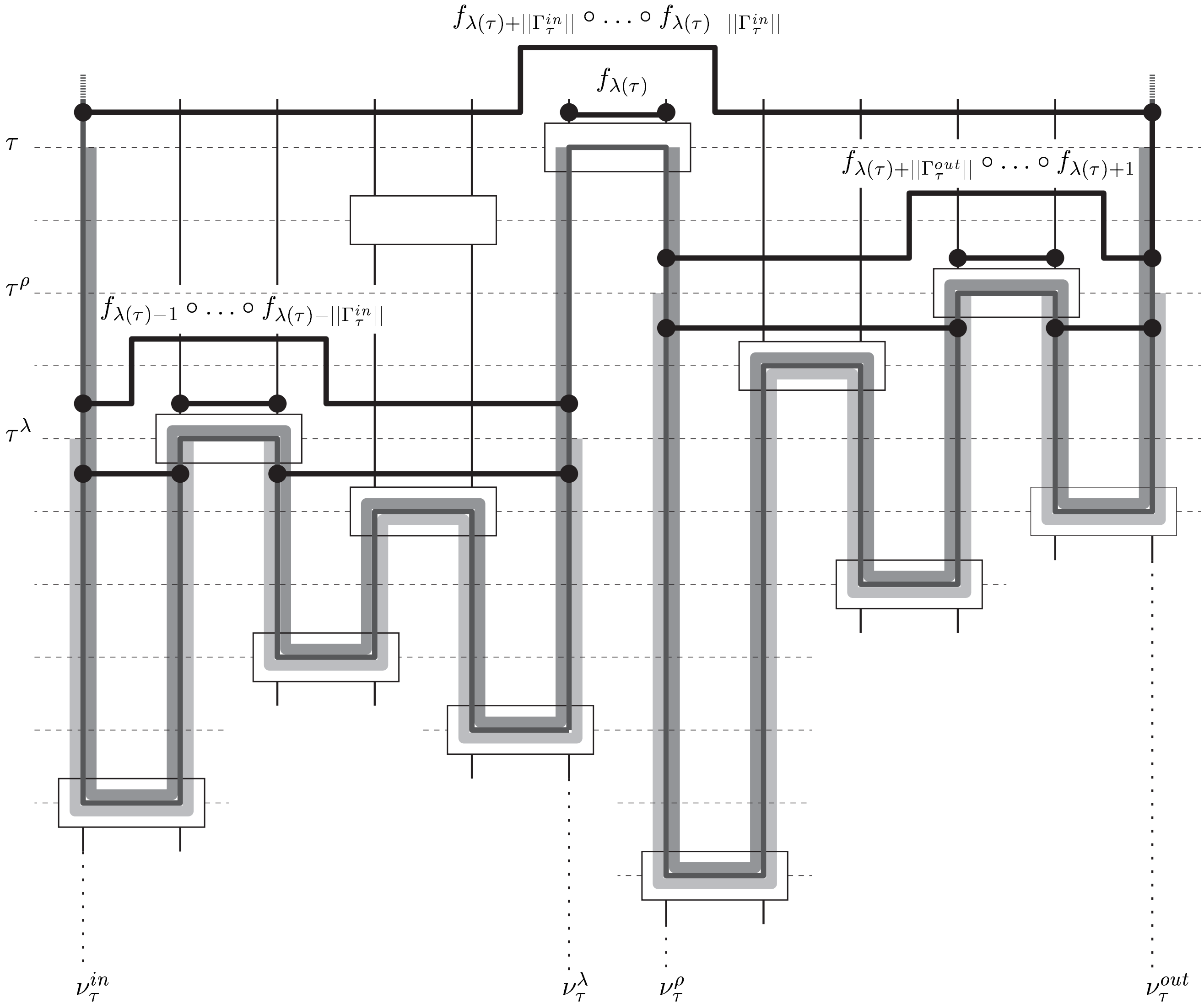
with

$$h^\lambda := f_{\gamma(\tau^\lambda)+||\Gamma^{in}_{\tau^\lambda}||}\circ\ldots\circ f_{\gamma(\tau^\lambda)-||\Gamma^{in}_{\tau^\lambda}||} \qquad\qquad h^\rho := f_{\gamma(\tau^\rho)+||\Gamma^{in}_{\tau^\rho}||}\circ\ldots\circ f_{\gamma(\tau^\rho)-||\Gamma^{in}_{\tau^\rho}||}\,.$$

Since we have

$$\nu^{in}_{\tau^\lambda} = \nu^{in}_\tau \qquad \nu^{out}_{\tau^\rho} = \nu^\rho_\tau \qquad \nu^{in}_{\tau^\lambda} = \nu^\lambda_\tau \qquad \nu^{out}_{\tau^\rho} = \nu^{out}_\tau$$

it follows that $\Psi^{\tau-1}$ will contain both of these factors. Again this becomes much clearer when providing a picture.



This establishes the first claim of the lemma since

$$\gamma(\tau^\lambda) + ||\Gamma^{in}_{\tau^\lambda}|| = \gamma(\tau) - 1 \qquad \gamma(\tau^\rho) - ||\Gamma^{in}_{\tau^\rho}|| = \gamma(\tau) + 1$$
$$\gamma(\tau^\rho) + ||\Gamma^{in}_{\tau^\rho}|| = \gamma(\tau) + ||\Gamma^{out}_\tau|| \qquad \gamma(\tau^\lambda) - ||\Gamma^{in}_{\tau^\lambda}|| = \gamma(\tau) - ||\Gamma^{in}_\tau||$$

such that $g^\lambda = h^\lambda$ and $h^\rho = g^\rho$. The second one then follows by applying Lemma 5.6. □

In the case that τ is the supremum of T_Γ, that is, when P_τ is the last **eP** (in physical time) of the network which is negative for Γ, this lemma yields the following.

Proposition 5.11 *For an output only path in an entanglement specification network we have the following shape of the outcome state:*

$$\Psi = \Big(\sum_{\alpha\beta} (f_{||\Gamma||} \circ \ldots \circ f_1)_{\alpha\beta} \cdot e^{(\nu_{in})}_\alpha \otimes e^{(\nu_{out})}_\beta\Big) \otimes \Phi\,.$$

5.3 Atomically singular maps

The main goal of this subsection is introducing the notion of an anti-projector.

Definition 5.12 Given a linear map $f : \mathcal{H}_1 \to \mathcal{H}_2$ or an anti-linear map $f : \mathcal{H}_1 \looparrowright \mathcal{H}_2$ its *kernel* is

$$\mathrm{Ker}(f) := \{\psi \in \mathcal{H}_1 \mid f(\psi) = \mho\}$$

and its *range* is

$$\mathrm{Range}(f) := \{f(\psi) \mid \psi \in \mathcal{H}_1\}\,.$$

Definition 5.13 A linear map $f : \mathcal{H}_1 \to \mathcal{H}_2$ or an anti-linear map $f : \mathcal{H}_1 \looparrowright \mathcal{H}_2$ is *atomically singular* iff

$$\mathrm{Dim}(\mathrm{Ker}(f)) = \mathrm{Dim}(\mathcal{H}_1) - 1\,.$$

By the well-known linear algebraic equality

$$\mathrm{Dim}(\mathrm{Ker}(f)) + \mathrm{Dim}(\mathrm{Range}(f)) = \mathrm{Dim}(\mathcal{H}_1)$$

we obtain the following alternative characterization of atomically singular maps.

Proposition 5.14 *A linear or an anti-linear map f is atomically singular iff*

$$\mathrm{Dim}(\mathrm{Range}(f)) = 1\,.$$

Proposition 5.15 *Let h be atomically singular.*

1. *If $h = f \circ g$ and f is injective then g is atomically singular.*
2. *If $h = g \circ f$ and f is surjective then g is atomically singular.*

Proof. If f is injective then $\mathrm{Dim}(\mathrm{Ker}(h)) = \mathrm{Dim}(\mathrm{Ker}(g))$ and if f is surjective then $\mathrm{Dim}(\mathrm{Range}(h)) = \mathrm{Dim}(\mathrm{Range}(g))$. The results then follows respectively by Definition 5.13 and Proposition 5.14. □

Since both $r_* : \mathcal{H} \looparrowright \mathcal{H}^*$ and $c : \mathcal{H} \to \mathcal{H}$ are bijections it follows that the maps

$$\tilde{r}_* := (- \circ r_*) : (\mathcal{H}_1 \looparrowright \mathcal{H}_2) \to (\mathcal{H}_1^* \to \mathcal{H}_2)$$

and

$$\tilde{c} := (- \circ c) : (\mathcal{H}_1 \looparrowright \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_2)$$

don't alter the range of their arguments. By Proposition 5.14 we then have the following.

Proposition 5.16 *The labeling transformations $\tilde{r}_*$ and $\tilde{c}$ preserve atomic singularity.*

We characterize the linear and the anti-linear atomically singular maps in terms of the shape of their matrix [Definition 4.7].

Lemma 5.17 *A linear or anti-linear map f is atomically singular iff in some well-chosen orthonormal base its matrix has the shape*

$$\begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

that is,

$$(f_{\alpha\beta})_{\alpha\beta} = r \cdot (\delta_{1\alpha}\delta_{1\beta})_{\alpha\beta}\,,$$

for some $r \in \mathbb{R}_0^+$.

Proof. Choose $\psi \perp \mathrm{Ker}(f)$ with $|\psi| = 1$ and let

$$\phi := \frac{f(\psi)}{|f(\psi)|} \in \mathrm{Range}(f)\,.$$

Set $\tilde{e}_1^{(1)} := \psi$ and $\tilde{e}_1^{(2)} := \phi$ and extend these to respective orthonormal bases $\{\tilde{e}_\alpha^{(1)}\}_\alpha$ and $\{\tilde{e}_\beta^{(2)}\}_\beta$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ by choosing

$$\tilde{e}_2^{(1)}, \ldots, \tilde{e}_{\mathrm{Dim}(\mathcal{H}_1)}^{(1)} \in \mathrm{Ker}(f) \quad \text{and} \quad \tilde{e}_2^{(2)}, \ldots, \tilde{e}_{\mathrm{Dim}(\mathcal{H}_2)}^{(2)} \perp \mathrm{Range}(f)\,.$$

In these bases the respective generic shapes of a linear or an anti-linear map are

$$|f(\psi)|\langle\psi \mid -\rangle \cdot \phi \qquad \text{and} \qquad |f(\psi)|\langle - \mid \psi\rangle \cdot \phi\,.$$

Hence we obtain a matrix of the above shape with $r = |f(\psi)|$. The converse is trivial. □

Relative to fixed bases we have the following.

Lemma 5.18 *Let $\{e_\alpha^{(1)}\}_\alpha$ be a base of $\mathcal{H}_1$ and let $\{e_\beta^{(2)}\}_\beta$ be a base of $\mathcal{H}_2$. A linear map $f : \mathcal{H}_1 \to \mathcal{H}_2$ or an anti-linear map $f : \mathcal{H}_1 \looparrowright \mathcal{H}_2$ is atomically singular iff in the bases $\{e_\alpha^{(1)}\}_\alpha$ and $\{e_\beta^{(2)}\}_\beta$ its matrix has the shape*

$$(f_{\alpha\beta})_{\alpha\beta} = (\psi_\alpha\phi_\beta)_{\alpha\beta}$$

for some tuples $(\psi_\alpha)_\alpha$ and $(\phi_\beta)_\beta$.

Proof. We explicitly prove the theorem for the anti-linear case. Since by Proposition 4.8 we know that

$$\tilde{c} : (\mathcal{H}_1 \looparrowright \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_2)$$

preserves matrices and since by Proposition 5.16 we know it also preserves atomic singularity the result follows for the linear case.

Let f be anti-linear and let $f_{\alpha\beta} = \psi_\alpha \phi_\beta$. For $\psi = \sum_\alpha \psi_\alpha \cdot e^{(1)}_\alpha$ and $\varphi \in \psi^\perp$ we have

$$0 = \langle \varphi \mid \psi \rangle = \sum_{\alpha\beta} \bar{\varphi}_\alpha \psi_\beta \langle e^{(1)}_\alpha \mid e^{(1)}_\beta \rangle = \sum_\alpha \bar{\varphi}_\alpha \psi_\alpha \,,$$

hence

$$f(\varphi) = \sum_{\alpha\beta} f_{\alpha\beta} \langle \varphi \mid e^{(1)}_\alpha \rangle \cdot e^{(2)}_\beta = \sum_{\alpha\beta} \psi_\alpha \phi_\beta \bar{\varphi}_\alpha \cdot e^{(2)}_\beta = \Big(\sum_\alpha \bar{\varphi}_\alpha \psi_\alpha \Big) \cdot \Big(\sum_\beta \phi_\beta \cdot e^{(2)}_\beta \Big) = \mho \,.$$

Thus we have $\mathrm{Dim}(\mathrm{Ker}(f)) = \mathrm{Dim}(\mathcal{H}_1) - 1$.

Conversely, first recall the two bases constructed in the proof of Lemma 5.17, that is,

$$\{\psi, \tilde{e}^{(1)}_2, \ldots, \tilde{e}^{(1)}_{\mathrm{Dim}(\mathcal{H}_1)}\} \quad \text{and} \quad \{\phi, \tilde{e}^{(2)}_2, \ldots, \tilde{e}^{(2)}_{\mathrm{Dim}(\mathcal{H}_2)}\} \,.$$

Since we know the image under f of all these vectors we will derive the coefficients $(f_{\alpha\beta})_{\alpha\beta}$ by inserting these vectors in the generic anti-linear form

$$f = \sum_{\alpha\beta} f_{\alpha\beta} \, \langle - \mid e^{(1)}_\alpha \rangle \cdot e^{(2)}_\beta \,.$$

By $f(\psi) = |f(\psi)| \cdot \phi$ we obtain

$$\forall \beta : \ \sum_\alpha f_{\alpha\beta} \bar{\psi}_\alpha = |f(\psi)| \phi_\beta$$

and by $f(\tilde{e}^{(1)}_i) = \mho$ for $i \in \{2, \ldots, \mathrm{Dim}(\mathcal{H}_1)\}$ we obtain

$$\forall \beta : \ \sum_\alpha f_{\alpha\beta} \bar{\xi}^i_\alpha = 0$$

where $\tilde{e}^{(1)}_i = \sum_\alpha \xi^i_\alpha \cdot e^{(1)}_\alpha$. Setting

$$(f_{\alpha\beta})_{\alpha\beta} := |f(\psi)| \cdot (\psi_\alpha \phi_\beta)_{\alpha\beta}$$

provides the (unique) solution for the above $\mathrm{Dim}(\mathcal{H}_1) \times \mathrm{Dim}(\mathcal{H}_2)$ equations. Indeed,

$$\sum_\alpha f_{\alpha\beta} \bar{\psi}_\alpha = \sum_\alpha |f(\psi)| \psi_\alpha \phi_\beta \bar{\psi}_\alpha = |f(\psi)| \phi_\beta \Big(\sum_\alpha \bar{\psi}_\alpha \psi_\alpha \Big) = |f(\psi)| \phi_\beta$$

since $|\psi| = 1$ and

$$\sum_\alpha f_{\alpha\beta} \bar{\xi}^i_\alpha = \sum_\alpha |f(\psi)| \psi_\alpha \phi_\beta \bar{\xi}^i_\alpha = |f(\psi)| \phi_\beta \Big(\sum_\alpha \bar{\xi}^i_\alpha \psi_\alpha \Big) = 0$$

since by $\psi \perp \mathrm{Ker}(f)$ and $\tilde{e}^{(1)}_i \in \mathrm{Ker}(f)$ we have $\langle \tilde{e}^{(1)}_i \mid \psi \rangle = 0$. $\square$

We recall the definition of pure tensors [Appendix A].

Definition 5.19 A vector $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is a *pure tensor* iff there exist $\psi \in \mathcal{H}_1$ and $\phi \in \mathcal{H}_1$ such that $\Psi = \psi \otimes \phi$.

Note that the pure tensors of $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be represented in $\mathcal{H}_1 \times \mathcal{H}_2$. However, $\mathcal{H}_1 \times \mathcal{H}_2$ cannot be conceived as a subset of $\mathcal{H}_1 \otimes \mathcal{H}_2$ since, for example, the pairs $(c \cdot \psi, \phi)$ and $(\psi, c \cdot \phi)$ are not equal in $\mathcal{H}_1 \times \mathcal{H}_2$ although

$$(c \cdot \psi) \otimes \phi = (c \cdot \psi \otimes \phi) = \psi \otimes (c \cdot \phi)$$

in $\mathcal{H}_1 \otimes \mathcal{H}_2$. When passing to one-dimensional subspaces the embedding becomes faithful.

Lemma 5.20 *A vector*

$$\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(1)} \otimes e_\beta^{(2)} \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

is a pure tensor iff there exist tuples $(\psi_\alpha)_\alpha$ and $(\phi_\beta)_\beta$ such that

$$(g_{\alpha\beta})_{\alpha\beta} = (\psi_\alpha \phi_\beta)_{\alpha\beta} \,.$$

Proof. It suffices to set $\psi = \sum_\alpha \psi_\alpha \cdot e_\alpha^{(1)}$ and $\phi = \sum_\beta \psi_\beta \cdot e_\beta^{(2)}$ in $\psi \otimes \phi$. □

The above results in a characterization of those functions which label pure tensors.

Proposition 5.21 *The following are equivalent:*

- *A vector $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is a pure tensor.*
- *The labeling function $\epsilon^{\{e_i\}}(\Psi) : \mathcal{H}_1 \to \mathcal{H}_2$ is atomically singular.*
- *The labeling function $\epsilon^*(\Psi) : \mathcal{H}_1 \nrightarrow \mathcal{H}_2$ is atomically singular.*
- *The labeling function $\epsilon(\Psi) : \mathcal{H}_1^* \to \mathcal{H}_2$ is atomically singular.*

Proof. Combining Lemmas 5.18 and 5.20 yields the result for the ϵ^*- and the $\epsilon^{\{e_i\}}$-labelings. By Proposition 5.16 this extends to the ϵ-labeling. □

Our constructive approach in the proofs of Lemma 5.17, Lemma 5.18 and Lemma 5.20 also indicates the explicit construction of the functions labeling pure tensors and vice versa for the ϵ^*-labeling. Using Proposition 4.4 on labeling interchangeability we obtain the corresponding results for the ϵ-labeling and the $\epsilon^{\{e_i\}}$-labelings.

Proposition 5.22 *Let $\Psi = \psi \otimes \phi$.*

- *If $f = \epsilon^{\{e_i\}}(\Psi)$ then* $\begin{cases} f(c(\psi)) = \phi \\ f(\varphi) = \mho \ \text{ for } \ \varphi \perp c(\psi)\,. \end{cases}$
- *If $f = \epsilon^*(\Psi)$ then* $\begin{cases} f(\psi) = \phi \\ f(\varphi) = \mho \ \text{ for } \ \varphi \perp \psi\,. \end{cases}$

- *If* $f = \epsilon(\Psi)$ *then* $\begin{cases} f(\bar{\psi}) = \phi \\ f(\bar{\varphi}) = \mho \ \text{ for } \ \varphi \perp \psi\,. \end{cases}$

Proof. Given $(\epsilon^*(\Psi))(\psi) = \phi$ we have $(\epsilon^{\{e_i\}}(\Psi))(c(\psi)) = \phi$ by $\epsilon^{\{e_i\}}(\Psi) \circ c = \epsilon^*(\Psi)$. Given $(\epsilon^*(\Psi))(\varphi) = \mho$ for $\varphi \perp \psi$ we have $(\epsilon^{\{e_i\}}(\Psi))(c(\varphi)) = \mho$ for

$$\varphi \perp \psi \ \Leftrightarrow\ \sum_\alpha \bar{\varphi}_\alpha \psi_\alpha = 0 \ \Leftrightarrow\ \sum_\alpha \bar{\varphi}_\alpha \psi_\alpha = 0 \ \Leftrightarrow\ c(\varphi) \perp c(\psi)$$

and thus by substitution of $c(\varphi)$ by φ we obtain $(\epsilon^{\{e_i\}}(\Psi))(\varphi) = \mho$ for $\varphi \perp c(\psi)$. The proof of the remaining $\epsilon(\Psi)$-case proceeds straightforwardly using $\epsilon(\Psi) \circ r^* = \epsilon^*(\Psi)$. $\square$

Proposition 5.23 *Let f be atomically singular.*

- *If* $f = \epsilon^{\{e_i\}}(\Psi)$ *then* $\Psi = c(\psi) \otimes f(\psi)$ *with* $\psi \perp \mathrm{Ker}(f)$.
- *If* $f = \epsilon^*(\Psi)$ *then* $\Psi = \psi \otimes f(\psi)$ *with* $\psi \perp \mathrm{Ker}(f)$.
- *If* $f = \epsilon(\Psi)$ *then* $\Psi = \psi \otimes f(\bar{\psi})$ *with* $\bar{\psi} \perp \mathrm{Ker}(f)$.

Proof. Given $\Psi = \psi \otimes (\epsilon^*(\Psi))(\psi)$ with $\psi \perp \mathrm{Ker}(\epsilon^*(\Psi))$ again by $\epsilon^{\{e_i\}}(\Psi) \circ c = \epsilon^*(\Psi)$ we have $\Psi = \psi \otimes (\epsilon^{\{e_i\}}(\Psi))(c(\psi))$ with

$$\begin{aligned}
\psi \perp \mathrm{Ker}(\epsilon^{\{e_i\}}(\Psi) \circ c) \ &\Leftrightarrow\ \psi \perp \{\varphi \in \mathcal{H}_1 \mid (\epsilon^{\{e_i\}}(\Psi))(c(\varphi)) = \mho\} \\
&\Leftrightarrow\ \psi \perp \{c(\varphi) \in \mathcal{H}_1 \mid (\epsilon^{\{e_i\}}(\Psi))(\varphi) = \mho\} \\
&\Leftrightarrow\ c(\psi) \perp \{\varphi \in \mathcal{H}_1 \mid (\epsilon^{\{e_i\}}(\Psi))(\varphi) = \mho\} \\
&\Leftrightarrow\ c(\psi) \perp \mathrm{Ker}(\epsilon^{\{e_i\}}(\Psi))
\end{aligned}$$

so by substitution of $c(\psi)$ by ψ we obtain

$$\Psi = c(\psi) \otimes (\epsilon^{\{e_i\}}(\Psi))(\psi) \qquad \text{with} \qquad \psi \perp \mathrm{Ker}(\epsilon^{\{e_i\}}(\Psi))\,.$$

The proof of the remaining $\epsilon(\Psi)$-case again proceeds straightforwardly using the fact that $\epsilon(\Psi) \circ r^* = \epsilon^*(\Psi)$ and $\psi \perp \varphi \Leftrightarrow \bar{\psi} \perp \bar{\varphi}$. $\square$

From Proposition 5.23 we can derive the generic shapes of the functions which label a pure tensor $\psi \otimes \phi$. We obtain, respectively for an $\epsilon^{\{e_i\}}$-, the ϵ- and the ϵ^*-labeling,

$$\langle c(\psi) \mid - \rangle \cdot \phi : \mathcal{H}_1 \to \mathcal{H}_2 \qquad \langle - \mid \psi \rangle \cdot \phi : \mathcal{H}_1 \nrightarrow \mathcal{H}_2 \qquad \langle \bar{\psi} \mid - \rangle \cdot \phi : \mathcal{H}_1^* \to \mathcal{H}_2\,.$$

In Subsection 4.3 we already encountered an important example of an atomically singular (endo)map namely the projector on the one-dimensional subspace spanned by a (unit) vector ψ, that is,

$$\mathrm{P}_\psi := \langle \psi \mid - \rangle \cdot \psi : \mathcal{H} \to \mathcal{H}\,.$$

It arises as the $\epsilon^{\{e_i\}}$-labeling of the pure tensor $c(\psi)\otimes\psi$. The corresponding **eP** which is labeled by P_ψ is itself also a projector, namely

$$\mathrm{P}_{\mathrm{P}_\psi} := \langle c(\psi)\otimes\psi \mid -\rangle\cdot c(\psi)\otimes\psi : \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}\,,$$

and thus also an atomically singular map, as are all **eP**'s, but in this paper the important ones are the atomically singular labeling functions. The atomic singularity of **eP**'s becomes important when axiomatizing the ideas of this paper [4].

Definition 5.24 By an *anti-projector* we mean the ϵ^*-labeling function

$$\mathrm{P}^*_\psi := \langle - \mid \psi\rangle\cdot\psi : \mathcal{H} \looparrowright \mathcal{H}$$

of a *symmetric pure tensor*

$$\psi\otimes\psi\in\mathcal{H}\otimes\mathcal{H}\,.$$

Symmetric pure tensors and anti-projectors are in canonical bijective correspondence. In particular does each anti-projector define an **eP** which projects on a symmetric pure tensor. Projectors do not necessarily define an **eP** which projects on a symmetric pure tensor due to the base-dependency of the $\epsilon^{\{e_i\}}$-labelings. The η-labeling does define a canonical pure tensor given a projector $\mathrm{P}_\psi : \mathcal{H}\to\mathcal{H}$ as labeling function, namely the *anti-symmetric pure tensor*

$$\bar{\psi}\otimes\psi\in\mathcal{H}^*\otimes\mathcal{H}\,.$$

Proposition 5.25 *An anti-linear map $f:\mathcal{H}_1\looparrowright\mathcal{H}_2$ is atomically singular iff its adjoint $f^\dagger:\mathcal{H}_2\looparrowright\mathcal{H}_1$ is atomically singular.*

Proof. By Proposition 4.14 the matrix of f takes the shape $(\psi_\alpha\phi_\beta)_{\alpha\beta}$ iff the matrix of $f^\dagger$ takes the shape $(\phi_\alpha\psi_\beta)_{\alpha\beta}$ from which the result follows. □

The state obtained when taking the adjoint of the labeling function which labels a state $\psi\otimes\phi$ is the state $\phi\otimes\psi$. This shows that for the anti-projector P_ψ, since it labels the state $\psi\otimes\psi$, we have $\mathrm{P}^\dagger_\psi=\mathrm{P}_\psi$, that is, *anti-linear self-adjointness.*

5.4 Proofs of compositionality theorems

i. *Full forward path compositionality.* In order to apply Proposition 5.11 to the case of full forward path compositionality we assume the existence of:

- An additional track labeled 0 of type $\mathcal{H}_0=\mathcal{H}_{in}$ with $\phi_{in}\in\mathcal{H}_0$ as initial state. The presence of this track will not change Ψ due to the independence lemma.
- An **eP** at time , namely

$$\mathrm{P}_{\mathrm{P}^*_{\phi_{in}};0,\nu_{in};} : \mathcal{H}_{in}\otimes\mathcal{H}_{in}\to\mathcal{H}_{in}\otimes\mathcal{H}_{in} :: \Phi\mapsto\langle\phi_{in}\otimes\phi_{in}\mid\Phi\rangle\cdot\phi_{in}\otimes\phi_{in}$$

where the labeling function is the anti-projector [Subsection 5.3]

$$\mathrm{P}^*_{\phi_{in}} : \mathcal{H}_{in} \nrightarrow \mathcal{H}_{in} :: \psi \mapsto \langle \psi \mid \phi_{in} \rangle \cdot \phi_{in} \,.$$

Setting $\phi_{in} = \sum_\alpha \phi^{in}_\alpha \cdot e_\alpha$ we have

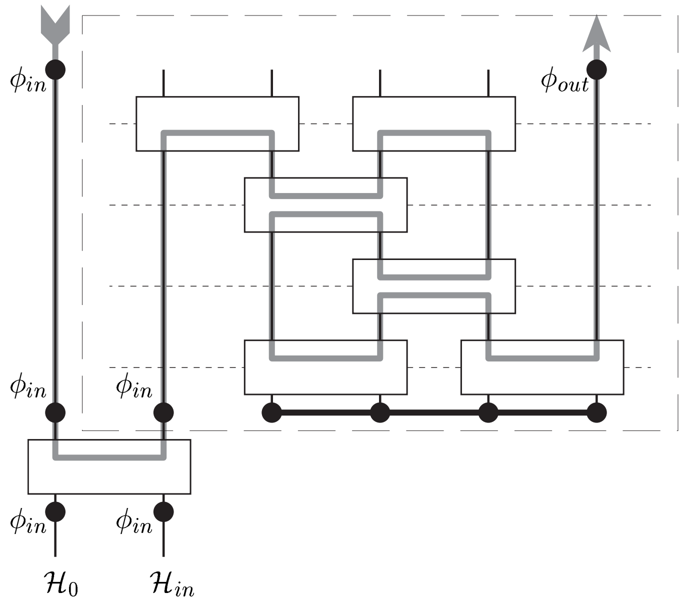
$$(\mathrm{P}^*_{\phi_{in}})_{\alpha\beta} = \phi^{in}_\alpha \phi^{in}_\beta \,.$$

Introducing this extra **eP** will merely change Ψ to $\phi_{in} \otimes \Psi$; it prepares the pure tensor $\phi_{in} \otimes \phi_{in}$ of which the righthandside vector ϕ_{in} is the one we assumed as being the input of the path and of which the lefthandside vector ϕ_{in} will not be acted on by any **eP** and consequently doesn't interact with the rest of the network. Hence it does not alter the qualitative content of Theorem 3.4.

We also extend the path Γ by adding

$$[(0,),(0,)[\cdot((0,),(\nu_{in},))\ \cdot\ \ldots$$

This results in the following picture.



We obtain an output only path to which we can apply Proposition 5.11. As $\phi_{in} \otimes \Psi$ we obtain

$$\Big(\sum_{\alpha\beta}(f_{||\Gamma||} \circ \ldots \circ f_1 \circ \mathrm{P}^*_{\phi_{in}})_{\alpha\beta} \cdot e^{(0)}_\alpha \otimes e^{(\nu_{out})}_\beta\Big) \otimes \Phi$$

where [Lemma 4.27]

$$(f_{||\Gamma||} \circ \ldots \circ f_1 \circ \mathrm{P}^*_{\phi_{in}})_{\alpha\beta} = \sum_{i_0 \ldots i_{||\Gamma||-1}} \phi^{in}_\alpha \phi^{in}_{i_0} \bar{f}^1_{i_0 i_1} f^2_{i_1 i_2} \ldots \bar{f}^{||\Gamma||-1}_{i_{||\Gamma||-2} i_{||\Gamma||-1}} f^{||\Gamma||}_{i_{||\Gamma||-1}\beta}$$

so

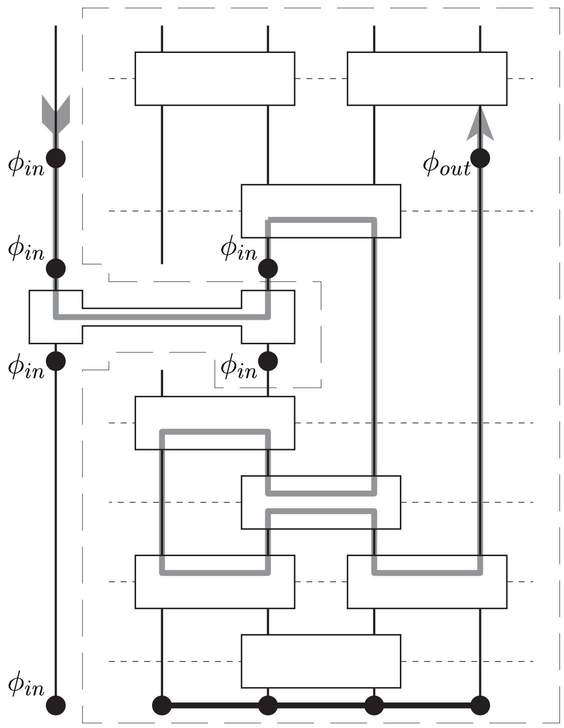
$$
\begin{aligned}
&\sum_{\alpha\beta}(f_{||\Gamma||}\circ\ldots\circ f_1\circ \mathrm{P}^*_{\phi_{in}})_{\alpha\beta}\cdot e^{(0)}_\alpha\otimes e^{(\nu_{out})}_\beta\\
&\quad= \sum_{\alpha i_0\ldots i_{||\Gamma||-1}\beta}\phi^{in}_\alpha\phi^{in}_{i_0}\bar f^1_{i_0 i_1}f^2_{i_1 i_2}\ldots \bar f^{||\Gamma||-1}_{i_{||\Gamma||-2}i_{||\Gamma||-1}}f^{||\Gamma||}_{i_{||\Gamma||-1}\beta}\cdot e^{(0)}_\alpha\otimes e^{(\nu_{out})}_\beta\\
&\quad= \Big(\sum_\alpha \phi^{in}_\alpha\cdot e^{(0)}_\alpha\Big)\otimes\Big(\sum_{\alpha i_1\ldots i_{||\Gamma||-1}\beta}\phi^{in}_\alpha\bar f^1_{\alpha i_1}f^2_{i_1 i_2}\ldots\bar f^{||\Gamma||-1}_{i_{||\Gamma||-2}i_{||\Gamma||-1}}f^{||\Gamma||}_{i_{||\Gamma||-1}\beta}\cdot e^{(\nu_{out})}_\beta\Big)\\
&\quad= \phi_{in}\otimes\phi_{out}
\end{aligned}
$$

with [Lemma 4.6]

$$\phi_{out}=(f_{||\Gamma||}\circ\cdots\circ f_{\gamma+1}\circ f_\gamma\circ f_{\gamma-1}\circ\cdots\circ f_1)(\phi_{in})\,.$$

This finishes the proof of full forward compositionality. □

ii. *Partial forward path compositionality.* The proof proceeds along the same lines as the one for full forward path compositionality. We illustrate this in a picture.



We omit an explicit proof here. □

iii. *Proof of Proposition 4.12.* Since all compositionality results follow by Lemma 5.6 it suffices to consider reversal of the **eP**'s for that result. Having an **eP** opposite to the direction of the path corresponds in the proof of Lemma 5.6 with the substitution

$$\sum_{\alpha_i\alpha_j} f_{\alpha_i\alpha_j} \cdot e^{(i)}_{\alpha_i} \otimes e^{(j)}_{\alpha_j} \rightsquigarrow \sum_{\alpha_i\alpha_j} f_{\alpha_i\alpha_j} \cdot e^{(j)}_{\alpha_j} \otimes e^{(i)}_{\alpha_i}\,.$$

By Proposition 4.14 we have

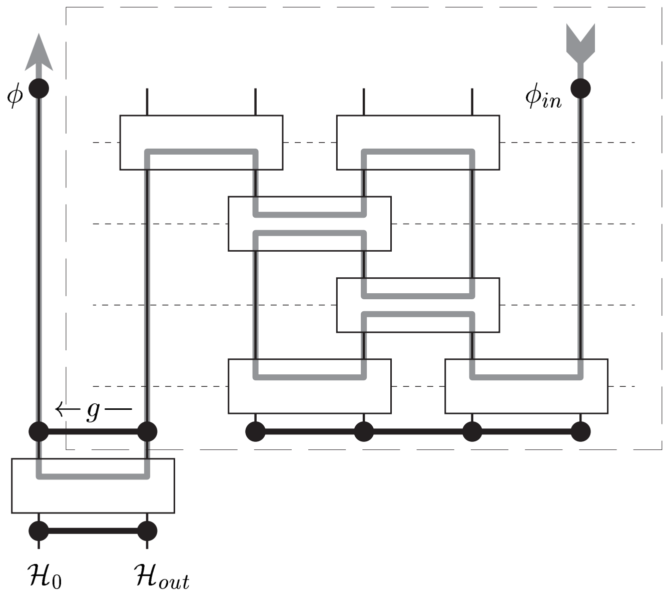
$$\sum_{\alpha_i\alpha_j} f_{\alpha_i\alpha_j} \cdot e^{(j)}_{\alpha_j} \otimes e^{(i)}_{\alpha_i} = \sum_{\alpha_i\alpha_j} f^{\dagger}_{\alpha_j\alpha_i} \cdot e^{(j)}_{\alpha_j} \otimes e^{(i)}_{\alpha_i}$$

what completes the proof. □

iv. *Partial freeness of the full backward path output when all labeling functions are surjective* [Interlude 4.22]. We demonstrate the sufficiency of surjectivity for deducing that ϕ_{out} is free in Ψ provided that:

1. ϕ_{in} is free in Ψ ;
2. ϕ_{out} is not entangled to any of the carriers that take part in the path.

Consider the following situation.



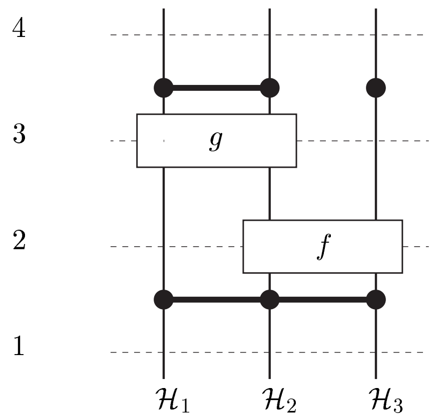
We have to prove that $\sum_{\alpha\beta} g_{\alpha\beta} \cdot e^{(\nu_{out})}_{\alpha} \otimes e^{(0)}_{\beta}$ is of the form $\phi_{out} \otimes \phi$ given that

$$\sum_{\alpha\beta} (g \circ f_{||\Gamma||} \circ \ldots \circ f_1)_{\alpha\beta} \cdot e^{(\nu_{in})}_{\alpha} \otimes e^{(0)}_{\beta} = \phi_{in} \otimes \phi$$

with the additional knowledge that all $f_1, \ldots, f_{||\Gamma||}$ are singular. Note here that since the path is backward we assume that the function g is directed from track ν_{out} to track

0, reflected in the exchange of the order of the base vectors. By Proposition 5.21 we have that $g \circ f_{||\Gamma||} \circ \ldots \circ f_1$ is atomically singular and by Proposition 5.15 it follows that whenever all f are surjective that also g is atomically singular. Again by Proposition 5.21 it then follows that ϕ_{out} is free in Ψ. $\square$

Counter example 5.26 Consider the configuration of the following picture.



By Lemma 5.2 on state preparation we know that carrier 1, 2 and 3 are all entangled in Ψ but that carrier 3 is not entangled to the other two in Ψ. Thus freeness of ϕ_{in} in Ψ does not guarantee ϕ_{out} to be free in Ψ.

Hence we can say nothing on whether ϕ_{out} is entangled to any of the carriers which take part in the path if this was not assumed in advance.

v. *Output only path compositionality.* By Proposition 5.11 we have

$$\sum_{\alpha\beta} (f_{||\Gamma||} \circ \ldots \circ f_1)_{\alpha\beta} \cdot e_\alpha^{(\nu_{in})} \otimes e_\beta^{(\nu_{out})}$$

as a component of the output state. Assuming that ϕ_{in} is free corresponds to requiring that the above component can be written as a pure tensor $\phi_{in} \otimes \phi_{out}$. Thus by Lemma 5.5 and Lemma 5.20 we know

$$\sum_{i_1 \ldots i_{||\Gamma||}} f^1_{\alpha i_1} \bar{f}^2_{i_1 i_2} \ldots \bar{f}^{||\Gamma||-1}_{i_{||\Gamma||-2} i_{||\Gamma||-1}} f^{||\Gamma||}_{i_{||\Gamma||-1}\beta} = \phi^{in}_\alpha \phi^{out}_\beta .$$

Using the fact that ϕ_{in} is normalized we obtain

$$\begin{aligned}
\phi^{out}_\beta &= \Big(\sum_\alpha \bar{\phi}^{in}_\alpha \phi^{in}_\alpha\Big) \phi^{out}_\beta \\
&= \sum_\alpha \bar{\phi}^{in}_\alpha (\phi^{in}_\alpha \phi^{out}_\beta) \\
&= \sum_\alpha \bar{\phi}^{in}_\alpha \sum_{i_1 \ldots i_{||\Gamma||}} f^1_{\alpha i_1} \bar{f}^2_{i_1 i_2} \ldots \bar{f}^{||\Gamma||-1}_{i_{||\Gamma||-2} i_{||\Gamma||-1}} f^{||\Gamma||}_{i_{||\Gamma||-1}\beta}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha i_1 \ldots i_{||\Gamma||}} \bar{\phi}^{in}_{\alpha} f^1_{\alpha i_1} \bar{f}^2_{i_1 i_2} \ldots \bar{f}^{||\Gamma||-1}_{i_{||\Gamma||-2} i_{||\Gamma||-1}} f^{||\Gamma||}_{i_{||\Gamma||-1}\beta} \\
&= ((f_{||\Gamma||} \circ \ldots \circ f_1)(\phi_{in}))_{\beta}
\end{aligned}
$$

by Lemma 4.27 what completes the proof. □

vi. *Extended compositionality.* After substituting Lemma 5.6 by Lemma 5.9 in Subsection 5.2 — what results in the obvious extension of Proposition 5.11 — the result follows from the other proofs in this subsection.

5.5 Solutions to the riddles

The solutions to the riddles rely on the preceding proofs in this section.

i. *Solution to Riddle 3.8.* Let $\Gamma_{\tau_{out}}$ be the subpath of Γ obtained by removing list elements, starting with the first one, until the list has no occurrences at a physical time later than τ_{out} anymore. The first element of the remaining list $\Gamma_{\tau_{out}}$ will then also be at time τ_{out}. Thus the resulting path $\Gamma_{\tau_{out}}$ will be an output only path. Since ϕ_{out} is free in $\Psi^{\tau_{out}}$ the composite of the labeling functions of all **eP**'s along $\Gamma_{\tau_{out}}$ has to be an atomically singular map, that is,

$$
\Big(\sum_{i_k \ldots i_{||\Gamma||}} f^k_{i_{k-1} i_k} \bar{f}^{k+1}_{i_k i_{k+1}} \ldots \bar{f}^{||\Gamma||-1}_{i_{||\Gamma||-2} i_{||\Gamma||-1}} f^{||\Gamma||}_{i_{||\Gamma||-1}\beta}\Big)_{i_{k-1}\beta} = (\phi_{i_{k-1}} \phi^{out}_{\beta})_{i_{k-1}\beta}
$$

with

$$
||\Gamma|| - k + 1 = ||\Gamma_{\tau_{out}}||\,.
$$

The range of this atomically singular map is the subspace spanned by ϕ^{out}. It then also follows that

$$
\Big(\sum_{i_1 \ldots i_{||\Gamma||}} \bar{f}^1_{\alpha i_1} f^2_{i_1 i_2} \ldots \bar{f}^{||\Gamma||-1}_{i_{||\Gamma||-2} i_{||\Gamma||-1}} f^{||\Gamma||}_{i_{||\Gamma||-1}\beta}\Big)_{\alpha\beta} = (\psi_{\alpha}\, \phi^{out}_{\beta})_{\alpha\beta}
$$

with

$$
\psi_{\alpha} = \sum_{i_1 \ldots i_{k-1}} \bar{f}^1_{\alpha i_1} f^2_{i_1 i_2} \ldots \bar{f}^{k-1}_{i_{k-2} i_{k-1}} \phi_{i_{k-1}}\,.
$$

Hence $f_{||\Gamma||} \circ \ldots \circ f_1$ is itself also atomically singular with as range the one-dimensional subspace spanned by ϕ^{out}. Since the outcome ϕ_{out} is independent of the other **eP**'s $f_1, \ldots, f_{k-1}$ which take no part in $\Gamma \setminus \Gamma_{\tau_{out}}$ it follows that we will have compositional behavior. The fact that in physical time $f_1, \ldots, f_{k-1}$ might not have taken place "yet" doesn't affect the outcome state ϕ^{out} at all. To summarize, when f is atomically singular we have for an arbitrary g that

$$
(f \circ g)(\psi) = f(\phi)
$$

ignoring normalization and provided that both sides of the equality are non-zero. We will discuss the issue of *disentanglement*, which is the functional counterpart to atomic singularity, in more detail in Subsection 6.5. □

ii. *Solution to Riddle 4.25.* The solution lies in the decreased degrees of freedom. In fact, once the network and the path is specified there are no degrees of freedom left anymore since both ϕ_{in} and ϕ_{out} are part of the physical output Ψ of which the part relevant for us is completely determined by the network itself, that is, there is no dependence on the physical input Ψ. Changing the notations to ϕ_{in}^{Γ} and ϕ_{out}^{Γ} would stress this fact. It then follows that the compositionality claim only applies to the particular input ϕ_{in}^{Γ} which we obtain for the chosen path, and not for any arbitrary vector $\phi \in \mathcal{H}_{in}$. The composite of the maps $f_1, \cdots, f_{||\Gamma||}$ has to be atomically singular in order to assure that ϕ_{in}^{Γ} and ϕ_{out}^{Γ} are free in Ψ. So we have, equivalently,

$$(f_{||\Gamma||} \circ \ldots \circ f_1)_{\alpha\beta} = \phi_\alpha^{\Gamma,in} \cdot \phi_\beta^{\Gamma,out} \quad \text{and} \quad (f_{||\Gamma||} \circ \ldots \circ f_1)_{\beta\alpha}^{\dagger} = \phi_\beta^{\Gamma,out} \cdot \phi_\alpha^{\Gamma,in}.$$

It is due to this that we can both have

$$\phi_{out}^{\Gamma} = (f_{||\Gamma||} \circ \cdots \circ f_1)(\phi_{in}^{\Gamma}) \quad \text{and} \quad (f_1^{\dagger} \circ \cdots \circ f_{||\Gamma||}^{\dagger})(\phi_{out}^{\Gamma}) = \phi_{in}^{\Gamma}$$

while neither

$$f_{||\Gamma||} \circ \cdots \circ f_1 \qquad \text{and} \qquad f_1^{\dagger} \circ \cdots \circ f_{||\Gamma||}^{\dagger}$$

admit an inverse. Given an entanglement specification network, this fact strongly restricts the output only paths for which a compositionality result in the sense of Theorem 4.24 will hold. We conclude that output only paths in the sense of Theorem 4.24 are somewhat pathological. However, as we will discuss below, output only paths do become interesting when we relax the condition on freeness by conceiving them "in context" [Subsection 6.1]. Again we refer to Subsection 6.5 for more details on the significance of *disentanglement* in perspective of the results of this paper. □

iii. *Solution to Riddle 4.26.* Here we cannot use the decreased degrees of freedom argument which we used above since both ϕ_1^{out} and ϕ_2^{in} are arbitrary. But the solution is very similar to the solution of Riddle 3.8. Denoting the global path by Γ we have

$$\begin{aligned}
&\Big(\sum_{i_1\ldots i_{||\Gamma||}} f_{\alpha i_1}^1 \bar{f}_{i_1 i_2}^2 \ldots \bar{f}_{i_{||\Gamma||-2} i_{||\Gamma||-1}}^{||\Gamma||-1} f_{i_{||\Gamma||-1}\beta}^{||\Gamma||}\Big)_{\alpha\beta} \\
&\qquad\qquad = \Big(\sum_{i_1\ldots i_{||\Gamma||}} f_{\alpha i_1}^1 \ldots \bar{f}_{i_{k-2} i_{k-1}}^{k-1} \phi_{i_{k-1}}^{1,out} \phi_{i_k}^{2,in} \bar{f}_{i_k i_{k+1}}^{k+1} \ldots f_{i_{||\Gamma||-1}\beta}^{||\Gamma||}\Big)_{\alpha\beta} \\
&\qquad\qquad = \left(\psi_\alpha \left((f_{||\Gamma||} \circ \ldots \circ f_{k+1})(\phi^{2,in})\right)_\beta\right)_{\alpha\beta}.
\end{aligned}$$

It then follows that we have

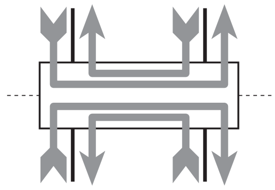
$$\phi_{out}^2 = (f_{||\Gamma||} \circ \ldots \circ f_{k+1})(\phi^{2,in})$$

for the global path Γ, which is (of course) the same expression as we obtain when considering the forward path only. Hence, due to the fact that the extra **eP** used to connect up the two paths is atomically singular the backward path and the **eP**'s therein play no role at all on ϕ_2^{out}. □

6 Functions as inputs and outputs

First we digest things a bit by means of a conclusion. Then we will analyze which facts brought us to this conclusion. We will investigate whether we can push things even further by exploiting these facts even more.

Conclusion 6.1 *We identified an (acausal) "***as if** *behavior" of the information flowing in an entanglement specification network which contains bipartite projectors and local unitary actions* [Subsections 3.2 and 3.1], *or, boldly put, the information flowing through bipartite entanglement itself:*



The necessity of the use of **as if** (*as compared to* **is**) *is enforced by*

- *the acausality of the flow* [Subsections 3.2 and 3.1], *and even more so,*
- *by the limitations of the interpretation* [Subsections 4.1 and 4.3].

All statements only refer to "global" behavior (the relation between inputs and outputs of paths) and not to "local" (intermediate) nodes of a path.

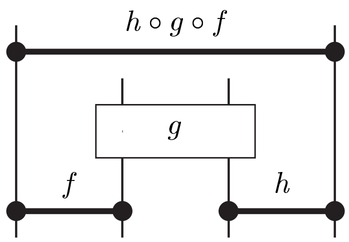
Subsection 5.2 reveals that the structural core of the compositional behavior of the virtual information flow through the **eP**'s rests in Lemma 5.6, which is not a lemma on

- how *a free state propagates* through the **eP**'s

but a lemma on

- how *two entangled states interact* when they pass an **eP**.

In the picture below f and h interact by means of the action of the g-labeled **eP** on them. This interaction results in a new entangled state $h \circ g \circ f$.



Considerations on *freeness* of the input ϕ_{in} and the output ϕ_{out} are of a secondary nature as compared to this interaction mechanism. This will allow some more refined interpretations of our paths [Subsection 6.1]. And these will allow at their turn for more sophisticated functions such as function-valued functions and higher-order functions to be encoded in terms of entanglement [Subsection 6.4].

The question on compositional behavior breaks up in two subquestions with corresponding subanswers namely one on compositionality and one on freeness.

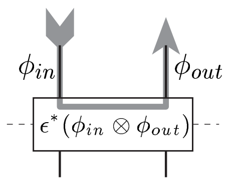
Corollary 6.2 *Consider an arbitrary path Γ of one of the shapes we have considered so far but without any a priori condition on the location of the input and the output concerning time-point and time-direction. Further we assume regularity.*

- *If both the input state ϕ_{in} and output state ϕ_{out} are free and if in addition the output points forward (in physical time) then ϕ_{out} depends compositionally on ϕ_{in}.*
- *If τ_{out} is later (in physical time) than any other point of the path then the freeness of the output state can be deduced from the freeness of the input state.*

Proof. The proof proceeds by case by case evaluation using Theorem 3.4, Riddle 3.8, Corollary 3.6 and Theorem 4.24. It still remains to be checked that if for an output only path freeness of an arbitrary located input guarantees freeness of the output whenever the latter is later than any other point of the path. We leave this to the reader. $\square$

For purely physical reasons only forward and output only paths can pass either of the conditions in the two statements in Corollary 6.2. Notice that for these two cases, given freeness of the input ϕ_{in}, the condition on compositionality provided ϕ_{out} is free is weaker than the one on freeness of the output ϕ_{out}.

Corollary 6.2 gives a unified view on forward and output only paths' compositional behavior. These two kinds of paths are however manifestly different. In particular is the output only case pathological in the sense that besides the output ϕ_{out} also the input ϕ_{in} is *produced by the network* [Riddle 4.25]. Having freeness of the input (and the output) then requires the composite of all labeling functions along the path to be atomically singular [Subsection 5.3]. This strongly restricts the paths for which any compositionality statement can be made. For example, the positive **eP** [Definition 3.13] of a one-**eP** output only path needs to be a projector on a pure tensor [Proposition 5.21] in order to have a compositionality statement of the kind of Theorem 4.24.



On the other hand, Lemma 5.6 and even more so Proposition 5.11 respectively apply to arbitrary 3-**eP** and n-**eP** output only paths. Below we will change the rules of the game and this will provide all output only paths with compositionality statements. As a matter of fact also backward and input only paths will become truly relevant.

6.1 Entanglement as a function

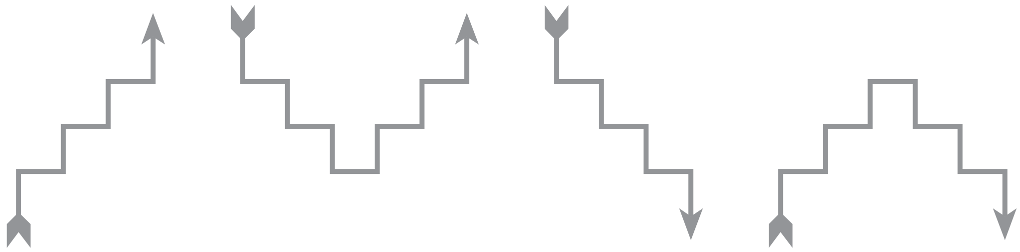
The relevant part of the physical output Ψ in Lemma 5.6 and Proposition 5.11 can and will in most cases be a proper entangled state (= not a pure tensor).

Question 6.3 *Can we formulate a compositionality statement for arbitrary output only paths while maintaining our information flow paradigm?*

To this aim we also answer a more specific question.

Question 6.4 *What is the operational significance of a bipartite entangled state?*

This question has several answers. We introduce the following graphical representation for the four kinds of paths we have considered so far, that is, forward, output only, backward and input only.



i. *Behavior under measurement.* Below we conceive g as being anti-linear.

- If we perform a measurement M_1 on carrier 1 of a bipartite system which is in state

$$\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(1)} \otimes e_\beta^{(2)}$$

  then, whenever after the measurement carrier 1 is in state $\phi_1 \in \mathcal{H}_1$, carrier 2 is in state $g(\phi_1) \in \mathcal{H}_2$; if we instead perform a measurement on carrier 2 then, whenever after the measurement carrier 2 is in state $\phi_2 \in \mathcal{H}_2$, carrier 1 is in state $g^\dagger(\phi_2) \in \mathcal{H}_1$.

This view is well-known and admits several extensions e.g. multipartite entanglement [17] and property lattices [18]. It however forces us to bring quantum measurements explicitly into the interpretation.

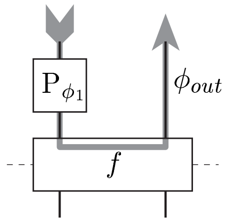
ii. *Effective input specification.* Bringing measurements into the picture can easily be avoided. Due to the spectral decomposition theorem [Subsection 2.1 and Theorem A.8] the action of a measurement reduces to that of a projector, namely the projector

$$\mathrm{P}_{\phi_1} : \mathcal{H} \to \mathcal{H} :: \psi \mapsto \langle \phi_1 \mid \psi \rangle \cdot \phi_1$$

which specifies the outcome state $\phi_1 = \sum_i \phi_i^1 \cdot e_i^{(1)}$ of the measurement. We have

$$\begin{aligned}(\mathrm{P}_{\phi_1} \otimes \mathrm{id}_{\mathcal{H}_2})\Big(\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(1)} \otimes e_\beta^{(2)}\Big) &= \sum_{\alpha\beta} g_{\alpha\beta} \cdot \mathrm{P}_{\phi_1}(e_\alpha^{(1)}) \otimes e_\beta^{(2)} \\ &= \sum_{\alpha\beta} g_{\alpha\beta} \, \langle \phi_1 \mid e_\alpha^{(1)} \rangle \cdot \phi_1 \otimes e_\beta^{(2)} \\ &= \phi_1 \otimes \Big(\sum_{\alpha\beta} \bar{\phi}_\alpha^1 \, g_{\alpha\beta} \cdot e_\beta^{(2)}\Big) \\ &= \phi_1 \otimes g(\phi_1)\end{aligned}$$

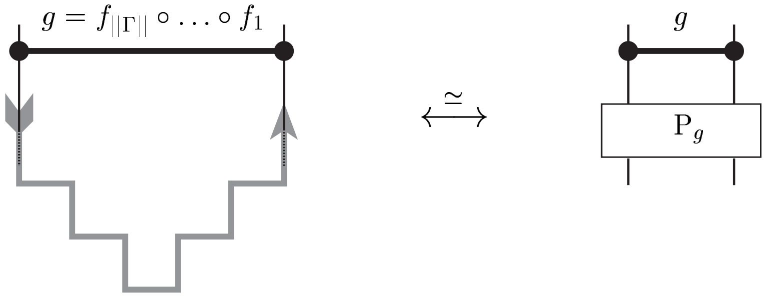
where the conjugation vanishes in the last step due to anti-linearity [Lemma 4.27]. In terms of linear maps this conjugation is the witness of the fact that the specification of the input is directed "backwardly" in time [Subsection 4.1]. Indeed, since an entangled state Ψ_f can be obtained as the physical bipartite output of the bipartite projector P_f we get the following picture.



Dually to the above calculation we also have

$$(id_{\mathcal{H}_1} \otimes \mathrm{P}_{\phi_2})\Big(\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(1)} \otimes e_\beta^{(2)}\Big) = g^\dagger(\phi_2) \otimes \phi_2 \, .$$

By Proposition 5.11 an entangle state Ψ_g can also be seen as the relevant part of the physical output of any output only path which is such that $g = f_{||\Gamma||} \circ \ldots \circ f_1$. Of course P_g defines itself also an output only path consisting of one **eP**. Hence the path-view generalizes the preparational one.



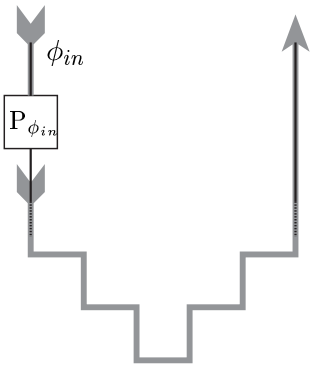
It then follows that we can read an output only path in the same way as we can interpret entanglement. This essentially consists in replacing

"If the input is free and equal to ϕ_{in} then ..."

by the *conditional* statement

"Whenever we effectively specify the input by a projector P_{ϕ_1} then ...".

In a picture this means that we evaluate compositionality by making the path first pass through one additional box which specifies (backwardly) that the input is indeed ϕ_{in}.

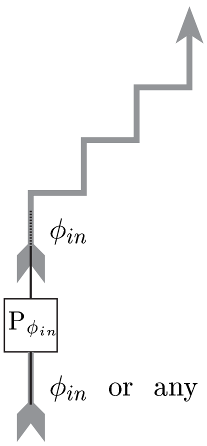


$$\begin{aligned}\phi_{out} &= (f_{||\Gamma||} \circ \ldots \circ f_1)(\phi_{in}) \\ &= (f_{||\Gamma||} \circ \ldots \circ f_1 \circ \mathrm{P}^{\dagger}_{\phi_{in}})(\phi_{in})\end{aligned}$$

Due to effective specification of ϕ_{in} by means of $\mathrm{P}_{\phi_{in}}$ we have that ϕ_{out} is now free in Ψ for any output only path. There is no harm in reading the projector $\mathrm{P}_{\phi_{in}}$ in the same way as we read local unitary operations acting on the path. Moreover, the direction of reading actually doesn't really matter since $\mathrm{P}^{\dagger}_{\phi_{in}} = \mathrm{P}_{\phi_{in}}$. All the above leads us to a second interpretation which views an entangled state as being the relevant part of the physical output of an output only path.

Conclusion 6.5 (Effective input specification) *If we effectively specify the input ϕ_{in} of any output only path by means of a projector $\mathrm{P}_{\phi_{in}}$ then $\phi_{out} = g(\phi_{in})$; the dual case $\phi_{out} = g^{\dagger}(\phi_{in})$ arises by reverting the direction of the path and hence replacing $\mathrm{P}_{\phi_{in}}$ by $\mathrm{P}_{\phi_{out}}$ now located at the other end of the path.*

Note that we can also add an input specification projector to each forward path.



$$\begin{aligned}\phi_{out} &= (f_{||\Gamma||} \circ \ldots \circ f_1)(\phi_{in}) \\ &= (f_{||\Gamma||} \circ \ldots \circ f_1 \circ \mathrm{P}_{\phi_{in}})(\phi_{in}) \\ &= (f_{||\Gamma||} \circ \ldots \circ f_1 \circ \mathrm{P}_{\phi_{in}})(\psi)\end{aligned}$$

This doesn't impose any change on the path's compositional properties.

Observe further that in contrast to the discussion in the solution of Riddle 4.25 effective input specification provides both forward an output only paths with the same number of available degrees of freedom, namely the choice of the projector by means of which we effectively specify the input.

The above clearly indicates that we can generalize Corollary 6.2 to arbitrary output only paths when dropping the requirement on freeness of the input and replacing it by effective input specification by means of a projector. It moreover at the same time gets rid of all the pathological cases [Riddles 3.8, 4.25 and 4.26]. We will not formulate the resulting corollary explicitly. It will be included in the main theorem of our story which we formulate at the end of this paper.

Convention 6.6 *We will read any input ϕ_{in} "as if" it is effectively specified by* $\mathrm{P}_{\phi_{in}}$.

This convention enables a unified treatment of many different kinds of paths.

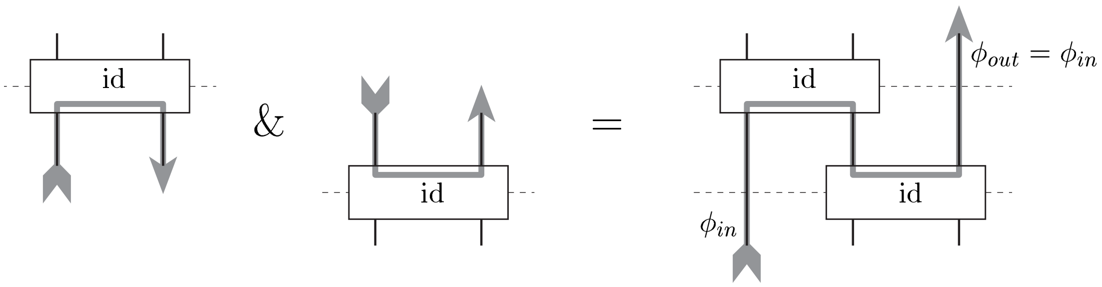
iii. *Entangled states as functions.* The above also indicates that we can interpret the entangled states themselves as functions. When passing from *effective input specification* to *potential input specification* the entangled state Ψ_f is waiting for the actual input ϕ_{in} to be specified by the projector $\mathrm{P}_{\phi_{in}}$. In terms of the λ-*calculus* [7] the entangled state acts as the λ-it term $\lambda x.f(x)$. Once the input input ϕ_{in} is specified this term becomes

$$(\lambda x.f(x))\phi_{in} \stackrel{\beta}{=} f(\phi_{in})\,.$$

As a consequence, we can think of an **eP** or more generally, an output only path, as producing a function which takes effective input specification as values. It follows that besides labeling entangled states by functions we can also view them as functions.

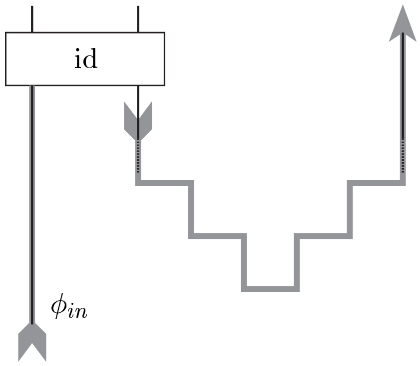
Conclusion 6.7 (Bipartite entanglement as a function) *Every bipartite entangled state can be interpreted as a function.*

iv. *Behavior in context.* Paths of the kinds we have studied until now can be conceived as parts of more complex collections of interacting paths. Paths which individually do not satisfy compositionality statements might do it within a larger *context.* This is for example the case for individual **eP**'s.



In the above picture neither the one-**eP** input only nor the one-**eP** output only path admits a compositionality statement in the sense of Section 3 but the joint path admits one. Hence one can define the compositional behavior for the output only and the input only path by their behavior within the concatenated path.

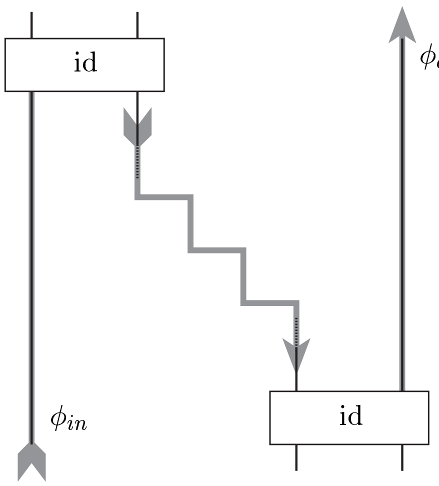
Consider now an output only path. We can turn it into a forward one by adding an additional identity **eP**.



$$\begin{aligned}\phi_{out} &= (f_{||\Gamma||} \circ \ldots \circ f_1 \circ \mathrm{id})(\phi_{in}) \\ &= (f_{||\Gamma||} \circ \ldots \circ f_1)(\phi_{in})\end{aligned}$$

We obtain a full forward path and thus compositional behavior for any input ϕ_{in}. This again provides an alternative interpretation of output only paths and thus also an alternative interpretation of entanglement — vs. the effective input specification interpretation. However, in both cases we evaluate compositional behavior in a larger network, for example by adding $\mathrm{P}_{\phi_{in}}$ in the case of effective input specification, and by adding an identity **eP** in the above case. The generic idea is captured by the utterance "behavior in context".

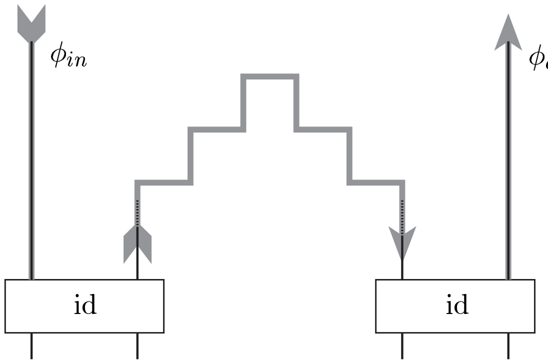
Any backward path can be turned into a forward one by adding two identity **eP**'s.



$$\begin{aligned}\phi_{out} &= (\mathrm{id} \circ f_{||\Gamma||} \circ \ldots \circ f_1 \circ \mathrm{id})(\phi_{in}) \\ &= (f_{||\Gamma||} \circ \ldots \circ f_1)(\phi_{in})\end{aligned}$$

The resulting behavior is perfectly compositional (without being pathological). Clearly this observation again involves an entertaining riddle. We encourage the reader to see what happens to the two provided counter examples for backward compositionality when placed in context, and how that context changes their non-compositional behavior into compositional behavior.

In an appropriate context an input only path becomes an output only path.



$$\phi_{out} = (\mathrm{id} \circ f_{||\Gamma||} \circ \ldots \circ f_1 \circ \mathrm{id})(\phi_{in}) = (f_{||\Gamma||} \circ \ldots \circ f_1)(\phi_{in})$$

This allows to attribute to each input only path a bipartite entangled state Ψ_g with $g = f_{||\Gamma||} \circ \ldots \circ f_1$. Rather abusively we can think of them as providing entanglement in a "backward" fashion (**as if** of course).

Conclusion 6.8 *Any path of the kinds we have considered so far, that is, forward, output only, backward and input only, exhibits (non-trivial) compositional behavior when placed in an appropriate context.*

v. *Both positive and negative* **eP***'s and both output only and input only paths produce functions.* An **eP** produces an entangled state which we can interpret as a function. This of course happens at its positive side and by physical causality it makes no sense to talk about producing an entangled state at its negative side. Nonetheless the above shows that its negative side acts as if it produces a function analogous to how its positive side produces one.

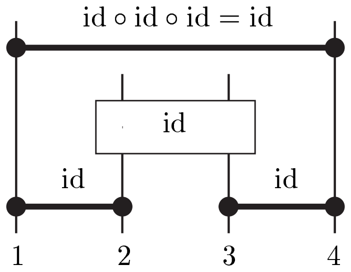
Conclusion 6.9 (Producing functions) *Each bipartite* **eP** *produces a function both at its positive and its negative side. Analogously, both input only and output only paths produce a function respectively at the side of their physical input and output.*

vi. *Functions vs. relations.* In a sense one could say that bipartite entanglement behaves rather as a relation than as function, or more precise, as a multirelation. Relations can be represented by $\{0,1\}$-valued matrices while in the case of multirelations we have $\mathbb{N}$-valued matrices. It is easy to extend this idea to $\mathbb{C}$-valued matrices and if we fix a base we can generate in this way all linear functions. In particular is the inverse of a (multi)relation obtained by transposing the matrix which is in harmony with the fact that reversal of the direction of passing an **eP** requires taking the adjoint. If one however tries to extend this line of thinking to multipartite entanglement then one ends up with something that could be called a *polymultirelation*. While a (multi)relation relates elements of two sets such a polymultirelation relates elements of many sets.

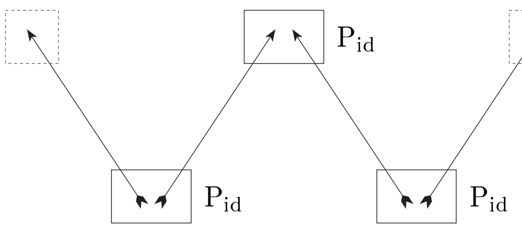
6.2 Example: entanglement swapping

The above discussed wider significance of Lemma 5.6 and of Proposition 5.11, and its implications for reading output only paths, allows us to discuss *entanglement swapping* [61] in a similar fashion as we discussed teleportation.

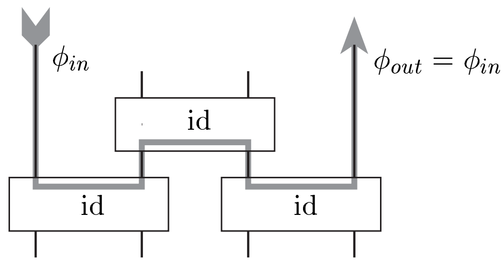
We consider four carriers, carrier 1 being id-entangled to carrier 2 and carrier 3 being id-entangled to carrier 4. How do we get carrier 1 id-entangled to carrier 4 without acting on either of them. The solution is provided by Lemma 5.6.



The spatial geometry associated with this setting is the following which also incorporates the preparation of the two initial entangled states.



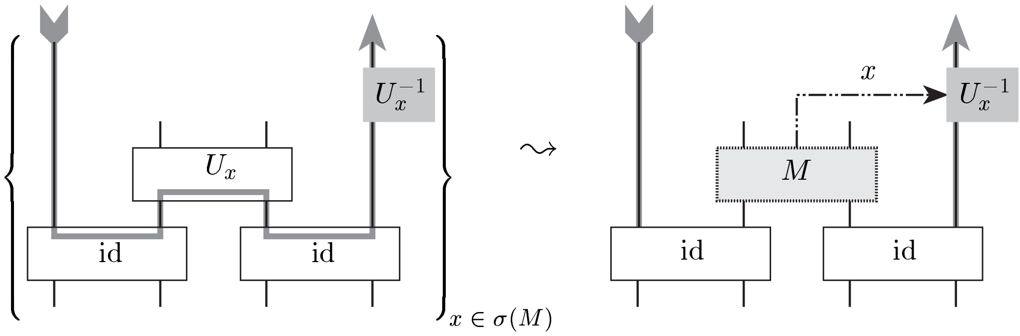
The corresponding path and information flow is depicted below.



By Conclusions 6.7 and 6.9 we can interpret this as merely being a composition of identity functions.

$$
\begin{tikzcd}
\mathcal{H} \arrow[rrr, "\mathrm{id}"] \arrow[dd, equal] & & & \mathcal{H} \arrow[dd, equal] \\
& \mathcal{H} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \mathcal{H} \arrow[d, equal] & \\
\mathcal{H} \arrow[r, "\mathrm{id}"] & \mathcal{H} & \mathcal{H} \arrow[r, "\mathrm{id}"] & \mathcal{H}
\end{tikzcd}
$$

The above is of course to be interpreted probabilistically with respect to the projector on carrier 3 and 4. Analogously to what we did in the case of the teleportation protocol we can produce the usual non-probabilistic swapping protocol by exploiting compositionality.



All symbols have the same significance as they had in the case of teleportation.

6.3 Example: preparation of entangled states

In view of the above presented analysis of entanglement swapping it should be obvious that the algorithm proposed in Subsection 3.4 can be modified into one for fault-tolerant preparation of entangled states from a small generating set of available ones, while imposing the same constraints as we did in Subsection 3.4. Hence we assume that the only available components are

- Local unitary operations in $\mathcal{G}_n$;
- Bell-base measurements ;
- Some prepared entangled states with as functional labels $f_1, \ldots, f_m \in \mathcal{CG}_n$.

The latter could be restricted to (bipartite) CNOT-gates, (unipartite) Hadamard gates and (unipartite) phase gates. In Subsection 3.4 we exposed how fault-tolerant "parallel composition" enables us to produce any gate in the subgroup of $\mathcal{CG}_n$ generated by $f_1, \ldots, f_m$ via composition, tensor and identities. If $g \in \mathcal{CG}_n$ is a unitary operation we can produce in that way, then we can also produce the entangled state

$$\Psi_g \in \mathcal{H}^n \otimes \mathcal{H}^n$$

which has g as labeling function. In order to obtain an algorithm which does this, it suffices to drop the following in the algorithm of Subsection 3.4:

- the first n carriers of states , and,
- the Bell-base measurements $M_i^{(\nu)}$ for which $\mathrm{I}(\nu, i) = 0$.

The "inputs" of the produced entangled state are the carriers

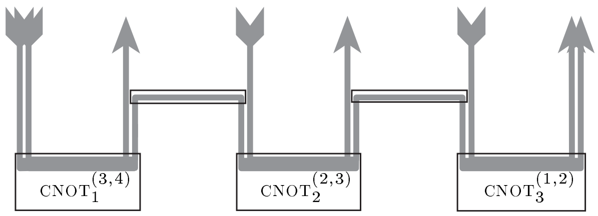
$$\left(2 \cdot \sum_{j=1}^{j=i-1} |\mathrm{Tracks}(f_j)|\right) + \mathrm{Order}(\nu, i)$$

for $\nu \in \{1, \ldots, n\}$ and i such that $\underline{\mathrm{I}}(\nu, i) = 0$. Its "outputs" are the carriers

$$\left(2 \cdot \sum_{j=1}^{j=\overline{\mathrm{I}}(\nu)-1} |\mathrm{Tracks}(f_j)|\right) + |\mathrm{Tracks}(f_{\overline{\mathrm{I}}(\nu)})| + \mathrm{Order}(\nu, \overline{\mathrm{I}}(\nu))$$

for $\nu \in \{1, \ldots, n\}$.

As an example, by means of the configuration



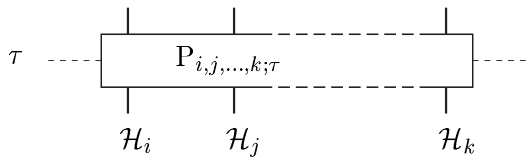
we realize the 8-qubit state labeled by the unitary map

$$\left(\mathrm{CNOT}_3^{(1,2)} \otimes \mathrm{id}_3^{(3,4)}\right) \circ \left(\mathrm{id}_2^{(1)} \otimes \mathrm{CNOT}_2^{(2,3)} \otimes \mathrm{id}_2^{(4)}\right) \circ \left(\mathrm{id}_1^{(3,4)} \otimes \mathrm{CNOT}_1^{(1,2)}\right).$$

Again, as it was the case for the construction in Subsection 3.4, this procedure can be extended from producing $\mathcal{CG}_n$-labeled entangled states to producing $\mathcal{CG}_n^\omega$-labeled entangled states. Note in particular that the carriers which make up the resulting entangled state have themselves not been acted on yet. Hence they can be used as prepared states for other algorithms.

6.4 Tri- and tetrapartite entanglement

So far we only considered single snake-like paths. We will now pass from the logic of bipartite entanglement specification to the logic of multipartite entanglement specification, that is, we will identify the information flow capabilities of **eP**'s of the shape



We cannot just label these **eP**'s via a unique labeling function since there are no (unique) multipartite analogues to the isomorphisms

$$\mathcal{H}_i \otimes \mathcal{H}_j \;\simeq\; \mathcal{H}_i^* \to \mathcal{H}_j \;\simeq\; \mathcal{H}_i \looparrowright \mathcal{H}_j\,.$$

Note in particular that while the tensor is associative and commutative the function arrow isn't. However, it is not completely true that in the bipartite case we had a unique obvious candidate to label $\mathcal{H}_i \otimes \mathcal{H}_j$. By commutativity of the tensor each of the types

$$\mathcal{H}_i \looparrowright \mathcal{H}_j \qquad \text{and} \qquad \mathcal{H}_j \looparrowright \mathcal{H}_i$$

provides a candidate labeling function — which is adjoint to the other one [Proposition 4.12]. We left the task of choosing one of these two candidates to the path — more specifically we left the task to the path's direction [Deceit 4.10]. Hence *each bipartite* **eP** *accepts two functional readings.*

In this subsection we study the *functional readings accepted by tripartite entanglement*, that is, accepted by **eP**'s of the shape

$$\mathrm{P}_\Psi : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \to \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 :: \Phi \mapsto \langle \Psi \mid \Phi \rangle \cdot \Psi$$

with

$$\Psi = \sum_{\alpha\beta\gamma} f_{\alpha\beta\gamma} \cdot e_\alpha^{(1)} \otimes e_\beta^{(2)} \otimes e_\beta^{(3)} \;\in\; \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3\,.$$

The binary choice of a direction in the bipartite case becomes in the multipartite case a 9-ary choice which consists of attributing a *type* to the **eP**.

Bipartite entanglement can be interpreted in terms of functional actions on an information flow. Here we will proceed by reducing the information flow interpretation of tripartite entanglement to that of bipartite entanglement. We enable this reduction by exploiting the associativity of the tensor product. We have

$$\begin{aligned} \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \;&\simeq\; (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3 \\ &\simeq\; \mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3)\,. \end{aligned}$$

Using the isomorphism between $\mathcal{H}_1 \looparrowright \mathcal{H}_2$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$ and (for now) ignoring commutation of the tensor we obtain two corresponding functional types namely

$$(\mathcal{H}_1 \looparrowright \mathcal{H}_2) \looparrowright \mathcal{H}_3 \quad \text{and} \quad \mathcal{H}_1 \looparrowright (\mathcal{H}_2 \looparrowright \mathcal{H}_3)\,.$$

Below we discuss these two cases separately.

Definition 6.10 The *order* of function types is

$$\left\{\begin{array}{l} \mathrm{order}(\mathcal{H}) \,=\, 0 \\ \mathrm{order}(\mathcal{H}_1 \times \mathcal{H}_2) \,=\, \max(\,\mathrm{order}(\mathcal{H}_1)\,,\, \mathrm{order}(\mathcal{H}_2)\,) \\ \mathrm{order}(\mathcal{H}_1 \to \mathcal{H}_2) \,=\, \max(\,\mathrm{order}(\mathcal{H}_1) + 1\,,\, \mathrm{order}(\mathcal{H}_2)\,) \\ \mathrm{order}(\mathcal{H}_1 \looparrowright \mathcal{H}_2) \,=\, \max(\,\mathrm{order}(\mathcal{H}_1) + 1\,,\, \mathrm{order}(\mathcal{H}_2)\,)\,. \end{array}\right.$$

We introduce the *size* of function types as

$$\begin{cases} \mathrm{size}(\mathcal{H}) \ = \ 1 \\ \mathrm{size}(\mathcal{H}_1 \times \mathcal{H}_2) \ = \ \mathrm{size}(\mathcal{H}_1) + \mathrm{size}(\mathcal{H}_2) \\ \mathrm{size}(\mathcal{H}_1 \to \mathcal{H}_2) \ = \ \mathrm{size}(\mathcal{H}_1) + \mathrm{size}(\mathcal{H}_2) \\ \mathrm{size}(\mathcal{H}_1 \looparrowright \mathcal{H}_2) \ = \ \mathrm{size}(\mathcal{H}_1) + \mathrm{size}(\mathcal{H}_2) \,. \end{cases}$$

Recall that *second order functions* such as

$$f : (\mathcal{H}_1 \to \mathcal{H}_2) \to \mathcal{H}_3 \qquad \text{or} \qquad f : (\mathcal{H}_1 \looparrowright \mathcal{H}_2) \looparrowright \mathcal{H}_3$$

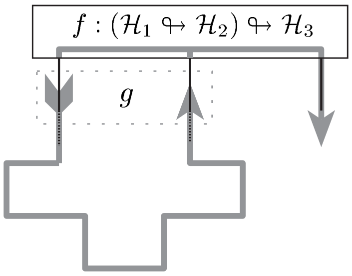
allow to accommodate things like *definite integrals* and *derivatives* and so on.

i. *Second order types.* The type

$$(\mathcal{H}_1 \looparrowright \mathcal{H}_2) \looparrowright \mathcal{H}_3$$

stands for a function which itself takes a function of type $\mathcal{H}_1 \looparrowright \mathcal{H}_2$ as input and outputs an element of $\mathcal{H}_3$. From Conclusion 6.7 we know that an entangled state can be interpreted as a function. Moreover, by Conclusion 6.9 we know that each output only and each input only path can be conceived as producing a function.

Let us consider the case of "feeding" the function produced by an output only path in a negative [Definition 3.13] tripartite **eP**.



Assume that the tripartite **eP** which we labeled by f projects on

$$\sum_{\alpha\beta\gamma} f_{\alpha\beta\gamma} \cdot e_\alpha^{(1)} \otimes e_\beta^{(2)} \otimes e_\beta^{(3)}$$

and that the function produced by the output only path is $g : \mathcal{H}_1 \looparrowright \mathcal{H}_2$ which has $(g_{\alpha\beta})_{\alpha\beta}$ as its matrix in base $\{e_\alpha^{(1)}\}_\alpha$ of $\mathcal{H}_1$ and base $\{e_\beta^{(2)}\}_\beta$ of $\mathcal{H}_2$, that is, the output only path produces the entangled state

$$\sum_{\alpha\beta} g_{\alpha\beta} \cdot e_\alpha^{(1)} \otimes e_\alpha^{(2)} \,.$$

When conceiving the tripartite **eP** as a bipartite one of type $(\mathcal{H}_1 \otimes \mathcal{H}_2) \looparrowright \mathcal{H}_3$ which projects on

$$\sum_{(\alpha\beta)\gamma} f_{(\alpha\beta)\gamma} \cdot (e_\alpha^{(1)} \otimes e_\beta^{(2)}) \otimes e_\beta^{(3)}$$

and which receives as input

$$\phi_{in} := \sum_{\alpha\beta} g_{\alpha\beta} \cdot e^{(1)}_\alpha \otimes e^{(2)}_\alpha$$

we can apply the compositionality results of Section 3. It then follows that it suffices to conceive the anti-linear (2nd order) function

$$f : (\mathcal{H}_1 \looparrowright \mathcal{H}_2) \looparrowright \mathcal{H}_3$$

as

$$f\Big(\sum_{\alpha\beta} g_{\alpha\beta}\langle - \mid e^{(1)}_\alpha\rangle \cdot e^{(2)}_\alpha\Big) \;=\; \sum_k\Big(\sum_{ij} \bar{g}_{ij} f_{ijk}\Big) \cdot e^{(3)}_k$$

in order to make the information flow interpretation hold.

In the case that a tripartite **eP** which we type in the same way acts positive [Definition 3.13] we obtain the same result. This can easily be seen by feeding the function which is "downwardly produced" by a bipartite **eP** [Conclusion 6.9] in the tripartite **eP**. Assuming that the bipartite **eP** projects on an entangled state $\sum_{\alpha\beta} g_{\alpha\beta} \cdot e^{(1)}_\alpha \otimes e^{(2)}_\alpha$, then by Conclusion 6.5 on effective input specification the above result follows again by setting $\phi_{in} := \sum_{\alpha\beta} g_{\alpha\beta} \cdot e^{(1)}_\alpha \otimes e^{(2)}_\alpha$ while conceiving the tripartite **eP** as bipartite.

Note that the indices i and j do not play an equivalent role given the type of the function g, and of course, neither does k. We can denote the *typed matrix* of f as

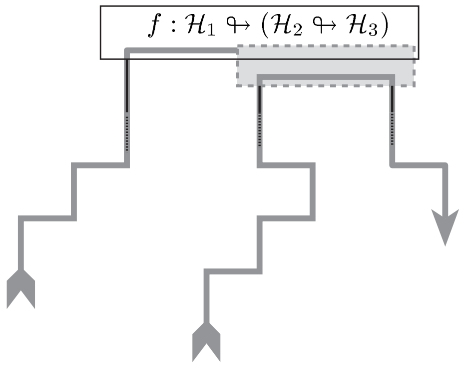
$$(f_{(i \looparrowright j)\looparrowright k})_{(i\looparrowright j)\looparrowright k}\,.$$

If we assume that the values f_{ijk} are defined such that the arrows $\looparrowright$ always point forward in this notation then we can simplify it as $(f_{(ij)k})_{(ij)k}$.

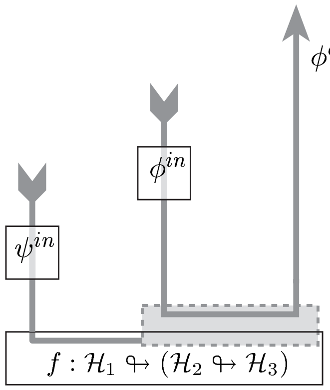
ii. *Virtual function boxes.* The type

$$\mathcal{H}_1 \looparrowright (\mathcal{H}_2 \looparrowright \mathcal{H}_3)$$

stands for a function which takes an element of $\mathcal{H}_1$ as input and outputs a function of type $\mathcal{H}_2 \looparrowright \mathcal{H}_3$. Since by [Conclusion 6.9] bipartite **eP**'s produce functions this tripartite **eP** should act as if it produces a "virtual" bipartite **eP** as output, of which labeling function depends (linearly) on the input of the tripartite **eP**.



In order to expose this kind of behavior consider the configuration below.



When conceiving the tripartite **eP** as a bipartite one which projects on

$$\sum_{\alpha(\beta\gamma)} f_{\alpha(\beta\gamma)} \cdot e_\alpha^{(1)} \otimes (e_\beta^{(2)} \otimes e_\gamma^{(3)})$$

after having applied the effective input specification $\mathrm{P}_{\psi^{in}}$ with $\psi^{in} = \sum_\alpha \psi_\alpha^{in}$ we obtain by Conclusion 6.5 that the entangled state of the second two carriers is

$$\sum_\alpha \bar{\psi}_\alpha^{in} f_{\alpha(\beta\gamma)} \cdot e_\beta^{(2)} \otimes e_\gamma^{(3)}$$

We know that an entangled state can be conceived as a function [Conclusion 6.7]. When applying the second effective input specification $\mathrm{P}_{\phi^{in}}$ the state of the third carrier becomes

$$\sum_{\alpha\beta} \bar{\phi}_\beta^{in} \bar{\psi}_\alpha^{in} f_{\alpha(\beta\gamma)} \cdot e_\beta^{(3)}$$

It follows that when conceiving the anti-linear (first order) function

$$f : \mathcal{H}_1 \looparrowright (\mathcal{H}_2 \looparrowright \mathcal{H}_3)$$

as

$$f(\psi) \;=\; \sum_{jk} \Big(\sum_i \bar{\psi}_i f_{ijk} \Big) \langle - \mid e_j^{(2)} \rangle \cdot e_k^{(3)}$$

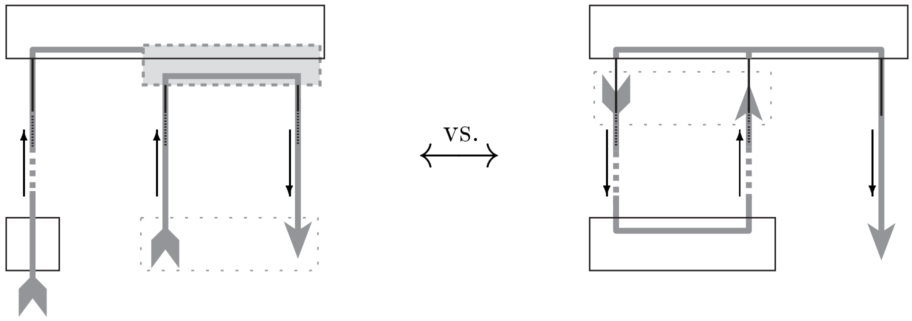
the information flow interpretation indeed holds. This shows that passing from bipartite to multipartite entanglement goes with passing from *virtual paths* to *virtual* **eP***'s*. The *typed matrix* of f is now

$$(f_{i \looparrowright (j \looparrowright k)})_{i \looparrowright (j \looparrowright k)} \,.$$

and simplifies under the assumption mentioned above to $(f_{i(jk)})_{i(jk)}$.

iii. *Consuming versus producing.* In the case discussed above where we used the type $\mathcal{H}_1 \looparrowright (\mathcal{H}_2 \looparrowright \mathcal{H}_3)$ to interpret a tripartite **eP** we *produced* a (virtual) bipartite **eP**, that is, we *produced* a function. In the $(\mathcal{H}_1 \looparrowright \mathcal{H}_2) \looparrowright \mathcal{H}_3$ case we *consumed* a function which itself was *produced* by an output only path. *Producing* and *consuming* a function are very different operations:

- *Consuming a function* stands for *producing some input value for that function* and then *consuming the output returned by the function*.
- *Producing a function* stands for *consuming some input value for that function* and then *producing the corresponding output*.



This producing-consuming dialectics is very typical for linear logic [29]. Much of the structure exposed here has striking similarities with that of Girard's linear logic and in particular with that of proof nets [31]. We will elaborate further on this matter below when we discuss general multipartite entanglement [Section 7].

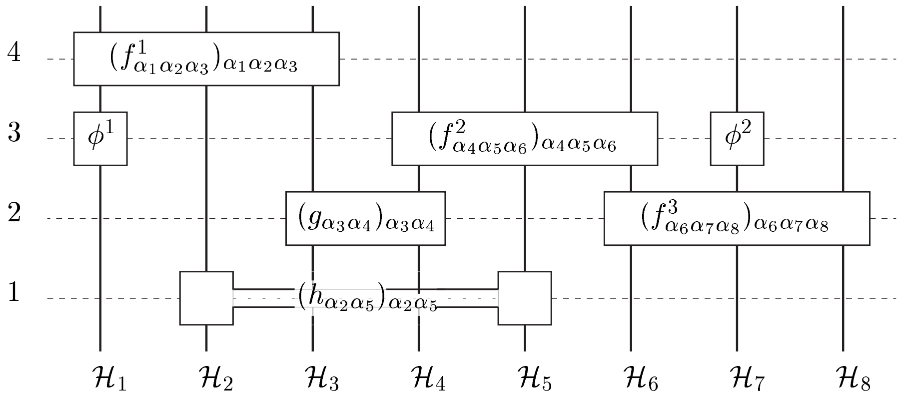
Note in particular the difference of the number of paths of size one which approach and which leave the **eP** in the two cases. This shows that one of them cannot be converted to the other by changing the paths "within the **eP**".

iv. *A network with tripartite* **eP***'s.* Passage from networks which contain only bipartite **eP**'s to those where we also have multipartite ones requires adequate denotational tools. We will restrict ourself here to a particular example of a network in order to demonstrate that a compositional interpretation indeed still holds in the tripartite case.

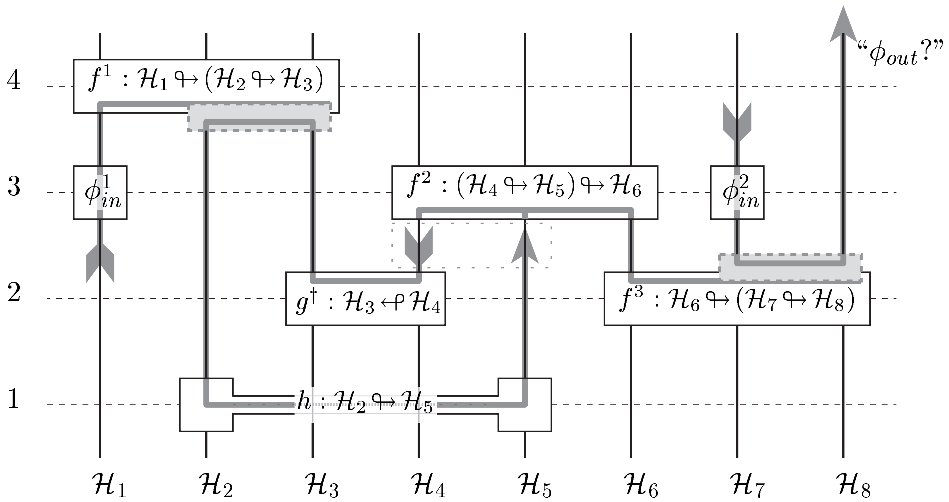
Consider the following configuration of **eP**'s where the matrices inside of them are the coefficients of the vector on which the corresponding **eP** projects, these matrices being expressed in the base of tensors made up of the bases

$$\{e^{(1)}_{\alpha_1}\}_{\alpha_1}, \ldots, \{e^{(8)}_{\alpha_8}\}_{\alpha_8}$$

of the respective carrier types $\mathcal{H}_1, \ldots, \mathcal{H}_8$.



We can define the types for the (tripartite) **eP**'s in such a way that we can draw the following "compound" information flow path.



Provided "compositionality would hold" we derive what ϕ_{out} should be. Due to the fact that some functions are of size three compositionality is for this "compound" path more sophisticated than in the case of functions of size two where we always can the composite as a list of the shape $f_k \circ \ldots \circ f_1$. Therefore we will keep track of the types in the derivation of the composite. At the output of the f^1-labeled **eP** typed $\mathcal{H}_1 \looparrowright (\mathcal{H}_2 \looparrowright \mathcal{H}_3)$ we obtain the function

$$f^1(\phi^1_{in}) : \mathcal{H}_2 \looparrowright \mathcal{H}_3\,.$$

At the input of the f^2-labeled **eP** we obtain a function which is itself a (ordinary) composite of functions

$$h \circ (f^1(\phi^1_{in}))^\dagger \circ g^\dagger : \mathcal{H}_4 \looparrowright \mathcal{H}_5\,.$$

At the output of the f^2-labeled **eP** typed $(\mathcal{H}_4 \looparrowright \mathcal{H}_5) \looparrowright \mathcal{H}_6$ we obtain the vector

$$f^2(h \circ (f^1(\phi^1_{in}))^\dagger \circ g^\dagger) : \mathcal{H}_6\,.$$

At the output of the f^3-labeled **eP** typed $(\mathcal{H}_6 \looparrowright \mathcal{H}_7) \looparrowright \mathcal{H}_8$ we obtain the function

$$(f^3 \circ f^2)(h \circ (f^1(\phi^1_{in}))^\dagger \circ g^\dagger) : \mathcal{H}_7 \looparrowright \mathcal{H}_8$$

where

$$f^3 \circ f^2 : (\mathcal{H}_4 \looparrowright \mathcal{H}_5) \looparrowright (\mathcal{H}_7 \looparrowright \mathcal{H}_8)$$

and hence

$$\phi_{out} = (f^3 \circ f^2)(h \circ (f^1(\phi^1_{in}))^\dagger \circ g^\dagger)(\phi^2_{in}) \,.$$

We can make this expression explicit by using the (typed) matrices of the functions involved. We also set

$$\phi^1_{in} = \sum_{\alpha_1} \phi^{in,1}_{\alpha_1} \qquad\qquad \phi^2_{in} = \sum_{\alpha_7} \phi^{in,2}_{\alpha_7} \qquad\qquad \phi_{out} = \sum_{\alpha_8} \phi^{out}_{\alpha_8} \,.$$

As matrix of the function $f := f^1(\phi^1_{in})$ we have

$$(f_{\alpha_2\alpha_3})_{\alpha_2\alpha_3} \;:=\; \Big(\sum_{\alpha_1} \bar{\phi}^{in,1}_{\alpha_1} f^1_{\alpha_1(\alpha_2\alpha_3)}\Big)_{\alpha_2\alpha_3}$$

and hence the matrix of the function $\tilde{f} := h \circ f^\dagger \circ g^\dagger$ is

$$(\tilde{f}_{\alpha_4\alpha_5})_{\alpha_4\alpha_5} \;:=\; \Big(\sum_{\alpha_2\alpha_3} g^\dagger_{\alpha_4\alpha_3} \bar{f}^\dagger_{\alpha_3\alpha_2} h_{\alpha_2\alpha_5}\Big)_{\alpha_4\alpha_5} = \;\Big(\sum_{\alpha_4\alpha_5} g_{\alpha_3\alpha_4} \bar{f}_{\alpha_2\alpha_3} h_{\alpha_2\alpha_5}\Big)_{\alpha_4\alpha_5} .$$

Applying the composite $f^3 \circ f^2$ to these then yields

$$\Big(\sum_{(\alpha_4\alpha_5)\alpha_6} \tilde{f}_{\alpha_4\alpha_5} \bar{f}^2_{(\alpha_4\alpha_5)\alpha_6} f^3_{\alpha_6(\alpha_7\alpha_8)}\Big)_{\alpha_7\alpha_8}$$

resulting in

$$\phi^{out}_{\alpha_8} = \sum_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7} \bar{\phi}^{in,2}_{\alpha_7} g_{\alpha_3\alpha_4} \phi^{in,1}_{\alpha_1} \bar{f}^1_{\alpha_1(\alpha_2\alpha_3)} h_{\alpha_2\alpha_5} \bar{f}^2_{(\alpha_4\alpha_5)\alpha_6} f^3_{\alpha_6(\alpha_7\alpha_8)} \,.$$

Now we will confront this prediction which we obtained by assuming compositionality with the quantum mechanical calculation of the physical output of the network. In order to do that we will use a lemma which applies to entanglement specification networks with an arbitrary number n of carriers, an arbitrary number of time-instances m at which we apply projectors and we allow these projectors to apply to any number of the n carriers such that we cover the case of arbitrary multipartite **eP**'s. We set

$$\Psi = \sum_{i_1 \ldots i_n} \Psi_{i_1 \ldots i_n} \cdot e^{(1)}_{i_1} \otimes \ldots \otimes e^{(n)}_{i_n} \qquad\qquad \Psi = \sum_{i_1 \ldots i_n} \Psi_{i_1 \ldots i_n} \cdot e^{(1)}_{i_1} \otimes \ldots \otimes e^{(n)}_{i_n}$$

and for $\tau \in \{1, \ldots, m\}$ we set

$$\Psi^\tau = \sum_{i_1 \ldots i_n} \Psi^\tau_{i_1 \ldots i_n} \cdot e^{(1)}_{i_1} \otimes \ldots \otimes e^{(n)}_{i_n} \,,$$

All these having the same significance as they had in Section 3. For

$$I \subseteq \{1, \ldots, n\}$$

let

$$\mathrm{P}_\Phi^I : \bigotimes_{i \in I} \mathcal{H}_i \to \bigotimes_{i \in I} \mathcal{H}_i :: \Psi \mapsto \langle \Phi \mid \Psi \rangle \cdot \Phi$$

be the projector which projects on the unit vector

$$\Phi = \sum_{i_\alpha | \alpha \in I} \Phi_{(i_\alpha | \alpha \in I)} \ \in \ \bigotimes_{i \in I} \mathcal{H}_i \,.$$

As in Section 3 we allow P_Φ^I to act on any state $\Psi \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$ by tensoring it with identities which act on the carriers labeled by indices in

$$\neg I := \{1, \ldots, n\} \setminus I \,.$$

Recall that we have $\Psi = \Psi^m$. For convenience we also set $\Psi^0 := \Psi$.

Lemma 6.11 (Multipartite projector action) *For $\tau \in \{1, \ldots, m\}$, given that*

$$\Psi^\tau = \mathrm{P}_\Phi^I(\Psi^{\tau-1}) \,,$$

then

$$\Psi^\tau_{i_1 \ldots i_n} = \sum_{j_\alpha | \alpha \in I} \Psi^{\tau-1}_{i_1 \ldots i_n [j_\alpha / i_\alpha | \alpha \in I]} \bar{\Phi}_{(j_\alpha | \alpha \in I)} \Phi_{(i_\alpha | \alpha \in I)}$$

where

$$i_1 \ldots i_n [j_\alpha / i_\alpha \mid \alpha \in I]$$

denotes that we substitute the indices i_α for which $\alpha \in I$ by j_α, this j_α being an index which ranges over the same values as i_α.

Proof. We have

$$\begin{aligned}
\mathrm{P}_\Phi^I\left(\Psi^{\tau-1}\right) &= \sum_{i_1 \ldots i_n} \Psi^{\tau-1}_{i_1 \ldots i_n} \left\langle \Phi \,\middle|\, \bigotimes_{\alpha \in I} e^{(\alpha)}_{i_\alpha} \right\rangle \cdot \Phi \otimes \left(\bigotimes_{\alpha \in \neg I} e^{(\alpha)}_{i_\alpha} \right) \\
&= \sum_{\substack{i_1 \ldots i_n \\ j_\alpha | \alpha \in I \\ k_\alpha | \alpha \in I}} \Psi^{\tau-1}_{i_1 \ldots i_n} \bar{\Phi}_{(j_\alpha | \alpha \in I)} \Phi_{(k_\alpha | \alpha \in I)} \prod_{\alpha \in I} \left\langle e^{(\alpha)}_{j_\alpha} \,\middle|\, e^{(\alpha)}_{i_\alpha} \right\rangle \cdot \left(\bigotimes_{\alpha \in I} e^{(\alpha)}_{k_\alpha} \right) \otimes \left(\bigotimes_{\alpha \in \neg I} e^{(\alpha)}_{i_\alpha} \right) \\
&= \sum_{\substack{i_\alpha | \alpha \in \neg I \\ j_\alpha | \alpha \in I \\ k_\alpha | \alpha \in I}} \Psi^{\tau-1}_{i_1 \ldots i_n [j_\alpha / i_\alpha | \alpha \in I]} \bar{\Phi}_{(j_\alpha | \alpha \in I)} \Phi_{(k_\alpha | \alpha \in I)} \cdot \left(\bigotimes_{\alpha \in I} e^{(\alpha)}_{k_\alpha} \right) \otimes \left(\bigotimes_{\alpha \in \neg I} e^{(\alpha)}_{i_\alpha} \right) \\
&= \sum_{i_1 \ldots i_n} \left(\sum_{j_\alpha | \alpha \in I} \Psi^{\tau-1}_{i_1 \ldots i_n [j_\alpha / i_\alpha | \alpha \in I]} \bar{\Phi}_{(j_\alpha | \alpha \in I)} \Phi_{(i_\alpha | \alpha \in I)} \right) \cdot e^{(1)}_{i_1} \otimes \ldots \otimes e^{(n)}_{i_n}
\end{aligned}$$

where the last step merely consists in substituting k_α by i_α for $\alpha \in I$, which is allowed since these i_α's are not in use anymore. Identification of the coefficients for the different base vectors completes the proof. $\square$

Setting $n := 8$ and $m := 4$ we can now apply this lemma to the example above. Note that by Proposition B.4 in Appendix B it poses no problem that more **EP**'s act simultaneously. We obtain

$$\begin{aligned}
\Psi^1_{i_1\ldots i_8} &= \sum_{j_2 j_5} \Psi_{i_1 j_2 i_3 i_4 j_5 i_6 i_7 i_8} \bar{h}_{j_2 j_5} h_{i_2 i_5} \\
\Psi^2_{i_1\ldots i_8} &= \sum_{\substack{j_2 j_5 \\ k_3 k_4 k_6 k_7 k_8}} \Psi_{i_1 j_2 k_3 k_4 j_5 k_6 k_7 k_8} \bar{h}_{j_2 j_5} h_{i_2 i_5} \bar{g}_{k_3 k_4} g_{i_3 i_4} \bar{f}^3_{k_6 k_7 k_8} f^3_{i_6 i_7 i_8} \\
\Psi^3_{i_1\ldots i_8} &= \sum_{\substack{j_2 j_5 \\ k_3 k_4 k_6 k_7 k_8 \\ l_1 l_4 l_5 l_6 l_7}} \Psi_{l_1 j_2 k_3 k_4 j_5 k_6 k_7 k_8} \bar{h}_{j_2 j_5} h_{i_2 l_5} \bar{g}_{k_3 k_4} g_{i_3 l_4} \bar{f}^3_{k_6 k_7 k_8} f^3_{l_6 l_7 i_8} \bar{\phi}^1_{l_1} \phi^1_{i_1} \bar{f}^2_{l_4 l_5 l_6} f^2_{i_4 i_5 i_6} \bar{\phi}^2_{l_7} \phi^2_{i_7}
\end{aligned}$$

Hence the coefficient $\Psi_{i_1\ldots i_8}$ factors into five components, one in which no index in $\{i_1, \ldots, i_8\}$ appears namely

$$\sum_{\substack{j_2 j_5 \\ k_3 k_4 k_6 k_7 k_8 \\ l_1}} \Psi_{l_1 j_2 k_3 k_4 j_5 k_6 k_7 k_8} \bar{h}_{j_2 j_5} \bar{g}_{k_3 k_4} \bar{f}^3_{k_6 k_7 k_8} \bar{\phi}^1_{l_1}$$

three with indices in $\{i_1, \ldots, i_7\}$ namely

$$f^2_{i_4 i_5 i_6} f^1_{i_1 i_2 i_3} \phi^2_{i_7}$$

and one which contains the index i_8 namely

$$\sum_{\substack{l_4 l_5 l_6 l_7 \\ m_1 m_2 m_3}} h_{m_2 l_5} g_{m_3 l_4} f^3_{l_6 l_7 i_8} \phi^1_{m_1} \bar{f}^2_{l_4 l_5 l_6} \bar{\phi}^2_{l_7} \bar{f}^1_{m_1 m_2 m_3}$$

After substituting the *bounded* indices

$$m_1, m_2, m_3, l_4, l_5, l_6, l_7, i_8 \qquad \text{by} \qquad \alpha_1, \ldots, \alpha_8$$

and permuting the coefficients we obtain

$$\sum_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7} \bar{\phi}^2_{\alpha_7} g_{\alpha_3 \alpha_4} \phi^1_{\alpha_1} \bar{f}^1_{\alpha_1 \alpha_2 \alpha_3} h_{\alpha_2 \alpha_5} \bar{f}^2_{\alpha_4 \alpha_5 \alpha_6} f^3_{\alpha_6 \alpha_7 \alpha_8}$$

which coincides exactly with our prediction for $\phi^{out}_{\alpha_8}$ and thus confirms our prediction of ϕ^{out}. This (random) example indicates that compositionally indeed extends beyond the case of bipartite entanglement specification.

Note here in particular that the negatively acting **eP**'s, including the downward acting effective input specifications, are the ones of which the matrix elements are conjugated. This clearly indicates that Lemma 4.6 on the necessary conjugations in the matricial shape of composites in bipartite entanglement specification networks generalizes to arbitrary multipartite entanglement specification networks with respect to which coefficients have to be conjugated.

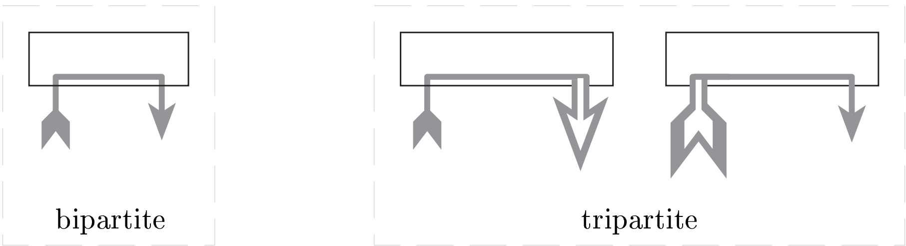
v. *The information flow capabilities of tripartite entanglement.* Above we have put some arguments forward in favour of an interpretation for tripartite entanglement which extends the one proposed for bipartite entanglement. While *bipartite entanglement* allows a (virtual) information flow by means of a function of size two of one of the <u>two</u> types

$$\mathcal{H}_{\pi(1)} \looparrowright \mathcal{H}_{\pi(2)} \quad \text{with} \quad \pi : \{1,2\} \to \{1,2\} \ \text{ a permutation,}$$

that *tripartite entanglement* allows a (virtual) information flow by means of a function of size three of one of the <u>twelve</u> types

$$\mathcal{H}_{\pi(1)} \looparrowright (\mathcal{H}_{\pi(2)} \looparrowright \mathcal{H}_{\pi(3)}) \quad \text{or} \quad (\mathcal{H}_{\pi(1)} \looparrowright \mathcal{H}_{\pi(2)}) \looparrowright \mathcal{H}_{\pi(3)}$$
$$\text{with} \quad \pi : \{1,2,3\} \to \{1,2,3\} \ \text{ a permutation.}$$

Six of them are first order functions and six of them are second order functions. Introducing a new graphical representation for "2nd order paths" while ignoring the permutations we can depict the distinction between bipartite and tripartite entanglement as follows:



Besides this distinction in functional actions in both cases **eP**'s agree on:

- *Time reversal* for any path passing through the **eP**.
- The capability to act both positively and negatively — even simultaneously.

The multiplicity of possible information flows follows by commutativity and associativity of the tensor resulting in

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \ \simeq \ (\mathcal{H}_{\pi(1)} \looparrowright \mathcal{H}_{\pi(2)}) \looparrowright \mathcal{H}_{\pi(3)} \ \simeq \ \mathcal{H}_{\pi(1)} \looparrowright (\mathcal{H}_{\pi(2)} \looparrowright \mathcal{H}_{\pi(3)})\,.$$

This degeneration will increase whenever we increase the size of the **eP**. The degeneration is however somewhat smaller than it seems to be at first sight. Via *Currying of arguments* [Subsection 6.5] we can reduce the six first-order typings to three:

$$\begin{array}{lll}\mathcal{H}_{\pi(1)} \looparrowright (\mathcal{H}_{\pi(2)} \looparrowright \mathcal{H}_{\pi(3)}) & \stackrel{Curry}{\longleftrightarrow} & (\mathcal{H}_{\pi(1)} \times \mathcal{H}_{\pi(2)}) \looparrowright \mathcal{H}_{\pi(3)} \\ & & (\mathcal{H}_{\pi(2)} \times \mathcal{H}_{\pi(1)}) \looparrowright \mathcal{H}_{\pi(3)} \stackrel{Curry}{\longleftrightarrow} \mathcal{H}_{\pi(2)} \looparrowright (\mathcal{H}_{\pi(1)} \looparrowright \mathcal{H}_{\pi(3)})\,.\end{array}$$

We can go even further and conceive the three inequivalent first-order typings as degenerated cases of the second-order typings arising due to *disentanglement* as we will discuss below [Subsection 6.5].

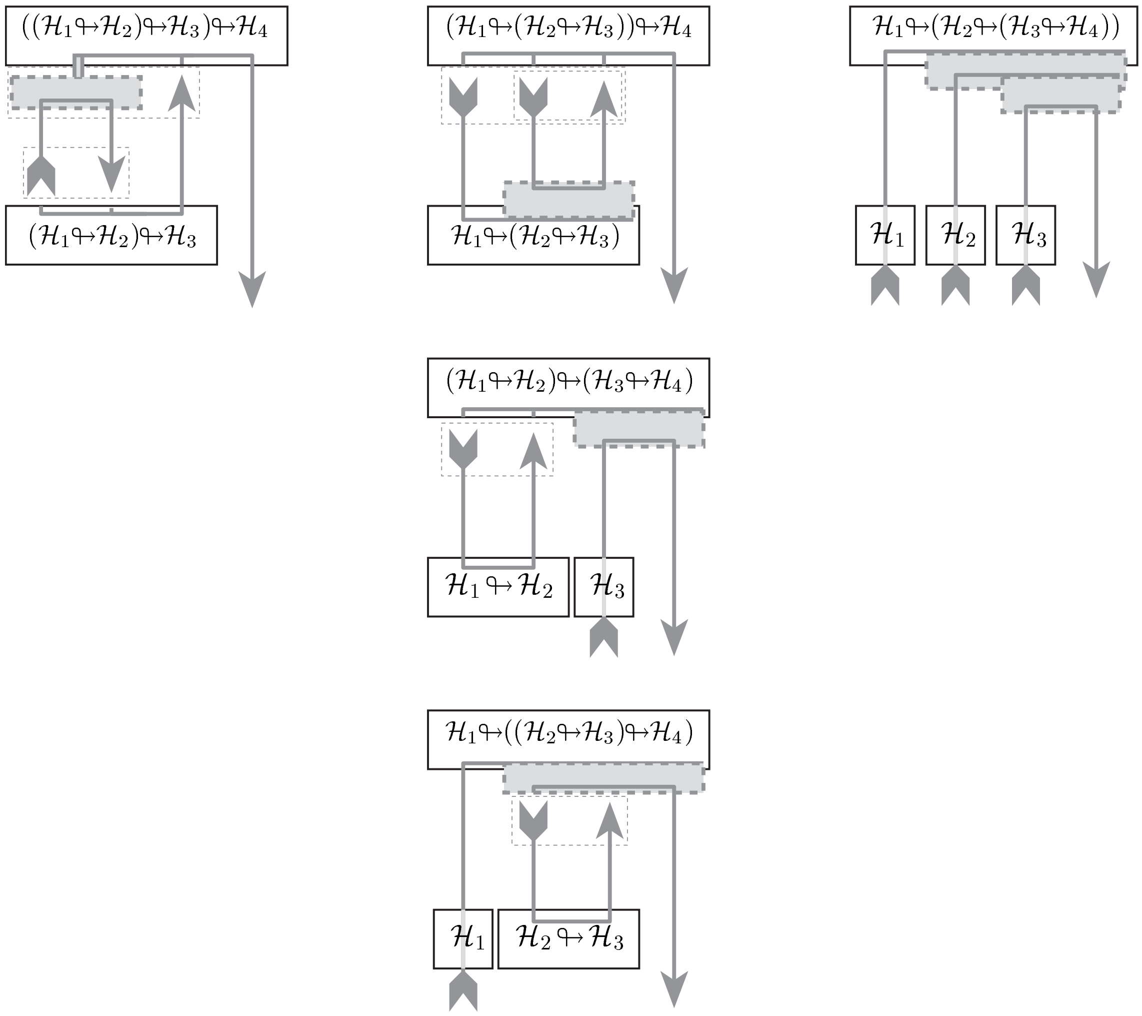
vi. *The information flow capabilities of tetrapartite entanglement.* We briefly mention how the picture changes when passing to *tetrapartite entanglement*. Again we rely on the fact that bipartite entanglement can be interpreted in terms of functional actions on an information flow. We reduce the information flow interpretation of tetrapartite entanglement to that of bipartite entanglement via

$$\begin{array}{rcl}\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 & \simeq & ((\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3) \otimes \mathcal{H}_4 \\ & \simeq & (\mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3)) \otimes \mathcal{H}_4 \\ & \simeq & (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes (\mathcal{H}_3 \otimes \mathcal{H}_4) \\ & \simeq & \mathcal{H}_1 \otimes ((\mathcal{H}_2 \otimes \mathcal{H}_3) \otimes \mathcal{H}_4) \\ & \simeq & \mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes (\mathcal{H}_3 \otimes \mathcal{H}_4))\,.\end{array}$$

so using $\mathcal{H}_1 \looparrowright \mathcal{H}_2 \simeq \mathcal{H}_1 \otimes \mathcal{H}_2$ we obtain

$$\begin{array}{lll}((\mathcal{H}_1 \looparrowright \mathcal{H}_2) \looparrowright \mathcal{H}_3) \looparrowright \mathcal{H}_4 & (\mathcal{H}_1 \looparrowright (\mathcal{H}_2 \looparrowright \mathcal{H}_3)) \looparrowright \mathcal{H}_4 & \mathcal{H}_1 \looparrowright (\mathcal{H}_2 \looparrowright (\mathcal{H}_3 \looparrowright \mathcal{H}_4)) \\ & (\mathcal{H}_1 \looparrowright \mathcal{H}_2) \looparrowright (\mathcal{H}_3 \looparrowright \mathcal{H}_4) & \\ & \mathcal{H}_1 \looparrowright ((\mathcal{H}_2 \looparrowright \mathcal{H}_3) \looparrowright \mathcal{H}_4) & .\end{array}$$

The type on the left is a third order one, those in the middle column are second order ones and the one on the right is a first order one. Hence our interpretation accepts five qualitatively different behaviors for tetrapartite entanglement which we depict below.



Note that there is no strict correspondence anymore between the order and the directions of the path (incoming vs. outgoing). In particular are these directions not sufficient to characterize the type — in the tripartite case they were sufficient. Hence passing from tripartite to tetrapartite entanglement reveals some new structural components.

Conclusion 6.12 (Information flow capabilities of multipartite entanglement) *Associativity of the tensor product gives rise to a variety of information flow capabilities for multipartite* **eP***'s, or boldly put, for multipartite entanglement itself. Each of these capabilities exposes itself depending on the context in which the corresponding* **eP** *is placed.*

6.5 Example: Currying and disentanglement

Passing from Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ to the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is not the innocent operation of conceiving two independent things as one whole as it is the case for the Cartesian product. It introduces many new (entangled) states for the joint system which have no counterpart in terms of *pairs of states*. In categorical terms [Appendix C] we pass from a *categorical product* which captures the operation of pairing two objects in terms of a "pairing bracket" and two "de-pairing projections" to a *monoidal tensor* which admits no such interpretation. In the extreme case of a *compact closed category* the tensor becomes the internalization of the morphism sets e.g.

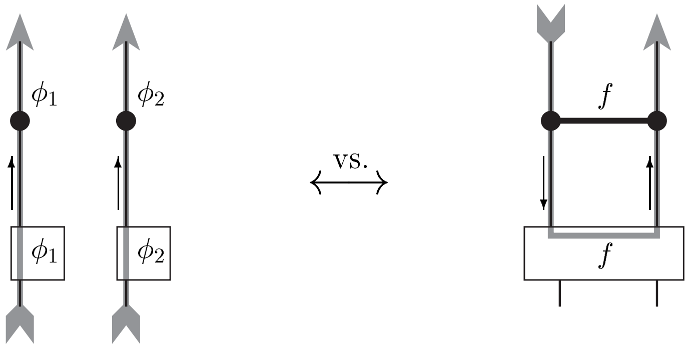
$$\mathcal{H}_1^* \otimes \mathcal{H}_2 \simeq \mathcal{H}_1 \to \mathcal{H}_2$$

for finite dimensional vector spaces [Appendix 1]. It makes therefore sense to distinguish between $- \otimes -$ and a pair of inputs. We introduce

$$\mathcal{H}_1 \times \mathcal{H}_2 := \{(\phi_1, \phi_2) \mid \phi_1 \in \mathcal{H}_1, \phi_1 \in \mathcal{H}_2\}.$$

One is tempted to refer to the elements of $\mathcal{H}_1 \times \mathcal{H}_2$ as pure tensors [Definition 5.19]. In view of the fact that in quantum theory it are the one-dimensional subspaces which make up the states and not the vectors themselves the distinction between $\phi_1 \otimes \phi_2$ and (ϕ_1, ϕ_2) is indeed essentially nihil. Adopting $- \times -$ will allow us to retain our classical picture in full extend and one of the goals of this paper is to provide an understanding of the **as if** flow of information through entanglement in classical functional terms, and this free from any circularities such as using $- \otimes -$ in the interpretation since $- \otimes -$ is itself the subject of the interpretation.

Obviously $\mathcal{H}_1 \times \mathcal{H}_2$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$ play a very different role within our interpretation. An element $(\phi_1, \phi_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ represents a pair of independently prepared states, respectively by projectors P_{ϕ_1} and P_{ϕ_2}, of which one or both might be acting backwardly on the path, while $\Psi_f \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is what comes out of a bipartite **eP**, that is, an anti-linear function $f : \mathcal{H}_1 \looparrowright \mathcal{H}_2$. Two pictures emerge.



Crucial is the difference in the direction of the (sub)paths for the corresponding carriers. These different *polarities* [Subsection 7.1] indicate that in our setting we where right to attribute a different type to the **eP** P_f and to the pair of **eP**'s $(\mathrm{P}_{\phi_1}, \mathrm{P}_{\phi_2})$ these types respectively being $\mathcal{H}_1 \looparrowright \mathcal{H}_2$ and $\mathcal{H}_1 \times \mathcal{H}_2$.

i. *Currying.* This is an operation in functional programming which consists of turning a function of two arguments into one of one argument [12]. It is named after Haskell Curry after whom also the functional programming language Haskell is named. For a function $f : X \times Y \to Z$ and a function $f_c : X \to (Y \to Z)$ we have a bijection of the function sets

$$(X \times Y) \to Z \quad \simeq \quad X \to (Y \to Z)$$

via identification of the prescriptions

$$f :: (x, y) \mapsto z \quad \Longleftrightarrow \quad f_c :: x \mapsto (f_c(x) :: y \mapsto z)\,.$$

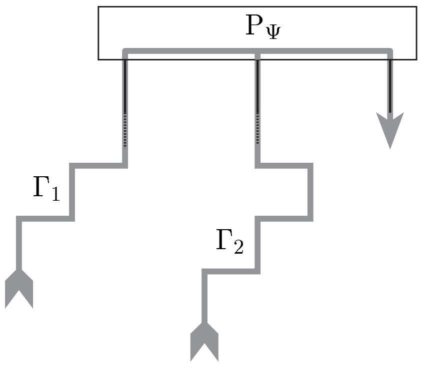
In logical perspective currying is the functional witness of the deduction rule

$$\frac{X \wedge Y \;\vdash\; Z}{X \;\vdash\; Y \Rightarrow Z}$$

something that can easily be seen in terms of categorical semantics [Appendix C].

Consider two paths Γ_1 and Γ_2 with $\tau^1_{out} = \tau^2_{out}$ which respectively for inputs $\phi^1_{in} \in \mathcal{H}^1_{in}$ and $\phi^2_{in} \in \mathcal{H}^2_{in}$ produce outputs $\phi^1_{out} \in \mathcal{H}^1_{in}$ and $\phi^1_{out} \in \mathcal{H}^2_{in}$. We can define $\Gamma_1 \times \Gamma_2$ which given input $(\phi^1_{in}, \phi^2_{in}) \in \mathcal{H}^1_{in} \times \mathcal{H}^2_{in}$ produces as output $(\phi^1_{out}, \phi^2_{out}) \in \mathcal{H}^1_{out} \times \mathcal{H}^2_{out}$. We feed the output of $\Gamma_1 \times \Gamma_2$ in a tripartite **eP** which projects on

$$\Psi := \sum_{ijk} \Psi_{ijk} \cdot e^{(1)}_i \otimes e^{(2)}_j \otimes e^{(3)}_k \;\in\; \mathcal{H}^1_{out} \otimes \mathcal{H}^2_{out} \otimes \mathcal{H}_3$$



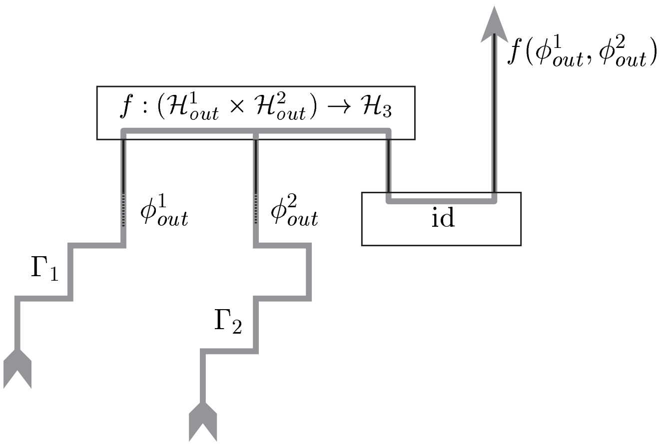
When conceiving the **eP** as a bipartite one of type $(\mathcal{H}_1 \otimes \mathcal{H}_2) \looparrowright \mathcal{H}_3$ which receives as input $\phi^1_{out} \otimes \phi^2_{out}$ and is placed in an appropriate context then it produces as output

$$\sum_{ijk} \Psi_{ijk} \langle \phi^1_{out} \otimes \phi^2_{out} \mid e^{(1)}_i \otimes e^{(2)}_j \rangle \cdot e^{(3)}_k$$

such that we can conceive the **eP** as being labeled by an *anti-bilinear function* which is defined as

$$f : (\mathcal{H}^1_{out} \times \mathcal{H}^2_{out}) \looparrowright \mathcal{H}_3 :: (\phi_1, \phi_2) \mapsto \sum_{ijk} \langle \phi_1 \mid e^{(1)}_i \rangle \langle \phi_2 \mid e^{(2)}_j \rangle\, \Psi_{ijk} \cdot e^{(3)}_k$$

since by the universality in the definition of the vector space tensor product [Proposition A.10] defining a anti-bilinear map of type $(\mathcal{H}_1 \times \mathcal{H}_2) \looparrowright \mathcal{H}_3$ is equivalent to defining a anti-linear map of type $(\mathcal{H}_1 \otimes \mathcal{H}_2) \looparrowright \mathcal{H}_3$.



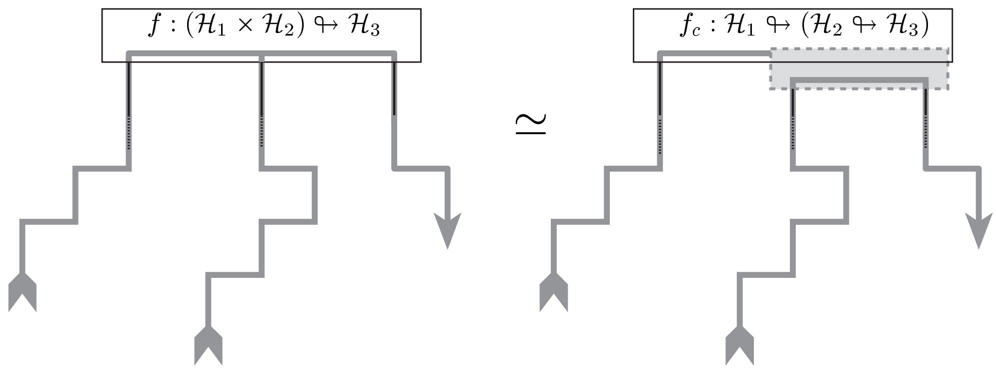
The action of an **eP** of this kind is completely analogous to that of one which is typed as $\mathcal{H}_1 \looparrowright (\mathcal{H}_2 \looparrowright \mathcal{H}_3)$. It merely suffices to redraw the internal paths within the tripartite **eP** since we have for $\phi_1 = \sum_i \phi_i \cdot e_i^{(1)}$ and $\phi_2 = \sum_i \phi_j \cdot e_j^{(2)}$ that

$$\begin{aligned} f(\phi_1, \phi_2) &= \sum_{ijk} \Psi_{ijk} \langle \phi_1 \mid e_i^{(1)} \rangle \langle \phi_2 \mid e_j^{(2)} \rangle \cdot e_k^{(3)} \\ &= \sum_{jk} \Big(\sum_i \bar{\phi}_i^1 \Psi_{ijk} \Big) \langle \phi_2 \mid e_j^{(2)} \rangle \cdot e_k^{(3)} \\ &= (f_c(\phi_1))(\phi_2) \end{aligned}$$

where

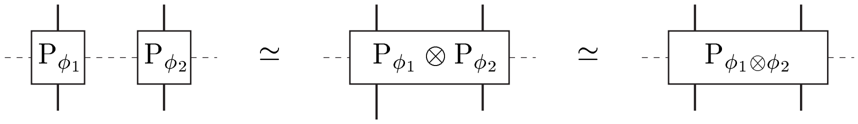
$$f_c(\phi_1) : \mathcal{H}_2 \looparrowright \mathcal{H}_3 :: \sum_{jk} \Big(\sum_i \bar{\phi}_i^1 \Psi_{ijk} \Big) \langle e_j^{(2)} \mid - \rangle \cdot e_k^{(3)} \,,$$

that is, we *Curried* the internal wiring of the bf eP typed as $(\mathcal{H}_1 \times \mathcal{H}_2) \looparrowright \mathcal{H}_3$ into one which is typed as $\mathcal{H}_1 \looparrowright (\mathcal{H}_2 \looparrowright \mathcal{H}_3)$.

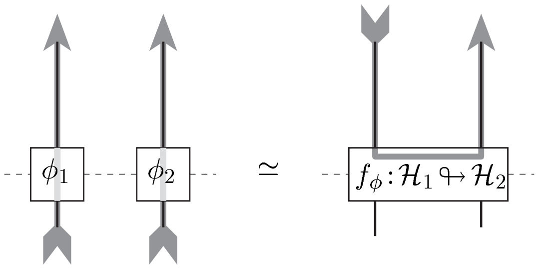


Conclusion 6.13 *The internal wiring of tripartite* **eP***'s admits Currying.*

ii. *Disentanglement.* The above also provides an interesting qualitative perspective on disentanglement. When consider the following correspondences [Proposition B.4]



it follows that we can conceive "the same **eP**" both as a pair of unipartite projectors and as well as a genuine functional action. This means that it admits at its positive side either two outgoing paths or one incoming and one outgoing one.



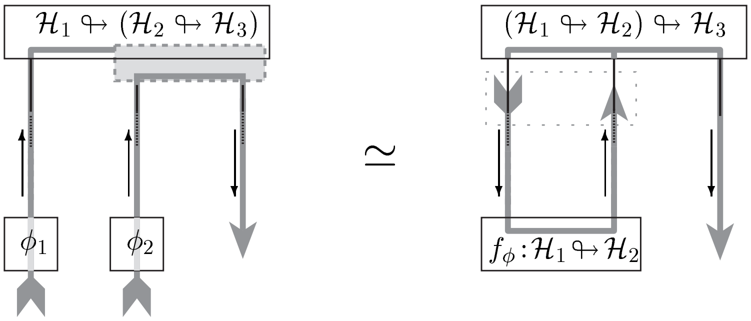
In this picture f_ϕ is the atomically singular map [Definition 5.13] defined by the prescription [Propositions 5.21 and 5.23]

$$\begin{cases} f_\phi(\phi_1) = \phi_2 \\ f_\phi(\varphi) = \mho \ \text{ for } \ \varphi \perp \phi_1 \,. \end{cases}$$

This twofoldness is not the case for bipartite **eP**'s which do not project on a pure tensor. Hence *disentanglement specification* introduces a degenerated **eP** which admits paths which are outgoing both at its input and its output, as well as normal functional behavior. In the case of normal functional behavior the functional action is however constant, that is, independent of the input. We recall here that it doesn't require a disentangled **eP** for an output only path to produce a disentangled state. It suffices that the composite of the functions is an atomically singular map [Riddles 3.8 and 4.25].

Conclusion 6.14 *An* **eP** *labeled by an atomically singular map does not admit any information flow. Neither does an output only or an input only path of which the composite of the labeling functions is an atomically singular map. Therefore it can be substituted by a pair of unipartite projectors without altering its action.*

Along the same lines one could argue that in some cases each typing of an **eP** such that $order < size - 1$ is a degeneration of one which satisfies $order = size - 1$.



The operations required to turn a large network into one where all **eP**'s satisfy

$$order = size - 1$$

are not obvious at all at first sight. It is also not clear when such a transformation is possible. Such a transformation moreover requires tuples of **eP**'s to be considered as one although they might be

- spatially separated ;
- act at a different time ;
- act in a different direction of time (positive vs. negative).

One could consider admitting some physical changes in the network to overcome these obstacles. This opens an interesting domain of study one could refer to as *entanglement specification network rewriting* in analogy the the *term-rewriting* in λ-calculus [7].

6.6 Example: non-local unitary maps, feedback and traces

We exposed different information flow capabilities for multipartite **eP**'s of which the exposure depends on the context in which the **eP** is placed [Conclusion 6.12]. However, we don't even need the notion of an **eP** at all to discover such a variety of possible behaviors. Recall that the multitude was essentially due to associativity and commutativity of the tensor combined with the use of the isomorphism(s)

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \ \simeq\ \mathcal{H}_1 \looparrowright \mathcal{H}_2 \ \simeq\ \mathcal{H}_1^* \to \mathcal{H}_2 \,.$$

Further we have

$$(\mathcal{H}^*)^* = \mathcal{H}$$

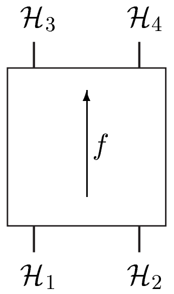
and one easily verifies that we also have

$$(\mathcal{H}_1 \otimes \mathcal{H}_2)^* = \ \mathcal{H}_1^* \otimes \mathcal{H}_2^* \,,$$

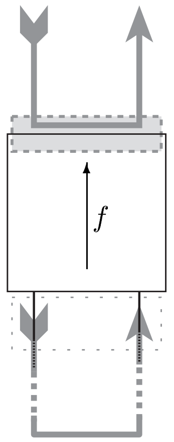
that is, the tensor *is self-dual* [Appendix C]. Consider a *bipartite non-local unitary operation*, or more general, a linear function of type

$$f : (\mathcal{H}_1 \otimes \mathcal{H}_2) \to (\mathcal{H}_3 \otimes \mathcal{H}_4) \,.$$

Physically one might want to assume $\mathcal{H}_1 = \mathcal{H}_3$ and $\mathcal{H}_2 = \mathcal{H}_4$ but this assumption is not crucial for the derivations we make below, neither is unitarity of f.



In view of the interpretation of bipartite entanglement as a function [Conclusion 6.7] we can think of f as consuming a function and producing another one [Subsection 6.4],



that is, we "refine" its type to

$$f : (\mathcal{H}_1 \looparrowright \mathcal{H}_2) \to (\mathcal{H}_3 \looparrowright \mathcal{H}_4)$$

such that its action becomes

$$f :: \sum_{ij} g_{ij}\langle - \mid e_i^{(1)}\rangle \cdot e_j^{(2)} \;\mapsto\; \sum_{ijkl} g_{ij} f_{ijkl}\langle - \mid e_k^{(3)}\rangle \cdot e_l^{(4)}\,.$$

Hence we can assign a *typed matrix* [Subsection 6.4]

$$\Big(f_{(\alpha_1\to\alpha_2)\to(\alpha_3\to\alpha_4)}\Big)_{(\alpha_1\to\alpha_2)\to(\alpha_3\to\alpha_4)}$$

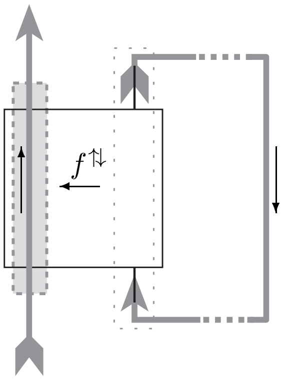
to f. This is however not the only manner how one can interpret the bipartite non-local unitary operation f. Unraveling the initial type

$$\begin{aligned}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to (\mathcal{H}_3 \otimes \mathcal{H}_4) &= (\mathcal{H}_1 \otimes \mathcal{H}_2)^* \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \\ &= \mathcal{H}_1^* \otimes \mathcal{H}_2^* \otimes \mathcal{H}_3 \otimes \mathcal{H}_4\end{aligned}$$

we obtain a commutative and associative tetrapartite tensor product for which we have

$$\begin{aligned}\mathcal{H}_1^* \otimes \mathcal{H}_2^* \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 &= (\mathcal{H}_4^* \otimes \mathcal{H}_2)^* \otimes (\mathcal{H}_1^* \otimes \mathcal{H}_3) \\ &= (\mathcal{H}_4 \to \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_3)\,,\end{aligned}$$

that is, we end up with an alternative typing and hence a different picture



which involves a *different function*

$$f^{\uparrow\downarrow} : (\mathcal{H}_4 \to \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_3)$$

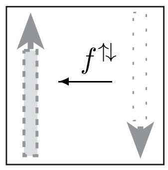
of which the typed matrix

$$\left(f^{\uparrow\downarrow}_{(\alpha_4\to\alpha_2)\to(\alpha_1\to\alpha_3)}\right)_{(\alpha_4\to\alpha_2)\to(\alpha_1\to\alpha_3)}\,.$$

satisfies

$$f^{\uparrow\downarrow}_{(\alpha_4\to\alpha_2)\to(\alpha_1\to\alpha_3)} = f_{(\alpha_1\to\alpha_2)\to(\alpha_3\to\alpha_4)}\,.$$

The new picture also involves a *different context*. Indeed, it consumes a function at "ports" $\mathcal{H}_2$ and $\mathcal{H}_4$ which has to be produced somewhere else by the environment. Note also that the direction of the function which it consumes is opposite to the one it produces. We can graphically represent this in a more concise manner.



Hence we obtain a different behavioral interpretation. As we show below such an alternative perspective can be very useful. And there are of course many of them.

i. *Calculating the trace using entanglement.* An (at least conceptually) interesting example is the *operational realization*, or even more, *computation*, of *partial traces*, or more precisely, computation of the *trace operation* of the *traced monoidal category* [41]

$(\mathbf{FDVec}_{\mathbb{C}}, \otimes, \mathrm{Tr})$ of finite dimensional vector spaces and linear maps with the vector space tensor product as tensor and with the trace defined as

$$\mathrm{Tr}^{\mathcal{H}_2}_{\mathcal{H}_1,\mathcal{H}_3}(f) : \mathcal{H}_1 \to \mathcal{H}_3 :: \phi \mapsto \sum_{ik\alpha} \phi_i f_{i\alpha k\alpha} \cdot e^{(3)}_k$$

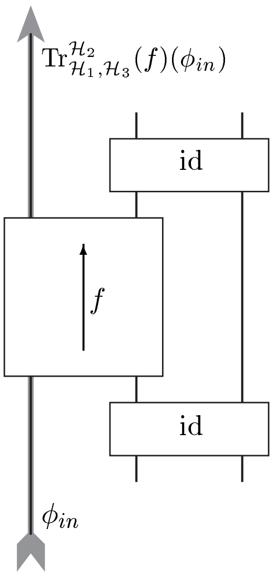
for

$$f : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3 \otimes \mathcal{H}_2 :: \sum_{ij} g_{ij} \cdot e^{(1)}_i \otimes e^{(2)}_j \mapsto \sum_{ijkl} g_{ij} f_{ijkl} \cdot e^{(3)}_k \otimes e^{(2)}_l$$

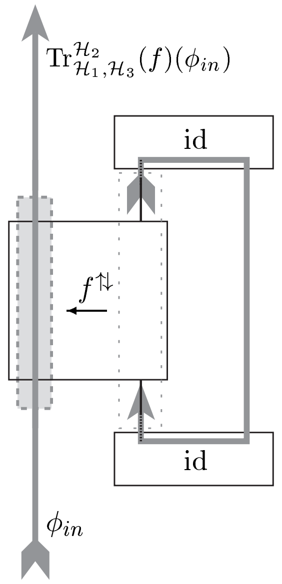
where $\{e^{(1)}_i\}_i$, $\{e^{(2)}_j\}_j$, $\{e^{(3)}_k\}_k$ are respective bases of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ and $\phi = \sum_i \phi_i \cdot e^{(1)}_i$. Hence the type of the trace operation is

$$\mathrm{Tr}^{\mathcal{H}_2}_{\mathcal{H}_1,\mathcal{H}_3} : (\mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3 \otimes \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_3)\,.$$

The interest in abstract modeling of traces can be seen as part of the *geometry of interaction* approach to logical models of functional programming [1, 30]. The construction of a physical realization of the $(\mathbf{FDVec}, \otimes)$ was initiated in [3] by S. Abramsky and the author where we showed that the network



realizes the functional action $\mathrm{Tr}^{\mathcal{H}_2}_{\mathcal{H}_1,\mathcal{H}_3}(f) : \mathcal{H}_1 \to \mathcal{H}_2$ on the input $\phi_{in} \in \mathcal{H}_1$, that is, it yields a *physical realization of the trace*. The discussion above places this result in a different light. Indeed, consider the following picture.

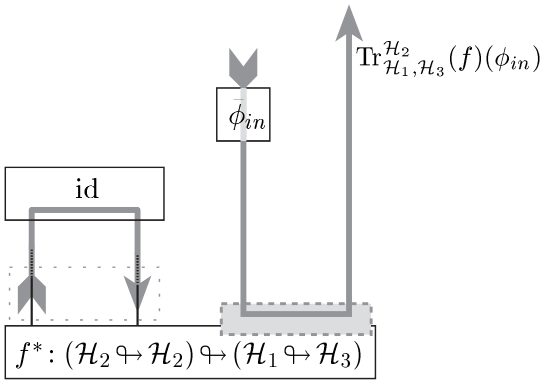


The action $\mathrm{Tr}^{\mathcal{H}_3}_{\mathcal{H}_1,\mathcal{H}_2}(f)$ is the result of feeding an identity into the now

$$f^{\updownarrows} : (\mathcal{H}_2 \to \mathcal{H}_2) \to (\mathcal{H}_1 \to \mathcal{H}_3)$$

labeled non-local unitary map. In particular does this provide an interpretation of the trace in terms of a *feedback loop*. This is exactly how traces arise in most classical settings [1, 6], that is, via feedback loops. Of course in our case the feedback loop is not a true iteration, but a *one-shot "sideways" injection of a "backward" identity.*

If all this is really true let us mess a bit further with the geometry of the above, while adding a complex conjugation whenever we reverse the direction of a path as compared to the picture above [Subsection 4.1]



where $\bar{\phi}_{in} := \sum_k \bar{\phi}^{in}_k \cdot e^{(3)}_k$. For the matrix of f^* we set

$$f^*_{(\alpha_2 \looparrowright \alpha_2) \looparrowright (\alpha_1 \looparrowright \alpha_3)} := f^{\updownarrows}_{(\alpha_2 \to \alpha_2) \to (\alpha_1 \to \alpha_3)} = f_{(\alpha_1 \to \alpha_2) \to (\alpha_3 \to \alpha_2)} \,.$$

Applying Lemma 6.11 we have for

$$\Psi_{f^*} = \sum_{ijkl} f_{(ij)(kl)} \cdot e_l^{(2)} \otimes e_j^{(2)} \otimes e_i^{(1)} \otimes e_k^{(3)} \qquad \text{and} \qquad \Psi_{\mathrm{id}} = \sum_{\alpha\beta} \delta_{\alpha\beta} \cdot e_\alpha^{(2)} \otimes e_\beta^{(2)}$$

that

$$\begin{aligned} \mathrm{P}_{\mathrm{id}}(\Psi_{f^*}) &= \sum_{ijkl}\sum_{\alpha\beta} f_{(i\beta)(k\alpha)}\bar{\delta}_{\alpha\beta}\delta_{lj} \cdot e_l^{(2)} \otimes e_j^{(2)} \otimes e_i^{(1)} \otimes e_k^{(3)} \\ &= \left(\sum_{lj} \delta_{lj} \cdot e_l^{(2)} \otimes e_j^{(2)}\right) \otimes \left(\sum_{ik\alpha} f_{(i\alpha)(k\alpha)} \cdot e_i^{(1)} \otimes e_k^{(3)}\right) \end{aligned}$$

and thus

$$\begin{aligned} (\mathrm{P}_{\bar{\phi}_{in}} \circ \mathrm{P}_{\mathrm{id}})(\Psi_{f^*}) &= \Psi_{\mathrm{id}} \otimes \bar{\phi}_{in} \otimes \sum_{ik\alpha} \phi_i^{in} f_{(i\alpha)(k\alpha)} \cdot e_k^{(3)} \\ &= \Psi_{\mathrm{id}} \otimes \bar{\phi}_{in} \otimes \mathrm{Tr}^{\mathcal{H}_2}_{\mathcal{H}_1,\mathcal{H}_3}(f)(\phi_{in})\,. \end{aligned}$$

Hence the entanglement specification network depicted above indeed provides a computation of the trace of f applied to ϕ^{in}. If we drop the projector $\mathrm{P}_{\bar{\phi}_{in}}$ in it we obtain $\mathrm{Tr}^{\mathcal{H}_2}_{\mathcal{H}_1,\mathcal{H}_3}(f)$ as an entangled state [Conclusion 6.7] up to conjugation of the input [Proposition 4.4] — since $\mathrm{Tr}^{\mathcal{H}_2}_{\mathcal{H}_1,\mathcal{H}_3}(f)$ is linear while an entangled state encodes an anti-linear map in the effective input specification view [Subsection 6.1].

ii. *Geometry of interaction.* In this section we consider a "slight" modification of the type which we used above for exposing a virtual notion "feedback". We define

$$f^{\upuparrows} : (\mathcal{H}_4 \to \mathcal{H}_2) \looparrowright (\mathcal{H}_1 \to \mathcal{H}_3)$$

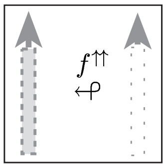
of which the typed matrix

$$\left(f^{\upuparrows}_{(\alpha_4\to\alpha_2)\looparrowright(\alpha_1\to\alpha_3)}\right)_{(\alpha_4\to\alpha_2)\looparrowright(\alpha_1\to\alpha_3)}\,.$$

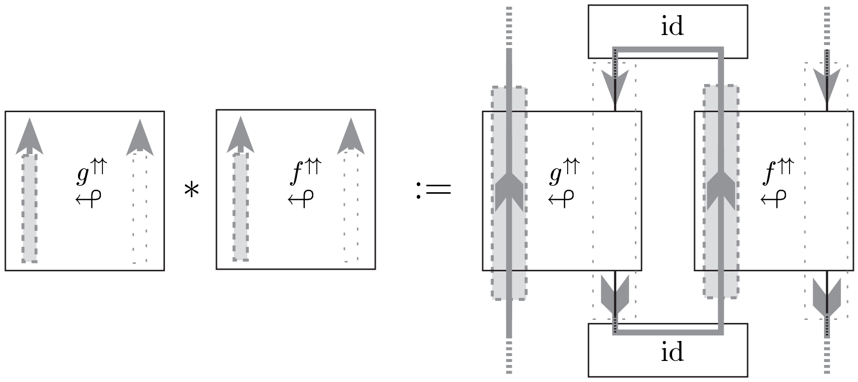
satisfies

$$f^{\upuparrows}_{(\alpha_4\to\alpha_2)\looparrowright(\alpha_1\to\alpha_3)} := f^{\uparrow\downarrow}_{(\alpha_2\to\alpha_2)\to(\alpha_1\to\alpha_3)} = f_{(\alpha_1\to\alpha_2)\to(\alpha_3\to\alpha_4)}\,.$$

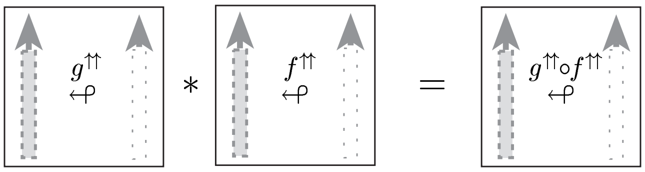
The corresponding change in the context is that the direction of the function to be consumed is now the same as that of the one which is to be produced.



Functions such as $f^{\upuparrows}$ admit a physically realizable notion of composition. The physical realization of this composition $*$ is not consecutive application but is of a *parallel* nature.



From the perspective of the functions $f^{\upuparrows}$ and $g^{\upuparrows}$ and the corresponding interpretation in terms of paths, $g^{\upuparrows}$ consumes the function produced by $f^{\upuparrows}$, that is, we obtain the concatenated function $g^{\upuparrows} \circ f^{\upuparrows}$ as the *typed functional label* for the composite.

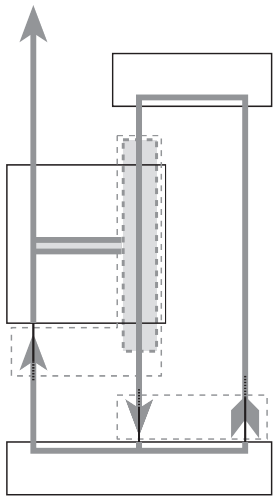


It should be clear to the reader that a similar construction can be used to *apply* the function $f^{\upuparrows}$ to some input.

Conclusion 6.15 *The type transformation $f \mapsto f^{\upuparrows}$ gives access to functions which are not physically present as a non-local unitary operation but which do allow manipulations such as composition and application to an input.*

There are of course other *type transformations* than $f \mapsto f^{\upuparrows}$ which at their turn admit different kinds of manipulations.

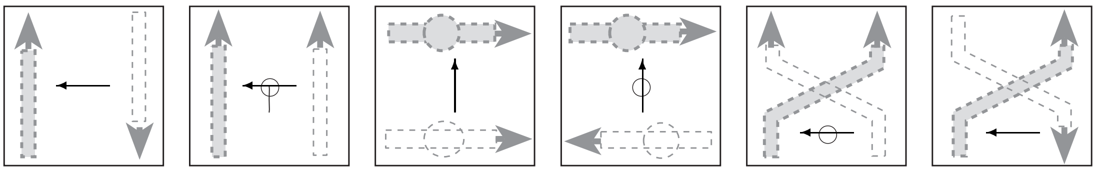
iii. *Types for non-local unitary actions.* Above we used two variants of possible typings for a non-local function with bipartite input and output. There are many other possible typings which might be exposed in a different context e.g. third-order behavior.



In Subsection 6.4 we showed for tetrapartite **eP**'s that there are five families of typings which cannot be turned into each other by commutation of the base types $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$. In the case of a non-local function the situation is somewhat different. In particular are not all base types equivalent since for the base types $\mathcal{H}_1$ and $\mathcal{H}_2$ it are their duals $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$ which appear in the tensor product from which we derive all types by bracketing and commutation. When we specify that f is a non-local unitary operation we have $\mathcal{H}_1 = \mathcal{H}_3$ and $\mathcal{H}_2 = \mathcal{H}_4$. As an example, the possible types of the shape

$$\left(\cdots\left\{\begin{array}{c}\rightarrow\\ \circ\!\!\rightarrow\end{array}\right\}\cdots\right)\left\{\begin{array}{c}\rightarrow\\ \circ\!\!\rightarrow\end{array}\right\}\left(\cdots\left\{\begin{array}{c}\rightarrow\\ \circ\!\!\rightarrow\end{array}\right\}\cdots\right)$$

are (using the graphical notation introduced above)



up to permutation of $\mathcal{H}_1$ and $\mathcal{H}_2$. The circles on the arrow indicate anti-linearity, that is, an arrow such as $\circ\!\!\rightarrow$ — which we failed to produce graphically.

7 General multipartite entanglement

In Section 6 we exposed the variety of emerging *capabilities* when passing from bipartite to multipartite entanglement [Subsection 6.4] and corresponding typings [Subsections 6.5 and 6.6]. It opens a wide field of study involving typing and network transformation issues. It also became clear how fast the complexity and sophistication of networks involving multipartite **eP**'s grows. Therefore the study of multipartite entanglement requires the *denotational tools* and *proof-theoretic methods* of modern logic. This interdisciplinary encounter will be initiated in this section. We also provide an indication of the proof of the general theorem on compositionality for entanglement specification networks involving multipartite entanglement. We conceive an elegant full proof and precise statement of the theorem as a subject for further research. Such a statement requires sophisticated denotational tools, and hence an entanglement of computational, logical and physical notions. Many existing results from these fields will be very useful, but it remains a matter to connect them up. This requires an encounter and endeavor towards mutual understanding between the scientists of these communities.

7.1 Types and polarities

Due to Currying [Subsection 6.5] the types of all **eP**'s can be captured by the syntax

$$T ::= \mathcal{H}_i \mid T_1 \multimap T_2$$

involving only *base types* $\mathcal{H}_i$ and (arbitrary) *function types* $T_1 \multimap T_2$.

Definition 7.1 By a *subtype* of a type T we mean any intermediate type which occurs during the formation of T according to the syntax defined above.

A type either refers either to the physical input or physical output of an **eP**, that is, refers either to the positive or negative action of the **eP** [Definition 3.13]. Hence each **eP** admits two distinct types.

Definition 7.2 By a *typed* **eP** we refer to a triple consisting of

- The **eP** itself;
- A choice of either the physical input or output;
- A type assigned to it.

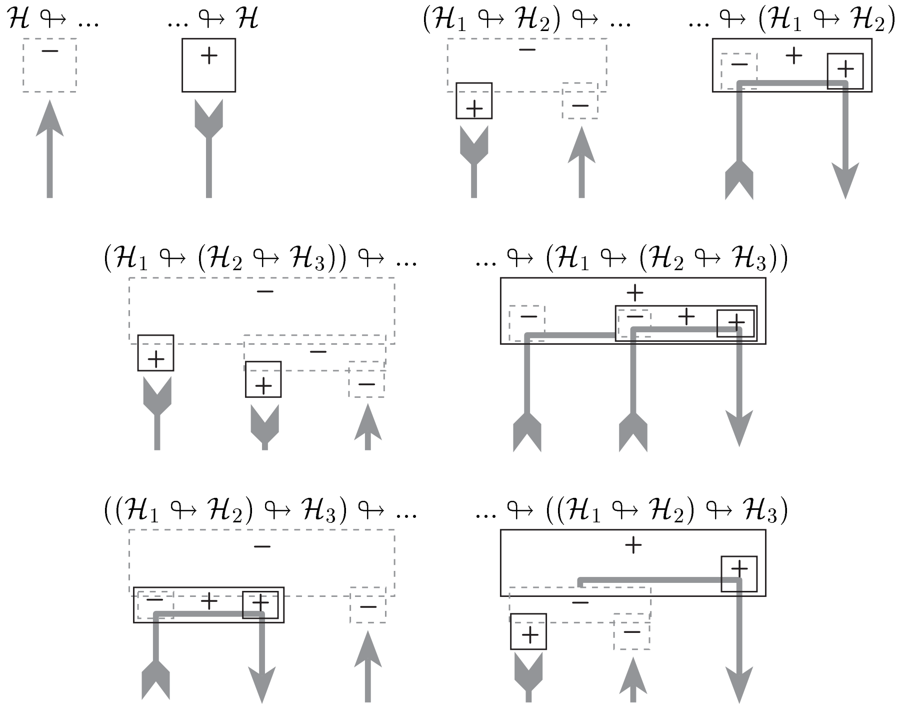
By the *ports* of a typed **eP** we refer to all the carriers that pass through it, either at the physical input or output depending on the chosen action.

When attributing a type to an **eP** we define its *internal wiring*, that is, we fix the internal behavior of paths. This internal wiring defines at which ports paths are either *incoming* or *outgoing*. More general, it defines for all subtypes whether the **eP** is either *consuming* or *producing* [Subsection 6.4] with respect to that subtype, incoming and outgoing paths being extremal cases of consuming and producing. This can easily be seen when introducing *polarities* for subtypes. Whenever something is *produced* we will attribute a "+"-sign and when something is *consumed* we attribute a "$-$"-sign. For T the type of an **eP** one easily verifies that these polarities obey the following rules.

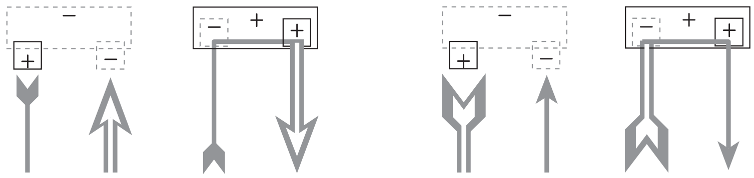
- The type T itself gets a "+" ;
- If $T_1 \multimap T_2$ has a "+" then T_1 gets a $-$ and T_2 gets a "+".

- If $T_1 \looparrowright T_2$ has a "$-$" then T_1 gets a $+$ and T_2 gets a "$-$".

This leads us to the following pictures where for each pair of pictures the one on the left shows the polarities and wiring if it appears on the left of $\looparrowright$ while the one on the right shows the *dual* polarities and internal wiring if it appears on the right of $\looparrowright$.



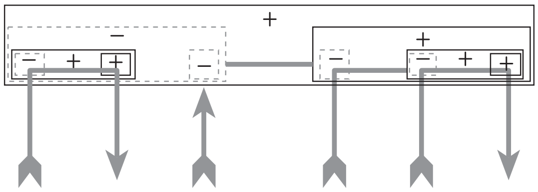
The last four pictures could also be represented omitting some of the size one paths.



Polarities also enable a rigorous definition of a virtual **eP**.

Definition 7.3 Given a typed **eP** a *virtual* **eP** (=: **veP**) is a non-exiting entity associated to (i) a proper subtype of the type of that **eP** which (ii) carries a "+"-sign and which (iii) has a size strictly larger than one.

As an example, the **eP**



of type

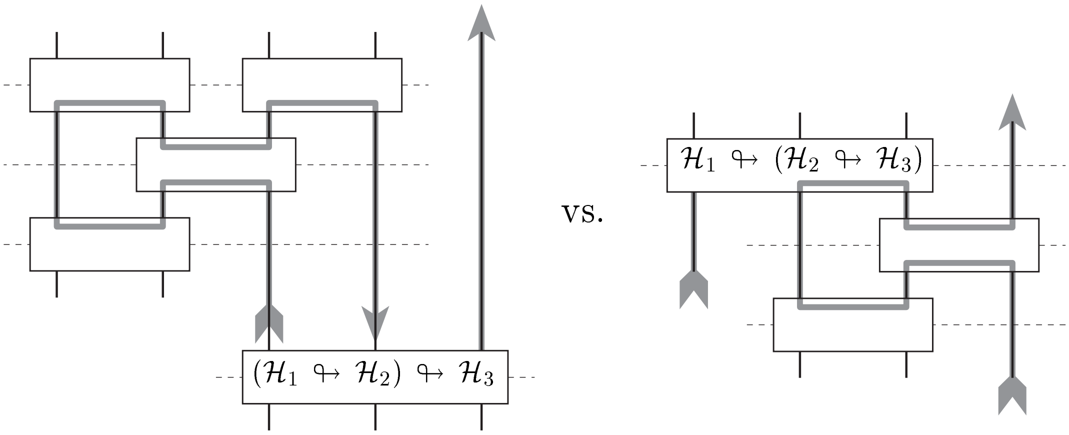
$$((\mathcal{H}_1 \multimap \mathcal{H}_2) \multimap \mathcal{H}_3) \multimap (\mathcal{H}_4 \multimap (\mathcal{H}_5 \multimap \mathcal{H}_6))$$

has three **veP**'s respectively being typed as

$$\mathcal{H}_1 \multimap \mathcal{H}_2 \qquad\qquad \mathcal{H}_5 \multimap \mathcal{H}_6 \qquad\qquad \mathcal{H}_4 \multimap (\mathcal{H}_5 \multimap \mathcal{H}_6)\,.$$

7.2 The logic of interacting paths

The type which one has to assign to an **eP** is determined by the context e.g.



Hence we need to deduce the types to be attributed to an **eP** from the geometry of the paths in a network. The syntax

$$T ::= \mathcal{H}_i \mid T_1 \multimap T_2$$

suffices to type **eP**'s. However,

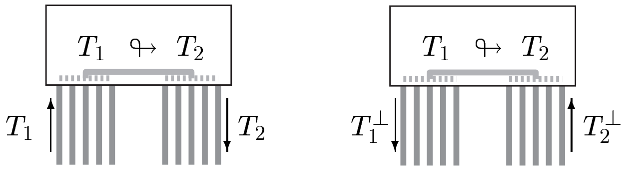
- the *bundle* of all incoming and outgoing paths at the ports contained in the type T_1 for an **eP** of type $T_1 \multimap T_2$, and,
- the same thing for an **eP** of type $T_2 \multimap T_1$,

cannot both have the same type T_1 since the internal wirings inside these **eP**'s at the ports of type T_1 are *dual* to each other. In order to be able to distinguish between these "dual" cases we have to extend our type system to

$$T ::= \mathcal{H}_i \mid T_1 \multimap T_2 \mid T^{\perp}\,.$$

We will attribute type

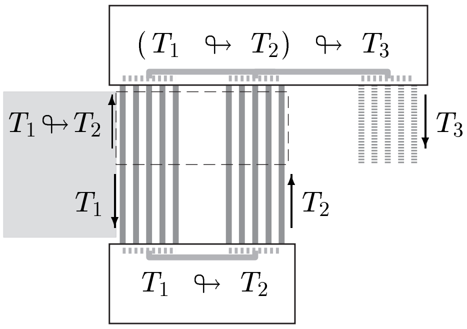
- T_1 to the *incoming bundle* of paths at the *input* of an **eP** of type $T_1 \looparrowright T_2$;
- $T_1^\perp$ to the *outgoing bundle* of paths at the *input* of an **eP** of type $T_1 \looparrowright T_2$;
- $T_2^\perp$ to the *incoming bundle* of paths at the *output* of an **eP** of type $T_1 \looparrowright T_2$;
- T_2 to the *outgoing bundle* of paths at the *output* of an **eP** of type $T_1 \looparrowright T_2$.



Hence the logic is

$$\textit{type outgoing} = (\textit{type incoming})^\perp \qquad \textit{type incoming} = (\textit{type outgoing})^\perp .$$

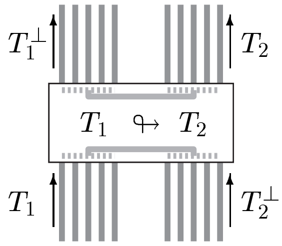
For paths of size one these two dual perspectives of incoming and outgoing bundle are superfluous since the path itself defines a unique direction and thus a preferred perspective. For bundles of paths of higher size this is not the case anymore since in general the direction indicated by the input and the output of **eP**'s might be conflicting e.g.



For convenience one could take the direction of the actual physical time as a reference and then we assign for an **eP** of type $T_1 \looparrowright T_2$

- T_1 to the *input* bundle if the **eP** acts *negative* ;
- $T_2^\perp$ to the *output* bundle if the **eP** acts *negative* ;
- $T_1^\perp$ to the *input* bundle if the **eP** acts *positive* ;
- T_2 to the *output* bundle if the **eP** acts *positive* .

In a picture this yields

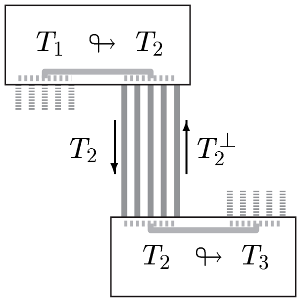


Whenever for $\mathbf{eP}(1)$ of type $T_1 \looparrowright T_2$ and $\mathbf{eP}(2)$ of type $T_2 \looparrowright T_3$ we state that

$$\mathit{type}(\mathit{outgoing\ bundle\ at\ }(1)\text{'}s\ \mathit{output}) = \mathit{type}(\mathit{incoming\ bundle\ at\ }(2)\text{'}s\ \mathit{input})$$

then we assure that we can effectively draw a bundle of paths between them such that

- The sizes and all polarities match, that is, a "+" at the source corresponds with a "−" at the target (and vice versa) for all subbundles of any size;
- Information gets consumed by the subbundle's source (= at the output of some ($\mathbf{v}$)$\mathbf{eP}$) and information is produced at the subbundle's target (= at the input of some ($\mathbf{v}$)$\mathbf{eP}$)) and thus all subbundles are virtual carriers of information.



This is a first (very simple) example of the following general idea:

"We can draw a bundle of paths which expresses the flow of information"
$\Updownarrow$
"The types of the bundles of paths match with those of the **eP**'s".

If we want to push this a bit further it is useful to introduce alternative connectives [29]

$$T_1^\perp \wp T_2 := T_1 \looparrowright T_2 \quad \text{and} \quad T_1 \times T_2^\perp := (T_1 \looparrowright T_2)^\perp .$$

It easily follows that $\times$ and $\wp$ are dual via De Morgan rules, that is,

$$(T_1 \wp T_2)^\perp = T_1^\perp \times T_2^\perp \quad \text{and} \quad (T_1 \times T_2)^\perp = T_1^\perp \wp T_2^\perp .$$

The resulting syntax

$$T ::= \mathcal{H}_i \mid T_1 \times T_2 \mid T_1 \wp T_2 \mid T^{\perp}$$

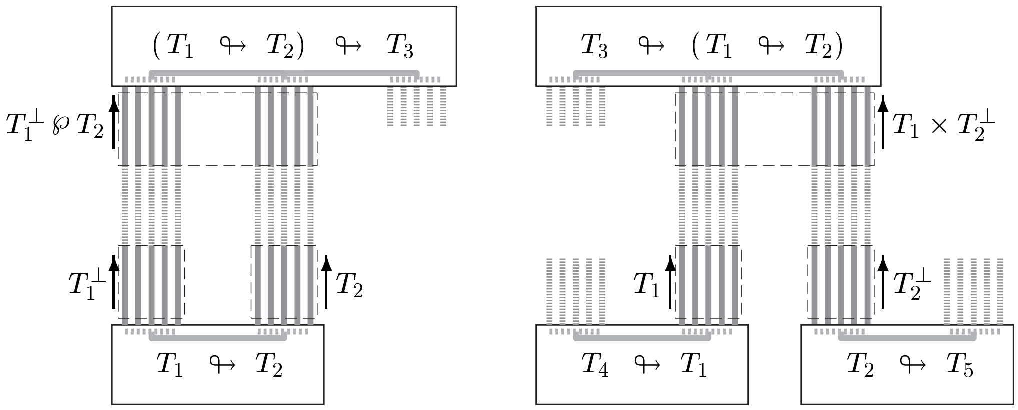
is that of *multiplicative linear logic* [29]. If we apply it to **eP**'s then *Curry-equivalence* [Subsection 6.5] of internal wirings becomes a consequence of associativity e.g.

$$\mathcal{H}_1 \hookrightarrow (\mathcal{H}_2 \hookrightarrow (\mathcal{H}_3 \hookrightarrow \mathcal{H}_4)) \quad \text{yields} \quad \mathcal{H}_1^{\perp} \wp \mathcal{H}_2^{\perp} \wp \mathcal{H}_3^{\perp} \wp \mathcal{H}_4$$

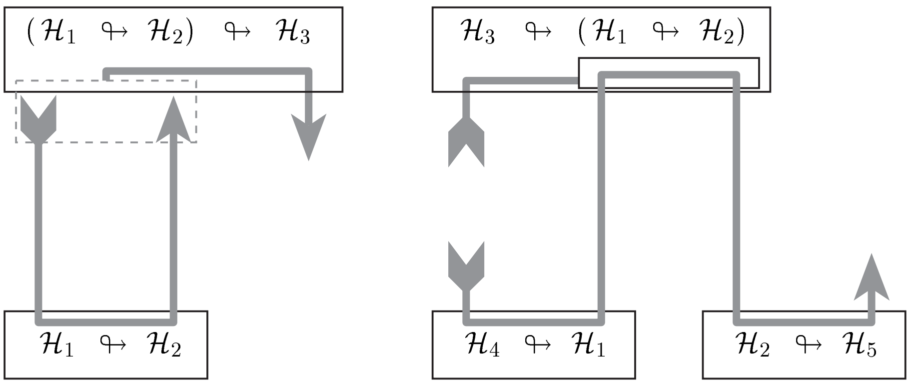
and thus allows permutation of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ while e.g.

$$((\mathcal{H}_1 \hookrightarrow \mathcal{H}_2) \hookrightarrow \mathcal{H}_3) \hookrightarrow \mathcal{H}_4 \quad \text{yields} \quad ((\mathcal{H}_1^{\perp} \wp \mathcal{H}_2) \times \mathcal{H}_3^{\perp}) \wp \mathcal{H}_4\,.$$

We will now expose how these operations arise. Consider the picture

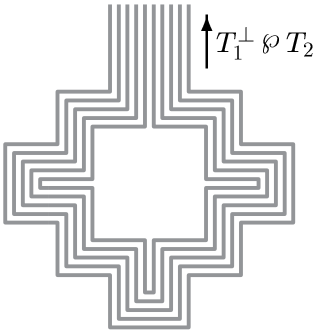


which we can simplify by substituting all types by base types.



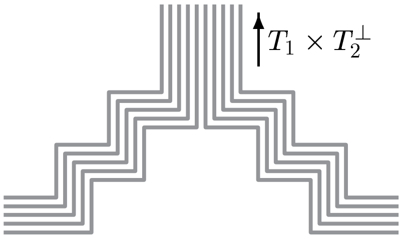
If for the ones on the left we make the types match the bundle at the input of the tripartite **eP** should have type $T_1 \hookrightarrow T_2$, that is, $T_1^{\perp} \wp T_2$. Hence

- $T_1^{\perp} \wp T_2$ expresses that two bundles of paths of respective types $T_1^{\perp}$ and T_2 are *functionally correlated* due to the presence of some common **eP** to be encountered when traveling backward along the bundles.



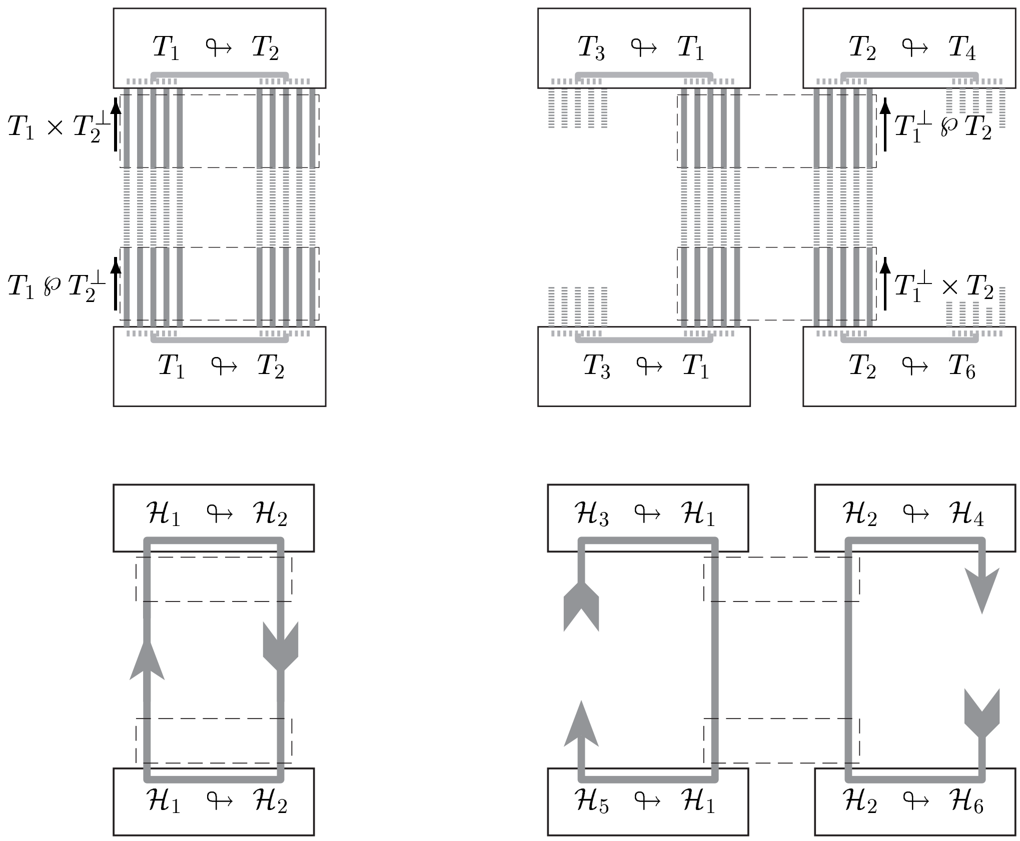
In other words, The symbol $\wp$ witnesses the *presence of entanglement*. Note that this case does include disentanglement [Subsection 6.5] due to the possibility of having atomically singular maps as labeling functions or as the composite of labeling functions. Dually

- $T_1 \times T_2^{\perp}$ expresses that two bundles of paths of respective types T_1 and $T_2^{\perp}$ are *not functionally correlated* so we will not encounter a common **eP** when traveling backwards along the bundles.



Hence the symbol $\times$ witnesses true *disentanglement*, that is, it witnesses a proper *pair* of paths. The distinction between $\wp$ and $\times$ allows us to encode how the paths in a bundle relate to each other, if they either have been produced by common **eP**'s or not.

In which way does the distinction between $\times$ and $\wp$ contribute to the shape of paths? Consider the following situations of conflicting types with respect to the operations $\times$ and $\wp$. We chose the typing relative to the closest **eP**.

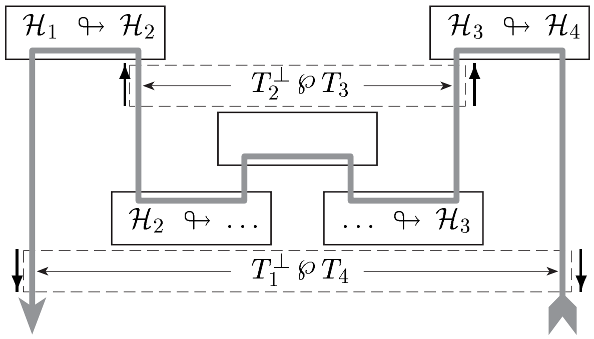


We conclude that matching types in the connectives $\times$ and $\wp$:

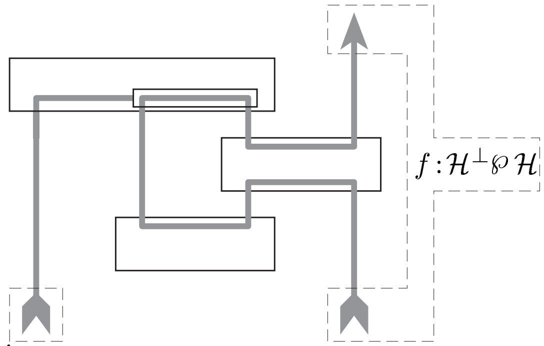
- *Forces connectedness* of the paths ;
- *Bans loops* from the paths .

This satisfies our intention that paths should carry information from a *source* to a *target*.

Note that types for bundles of paths can be *non-local* both in *space* and *time*. That is, two bundles of respective types $T_1^\perp$ and T_2 which are located in different regions of the network but which are correlated due to a common **ep** when one travels backward along these bundles should be conceived as having joint type $T_1^\perp \wp T_2$.



E.g. if one wants to designs a network which has a function $f : \mathcal{H}_1 \to \mathcal{H}_2$ as output, that is, a bundle of type $\mathcal{H}_1^\perp \wp \mathcal{H}_2$, then it makes a lot of sense to locate this bundle as in the following example.

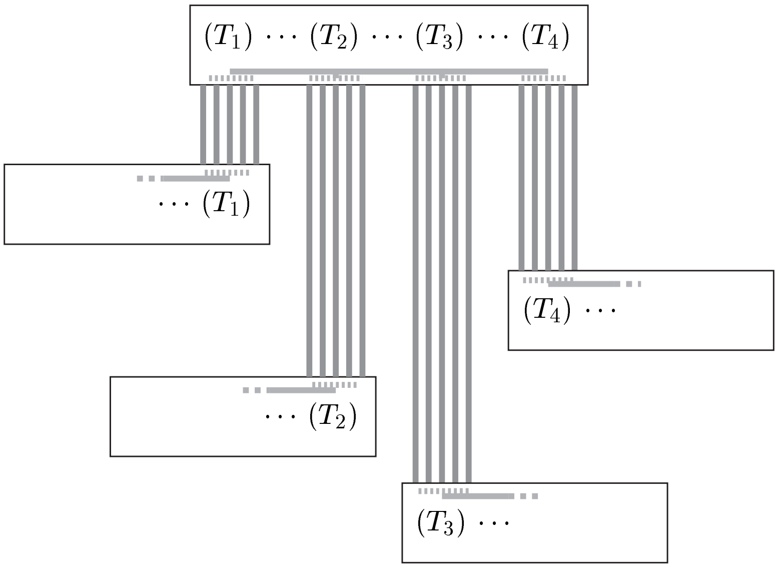


This then allows to "physically" feed an input in the function f (= output of the network) at the same time as one delivers the input to the network which produces the function.

7.3 The hypercompositionality lemma

Bipartite compositionality essentially relied on the compositionality lemma [Lemma 5.6]. Its use in the inductive argument which relied on the geometry of (bipartite) paths [Subsection 5.2] then resulted in Proposition 5.11 from which we derived case by case the validity of compositionality for the different kinds of paths. The same methodology can be used to generalize the bipartite compositionality theorems to the multipartite case. We provide the analogue of Lemma 5.6 for *multipartite entanglement*.

Just as lemma [Lemma 5.6] exposes how two bipartite entangled states interact when a bipartite **eP** acts on them, the lemma below expresses how many multipartite entangled states interact when a multipartite **eP** acts on them. Consider a negatively acting typed **eP** P_f. For each port $\mathcal{H}_i$ of P_f there should be a positive **eP** P_{g^i} which is the last **eP** through which the i'th carrier passed before passing through P_f. We say that port $\mathcal{H}_i$ of P_{g^i} is in the *scope* of P_f.



Let $T^i_1,\ldots,T^i_{l(i)}$ be disjoint subtypes of g^i's type $\mathcal{T}_g(T^i_1,\ldots,T^i_{l(i)})$ which all are in the scope of P_f and which form a complete set in the sense that they contain all ports $\mathcal{H}_i$ of P_{g^i} which are in the scope of P_f. Let $\mathrm{P}_{g^1},\ldots,\mathrm{P}_{g^k}$ be the positive **eP**'s which have ports that are in the scope of P_f. Then the type

$$\mathcal{T}_f(T^1_1,\ldots,T^1_{l(1)};\ldots;T^k_1,\ldots,T^k_{l(k)})$$

of f is made up of the subtypes $T^1_1,\ldots,T^1_{l(1)};\ldots;T^k_1,\ldots,T^k_{l(k)}$ and arrows $(-\looparrowright-)$. Let

$$\Big(f_{\mathcal{A}(\alpha^1_1,\ldots,\alpha^1_{l(1)};\ldots;\alpha^k_1,\ldots,\alpha^k_{l(k)})}\Big)_{\mathcal{A}(\alpha^1_1,\ldots,\alpha^1_{l(1)};\ldots;\alpha^k_1,\ldots,\alpha^k_{l(k)})} \qquad \text{and} \qquad \Big(g^i_{\mathcal{A}(\alpha^i_1,\ldots,\alpha^i_{l(i)})}\Big)_{\mathcal{A}(\alpha^i_1,\ldots,\alpha^i_{l(i)})}$$

respectively be the typed matrices of f and g^i where α^i_j is the index labeling a base $\{e_{\alpha^i_j}\}_{\alpha^i_j}$ of the tensor product $\bigotimes T^i_j$ of the Hilbert spaces contained in T^i_j. We assume the typed indices $\mathcal{A}(\alpha^i_1,\ldots,\alpha^i_{l(i)})$ to also include some other indices $\mathcal{B}_i$ besides $\alpha^i_1,\ldots,\alpha^i_{l(i)}$. The state produced by P_{g^i} is then of the shape

$$\sum_{\alpha^i_1\ldots\alpha^i_{l(i)}\mathcal{B}_i} g^i_{\mathcal{A}(\alpha^i_1,\ldots,\alpha^i_{l(i)})}\cdot e_{\alpha^i_1}\otimes\ldots\otimes e_{\alpha^i_{l(i)}}\otimes e_{\mathcal{B}_i}\,.$$

Before this state reaches P_f it will interact with other states due to the action of other negative **eP**'s. By Lemma 6.11 we know that after this interaction the resulting state is of the shape

$$\sum_{\alpha^i_1\ldots\alpha^i_{l(i)}\Gamma_i\mathcal{D}_i} g^i_{\mathcal{A}(\alpha^i_1,\ldots,\alpha^i_{l(i)})[\Gamma_i/\Delta_i]}\Phi_{\Gamma_i\mathcal{D}_i}\cdot e_{\alpha^i_1}\otimes\ldots\otimes e_{\alpha^i_{l(i)}}\otimes e_{\mathcal{D}_i}$$

where $\mathcal{A}(\alpha^i_1, \ldots, \alpha^i_{l(i)})[\Gamma_i/\Delta_i]$ denotes substitution of the indices $\Delta_i \subseteq \mathcal{B}^i$ (hence they are distinct from $\alpha^i_1, \ldots, \alpha^i_{l(i)}$) by Γ_i. Set

$$e_{\alpha^1_1 \ldots \alpha^1_{l(1)} \ldots \alpha^k_1 \ldots \alpha^k_{l(k)}} := e_{\alpha^1_1} \otimes \ldots \otimes e_{\alpha^1_{l(1)}} \otimes \ldots \otimes e_{\alpha^k_1} \otimes \ldots \otimes e_{\alpha^k_{l(k)}} .$$

Lemma 7.4 (Hypercompositionality) *The result of the action of* P_f *on*

$$\sum_{\substack{\alpha^i_1 \ldots \alpha^i_{l(i)} \\ \Gamma_1 \ldots \Gamma_k \mathcal{D}}} g^1_{\mathcal{A}(\alpha^1_1, \ldots, \alpha^1_{l(1)})[\Gamma_1/\Delta_1]} \cdots g^k_{\mathcal{A}(\alpha^k_1, \ldots, \alpha^k_{l(k)})[\Gamma_k/\Delta_k]} \Phi_{\Gamma_1 \ldots \Gamma_k \mathcal{D}} \cdot e_{\alpha^1_1 \ldots \alpha^1_{l(1)} \ldots \alpha^k_1 \ldots \alpha^k_{l(k)}} \otimes e_{\mathcal{D}}$$

contains

$$\sum_{\substack{\beta^1_1, \ldots, \beta^1_{l(1)} \\ \vdots \\ \beta^k_1, \ldots, \beta^k_{l(k)} \\ \Gamma_1 \ldots \Gamma_k \mathcal{D}}} g^1_{\mathcal{A}(\beta^1_1, \ldots, \beta^1_{l(1)})[\Gamma_1/\Delta_1]} \cdots g^k_{\mathcal{A}(\beta^k_1, \ldots, \beta^k_{l(k)})[\Gamma_k/\Delta_k]} \bar{f}_{\mathcal{A}(\beta^1_1, \ldots, \beta^1_{l(1)}; \ldots; \beta^k_1, \ldots, \beta^k_{l(k)})} \Phi_{\Gamma_1 \ldots \Gamma_k \mathcal{D}} \cdot e_{\mathcal{D}}$$

as a factor with respect to the tensor product.

Proof. The result straightforwardly follows from Lemma 6.11. $\square$

Note that this lemma embodies the abstraction of the explicit calculation which we did for the tripartite example in Subsection 6.4.

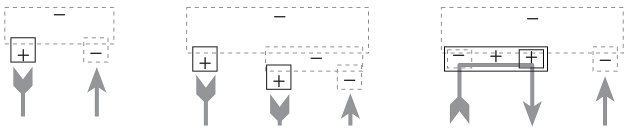
7.4 The grand theorem

Above we provided the generic ingredients for a theorem on compositionality for general entanglement specification networks involving

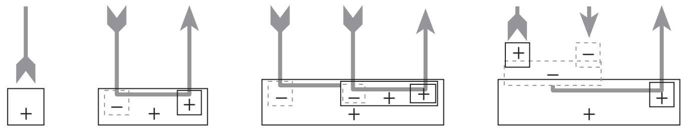
- arbitrary multipartite **eP**'s, and,
- both local and non-local unitary actions.

Formulation of the theorem itself requires some decennia of research in logical syntax, a development which (unfortunately) has happened very much independently of that of the development of tools considered as "relevant" by the majority of the physics community. We state the theorem we have in mind as a conjecture since we will not provide a proof nor an exact statement. We postpone an elaborated presentation to future (co)writings.

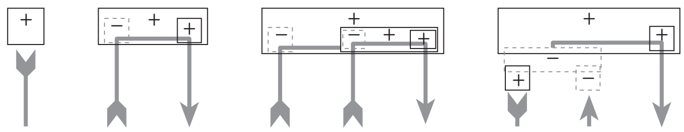
By a *typed output of a network* we mean a bundle of paths of any size which ends without "globally" entering an **eP**. The wiring at a typed output of a network is negative in terms of polarities but this in general does involve real physical **eP**'s e.g.



We moreover allow outputs to be non-local both in space and time. By a *typed input of a network* we mean a bundle of paths of any size which starts in a projector typed in the same syntax as the **eP**'s. The wiring at an input is of course positive e.g.



but we do also allow inputs to point downward e.g.



analogous to output only paths where we effectively specified ϕ^{in} by a projector. Inputs can be conceived as being local in space and time since they constitute a single projector.

Definition 7.5 A *semimultilinear function* is a function

$$f : T_1^{in} \times \ldots \times T_\kappa^{in} \to T^{out}$$

which is either linear or anti-linear in its arguments.

Conjecture 7.6 (Compositionality for arbitrary networks) *Given is a network which contains multipartite* **eP**'s *and both local and non-local unitary operators. Assume that we can define:*

1. *Types for all* **eP**'s;
2. *A subset of these* **eP**'s *of respective types* $T_1^{in}, \ldots, T_\kappa^{in}$ *which we conceive as inputs*;
3. *A subset of these* **eP**'s *which we conceive as output to which we attribute type* T^{out};
4. *A complete family of paths for these* **eP**'s;

Assume further that the following conditions are satisfied:

- *Types match for all bundles of paths and all* **eP**'s;
- *The paths contained in the output which are of a negative signature point upward and end later (in physical time) than all other* **eP**'s *in the network.*

Then the state at this output is not entangled to other carriers and can be expressed as the result of applying a semimultilinear function

$$f : T_1^{in} \times \ldots \times T_\kappa^{in} \to T^{out}$$

to the input values. The multilinear function is obtained via sequential reading of paths from the inputs to the outputs in such a way that the action of a **eP/veP** *consists of producing an output by applying its typed labeling function to its input.*

7.5 Entanglement measures from information flow capabilities

This paper provides an answer to the following question.

Question 7.7 *What is the main capability of entanglement?*

Our (somewhat controversial) answer to this question is

> "It enables information flow between the subsystems."

It is then obvious to *measure entanglement* in terms of its *information flow capabilities.* This principle goes both for bipartite and multipartite entanglement. In particular, whenever a *preorder* on bipartite entanglement is assumed, e.g. *majorozation* [48, 51], we can lift this preorder to one for multipartite entanglement exactly in the same way as we were able to reduce the information flow capabilities for multipartite entanglement to those of bipartite entanglement. Indeed, whenever a context is specified the information flow capabilities are given by a function of type $T_1 \looparrowright T_2$ to which the given preorder applies via the natural isomorphism

$$\left(\bigotimes T_1\right) \looparrowright \left(\bigotimes T_2\right) \;\simeq\; \left(\bigotimes T_1\right) \otimes \left(\bigotimes T_2\right) .$$

Definition 7.8 A (multipartite) entangled state $\Psi \in \bigotimes_i \mathcal{H}_i$ is *above* another entangled state $\Phi \in \bigotimes_i \mathcal{H}_i$ (hence of the same size as Ψ) iff in every context the information flow capabilities of the **eP** P_Ψ are above the information flow capabilities of the **eP** P_Φ.

The reader might want to verify that this definition provides a clear qualitative distinction between the tripartite *GHZ-state* [34] and the *W-state* [25] in terms of their respective information flow capabilities.

The idea of relating measures (of content) to (pre-/partial) orders has been intensively studied in [47]. An *operational* view for producing orders similar to the above one has been proposed by K. Martin and the author for the case of ordering mixed quantum states in terms of their *(static) informative capabilities* [19].

We postpone a detailed study of these matters to future (co)writings.

8 Significance for computing and physics

We conclude this paper by pointing at the significance of our results.

8.1 Practical use

We exposed the *logic of entanglement.* Indeed, logic stands for a way to reason about something, and this paper provides a way to reason about entanglement. That this is a good way of reasoning about entanglement is affirmed by the fact that it allowed us to reproduce many existing protocols in a trivial manner [Subsection 2.3, 3.3 and 6.2] and that it moreover allowed us to produce some useful new ones [Subsection 3.4 and 6.3]. In particular does this logic of entanglement provide a handle to truly tackle applications involving multipartite entanglement [Subsection 6.4, 6.5, 6.6 and 7.5].

More precisely, our logic of entanglement allows a classical functional interpretation for networks containing measurements and unitary transformations on quantum systems [Sections 3 and 7]. By this we mean that we can "read" the action of such a network on an input in terms of an *(acausal) traveling token* which carries the information around the network along paths and is acted on by function boxes. Obviously this strongly simplifies methods of network and protocol design. The crucial step is the passage to unitary transformations and projectors on one-dimensional subspaces as the primitive ingredients of quantum theory — a measurement M should then be conceived as the observer getting informed about which projector P_i in the spectral decomposition

$$M = \sum_i a_i \cdot \mathrm{P}_i$$

actually "took place" [Subsection 2.1].

Conjecture 7.6, the most general result in this paper, has the following shape:

"If a network satisfies bla then we can do bla."

In practice the application of the theorem would be the converse of this, namely, given a *hypercompositional expression* how can we physically realize it using entanglement, possibly with some other constraints attached to it related to efficiency, fault-tolerance etc. We indeed conceive *fault-tolerance* [55, 58] as an important application of the results of this paper. The passage from unitary actions to entanglement specification allows a compilation from sequential to parallel composition [Subsections 3.4, 6.3 and 6.6], preventing accumulation of inaccuracies in long computations. We stress that in the quantum computing literature there has been a growing interest in models of computation which rather rely on measurements than on unitary actions e.g. [16, 56, 33, 50], the motivation for these exactly being fault-tolerance.

Calculating with functions of non-trivial higher order types requires denotational tools such as the λ-*calculus* [7]. On the other hand, it should be clear that our network enables effective interpretation of arbitrary typed (linear) λ-*terms*. Hence a profound argument for an interdisciplinary encounter of computing and physics has been made.

The ultimate output of this paper is the initiation of a two-way compilation scheme which turns a setting of unitary transformations, measurements and classical communication in a family of entanglement specification networks. We illustrated how this tool works for some examples [Subsections 2.3, 3.3, 3.4, 6.2 and 6.3]. The main ingredients were Corollary 3.11 and Corollary 3.12. This job is however not completely finished yet. The correction of "unwanted measurement outcomes" is itself a subject that should be further developed and which might strongly benefit from existing theories such as *quantum error correction* e.g. [32].

8.2 Paper or proposal?

Much of this paper is only an introduction, or, *an invitation*, to a certain line of research. Some of this research is currently ongoing — involving several researchers including myself. The following topics are of major interest to us.

- Analyzing other existing protocols using the results of this paper.
- The design of new protocols using the results of this paper.
- A rigorous mathematical account on the interplay between physical time, progressing paths, complex conjugation and types of paths. A syntax which captures this interplay would be very desirable.
- The study of the connection between our paths and other existing geometric systems for reasoning such as proof-nets e.g. [29, 31, 43]. Some attempts in that direction have already been initiated [24, 38].
- The study of *entanglement specification network rewriting* in analogy the the *term-rewriting* in λ-calculus [7] — as hinted at at the end of the "disentanglement" paragraph in Subsection 6.5.
- An axiomatic understanding on the flow of information through entanglement. This includes a characterization of other mathematical categories besides Hilbert spaces and linear maps which allow the same constructions as the ones of this paper. Also here some attempts have been initiated [4].
- A concise formal presentation of the information flow through multipartite entanglement including a simple elegant proof of "Theorem" 7.6. This should lead to a syntax which automates reasoning about multipartite entanglement.
- A quantitative account on information flow through entanglement and the connection between this and measures of entanglement. This can be pushed towards a qualitative study of entanglement itself — this includes an elaboration on the ideas of Subsection 6.5 on qualitative disentanglement and of Subsection 7.5 on measures of entanglement.

And last, but not least.

- An elaborated study of compilation protocols which translate families of entanglement specification networks into unitary transformations, measurements and classical communication. An account on complexity issues concerning would be desirable. Balancing probabilistic protocols and deterministic ones and the study of the computational merit of our approach would also be desirable.

8.3 On mathematics and physics

We now turn our attention again to the isomorphism

$$\mathcal{H}_1 \to \mathcal{H}_2 \simeq \mathcal{H}_1^* \otimes \mathcal{H}_2$$

between linear maps and the tensor product of finite-dimensional vector spaces — which constitutes the core of this paper. When we say that this is an isomorphism we mean that both are vector spaces and it is with respect to the vector space structure that they are isomorphic. But there is more to $\mathcal{H}_1 \to \mathcal{H}_2$ than merely being a Hilbert/vector space. It is a morphism set in the *category* $\mathbb{C}$ of finite dimensional complex vector spaces with linear functions as morphisms. This means that elements of $\mathcal{H}_1 \to \mathcal{H}_2$ and $\mathcal{H}_2 \to \mathcal{H}_3$ admit a *composition*

$$- \circ - : (\mathcal{H}_1 \to \mathcal{H}_2) \times (\mathcal{H}_2 \to \mathcal{H}_3) \to (\mathcal{H}_1 \to \mathcal{H}_3)\,.$$

The Hilbert space structure of $\mathcal{H}_1 \to \mathcal{H}_2$ is to be conceived as an *enrichment* [14] of the categorical structure. Hence one can ask whether there exists a counterpart for $\mathcal{H}_1^* \otimes \mathcal{H}_2$ to this compositional property of $\mathcal{H}_1 \to \mathcal{H}_2$. The answer is provided in this paper. There is a counterpart to composition of functions in terms of entanglement specification. Indeed, our notion of path for entanglement specification networks realizes a compositional action of the labeling functions on inputs of the path. Hence the answer provided by entanglement specification admits a truly physical implementation. However, mathematically this answer is highly non-trivial since it involves the notions of projection, of before and after (in time), of positive and negative **eP**'s etc. It would be useful to have a better understanding of what is really going on here (mathematically!). This observation adds one more topic to the proposed topics for further research.

Let us also recall that the structural features of quantum theory don't involve probability as an *a priori*. This was well-known to many authors e.g. [21, 39, 54, 59]. In the light of this we point to the fact that the results of this paper didn't involve probability. In particular did we make no mention of *mixed states* and *mixed measurements*. Those would only have blurred our developments.

Conclusion 8.1 *Generality doesn't always serve the essential.*

The primal ingredients in this paper are *unitary transformations* and *projectors*, that is, respectively *isomorphisms* and (closed) *subspaces* of a Hilbert space. This paper shows that such a "restrictive" view can be very beneficial to produce an accurate understanding of the behavior of quantum systems.

Finally, let us stress that the results in this paper, which were initiated in [3], continue to confirm those of [17, 18, 20] in the sense that an understanding of compoundness for quantum systems is not necessarily a matter of so-called subsystem recognition but rather one of producing models of interaction.

8.4 ... and on Hollywood

We can now apply the Hollywood transformation:

$$\textbf{as if} \mapsto \textbf{is}\,.$$

Note that the Hollywood transformation consists of "discarding information", namely $\{\mathrm{a}, \mathrm{f}\}$, and swapping things, namely $\mathrm{s} \leftrightarrow \mathrm{i}$. The Hollywood version of the above then would sound like:

"Time goes backward at the other side of the bridge".

Indeed, using the metaphor of the Einstein-Podolski-Rosen bridge for the fact that entanglement allows instantaneous communication over a large distance, the above teaches us that each passage of a bridge goes with time reversal! Fortunately, the restrictions of the functionality theorem teach us that when going back in time "whatever the future was" doesn't exist anymore since the existence of the plot requires it being completely effectuated. Bye bye Hollywood.

A Hilbert spaces and projectors

Definition A.1 A finite-dimensional *Hilbert space* is a complex vector space $(\mathcal{H}, \mho, \cdot, +, \mathbb{C})$ equipped with an inner product $\langle - | - \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that satisfies

$$\langle \phi \mid c_1 \cdot \psi_1 + c_2 \cdot \psi_2 \rangle = c_1 \langle \phi \mid \psi_1 \rangle + c_2 \langle \phi \mid \psi_2 \rangle$$

$$\langle c_1 \cdot \phi_1 + c_2 \cdot \phi_2 \mid \psi \rangle = \bar{c}_1 \langle \phi_1 \mid \psi \rangle + \bar{c}_2 \langle \phi_2 \mid \psi \rangle$$

$$\overline{\langle \phi \mid \psi \rangle} = \langle \psi \mid \phi \rangle \qquad \langle \phi \mid \phi \rangle = 0 \Rightarrow \phi = \mho \qquad \langle \phi \mid \phi \rangle \geq 0$$

The latter allows us to define a norm on $\mathcal{H}$ as $|-| := \sqrt{\langle - \mid - \rangle}$. We introduce an *orthogonality relation* $\perp \subseteq \mathcal{H} \times \mathcal{H}$ such that $\psi \perp \phi \Leftrightarrow \langle \psi \mid \phi \rangle = 0$ and given a subspace A of $\mathcal{H}$ its *orthocomplement* is

$$A^{\perp} := \{\phi \in \mathcal{V} \mid \forall \psi \in A : \psi \perp \phi\}\,.$$

Every finite-dimensional complex vector space extends to a Hilbert space via choice of an inner product. An orthonormal base is a set of vectors $\{e_i\}_i$ such that $\langle e_i \mid e_j \rangle = \delta_{ij}$ where δ_{ij} is the *Kronecker delta function*, that is,

$$\delta_{ij} = 1 \ \text{ when } \ i = j \ \text{ and } \ \delta_{ij} = 0 \ \text{ when } \ i \neq j\,.$$

In much of the study of Hilbert spaces the canonical morphisms are still linear maps

$$f(c_1 \cdot \psi_1 + c_2 \cdot \psi_2) = c_1 \cdot f(\psi_1) + c_2 \cdot f(\psi_2)$$

ignoring the in-product part of the Hilbert space structure. For a linear map $f : \mathcal{H}_1 \to \mathcal{H}_2$ and orthonormal bases $\{e_\alpha\}_\alpha$ and $\{e_\beta\}_\beta$ respectively of $\mathcal{H}_1$ and $\mathcal{H}_2$ we have

$$f = \sum_\alpha f_{\alpha\beta} \langle e_\alpha \mid - \rangle \cdot e_\beta \quad \text{ given that } \quad f(e_\alpha) = f_{\alpha\beta} \cdot e_\beta\,.$$

We then have for $\psi = \sum_\alpha \psi_\alpha \cdot e_\alpha$ that

$$\begin{aligned} f\Big(\sum_\alpha \psi_\alpha e_\alpha\Big) &= \sum_\alpha \psi_\alpha \cdot f(e_\alpha) \\ &= \sum_\alpha \psi_\alpha f_{\alpha\beta} \cdot e_\beta \end{aligned}$$

resulting in

$$(f(\psi))_\beta = \sum_\alpha \psi_\alpha f_{\alpha\beta}$$

and also

$$(g \circ f)_{\alpha\beta} = \sum_i f_{\alpha i} g_{i\beta} \,.$$

Definition A.2 A linear map $f^\dagger : \mathcal{H}_2 \to \mathcal{H}_1$ is *adjoint* to a linear map $f : \mathcal{H}_1 \to \mathcal{H}_2$ iff

$$\langle \psi \mid f(\phi) \rangle = \langle f^\dagger(\psi) \mid \phi \rangle$$

for all $\phi \in \mathcal{H}_1$ and all $\psi \in \mathcal{H}_2$.

These adjoints relate to usual Galois adjoints in the following way. If one considers the pointwise extension of a linear function to the orthocomplemented lattice of closed subspaces $\mathbb{L}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ then each linear map f induces a map $\tilde{f}$ on these lattices. These induced maps preserve arbitrary joins. One verifies that the extension of $f^\dagger$ to these lattices is given by $(-)' \circ \tilde{f}^* \circ (-)'$ where $(-)'$ stands for orthocomplementing and $\tilde{f}^*$ is the right Galois adjoint of $\tilde{f}$ — which preserves arbitrary meets. For details see for example [28]. Adjoints make the category of finite-dimensional Hilbert spaces and linear maps self-dual.

Proposition A.3 *For a linear function $f : \mathcal{H}_1 \to \mathcal{H}_2$ we have:*

1. *Adjoints always exist and are unique.*
2. *Adjunction is involutive.*
3. *The matrix of $f^\dagger$ is the complex conjugated transposed matrix of that of f.*
4. *f is surjective iff $f^\dagger$ is injective and vice versa.*
5. *If $g : \mathcal{H}_2 \to \mathcal{H}_3$ is also linear then $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.*

Proof. Setting

$$\phi = \sum_j \phi_j \cdot e_j^{(1)} \in \mathcal{H}_1 \qquad \text{and} \qquad \psi = \sum_i \psi_i \cdot e_i^{(2)} \in \mathcal{H}_2$$

it follows that

$$f(\phi) = \sum_{jk} \phi_j f_{jk} \cdot e_i^{(2)} \in \mathcal{H}_2$$

and thus we have

$$\begin{aligned}
\langle \psi \mid f(\phi) \rangle &= \sum_{ijk} \bar{\psi}_i \phi_j f_{jk} \langle e_i^{(1)} \mid e_k^{(2)} \rangle \\
&= \sum_{ijk} \bar{\psi}_i \phi_j f_{jk} \delta_{ik} \\
&= \sum_{ij} \bar{\psi}_i f_{ji} \phi_j \\
&= \langle \sum_i \psi_i \bar{f}_{ji} \mid \phi \rangle \\
&= \langle f^\dagger(\psi) \mid \phi \rangle \,.
\end{aligned}$$

so we have $(f^\dagger)_{ij} = \bar{f}_{ji}$. Moreover, the above equations uniquely define the mutual adjoints when taking as ψ each of the base vectors $\{e_i^{(2)}\}_i$ and as ϕ each of the base vectors $\{e_j^{(1)}\}_j$. Further we have that f is surjective

$$\begin{aligned}
&\Leftrightarrow \ \text{Range}(f) = \mathcal{H}_2 \\
&\Leftrightarrow \ (\text{Range}(f))^\perp = \mho \\
&\Leftrightarrow \ \forall \psi \in \mathcal{H}_2 \setminus \{\mho\}, \exists \phi \in \mathcal{H}_1 : \langle \psi \mid f(\phi) \rangle \neq 0 \\
&\Leftrightarrow \ \forall \psi \in \mathcal{H}_2 \setminus \{\mho\}, \exists \phi \in \mathcal{H}_1 : \langle f^\dagger(\psi) \mid \phi \rangle \neq 0 \\
&\Leftrightarrow \ \forall \psi \in \mathcal{H}_2 \setminus \{\mho\} : f^\dagger(\psi) \neq \mho
\end{aligned}$$

that is if $f^\dagger$ is injective. The shape that the adjoint of composed functions takes follows by the matrix representation. □

Definition A.4 A linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a *unitary operator* if it preserves the inner product, that is

$$\langle U(\psi) \mid U(\phi) \rangle = \langle \psi \mid \phi \rangle$$

for all $\phi, \psi \in \mathcal{H}_1$.

Proposition A.5 *Every unitary operator has the following properties:*

1. *U preserves orthogonality.*
2. *U has an inverse U^{-1}.*
3. *$U^\dagger = U^{-1}$.*

Proof. The first two statements are obvious so we only prove the third one. We have

$$\begin{aligned}\langle\psi \mid U(\phi)\rangle &= \langle(U\circ U^{-1})(\psi) \mid U(\phi)\rangle \\ &= \langle U(U^{-1}(\psi)) \mid U(\phi)\rangle \\ &= \langle U^{-1}(\psi) \mid \phi\rangle\end{aligned}$$

What completes the proof. □

Definition A.6 A *projector* is an idempotent self-adjoint linear endomap $\mathrm{P} : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$, that is respectively, for $\psi, \phi \in \mathcal{H}$,

$$\mathrm{P}(\mathrm{P}(\psi)) = \mathrm{P}(\psi) \quad \text{and} \quad \langle \mathrm{P}(\psi) \mid \phi\rangle = \langle\psi \mid \mathrm{P}(\phi)\rangle\,.$$

As an example, given a unit vector $\psi \in \mathcal{H}$, that is $|\psi| = 1$, the map

$$\mathrm{P}_\psi : \mathcal{H} \to \mathcal{H} :: \phi \mapsto \langle\psi \mid \phi\rangle \cdot \psi$$

defines a projector. The fixed points of P_ψ are the vectors in the one-dimensional subspace spanned by ψ. More general, for a finite-dimensional Hilbert space, the set of all projectors $\mathbb{P}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ is in bijective correspondence with the set of all subspaces $\mathbb{L}(\mathcal{H})$ of that Hilbert space, the corresponding subspace then being the projector's set of fixed points. When ordering these subspaces by inclusion, this bijective correspondence

$$\mathbb{L}(\mathcal{H}) \simeq \mathbb{P}(\mathcal{H})$$

lifts to one between complete lattices when ordering the projectors according to

$$\mathrm{P} \le \mathrm{Q} \iff \mathrm{Q}\circ\mathrm{P} = \mathrm{P}\,.$$

Definition A.7 Two projectors P and Q are (mutually) *orthogonal* iff, equivalently,

- $\mathrm{P}\circ\mathrm{Q}$ is the identical map with range $\mho$;
- The respective sets A_P and A_Q of fixed points of P and Q are orthogonal.

In that case we write $\mathrm{P} \perp \mathrm{Q}$. We denote by $\mathrm{P}^\perp$ the (unique) orthocomplement in $\mathbb{P}(\mathcal{H})$ to P, that is, the projector of which the subspace of fixed points is the orthocomplement in $\mathbb{L}(\mathcal{H})$ to the subspace of fixed points of P.

Theorem A.8 (Finitary spectral decomposition) *Every self-adjoint operator $M : \mathcal{H} \to \mathcal{H}$ decomposes as a linear combination of projectors:*

$$M = \sum_{x\in\sigma(M)} x \cdot \mathrm{P}_x$$

where $\sigma(M)$ is the spectrum of M, that is, the set of its eigenvalues and where that set $\{\mathrm{P}_x \mid x \in \sigma(M)\}$ satisfies

$$x \neq y \Rightarrow \mathrm{P}_x \perp \mathrm{P}_y\,.$$

As usual in linear algebra, given a self-adjoint operator, via diagonalization one obtains the eigenvalues, that is the spectrum $\sigma(M)$, and the corresponding eigenspaces

$$\mathcal{A}(M) := \{A_x \mid x \in \sigma(M)\}\,.$$

The projector P_x for $x \in \sigma(M)$ in the spectral decomposition theorem is then the one which has as fixed points $A_x \in \mathcal{A}(M)$.

Definition A.9 A map f between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is *anti-linear* iff

$$f(c_1 \cdot \psi_1 + c_2 \cdot \psi_2) = \bar{c}_1 \cdot f(\psi_1) + \bar{c}_2 \cdot f(\psi_2)\,.$$

We specify that a map is anti-linear by denoting its type by $f : \mathcal{H}_1 \looparrowright \mathcal{H}_2$.

The linear maps $f : \mathcal{H} \to \mathbb{C}$, that is, the *linear functionals* $\mathcal{H}^*$ associated to $\mathcal{H}$, constitute a vector space isomorphic to $\mathcal{H}$. However, there is in general no canonical isomorphism that connects them. Indeed, since we have $\langle c \cdot \psi \mid - \rangle = \bar{c}\,\langle \psi \mid - \rangle$ the canonical correspondence is anti-linear instead of linear. Thus, given a base $\{e_i\}_i$ of $\mathcal{H}$, specification of an isomorphism as

$$h : \mathcal{H} \looparrowright \mathcal{H}^* :: e_i \mapsto \langle e_i | - \rangle$$

depends on the choice of the base. A functional $\langle \psi \mid - \rangle : \mathcal{H} \to \mathbb{C}$ defines a projector

$$\mathrm{P}_\psi : \left\langle \frac{\psi}{|\psi|} \,\middle|\, - \right\rangle \cdot \frac{\psi}{|\psi|} :: \mathcal{H} \to \mathcal{H}$$

via composition with the injection

$$\iota_\psi : \mathbb{C} \to \mathcal{H} :: c \mapsto \frac{c \cdot \psi}{|\psi|^2}\,.$$

Any pair of complex vector spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ admits a tensor product, that is, a pair consisting of a vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and a bilinear map $h : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ such that for any other bilinear map $f : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_3$ there exists a unique $g : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3$ with $f = g \circ h$. This tensor product equips the category of finite-dimensional complex vector spaces and linear maps with a monoidal structure with $\mathbb{C}$ as unit, since given

$$h : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 \quad \text{and} \quad h' : \mathcal{H}_3 \times \mathcal{H}_4 \to \mathcal{H}_3 \otimes \mathcal{H}_4\,,$$

and two linear maps $f : \mathcal{H}_1 \to \mathcal{H}_3$ and $g : \mathcal{H}_2 \to \mathcal{H}_4$, their tensor product

$$f \otimes g : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3 \otimes \mathcal{H}_4$$

is uniquely defined due to universality of h with respect to $h' \circ (f \times g)$. We can construct a tensor product for vector spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as

$$h : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 :: (e_\alpha, e_\beta) \mapsto e_\alpha \otimes e_\beta$$

with $\mathcal{H}_1 \otimes \mathcal{H}_2$ being the vector space spanned by $\{e_\alpha \otimes e_\beta\}_{\alpha\beta}$. Identifying

$$(\sum_\alpha \psi_\alpha \cdot e_\alpha) \otimes (\sum_\beta \phi_\beta \cdot e_\beta) \quad \text{and} \quad \sum_{\alpha\beta} \psi_\alpha \phi_\beta \cdot (e_\alpha \otimes e_\beta)\,,$$

this construction does not depend on the choice of orthonormal base, in particular, for arbitrary $\psi \in \mathcal{H}_1$ and $\phi \in \mathcal{H}_2$ we have that

$$h(\psi, \phi) = \psi \otimes \phi\,.$$

Proposition A.10 *The above construction of a vector space tensor product induces a bijective correspondence between bilinear maps and linear maps that is*

$$(\mathcal{H}_1 \times \mathcal{H}_2) \to \mathcal{H}_3 \quad \simeq \quad \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3\,.$$

Proof. Each linear map $f : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3$ induces a bilinear map

$$g = f \circ h : (\mathcal{H}_1 \times \mathcal{H}_2) \to \mathcal{H}_3 :: (\phi, \psi) \mapsto f(\phi \otimes \psi)$$

since h is bilinear and f is linear. Conversely, each bilinear map $g : (\mathcal{H}_1 \times \mathcal{H}_2) \to \mathcal{H}_3$ induces a linear map $f : (\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{H}_3$ by the universal property in the definition of the vector space tensor product. $\square$

We can define an inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ via

$$\langle \psi \otimes \psi' \mid \phi \otimes \phi' \rangle := \langle \psi \mid \phi \rangle \langle \psi' \mid \phi' \rangle\,,$$

hence

$$\begin{aligned}\Big\langle \sum_{\alpha\beta} f_{\alpha\beta} \cdot (e_\alpha \otimes e_\beta) \,\Big|\, \sum_{\gamma\lambda} g_{\gamma\lambda} \cdot (e_\gamma \otimes e_\lambda) \Big\rangle &= \sum_{\alpha\beta\gamma\lambda} \bar{f}_{\alpha\beta} g_{\gamma\lambda} \langle e_\alpha \mid e_\gamma \rangle \langle e_\beta \mid e_\lambda \rangle \\ &= \sum_{\alpha\beta} \bar{f}_{\alpha\beta} g_{\alpha\beta}\,.\end{aligned}$$

The following is obvious.

Proposition A.11 $\Psi_1 \otimes \Psi_2 \perp \Phi_1 \otimes \Phi_2 \iff \Psi_1 \perp \Phi_1$ <u>or</u> $\Psi_2 \perp \Phi_2$.

It then also follows that when both $\{e_\alpha\}_\alpha$ and $\{e_\beta\}_\beta$ are orthonormal bases then $\{e_\alpha \otimes e_\beta\}_{\alpha,\beta}$ is again orthonormal with respect to this inner product.

The general form of elements of $\mathcal{H}_1^* \otimes \mathcal{H}_2$ and $\mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ respectively is

$$\sum_{\alpha\beta} f_{\alpha\beta} \cdot (\langle e_\alpha \mid - \rangle \otimes e_\beta) \qquad \text{and} \qquad \sum_{\alpha\beta} f_{\alpha\beta} \, \langle e_\alpha \mid - \rangle \cdot e_\beta$$

and thus we obtain an isomorphism of vector spaces when providing the set $\mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ with its canonical vector space structure. Note also that we have

$$\mathcal{H}_1^* \otimes \mathcal{H}_2^* \;\simeq\; (\mathcal{H}_1 \otimes \mathcal{H}_2)^*$$

via identification of $\varepsilon_\alpha \otimes \varepsilon_\beta$, with $\{\varepsilon_\alpha : \mathcal{H}_1 \to \mathbb{C}\}_\alpha$ and $\{\varepsilon_\beta : \mathcal{H}_1 \to \mathbb{C}\}_\beta$ respective bases for $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$, and the unique functional

$$\varepsilon_\alpha * \varepsilon_\beta : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathbb{C}$$

that arises due to universality of h within

$$\begin{tikzcd}
\mathcal{H}_1 \otimes \mathcal{H}_2 \arrow[r, "\varepsilon_\alpha * \varepsilon_\beta"] & \mathbb{C} \\
\mathcal{H}_1 \times \mathcal{H}_2 \arrow[u, "h"] \arrow[ur, "\varepsilon_\alpha \cdot \varepsilon_\beta"'] &
\end{tikzcd}$$

where

$$\varepsilon_\alpha \cdot \varepsilon_\beta : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C} :: (\psi, \phi) \mapsto \varepsilon_\alpha(\psi)\varepsilon_\beta(\phi) .$$

The definition for an inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ then embodies this fact as

$$\langle \psi \otimes \phi \mid - \otimes - \rangle := \langle \psi \mid - \rangle \langle \phi \mid - \rangle$$

when expressing functionals in terms of the inner product.

B Quantum theory

We restrict our discussion to elementary (superselection free) finite-dimensional quantum theory. As a general reference we propose [49, 37]. As more structurally focused references we recommend [39, 54, 59]. For an overview of some current abstract perspectives we refer to [21]. For an overview on quantum information processing we refer to [15]. Let $\mathcal{H}$ be a finite-dimensional (complex) Hilbert space. The description of a quantum system constitutes:

1. Description of the states of the system = kinematics.
2. The description of evolution = reversible dynamics.
3. The description of measurements = the "new" irreversible observational feature.

The states of a quantum system *which is not in the presence of any other quantum system* has the set $\Sigma(\mathcal{H})$ of one-dimensional subspaces of a Hilbert space $\mathcal{H}$ as formal counterpart. In other words, a state of a quantum system is represented by a vector $\psi \in \mathcal{H}$ up to normalization and phase factor. So $\psi, \phi \in \mathcal{H}$ represent the same state whenever

$$\psi = c \cdot \phi \quad \text{with} \quad c \in \mathbb{C}_0 .$$

For calculations it is obviously easier to represent states as vectors while structural considerations benefit from the subspace perspective.

Measurements are represented by self-adjoint operators $M : \mathcal{H} \to \mathcal{H}$. When performing a measurement on the system in state ψ (a notational abuse that we from now on will use freely) where the corresponding self-adjoint operator has $\sigma(M)$ as its spectrum of eigenvalues then we obtain as outcome of the measurement a value $x \in \sigma(M)$ with corresponding probability

$$\mathrm{Prob}_x^H(\psi) = \langle \psi \mid \mathrm{P}_x(\psi) \rangle = |\mathrm{P}_x(\psi)|^2 ,$$

where ψ is assumed to be normalized and P_x is the projector on the subspace A_x of eigenvectors with eigenvalue x, that is, the projector which exactly has A_x as fixed points. Note that

$$\sum_{x\in\sigma(H)} \mathrm{P}_x(\psi) = \psi \qquad \text{so} \qquad \sum_{x\in\sigma(H)} \mathrm{Prob}_x^H(\psi) = 1$$

since all eigenspaces

$$A_x = \{\psi \in \mathcal{H} \mid \mathrm{P}_x(\psi) = \psi\}$$

are mutually orthogonal and span $\mathcal{H}$.

Definition B.1 We call a quantum measurement *finitary* if its spectrum is finite. We call a quantum measurement *non-degenerated* if the size of its spectrum is equal to the dimension of the Hilbert space.

Sequential measurements obey von Neumann's projection postulate [49].

Postulate B.2 *If a measurement yields $x \in \sigma(H)$ as outcome then the state of the system changes from its initial state ψ to $\mathrm{P}_x(\psi)$; thus an immediate next measurement gives again x as outcome — since $\mathrm{P}_x(\psi)$ is itself an eigenvector with eigenvalue x.*

It follows projectors encode true state transitions, explicitly

$$\varphi_x : \Sigma(\mathcal{H}) \setminus K \to \Sigma(\mathcal{H}) :: ray(\psi) \mapsto ray(\mathrm{P}_x(\psi))$$

where

$$K = \{ray(\psi) \in \Sigma(\mathcal{H}) \mid \mathrm{Prob}_x^H(\psi) = 0\} .$$

So far we conceived projectors either as components of measurements or as representative for subspaces. However, any projector, being self-adjoint, encodes itself a measurement with spectrum $\sigma(\mathrm{P}) = \{0, 1\}$ and eigenspaces $A_0 \perp A_1$ and probabilities

$$\mathrm{Prob}_1^{\mathrm{P}}(\psi) = |\mathrm{P}(\psi)|^2 \quad \text{and} \quad \mathrm{Prob}_0^{\mathrm{P}}(\psi) = 1 - |\mathrm{P}(\psi)|^2 .$$

In view of the spectral decomposition theorem (Theorem A.8 in Appendix A) they can actually be seen as primitive since all other measurements encode as linear combinations of them. In observational terms they correspond to *yes/no*-questions about the system. As an example consider polarization of photons. Let Z be the axis of propagation. Consider as projector a light analyzer that allows only vertically polarized light to pass, say polarized along the X-axis. If the in-coming light is polarized along the X-axis it passes (outcome 1). If the in-coming light is polarized along the Y-axis nothing passes (outcome 0). If it is polarized along an axis that makes an angle θ with the X-axis then some light will pass, with relative amplitude $\cos^2\theta$ and the light that passed will be vertically polarized. The amplitude reflects the quantum probability to pass that is to obtain an outcome 1 of the projector. The change of polarization angle from θ to 0 is then the transition according to the projection postulate.

But there is however more to this. The primitive propositions for a classical system are the subsets of the state space Σ. Indeed, for each observable $f : \Sigma \to \mathbb{R}$ of a classical system (e.g. energy, position, momentum) the *property*

$$f^{-1}[E] \subseteq \Sigma \quad \text{for} \quad E \subseteq \sigma(f) := Range(f)$$

expresses for a state that:

> "The value of observable f is contained in $E \subseteq \sigma(f)$".

It follows that all subsets of Σ potentially encode a property of that classical system. In the quantum theory we have that all statements of the kind become:

> "The value of observable M is contained in $E \subseteq \sigma(H)$"

and can be represented by

$$\text{the projector} \quad \mathrm{P}_E^H := \sum_{x\in E} \mathrm{P}_x^H \quad \text{with} \quad \bigoplus_{x\in E} A_x \quad \text{as fixpoints.}$$

Therefore, it follows that the subspaces $\mathbb{L}(\mathcal{H})$ encode the physical properties attributable to a quantum system. The projectors $\mathbb{P}(\mathcal{H})$ can then be conceived as the observational verifications of properties. It can then be argued that the state space of a quantum structure goes naturally equipped with the structure $\mathbb{L}(\mathcal{H})$, that is, the complete lattice of all the properties the system can posses and which has the system's states as its atoms. Also the orthogonality relation on $\mathcal{H}$ which pointwisely extends to $\mathbb{L}(\mathcal{H})$ has an observational counterpart. Two subspaces $B, C \subseteq \mathcal{H}$ are orthogonal if there exists a projector P such that in case of verification we obtain a positive answer whenever the system possesses property B and a negative answer whenever the system possesses property C. Formally this means $B \subseteq A_1$ and $C \subseteq A_1$. Thus orthogonality encodes a notion of distinguishability. From the above it follows that the reversible dynamics of a quantum systems is *structure preserving*. It preserves both $\mathbb{L}(\mathcal{H})$ and the orthogonality relation. For details on unitary dynamics we refer to the relevant literature. We choose to keep this paper focussed on structural non-numerical issues.

A bipartite compound quantum system is described in the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the Hilbert spaces in which we describe the respective subsystems. Whereas in classical physics two systems are described by pairing states — the cartesian product — in quantum theory we also have to consider superpositions of such pairs. Examples of projectors on $\mathcal{H}_1 \otimes \mathcal{H}_2$ are those of the form $\mathrm{P}_1 \otimes \mathrm{P}_2$, explicitly definable as

$$(\mathrm{P}_1 \otimes \mathrm{P}_2)(\psi \otimes \phi) = \mathrm{P}_1(\psi) \otimes \mathrm{P}_2(\phi)\,.$$

Even though the values for general self-adjoint operators of this form should be conceived as pairs

$$(x_1, x_2) \in \sigma(M_1) \times \sigma(M_2)$$

with corresponding probabilities

$$\mathrm{Prob}_{(x_1,x_2)}^{M_1\otimes M_2}(\Psi) = \langle \Psi \mid (\mathrm{P}_{x_1} \otimes \mathrm{P}_{x_2})(\Psi)\rangle\,,$$

projectors compose conjunctively under $\otimes$ that is

$$(0,1) \sim (0,0) \sim (1,0) \sim 0 \quad \text{and} \quad (1,1) \sim 1\,.$$

Other examples of projectors on $\mathcal{H}_1 \otimes \mathcal{H}_2$ are those of the shape

$$\mathrm{P}_\Psi = \langle \Psi \mid - \rangle \cdot \Psi$$

where Ψ cannot be written as a pure tensor.

Definition B.3 A(n) (anti-)linear [Definition A.9] operator

$$f : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$$

is called *non-local* if there exist no (anti-)linear operators

$$f_1 : \mathcal{H}_1 \to \mathcal{H}_1 \qquad \text{and} \qquad f_2 : \mathcal{H}_2 \to \mathcal{H}_2$$

such that

$$f \equiv f_1 \otimes f_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2\,.$$

For the particular case of projectors we have that if $\Psi = \Psi_1 \otimes \Psi_2$ then

$$\begin{aligned}
\mathrm{P}_\Psi &= \langle \Psi_1 \otimes \Psi_2 \mid - \rangle \cdot \Psi_1 \otimes \Psi_2 \\
&= \langle \Psi_1 \mid - \rangle \langle \Psi_2 \mid - \rangle \cdot \Psi_2 \otimes \Psi_2 \\
&= \langle \Psi_1 \mid - \rangle \cdot \Psi_1 \,\otimes\, \langle \Psi_2 \mid - \rangle \cdot \Psi_2 \\
&= \mathrm{P}_{\Psi_1} \otimes \mathrm{P}_{\Psi_2}\,.
\end{aligned}$$

Hence we have the following.

Proposition B.4 $\mathrm{P}_{\Psi_1\otimes\Psi_2} = \mathrm{P}_{\Psi_1} \otimes \mathrm{P}_{\Psi_2}$.

One verifies that if Ψ is not a pure tensor it cannot be written as a tensor of projectors. It then follows most projectors in this paper are of this so-called non-local kind. The term non-local should in this context not necessarily be conceived in space-like terms. Non-local unitary operations are considered in quantum control theory, quantum computation and quantum information [10] when the system evolves according to a so-called non-local Hamiltonian i.e.

$$U(t) = e^{\frac{i}{\hbar}Ht} \neq U_1(t) \otimes U_2(t) = e^{\frac{i}{\hbar}H_1 t} \otimes e^{\frac{i}{\hbar}H_2 t}\,.$$

They allow to obtain non-local projectors as

$$U \circ (\mathrm{P}_1 \otimes \mathrm{P}_2) \circ U^{-1} \quad \text{when} \quad U \neq U_1 \otimes U_2 \,.$$

Any projector P which projects on a one-dimensional subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be obtained in that way as

$$U \circ \mathrm{P}_{\psi_1 \otimes \psi_2} \circ U^{-1}$$

taking U such that $U^{-1}(\psi_1 \otimes \psi_2)$ is a fixed point of P since

$$\mathrm{P}_{\psi_1 \otimes \psi_2} = \mathrm{P}_{\psi_1} \otimes \mathrm{P}_{\psi_1} \,.$$

Note that one cannot obtain arbitrary projectors in this way due to the simple fact that the dimension of the projector on the global space should factor in a product of the dimensions of the underlying ones, e.g a projector on a 5-dimensional subspace (with 7-dimensional orthocomplement) in case of $dim(\mathcal{H}_1) = 3$ and $dim(\mathcal{H}_2) = 4$.

C Linear logic and its categorical semantics

As compared to the previous two appendices this one is not crucial for understanding this paper. It however provides a perspective which places the results of this paper in a very different light and might be useful for further elaborations.

One could wonder why linear logic was discovered in a categorical logic and a proof theory context and not in physics, the machinery of quantum mechanics being much older than either category theory or the tools of modern proof theory — the categorical semantics of linear logic was discovered in 1968 by Lambek [44] and its full-blown syntax was introduced by Girard in 1987 [29]. The main idea behind linear logic is that the *structural rules* of *weakening* and *contraction* seize to hold and the motivation for doing so is to achieve an accountancy of the available resources. An example in physics of such a resource sensitive behavior is captured by the *no-cloning* and the *no-deleting* principles with respect to tensored states [42, 52, 60]. These two principles do not require any sophisticated mathematics in their derivations, they only use some very basic linear algebra. However, the logical content of these principles seems not to have been fully exposed nor does this logical content seem to have caused a lot of interest so far.

In terms of Gentzen's sequence calculus the structural rules of contraction and weakening respectively stand for

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \qquad\qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B} \,.$$

Respectively setting $\Gamma := \emptyset$ and $\Gamma := A$ we obtain

$$\frac{A, A \vdash B}{A \vdash B} \qquad\qquad \frac{A \vdash B}{A, A \vdash B} \,.$$

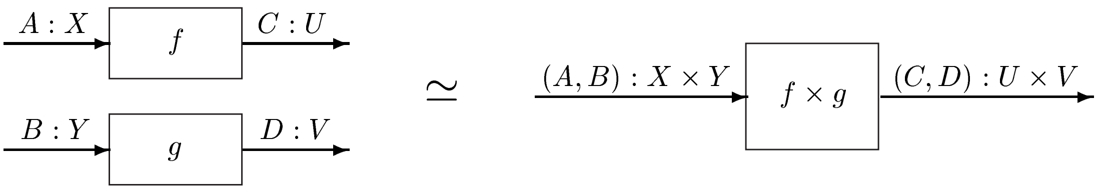
These rules should be read as follows:

- If we can derive B from a pair of A's then we can also derive B from a single A.
- If we can derive B from a single A then we can also derive B from a pair of A's.

Hence they respectively express the ability to delete and copy propositions in the assumptions of a derivation. If they seize to hold it means that the number of occurrences of a proposition becomes essential for the ability to perform a derivation. Girard's justification for this is that a certain derivation might *consume* its assumptions while it *produces* its results. The general full blown version of linear logic however admits a full and faithful embedding of classical non-linear logic by introducing a new operation $!(-)$. For a proposition A the expression $!A$ relieves A from its linear constraints, that is, we are free to either use A any number of times or not to use it at all. Hence linear logic should not be seen as a modification of classical logic but as a refinement of it.

The reasoning below is much influenced by [2]. We conceive linear logic in terms of the categorical semantics of its multiplicative fragment which preceded its full-blown syntactic version by two decades [44].

Recall that a *categorical product* $-\underline{\times}-$ captures the notion of *pairing* and *de-pairing* which is typical for the Cartesian product $- \times -$.



It does this by only referring to morphisms. Indeed, $-\underline{\times}-$ goes equipped with two de-pairing *projections*

$$p_A^{A,B} : A\underline{\times}B \to A \qquad \text{and} \qquad p_B^{A,B} : A\underline{\times}B \to B$$

which in the symmetric case both yield

$$p_A := p_A^{A,A} : A\underline{\times}A \to A$$

and a pairing *bracket*

$$[-,-] : (C \to A) \times (C \to B) \longrightarrow C \to A\underline{\times}B$$

which for

$$id_A : A \to A$$

provides a diagonal

$$\Delta_A := [id_A, id_A] : A \to A\underline{\times}A\,.$$

When conceiving morphisms $f : A \to B$ as derivations (=proofs) of B from A then p_A and Δ_A respectively enable contraction and weakening. In order to obtain a category which makes sense as a semantics for logical derivations/proofs we assume some additional rules for the *conjunction* $-\underline{\times}-$ and the *implication* $- \to -$ and it is well-known that this results in Lawvere's *Cartesian closed categories* [46].

There exists a generalization of these Cartesian closed categories, namely *symmetric monoidal closed categories*, which (roughly stated) are Cartesian closed categories "without pairing and de-pairing morphisms". They still have a tensor $-\otimes-$ and an implication $- \multimap -$ which behave as in ordinary (intuitionistic) logic except for the inability to copy and delete propositions with respect to these generalized connectives. The absence of copying and deleting implies that in general $- \otimes -$ is not the categorical product (if there is any). Hence these symmetric monoidal closed categories provide an appropriate categorical semantics for linear logic.

Of course these symmetric monoidal closed categories include Cartesian closed categories as a special case, namely those for which the tensor $- \otimes -$ coincides with the categorical product. That is, for a Cartesian closed category we have

$$- \otimes - := -\underline{\times}- \,.$$

There is however another kind of special symmetric monoidal closed categories, namely *compact closed categories*, that is, those for which we have

$$- \otimes - := (-)^* \multimap -$$

where * turns the symmetric monoidal closed category into a $*$-autonomous one [8]. In this case the tensor represents *functions*. Note that this is exactly what we have for $\mathbb{C}$, the category of finite dimensional Hilbert spaces and linear maps. More importantly this property is what allowed us to produce all the results of this paper.

The bottom line is then that in the *universe of symmetric monoidal closed categories* it seems that the *classically* behaving ones are the Cartesian closed ones while the *quantum-like* behaving ones are the compact closed ones. To put this *boldly*,

- "*Classical logic*" stands for the tensor encoding "pairing".
- "*Quantum logic*" stands for the tensor encoding "functionality".

In this view classical logic is not a limit of quantum logic but these two both are extreme cases of a more general kind of logic. This view is radically different from the usual view on quantum logic in which classical logic encodes as the Boolean limit of the non-distributive lattices which encode quantum-like logic e.g. [21].

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