Programming Research Group

TRACTABLE CONSERVATIVE CONSTRAINT SATISFACTION PROBLEMS

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Abstract

In a constraint satisfaction problem (CSP) the aim is to find an assignment of values to a given set of variables, subject to specified constraints. The CSP is known to be NP-complete in general. However, certain restrictions on the form of the allowed constraints can lead to problems solvable in polynomial time. Such restrictions are usually imposed by specifying a constraint language. The principal research direction aims to distinguish those constraint languages which give rise to tractable CSPs from those which do not.

We achieve this goal for the important version of the CSP, in which the set of values for each individual variable can be restricted arbitrarily. Restrictions of this type can be expressed by including in a constraint language all possible unary constraints. Constraint languages containing all unary constraints will be called *conservative*. We completely characterise conservative constraint languages that give rise to CSP classes polynomial time solvable. In particular, this result allows us to obtain a complete description of those (directed) graphs H for which the List H-Coloring problem is solvable in polynomial time. The result, the solving algorithm, and the proofs heavily use the algebraic approach to CSP developed in [33, 31, 13, 8, 9].

1 Introduction

In a Constraint Satisfaction Problem (CSP) the aim is to find an assignment of values to a given set of variables, subject to constraints on the values which can be assigned simultaneously to certain specified subsets of variables. A CSP can also be expressed as the problem of deciding whether a given conjunctive formula has a model, or as the problem of deciding whether there exists a homomorphism between two relational structures.

CSP provides a generic framework to various combinatorial problems appearing in computer science and artificial intelligence, including the propositional satisfiability problem, in which the variables must be assigned Boolean values, graph-theoretical problems, scheduling problems, temporal and spatial reasoning, database theory, etc.

The general CSP is NP-complete [39]. However, many practical and theoretical problems can be expressed in terms of CSP using constraints of a certain restricted form. Such restricted CSPs often appear to be tractable (i.e., in **PTIME**). Constraints are usually specified by relations, and the constraint satisfaction problem can therefore be restricted by specifying a *constraint language*, that is a set of relations that are allowed to be used as constraints. In the homomorphism form of the CSP this is equivalent to restricting of a possible form of the target structure. The problem of distinguishing those constraint languages which give rise to a tractable problems from those which do not is acknowledged to be very deserving and attracts much attention (see e.g., [3, 16, 26]). We shall call this problem the *classification problem*.

Boolean constraint satisfaction problem is equivalent to the propositional Satisfiability problem, and provides one of the most important and the first investigated particular cases of the CSP. In this case the classification problem mentioned above is completely solved for the standard CSP [44, 36], and for various related problems (see

[16, 11] for a survey and further references). In [3], this problem was solved for CSPs over a 3-element set of values.

The analogous problem in the case of a bigger domain is believed to be very hard and requires more advanced approaches. Several such approaches based on logic, algebra, graph theory, and databases have been developed [3, 9, 8, 13, 17, 26, 29, 30, 33, 31, 36]. In spite of substantial progress achieved during the last decade, the problem withstands all attacks.

Remarkably, all studied CSP classes turn out to be either tractable or NP-complete. Dichotomy results of this type are of particular interest in the study of CSP, because, on the one hand, they determine the precise complexity of a problem, and on the other hand, the a priory existence of a dichotomy result cannot be taken for granted. In [26], it was conjectured that a dichotomy theorem holds for the complexity of an arbitrary restricted CSP, and in [13], a possible criteria for tractable problems has been suggested. For a short survey of dichotomy results the reader is referred to [16].

In this paper we study the classification problem for the important and widely used variant of the CSP, in which the set of values for each individual variable can be restricted arbitrarily. As is easily seen, the availability of this type of restrictions is equivalent to including into a constraint language all possible unary relations, or including into the target structure all possible unary predicates. A constraint language satisfying this condition is said to be *conservative*.

Constraint satisfaction problems related to some particular types of conservative constraint languages have already been studied. In [14], a dichotomy theorem has been proved for such languages containing all permutation relations; and in [7], conservative constraint languages related to binary operations have been studied.

We completely characterize conservative constraint languages that give rise to tractable constraint satisfaction problems, and provide a polynomial time algorithm (which appears to be quite non-trivial) solving the problem in the tractable cases. In particular, this algorithm allows one to solve numerous problems that cannot be solved by known algorithms. We also prove that in all other cases the constraint satisfaction problem is NP-complete. This result yields, probably, the first dichotomy theorem for a really wide class of constraint satisfaction problems, involving constraint of all types subject to no restriction (but finiteness) on the size of the set of values.

It is somewhat unexpected that, despite the result is applicable to CSPs over sets of arbitrary size, the form of the criteria it states is quite similar to that found by T.Schaefer [44] for SATISFIABILITY problem. By contrast with the earlier results, which are concerned with CSPs either over a very small set of values [44, 3, 16], or using constraints of a very restricted form [26, 30, 25, 33, 31, 7, 10, 4], or solvable by some particular algorithm [26, 18], the result of this paper yields, probably, the first dichotomy theorem for a really wide class of constraint satisfaction problems, involving constraints of all types subject to no restriction (but finiteness) on the size of the set of values.

We apply the main result to H-COLORING problem, an intensively studied combinatorial problem that can be naturally formulated as a restricted CSP [30, 27, 21]. In H-COLORING problem [30, 27, 21] the question is whether there exists a homomorphism of a given graph G to the fixed graph H. This problem is equivalent to the CSP for a constraint language consisting of one binary relation. The conservative version of H-Coloring problem is called List H-Coloring [37, 23, 24, 25]. In this problem the homomorphism sought is subject to restrictions on possible images of vertices of the input graph. Therefore, a complete description of those directed graphs H for which List H-Coloring is solvable in polynomial time can be deduced from our result.

The result, the solving algorithm, and the proofs heavily use the algebraic approach to CSP developed in [33, 31, 13, 9, 8]. This method builds on the fact that one can extract much information about the complexity of a restricted constraint satisfaction problem from knowing certain operations, called polymorphisms, connected with the constraint relations.

The paper is organised as follows. In Section 2.1 we give basic definitions and examples, and also pose the research problems we solve. Sections 2.2 contains a short introduction to invariance properties of constraints. In Section 2.3 we formulate the main result in a weak form that does not require extended terminology and apply it to List H-Coloring. In Section 3.1 we introduce some algebraic terminology and connect the main result with the general conjecture on tractable CSPs. Then, in Section 3.2 we recall a more general framework for CSP, multi-sorted CSP, and algebraic methods of studying the CSPs of this form. This provides the technique used in this paper and also allows us to state our results in a more general form, which is done in Section 3.3. In Section 3.4 we recall the notion of problems of bounded relational width and characterize those conservative CSPs which have bounded relational width. Sections 4,5,6,7 are devoted to auxiliary algebraic results. Finally, in Section 8 we describe a solving algorithm for tractable cases of the conservative CSP.

2 Definitions and the result

2.1 Constraint Satisfaction Problem

Let A be a finite set. An *n-ary relation* on A is a set of n-tuples of elements from A; we use R_A to denote the set of all finitary relations on A. Components of an n-tuple $\mathbf{a}$ are denoted by $\mathbf{a}[1], \ldots, \mathbf{a}[n]$. A *constraint language* is a subset of R_A (finite or infinite).

Definition 1 *The* constraint satisfaction problem over a constraint language $\Gamma \subseteq R_A$, *denoted* $\mathrm{CSP}(\Gamma)$, *is defined to be the decision problem with instance* $\mathcal{P} = (V; A; \mathcal{C})$, *where*

- V *is a finite set of* variables,
- A *is a* set of values *(sometimes called a* domain*), and*
- $\mathcal{C}$ *is a set of* constraints, *in which each constraint* $C \in \mathcal{C}$ *is a pair* $\langle s, \varrho \rangle$ *with* s *a list of variables of length* m_C, *called the* constraint scope, *and* ϱ *an* m_C*-ary relation on* A, *belonging to* Γ, *called the* constraint relation.

The question is whether there exists a solution *to* $\mathcal{P}$, *that is, a mapping* $\varphi: V \to A$ *such that, for each constraint in* $\mathcal{C}$, *the image of the constraint scope is a member of the constraint relation.*

The size of a problem instance is the length of encoding of all tuples in all constraints.

Example 1 (SATISFIABILITY) An instance of the standard propositional SATISFIABILITY problem [28, 40] is specified by giving a propositional formula $\mathcal{F} = X_1 \wedge \cdots \wedge X_n$ in CNF and asking whether $\mathcal{F}$ is satisfiable. As is easily seen, it is equivalent to the CSP instance $(V, \{0,1\}, \mathcal{C})$ where V is the set of all variables appearing in $\mathcal{F}$ and $\mathcal{C} = \{\langle s_1, \varrho_1\rangle, \ldots, \langle s_n, \varrho_n\rangle\}$ where s_i is the list of variables underlying the literals from X_i, and ϱ_i consists of all tuples that make X_i true. $\square$

Example 2 (H-COLORING) Let H be a (directed) graph. In the H-COLORING problem we are asked whether there is a homomorphism from a given graph G to H.

Every (directed) graph $H = (V(H), E(H))$ corresponds to a binary relation ϱ_H: $(a, b) \in \varrho_H$ iff (a, b) is an edge of H. Then every instance $G = (V(G); E(G))$ of H-COLORING problem corresponds to the instance $(V(G); V(H); \{\langle (a,b), \varrho_H\rangle \mid (a,b) \in E(G)\})$ of CSP(ϱ_H). $\square$

We shall be concerned with distinguishing between those constraint languages which give rise to *tractable* problems (i.e., problems for which there exists a polynomial-time solution algorithm), and those which do not.

Definition 2 *A constraint language,* Γ *is said to be* tractable, *if* CSP(Γ') *is tractable for each finite subset* $\Gamma' \subseteq \Gamma$. *It is said to be* NP-complete, *if* CSP(Γ') *is NP-complete for some finite subset* $\Gamma' \subseteq \Gamma$.

Note that algorithms solving CSP(Γ') for different finite subsets of a tractable constraint language Γ can be quite different. If there exists a uniform polynomial time algorithm for CSP(Γ) then we say that Γ is *globally tractable.* Clearly, every finite tractable constraint language is globally tractable. Normally, in applications, such strong version of tractability is not required; however, from a theoretical perspective, this stronger version is even more natural. The algebraic approach (see Sections 2.2,3.1) works for 'local' notion of tractability. This means that we should invest additional efforts in proving global tractability. In all known cases every tractable constraint language is also globally tractable.

Several different definitions of the Constraint Satisfaction Problem have appeared in literature, some of them are equivalent to that given above, some are not. In this paper we concentrate on one of the standard variants of the CSP (see, e.g., [20]) which we will call the *conservative CSP*.

Definition 3 *The* conservative constraint satisfaction problem over a constraint language $\Gamma \subseteq R_A$, *denoted* c-CSP(Γ), *is defined to be the decision problem with instance* $\mathcal{P} = (V; A; \mathcal{L}; \mathcal{C})$, *where* $V, A, \mathcal{C}$ *are defined as for the usual CSP, and* $\mathcal{L}$ *is a list of subsets* L_v $(v \in V)$ *from* A. *The question is whether there exists a solution* φ *to* $\mathcal{P}$, *defined as in Definition 1, such that* $\varphi(v) \in L_v$ *for all* $v \in V$.

Example 3 (LIST-H-COLORING) Let H be a (directed) graph. In LIST H-COLORING problem we are given a graph G and, for each vertex v of G, a set L_v of vertices of H.

The question is whether there is a homomorphism φ from G to H such that $\varphi(v) \in L_v$ for every vertex v of G. Clearly, LIST H-COLORING can be represented in the form of the conservative CSP in a similar way as in Example 2. □

Notice that, in fact, for any constraint language $\Gamma \subseteq R_A$, the problem c-CSP(Γ) is equivalent to CSP($\Gamma \cup 2^A$), where 2^A denotes the set of all subsets from A. A constraint language Γ with $2^A \subseteq \Gamma$ will be called *conservative*. Therefore, we may get rid of conservative constraint satisfaction problems, and, instead, consider problems of the form CSP(Γ) for conservative constraint languages Γ.

We are concerned with the following research problems.

Problem 1 (Tractability problem) *Characterize tractable conservative constraint languages,* Γ.

Problem 2 (Algorithmic problem) *Find efficient (polynomial time) algorithms for* CSP(Γ), Γ *tractable conservative.*

Problem 3 (Meta-problem) *Find an efficient (polynomial time) algorithm, if any, that decides whether a conservative constraint language is tractable or not.*

In the important particular case of Boolean CSP (i.e., SATISFIABILITY problem), and in the case of CSP over a 3-element domain, all these problems have been completely solved [44, 36, 3, 5] for arbitrary constraint languages, not only conservative. The result for SATISFIABILITY is known as Schaefer's Dichotomy Theorem.

Theorem 1 ([44]) *A Boolean constraint language,* Γ, *is tractable if and only if one of the following conditions holds:*

1. *every* R *in* Γ *contains* $(0, \ldots, 0)$.
2. *every* R *in* Γ *contains* $(1, \ldots, 1)$.
3. *every* R *in* Γ *is definable by a CNF formula in which each clause has at most one negated variable.*
4. *every* R *in* Γ *is definable by a CNF formula in which each clause has at most one unnegated variable.*
5. *every* R *in* Γ *is definable by a CNF formula in which each clause has at most two literals.*
6. *every* R *in* Γ *is the solution space of a linear system over* GF(2).

Otherwise Γ *is NP-complete.*

We revisit those results later on. Now we just notice that in both cases every constraint language Γ is either tractable or NP-complete, that allows us to pose another research problem.

Problem 4 (Dichotomy problem) *Is it true that every conservative constraint language is either tractable or NP-complete?*

T.Feder and M.Vardi [26] conjectured that Dichotomy problem is answered positively for all constraint languages, not only conservative.

2.2 Polymorphisms and tractability

The algebraic approach to the study of CSP uses, in addition to relations, arbitrary *operations* on the set of values. The set of all finitary operations on A (that is mappings $f\colon A^n \to A$, $n \in \mathbb{N}$) will be denoted by O_A.

Any operation on A can be extended in a standard way to an operation on tuples over A, as follows. For any n-ary operation $f \in O_A$, and any collection of tuples $\mathbf{a}_1, \dots, \mathbf{a}_n \in A^m$, define

$$f(\mathbf{a}_1, \dots, \mathbf{a}_n) = (f(\mathbf{a}_1[1], \dots, \mathbf{a}_n[1]), \dots, f(\mathbf{a}_1[m], \dots, \mathbf{a}_n[m])).$$

Definition 4 *For any m-ary relation $\varrho \in R_A$, and any n-ary operation $f \in O_A$, if $f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in \varrho$ for all choices of $\mathbf{a}_1, \dots, \mathbf{a}_n \in \varrho$, then ϱ is said to be* invariant *under f, and f is called a* polymorphism *of ϱ.*

The set of all relations that are invariant under each operation from some set $C \subseteq O_A$ will be denoted by $\mathsf{Inv}(C)$. The set of all operations that are polymorphisms of every relation from some set $\Gamma \subseteq R_A$ will be denoted by $\mathsf{Pol}(\Gamma)$. The operators Inv and Pol form a Galois correspondence between R_A and O_A (see Proposition 1.1.14 of [42]). A basic introduction to this correspondence can be found in [41], and a comprehensive study in [42]. The following result describes a connection between polymorphisms and the complexity of CSP.

Theorem 2 ([31]) *Let Γ_1 and Γ_2 be constraint languages over a finite set, such that Γ_2 is finite. If $\mathsf{Pol}(\Gamma_1) \subseteq \mathsf{Pol}(\Gamma_2)$ then $\mathrm{CSP}(\Gamma_2)$ is polynomial time reducible to $\mathrm{CSP}(\Gamma_1)$.*

Corollary 1 *A constraint language Γ is tractable [NP-complete] if and only if $\mathsf{Inv}(\mathsf{Pol}(\Gamma))$ is tractable [NP-complete].*

The theorem amounts to say that the complexity of $\mathrm{CSP}(\Gamma)$, Γ finite, is completely determined by the polymorphisms of Γ. Moreover, this is true for all infinite constraint languages studied.

A number of results on the complexity of constraint satisfaction problems have been obtained via this approach (e.g., [3, 4, 7, 9, 13, 10, 17, 33, 31]. In particular, it was proved that certain types of operations yields the tractability.

Theorem 3 ([33, 31, 7, 4]) *If one of the following operations is a polymorphism of a constraint language Γ over a finite set A, then $\mathrm{CSP}(\Gamma)$ is tractable:*

- *a* semilattice *operation, that is a binary operation f satisfying the conditions: (a) $f(a,a) = a$ (idempotency); (b) $f(a,b) = f(b,a)$ (commutativity); (c) $f(f(a,b),c) = f(a,f(b,c))$ (associativity), for all $a, b \in A$;*

- *a* conservative commutative *operation, that is a binary operation f such that $f(a,b) = f(b,a)$ and $f(a,b) \in \{a,b\}$;*

- *a* majority *operation, that is a ternary operation g such that $g(a,a,b) = g(a,b,a) = g(b,a,a) = a$, for all $a,b \in A$.*

- *a* Mal'tsev *operation, that is a ternary operation h such that $h(a,a,b) = h(b,a,a) = b$, for all $a,b \in A$.*

Moreover, Schaefer's Dichotomy Theorem [44], when appropriately re-stated, easily follows from Theorems 2,3 and well known algebraic results [43] (see [31]).

Theorem 4 ([44, 31]) *For any $\Gamma \subseteq R_{\{0,1\}}$, $\mathrm{CSP}(\Gamma)$ is tractable if and only if $\mathsf{Pol}(\Gamma)$ contains one of the following:*

- *the constant 0 or constant 1 operations;*

- *the conjunction or disjunction operations (which are semilattice);*

- *the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$;*

- *the Mal'tsev operation $x - y + z \pmod 2$ (it is also called the* affine *or* minority *operation).*

In all other cases $\mathrm{CSP}(\Gamma)$ is NP-complete.

Schaefer's Dichotomy Theorem completely solves Problems 1–3 for Boolean constraint languages. Problem 4 has been explicitly solved in [36].

In the non-Boolean case, we need further algebraic machinery even to pose a plausible conjecture for Problem 1. It will be introduced in Section 3.1.

2.3 A dichotomy theorem for conservative constraint languages

First, we show that the NP-completeness part of our dichotomy theorem can be easily deduced from the existing results.

It has been observed [33, 34, 13] that there are two benchmark problems that explain the NP-completeness of all known NP-complete CSPs. These are the problem $\mathrm{CSP}(\{\varrho\})$,

$$\varrho = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix},$$

in Boolean case (the tuple are written vertically), and $\mathrm{CSP}(\{\neq_A\})$ in non-Boolean case.

An operation f is a polymorphism of one those relations if and only if f is an *essentially unary surjective operation*, that is $f(x_1,\ldots,x_n) = g(x_i)$ where g is a bijection [33, 34]. Therefore, by Theorem 2, if every polymorphism of a constraint language, Γ, is an essentially unary surjective operation, then $\mathrm{CSP}(\Gamma)$ is NP-complete.

Note that every (n-ary) polymorphism f of a conservative constraint language satisfies the condition: $f(a_1,\ldots,a_n) \in \{a_1,\ldots,a_n\}$, for all $a_1,\ldots,a_n \in A$. Such an operation is said to be *conservative*. For example, the operations listed in Theorem 4, excluding the constant ones, are conservative. By $f|_B$ we denote the restriction of an operation f onto a set B. The following result from [13] provides a necessary condition for the tractability of $\mathrm{CSP}(\Gamma)$.

Proposition 1 ([13]) *Let Γ be a constraint language on a finite set A, and a subset $B \subseteq A$, $|B| \geq 2$, treated as a unary relation belongs to Γ. If, for any $f \in \mathsf{Pol}\Gamma$, $f|_B$ is an essentially unary surjective operation, then* $\mathrm{CSP}(\Gamma)$ *is NP-complete.*

A conservative language Γ satisfies the conditions of Proposition 1 for any $B \subseteq A$. Therefore, if $\mathrm{CSP}(\Gamma)$ is tractable then, for any 2-element subset $B = \{0,1\} \subseteq A$, there exists an operation f^B such that $f^B|_B$ is not an essentially unary surjective operation. Well known properties of Boolean operations [43], imply that f^B can be chosen such that $f^B|_B$ is either a semilattice (that is conjunction or disjunction) operation, or the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, or the Mal'tsev operation $x - y + z \pmod 2$. The main result of this paper asserts that this property is also sufficient for the tractability of $\mathrm{CSP}(\Gamma)$.

Theorem 5 *The problem* $\mathrm{CSP}(\Gamma)$ *for a conservative constraint language Γ is globally tractable if and only if, for any 2-element subset $B \subseteq A$ (without loss of generality, we may assume $B = \{0,1\}$), there exists an operation $f^B \in \mathsf{Pol}(\Gamma)$ such that $f^B|_B$ is either a semilattice operation, or the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, or the Mal'tsev operation $x - y + z \pmod 2$. Otherwise* $\mathrm{CSP}(\Gamma)$ *is NP-complete.*

In [13, 9] (see also Section 3.1) a possible criteria has been conjectured characterising tractable problems $\mathrm{CSP}(\Gamma)$ for an arbitrary constraint language, Γ, over a finite set. It can be shown that, the criteria stated in Theorem 5 is equivalent to that from [13, 9] in the particular case of conservative constraint languages.

Theorem 5 completely solves the classification problem, and give a positive solution to the dichotomy problem. It also provides a polynomial time algorithm to recognise tractable conservative constraint satisfaction problems. Indeed, to verify the condition stated in the theorem for a conservative constraint language, Γ, one just have to calculate all binary and ternary polymorphisms of Γ, their number is bounded with a constant when the size of A is fixed; and then, for each 2-element subset $B \subseteq A$, check all the computed polymorphisms looking for an operation f such that $f|_B$ is of the specified types. Finally, in Section 8 we provide a polynomial time algorithm solving $\mathrm{CSP}(\Gamma)$ for a conservative Γ whenever it is tractable.

In fact, Theorem 5 can be stated for a wider class of constraint languages than conservative languages. Suppose that Γ is a constraint language on A containing all 2- and 3-element unary relations (subsets of A). If the conditions of Theorem 5 do not hold then $\mathrm{CSP}(\{\varrho\})$ is reducible to $\mathrm{CSP}(\Gamma)$ and, hence, the latter problem is NP-complete. Otherwise, let C denote the set of all binary and ternary polymorphisms of Γ. Then the conditions of Theorem 5 hold for the language $\mathsf{Inv}(C) \supseteq \Gamma$. Finally, if a binary or ternary

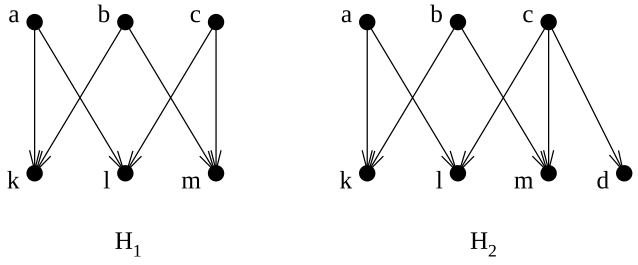

Figure 1:

operation is a polymorphism of all 3-element unary relations then it is also a polymorphisms of all unary relations. Therefore, $\mathsf{Inv}(C)$ is conservative and, by Theorem 5, is tractable.

A constraint language containing all k-element unary relations will be called *k-conservative.*

Corollary 2 *If* Γ *is a 3-conservative constraint language then* $\mathrm{CSP}(\Gamma)$ *is globally tractable if and only if the conditions of Theorem 5 holds. Otherwise it is NP-complete.*

Corollary 2 has been independently observed by T.Feder and P.Hell in [22].

2.4 List H-Coloring problem

We apply Theorem 5 to characterise those digraphs H for which List H-Coloring problem is tractable. Such a characterisation has been obtained in [24, 25] for undirected graphs. For an undirected graph H, List H-Coloring is tractable if and only if H is *bi-arc*. Remarkably, bi-arc graphs can be described in terms of polymorphisms: a graph H is bi-arc if and only if ϱ_H is symmetric and has a majority polymorphism [1, 25]. To our best knowledge, up to now, no progress has been made in studying List H-Coloring for directed graphs. Theorem 5 allows us to obtain a complete classification of the complexity of List H-Coloring problem in this case.

Corollary 3 List H-Coloring *problem for a digraph H is tractable if and only if ϱ_H satisfies the conditions of Theorem 5. Otherwise it is NP-complete.*

The problem of describing the class of digraphs specified in Corollary 3 in graph-theoretic terms remains open.

Example 4 Consider the List H-Coloring problems for the graphs shown in Fig. 1. It is not hard to see that $\mathsf{Pol}(\{\varrho_{H_1}\})$ contains a binary commutative operation f, such that the restriction of f onto any 2-element set is a semilattice operation. For example, f can be chosen to be defined by the equalities: $f(a,b)=b$, $f(b,c)=c$, $f(c,a)=a$, $f(k,l)=l$, $f(l,m)=m$, $f(m,k)=k$, and $f(x,y)=f(y,x)=x$ if $x\in\{a,b,c\}$, $y\in\{k,l,m\}$. By Corollary 3, List H_1-Coloring is tractable. In the Section 4.1 we show that List H_2-Coloring is NP-complete. □

3 Algebraic structure of CSP and further results

In this section we recall the algebraic machinery introduced in [13, 8, 9]. It allows us to generalise Theorem 5 and Corollary 2. We shall heavily use these methods to establish the results of the paper.

3.1 Algebraic definitions

In our algebraic definitions we mainly follow [45, 38].

Definition 5 *An* algebra *is an ordered pair* $\mathbb{A} = (A, F)$ *such that* A *is a nonempty set and* F *is a family of finitary operations on* A. *The set* A *is called the* universe *of* $\mathbb{A}$, *the operations from* F *are called* basic. *An algebra with a finite universe is referred to as a* finite algebra.

Every constraint language on a set A can be assigned an algebra with the universe A.

Given a constraint language, Γ, on A, the algebra $(A, \mathsf{Pol}(\Gamma))$ is called the *algebra associated with* Γ, and is denoted $\mathbb{A}_\Gamma$. Conversely, for any finite algebra $\mathbb{A} = (A; F)$, there is a constraint language associated with $\mathbb{A}$, the language $\mathsf{Inv}(F)$, and the associated problem class $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathsf{Inv}(F))$. Notice that $\mathsf{Inv}(F)$ is the largest constraint language Γ such that $\mathbb{A}_\Gamma = \mathbb{A}$, see, e.g. [42, 45]. A connection between the complexity of a constraint language and the corresponding set of operations can be easily extended to the associated algebra: if $\mathbb{A}_{\Gamma_1} = \mathbb{A}_{\Gamma_2}$ then Γ_1, Γ_2 are tractable [NP-complete] simultaneously. We, therefore, make the following definition: for a constraint language Γ, the algebra $\mathbb{A}_\Gamma$ is said to be *tractable* [*NP-complete*] if Γ is tractable [NP-complete].

An algebra is said to be a *G-set* if every term operation is an essentially unary surjective operation.

Proposition 2 ([32, 33]) *A finite G-set is NP-complete.*

By combining this results, Theorem 3 and the classical result of E.Post [43], even more concise version of Schaefer's theorem can be derived [12, 13].

Theorem 6 ([44, 12]) *A constraint language* Γ *on a 2-element set is tractable if and only if* $\mathbb{A}_\Gamma$ *is not a G-set. Otherwise* Γ *is NP-complete.*

Certain transformations of constraint languages preserve the complexity, and may lead to languages with certain desirable properties. Let Γ be a constraint language on A, and g a unary polymorphism of Γ such that $g(g(x)) = g(x)$. By $g(\Gamma)$ we denote the set $\{g(\varrho) \mid \varrho \in \Gamma\}$ where $g(\varrho) = \{(g(\mathbf{a}[1]), \ldots, g(\mathbf{a}[n])) \mid (\mathbf{a}[1], \ldots, \mathbf{a}[n]) \in \varrho\}$; and by Γ^+ the set $\Gamma \cup \{\{(a)\} \mid a \in A\}$. If $g \in \mathsf{Pol}(\Gamma)$ is a unary operation whose range is minimal among ranges of the unary operations from $\mathsf{Pol}(\Gamma)$ with the property $g(g(x)) = g(x)$, then the constraint language $g(\Gamma)^+$ will be denoted Γ^{id}_g.

Proposition 3 ([13]) *Let* Γ *be a constraint language on* A, *and* $g \in \mathsf{Pol}(\Gamma)$ *a unary operation on* A *with a minimal range and such that* $g(g(x)) = g(x)$. *Then* Γ *is tractable [NP-complete] if and only if* Γ^{id}_g *is tractable [NP-complete].*

An operation f on A is called *idempotent* if $f(x,\dots,x) = x$ for any $x \in A$. If Γ and g satisfy the conditions of Proposition 3 then the algebra $\mathbb{A}_{\Gamma_g^{\mathrm{id}}}$ is *idempotent*, that is, all its basic operations are idempotent. The complexity of the constraint language Γ_g^{id} does not depend on the choice of g, and we shall denote every such language by Γ^{id}.

Due to Theorem 2 and Proposition 3 the study of the complexity of constraint languages is completely reduced to the study of properties of idempotent algebras.

We shall also need some constructions on algebras.

Definition 6 *Let $\mathbb{A} = (A, F)$ be an algebra, and B a subset of A such that, for any $f \in F$ (n-ary), and for any $b_1,\dots,b_n \in B$, we have $f(b_1,\dots,b_n) \in B$. Then the algebra $\mathbb{B} = (B, F|_B)$, where $F|_B$ consists of restrictions of operations from F onto B, is called a* subalgebra *of $\mathbb{A}$. The universe of a subalgebra of $\mathbb{A}$ is called a* subuniverse *of $\mathbb{A}$.*

An equivalence relation $\theta \in \mathsf{Inv}(F)$ is said to be a congruence *of $\mathbb{A}$. The θ-class containing $a \in A$ is denoted a^θ, the set $A/_\theta = \{a^\theta \mid a \in A\}$ is said to be the* factor-set, *and the algebra $\mathbb{A}/_\theta = (A/_\theta; F^\theta)$, $F^\theta = \{f^\theta \mid f \in F\}$ where $f^\theta(a_1^\theta,\dots,a_n^\theta) = (f(a_1,\dots,a_n))^\theta$, is said to be the* factor-algebra.

Proposition 4 ([13]) *Let Γ be a tractable constraint language on A, $B \subseteq A$ a subuniverse of $\mathbb{A}_\Gamma$, and θ an equivalence relation invariant under $\mathsf{Pol}(\Gamma)$. Then*
(1) the subalgebra $\mathbb{B} = (B; (\mathsf{Pol}(\Gamma))|_B)$ of $\mathbb{A}_\Gamma$ is tractable;
(2) $\mathbb{A}_\Gamma/_\theta$, and the set $\Gamma^\theta = \{\varrho^\theta \mid \varrho \in \Gamma\}$ where $\varrho^\theta = \{(\mathbf{a}[1]^\theta,\dots,\mathbf{a}[n]^\theta) \mid (\mathbf{a}[1],\dots,\mathbf{a}[n]) \in \varrho\}$, are tractable.

Hence, every subalgebra and every factor-algebra of a tractable algebra are tractable. Furthermore, every *factor* of a tractable algebra, that is, a factor-algebra of a subalgebra, is tractable and cannot be a G-set. Thus, every tractable algebra $\mathbb{A}$ satisfies the condition

$$\text{none of the factors of } \mathbb{A} \text{ is a } G\text{-set.} \qquad (\textsc{No-G-Set})$$

Moreover, all known examples of NP-complete subclasses of the CSP have a G-set associated in some way. We, therefore, make the following conjecture.

Conjecture 1 ([13, 12]) *A constraint language Γ on a finite set A is tractable if $\mathbb{A}_{\Gamma^{\mathrm{id}}}$ satisfies (*NO-G-SET*). Otherwise it is NP-complete.*

Now we show that for conservative constraint languages the conditions stated in Theorem 5 are equivalent to (NO-G-SET). In particular, Conjecture 1 holds for conservative constraint languages.

Since every 1-element subset is invariant under any conservative operation f, we have $f(a,\dots,a) = a$ for any $a \in A$, which means that any conservative operation is idempotent. Therefore, for any conservative constraint language Γ, we have $\Gamma = \Gamma^{\mathrm{id}}$, and Conjecture 1 can be applied to $\mathbb{A}_\Gamma$ itself.

Lemma 1 *Let Γ be a conservative constraint language on a finite set A. Then $\mathbb{A}_\Gamma$ satisfies (*NO-G-SET*) if and only if every 2-element subalgebra of $\mathbb{A}_\Gamma$ has a term operation which is either a semilattice or majority or minority operation.*

Proof: If (No-G-Set) holds then every 2-element subalgebra $\mathbb{B}$ is idempotent and is not a G-set. By Post's theorem [43], it has one of the listed term operations.

Conversely, let $\mathbb{C} = \mathbb{B}/_\theta$ be a factor-algebra of a subalgebra $\mathbb{B}$ of $\mathbb{A}_\Gamma$, and $D = \{a, b\} \subseteq \mathbb{B}$ such that $a^\theta \neq b^\theta$. Let also f be a term operation such that $f|_D$ is either a semilattice or majority or minority operation. Then f^θ is not an essentially unary surjective operation and, therefore, $\mathbb{C}$ is not a G-set. $\square$

Corollary 4 *A conservative constraint language Γ on a finite set A is tractable if and only if $\mathbb{A}_\Gamma$ satisfies (*No-G-Set*). Otherwise it is NP-complete.*

An algebra corresponding to a k-conservative constraint language, that is such that every k-element subset of the universe form a subalgebra, is said to be *k-conservative.*

Corollary 5 *A 3-conservative constraint language Γ on a finite set A is tractable if and only if $\mathbb{A}_\Gamma$ satisfies (*No-G-Set*). Otherwise it is NP-complete.*

3.2 Multi-sorted constraints satisfaction problem

In [8], an algebraic approach to a generalised version of the constraint satisfaction problem was developed. In this generalised version every variable is allowed to have its own domain. In the following definitions we follow [8, 6].

In multi-sorted constraint satisfaction problems we allow *multi-sorted relations.*

Definition 7 *For any collection of sets $\mathcal{A} = \{A_i \mid i \in I\}$, and any list of indices $(i_1, i_2, \ldots, i_m) \in I^m$, a subset ϱ of $A_{i_1} \times A_{i_2} \times \cdots \times A_{i_m}$, together with the list $(i_1, i_2, \ldots, i_m)$, will be called a* multi-sorted relation *over $\mathcal{A}$ with arity m and* signature *$(i_1, i_2, \ldots, i_m)$. For any such relation ϱ, the signature of ϱ will be denoted $\sigma(\varrho)$.*

Any set of multi-sorted relations over $\mathcal{A}$ is called a multi-sorted constraint language *over $\mathcal{A}$.*

As usual, the ith component of a tuple $\mathbf{a}$ will be denoted by $\mathbf{a}[i]$.

Definition 8 *Let Γ be a multi-sorted constraint language over a collection of sets $\mathcal{A} = \{A_i \mid i \in I\}$. The* multi-sorted constraint satisfaction problem $\mathrm{MCSP}(\Gamma)$ *is defined to be the decision problem with*

Instance: *a quadruple $(V; \mathcal{A}; \delta; \mathcal{C})$ where*

- *V is a set of* variables*;*
- *δ is a mapping from V to I, called the* domain function*;*
- *$\mathcal{C}$ is a set of* constraints *where each constraint $C \in \mathcal{C}$ is a pair $\langle s, \varrho\rangle$, such that*
  - *$s = (v_1, \ldots, v_{m_C})$ is a tuple of variables of length m_C, called the* constraint scope*;*
  - *ϱ is an m_C-ary relation over $\mathcal{A}$ with signature $(\delta(v_1), \ldots, \delta(v_{m_C}))$, called the* constraint relation.

QUESTION: *Does there exist a* solution, *i.e. a function* φ, *from* V *to* $\bigcup_{A \in \mathcal{A}} A$, *such that, for each variable* $v \in V$, $\varphi(v) \in A_{\delta(v)}$, *and for each constraint* $\langle s, \varrho \rangle \in \mathcal{C}$, *with* $s = (v_1, \ldots, v_m)$, *the tuple* $(\varphi(v_1), \ldots, \varphi(v_m))$ *belongs to* ϱ?

Tractable and NP-complete multi-sorted constraint languages are defined in the same way as one-sorted ones.

It is possible to introduce the algebraic structure of the multi-sorted CSP in a very similar way as for the usual one.

To extend the algebraic approach to the multi-sorted case, we need to define a suitable extension of the notion of a polymorphism. It is not hard to see that we cannot simply separate out different domains and consider polymorphisms on each one separately; we must ensure that all of the domains are treated in a co-ordinated way. In the following definition, this is achieved by defining different interpretations for the same operation applied to different sets.

Definition 9 *Let* $\mathcal{A}$ *be a collection of sets. An* n*-ary* multi-sorted operation t *on* $\mathcal{A}$ *is defined by a collection of* interpretations $\{t^A \mid A \in \mathcal{A}\}$, *where each* t^A *is an* n*-ary operation on the corresponding set* A.

The multi-sorted operation t *is said to be a* polymorphism *of an* m*-ary multi-sorted relation* ϱ *over* $\mathcal{A}$ *with signature* $(\delta(1), \ldots, \delta(m))$ *or* ϱ *is said to be an* invariant *of* t *if, for any* $(a_{11}, \ldots, a_{m1}), \ldots, (a_{1n}, \ldots, a_{mn}) \in \varrho$ *we have*

$$t \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} t^{A_{\delta(1)}}(a_{11}, \ldots, a_{1n}) \\ \vdots \\ t^{A_{\delta(m)}}(a_{m1}, \ldots, a_{mn}) \end{pmatrix} \in \varrho.$$

For a multi-sorted constraint language Γ, the set all those multi-sorted operations which are polymorphisms of every relation in Γ is denoted $\mathsf{MPol}(\Gamma)$.

We need to introduce a little more standard algebraic terminology. First we note that algebras can be grouped into families which share the same set of basic operations.

Definition 10 *The collection of algebras* $\mathfrak{A} = \{(A_i; F^{A_i}) \mid i \in I\}$ *is said to be a collection of* similar *algebras if there exists some fixed set* F *of multi-sorted operations over the sets* $\{A_i \mid i \in I\}$, *such that each set of basic operations* F^{A_i} *is the set of interpretations of the operations in* F *on the set* A_i. *The set* F *is called the set of basic operations of the collection* $\mathfrak{A}$.

Definition 11 *For any collection of similar algebras* $\mathfrak{A} = \{(A_i; F^{A_i}) \mid i \in I\}$ *with basic operations* F, *we define* $\mathsf{MInv}(\mathfrak{A})$ *to be the constraint language* Γ *of all multi-sorted relations over the sets* $\{A_i \mid i \in I\}$ *such that* $\mathsf{MPol}(\Gamma) = F$.

A collection of similar algebras $\mathfrak{A}$ will called *tractable* if the multi-sorted constraint language $\mathsf{MInv}(\mathfrak{A})$ is tractable. Similarly, $\mathfrak{A}$ will be called NP-complete if $\mathsf{MInv}(\mathfrak{A})$ is NP-complete. We also define the associated multi-sorted CSP just by setting $\mathrm{MCSP}(\mathfrak{A}) = \mathrm{MCSP}(\mathsf{Inv}(\mathfrak{A}))$ in which the domains of the variables are the universes of these algebras.

The notion of global tractability is a little more sophisticated. The collection $\mathfrak{A}$ is said to be *globally tractable* if, for any finite $\mathfrak{A}' \subseteq \mathfrak{A}$, the problem MCSP($\mathfrak{A}'$) is solvable in polynomial time. Clearly, this version of tractability is weaker than the natural one: tractability of MCSP($\mathfrak{A}$). The reason why we use this weakened version is that algorithms solving problems from MCSP($\mathfrak{A}$) often use certain structural properties and parameters of algebras in $\mathfrak{A}$. In many cases, the problem of recognising these properties and of calculating the parameters are harder than solving constraint satisfaction problems assuming the parameters are known. Generally, they cannot be assumed to be known if $\mathfrak{A}$ is an infinite class.

Conversely, using polymorphisms of a multi-sorted constraint language Γ over $\mathcal{A}$ we can turn every set A in $\mathcal{A}$ into an algebra by setting $\mathbb{A} = (A; (\mathsf{MPol}(\Gamma))^A)$. We shall use the notation $\mathfrak{A}_\Gamma$ for the obtained class of algebras. Note that the obtained algebras are similar.

It can be show that Γ is tractable [NP-complete] if and only if $\mathfrak{A}_\Gamma$ is tractable; for details see [8, 6].

Definition 12 *Let $\mathfrak{A} = \{(A_i; F^{A_i}) \mid i \in I\}$ be a collection of similar algebras. A multi-sorted operation f over $\mathcal{A} = \{A_i \mid i \in I\}$ is said to be a* term operation *of $\mathfrak{A}$ if it can be derived from operations in F by means of substitution.*

Definition 13 *Let $\mathbb{A}_1, \mathbb{A}_2$ be similar algebras with universes A_1, A_2 and basic operations F. A mapping $\varphi: A_1 \to A_2$ is called a* homomorphism *from $\mathbb{A}_1$ to $\mathbb{A}_2$ if $\varphi f^{\mathbb{A}_1}(a_1, \dots, a_k) = f^{\mathbb{A}_2}(\varphi(a_1), \dots, \varphi(a_k))$ for all $f \in F$ and all $a_1, \dots, a_k \in A_1$, where k is the arity of f.*

If the map φ is surjective, then $\mathbb{A}_2$ is called a homomorphic image *of $\mathbb{A}_1$.*

A factor-algebra $\mathbb{A}_1/_\theta$ is a homomorphic image of $\mathbb{A}_1$. The homomorphism is defined through the rule $a \mapsto a^\theta$.

Definition 14 *Let $\mathbb{A}_1, \dots, \mathbb{A}_n$ be similar algebras, and F a set of basic operations. The* direct product *of $\mathbb{A}_1, \dots, \mathbb{A}_n$ is the algebra $\mathbb{B} = (B; F^{\mathbb{B}})$ where $B = A_1 \times \ldots \times A_n$ and the $\mathbb{B}$-interpretation of a basic operation $f \in F$ with arity k is defined as follows*

$$f^{\mathbb{B}}\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots, \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix}\right) = \begin{pmatrix} f^{\mathbb{A}_1}(a_{11}, \dots, a_{1k}) \\ \vdots \\ f^{\mathbb{A}_n}(a_{n1}, \dots, a_{nk}) \end{pmatrix}$$

where $a_{11}, \dots, a_{1k} \in A_1, \dots, a_{n1}, \dots, a_{nk} \in A_n$. The direct product of $\mathbb{A}_1, \dots, \mathbb{A}_n$ is denoted by $\mathbb{A}_1 \times \ldots \times \mathbb{A}_n$ or $\prod_{i \in \{1,\dots,n\}} \mathbb{A}_i$.

Theorem 7 ([9]) *For any set of finite similar algebras, $\mathbb{A}_1, \dots, \mathbb{A}_n$, the following are equivalent:*

1. *$\mathbb{A}_1, \dots, \mathbb{A}_n$ is tractable [NP-complete];*
2. *$\mathbb{B}, \mathbb{A}_1, \dots, \mathbb{A}_n$ where $\mathbb{B}$ is a subalgebra of $\mathbb{A}_1$ is tractable [NP-complete];*
3. *$\mathbb{C}, \mathbb{A}_1, \dots, \mathbb{A}_n$ where $\mathbb{C}$ is a homomorphic image of $\mathbb{A}_1$ is tractable [NP-complete];*

4. $\mathbb{A}_1 \times \mathbb{A}_2, \mathbb{A}_1, \ldots, \mathbb{A}_n$ *is tractable [NP-complete].*

A class of algebras $\mathfrak{A}$ is said to be a *variety* if it contains all subalgebras, homomorphic images and direct products (including infinite) of algebras from $\mathfrak{A}$. It is said to be a *pseudo-variety* if it contains all subalgebras, homomorphic images and finite direct products of algebras from $\mathfrak{A}$. So, there exists non-trivial pseudo-varieties consisting of finite algebras, while every variety that contains an algebra with more than 1 element contains algebras of any cardinality. For a class of algebras, $\mathfrak{A}$, the least [pseudo-] variety containing $\mathfrak{A}$ is denoted by [$\mathsf{pvar}(\mathfrak{A})$] $\mathsf{var}(\mathfrak{A})$ and is called the [*pseudo-*] *variety generated by* $\mathfrak{A}$.

Corollary 6 *A class of finite algebras, $\mathfrak{A}$, is tractable if and only if* $\mathsf{pvar}(\mathfrak{A})$ *is tractable.*

As is well known, if $\mathfrak{A}$ is a finite class of finite algebras, then the pseudo-variety generated by $\mathfrak{A}$ equals the class of finite algebras from the variety generated by $\mathfrak{A}$.

Corollary 7 *A finite class of finite algebras, $\mathfrak{A}$, is tractable if and only if the class of finite algebras from* $\mathsf{var}(\mathfrak{A})$ *is tractable.*

Corollary 8 *If $\mathfrak{A}$ is a finite class of finite algebras and* $\mathsf{var}(\mathfrak{A})$ *contains a finite NP-complete algebra, then $\mathfrak{A}$ is NP-complete.*

3.3 The general form of the result

Now we are ready to formulate the result that we shall prove in this paper.

Theorem 8 *A class of finite 3-conservative algebras is globally tractable if and only if the pseudo-variety it generates contains no finite G-sets.*

Corollaries 4 and 5 are particular cases of Theorem 8 in which $\mathfrak{A}$ is one-element. Moreover, as is shown in [9], if the variety generated by a finite algebra contains a finite G-set, then this G-set is a factor of the algebra.

Theorem 8 can be stated for multi-sorted constraint languages.

Corollary 9 *A multi-sorted 3-conservative constraint language, Γ, over a collection of sets $\mathcal{A}$ is tractable and, if $\mathcal{A}$ is finite, globally tractable, if and only if for any $A \in \mathcal{A}$ and any 2-element subset $B \subseteq A$ (without loss of generality, we may assume $B = \{0,1\}$), there exists an multi-sorted operation $f \in \mathsf{MPol}(\Gamma)$ such that $f|_B$ is either a semilattice operation, or the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, or the Mal'tsev operation $x - y + z \pmod 2$. Otherwise $\mathrm{CSP}(\Gamma)$ is NP-complete.*

A weaker version of Corollary 9 (involving only finite constraint languages) has been independently derived from Theorem 5 by T.Feder and P.Hell [22].

Input. A problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.

Output. A k-minimal problem instance $\mathcal{P}' = (V; \mathcal{A}; \delta; \mathcal{C}')$ equivalent to $\mathcal{P}$.

Step 1 /* initialisation
Set $\mathcal{P}' := (V; \mathcal{A}; \delta; \mathcal{C}')$ where
$$\mathcal{C}' = \mathcal{C} \cup \{\langle W, \prod_{v \in W} \mathbb{A}_{\delta(v)} \rangle \mid W \subseteq V,\ |W| = k\};$$
and $\mathcal{P}'' := \mathcal{P}'$.
Step 2 **Do**
Step 3 **Set** $\mathcal{P}' := \mathcal{P}''$.
Step 4 **For each** $W \subseteq V$ with $|W| = k$ **do**
Step 5 **Solve** the restricted problem $\mathcal{P}'_W$.
Step 6 **For each** constraint $C = \langle s, \mathbb{C} \rangle \in \mathcal{C}''$, **replace** C with $\langle s, \mathbb{C}' \rangle$ where $\mathbb{C}' = \{\mathbf{a} \in \mathbb{C} \mid \mathrm{pr}_{s \cap W} \mathbf{a} \in \mathrm{pr}_{s \cap W} \mathcal{S}'_W\}$.
Step 7 **Until** $\mathcal{P}'' = \mathcal{P}'$.
Step 8 **Output** $\mathcal{P}'$.

Figure 2: Algorithm k-minimality

3.4 Partial solutions and bounded relational width

We recall a property of a subclass of the CSP that provides a polynomial time solving algorithm.

We use $\underline{n}$ to denote the set $\{1, \ldots, n\}$. For an n-ary relation ϱ, $\mathbf{a} \in \varrho$, and $J = \{i_1, \ldots, i_k\} \subseteq \underline{n}$, by $\mathrm{pr}_J \mathbf{a}$ we denote the tuple $(\mathbf{a}[i_1], \ldots, \mathbf{a}[i_k])$, and by $\mathrm{pr}_J \varrho$ the set $\{\mathrm{pr}_J \mathbf{b} \mid \mathbf{b} \in \varrho\}$. We sometimes will also write $\mathrm{pr}_{i_1, \ldots, i_k} \varrho$ instead of $\mathrm{pr}_J \varrho$. Clearly, $\varrho \subseteq \mathrm{pr}_1 \varrho \times \ldots \times \mathrm{pr}_n \varrho$; we say that ϱ is a *subdirect product* of $A_1, \ldots, A_n$ if $\mathrm{pr}_i \varrho = A_i$ for any i.

Every multi-sorted relation over a collection of sets $\mathcal{A} = \{A_i \mid i \in I\}$ invariant with respect to a set of multi-sorted operations F on $\mathcal{A}$ can also be treated as a subalgebra of a direct product of certain algebras from $\{(A_i; F^{A_i} \mid i \in I\}$. We, therefore, will use for relations the same notation as for algebras.

Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be an instance of the multi-sorted CSP, and $W \subseteq V$. By $\mathcal{P}_W$ we denote the *restricted problem*, that is the problem instance defined as $(W; \mathcal{A}; \delta|_W; \mathcal{C}')$ where, for every $\langle s, \mathbb{C} \rangle \in \mathcal{C}$, there is $\langle s', \mathbb{C}' \rangle \in \mathcal{C}'$ with $s' = s \cap W$, and $\mathbb{C}' = \mathrm{pr}_{s'} \mathbb{C}$. Every solution to $\mathcal{P}_W$ is said to be a *partial solution* to $\mathcal{P}$ on W. Let us denote the set of all partial solutions on W by $\mathcal{S}_W$. Notice that $\mathcal{S}_W$ can be treated as a $|W|$-ary relation.

We call the problem instance $\mathcal{P}$ *k-minimal* if, for any k-element subset W of V, there is a constraint $\langle s, \mathbb{C} \rangle \in \mathcal{C}$ such that $W \subseteq s$, and for any $\langle s, \mathbb{C} \rangle \in \mathcal{C}$, we have $\mathrm{pr}_{W \cap s} \mathbb{C} = \mathrm{pr}_{W \cap s} \mathcal{S}_W$.

Any problem instance $\mathcal{P}$ can be transformed to an equivalent k-minimal instance $\mathcal{P}'$. To do this we employ the algorithm k-minimality, see Fig. 2.

As is easily seen, the time complexity of this algorithm is $O(m^3 n^k)$ where n is the number of variables and m is the total number of tuples in the constraint relations.

A class of constraint satisfaction problems, $\mathbf{K}$, is said to be of *relational width* k if, for any $\mathcal{P} \in \mathbf{K}$, the corresponding k-minimal problem $\mathcal{P}'$ is also in $\mathbf{K}$, and $\mathcal{P}$ has a solution if and only if $\mathcal{P}'$ has no empty constraint relation. As is easily seen, any problem class of finite relational width is tractable. Problem classes of bounded relational width have been widely studied in computer science, see, for example, [15, 19, 26, 32].

In the case of conservative and 3-conservative constraint languages it is possible to characterize those languages that give rise to a problem of bounded relational width.

Theorem 9 *Let Γ be a 3-conservative constraint language over a set A. Then $\mathrm{CSP}(\Gamma)$ has the relational width 3 if and only if for any 2-element subset $B \subseteq A$ (without loss of generality, we may assume $B = \{0, 1\}$), there exists an operation $f \in \mathsf{Pol}(\Gamma)$ such that $f\big|_B$ is either a semilattice operation, or the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. Otherwise $\mathrm{CSP}(\Gamma)$ is not of bounded relational width.*

Proof: If Γ satisfies the conditions specified in the corollary then $\mathrm{CSP}(\Gamma)$ is of relational width 3 by Propositions 10 and 11.

Conversely, as $\mathrm{CSP}(\Gamma)$ is tractable, it satisfies the conditions of Theorem 5. Suppose that there exist a 2-element $B \subseteq A$ (we denote the elements of B by 0,1) such that $f\big|_B$ is a semilattice or majority operation for no $f \in \mathsf{Pol}(\Gamma)$. Then by the results of [43] and Theorem 2 (see also [31]) the problem of solving systems of linear equations over the 2-element field can be simulated by $\mathrm{CSP}(\Gamma)$. However, this problem is not of bounded relational width; hence, the same holds for $\mathrm{CSP}(\Gamma)$. $\Box$

4 Structure of conservative algebras

4.1 Red, yellow, and blue

Let $\mathbb{A} = (A; F)$ be a finite conservative algebra, satisfying the conditions of Theorem 8 or, equivalently, $\mathsf{Inv}(F)$ satisfies the conditions of Corollary 9. We consider the graph $\mathcal{G}(\mathbb{A})$, an edge-coloured digraph with vertex set A. An edge (a, b) is present and is coloured *red* if there is a term operation $f_{a,b}$ of $\mathbb{A}$ such that $f_{a,b}\big|_{\{a,b\}}$ is a semilattice operation with $f_{a,b}(a, b) = f_{a,b}(b, a) = f_{a,b}(b, b) = b$, $f_{a,b}(a, a) = a$. Edges $(a, b), (b, a)$ are present and are coloured *yellow* if neither (a, b) nor (b, a) can be coloured red and there is a term operation $f_{a,b}$ of $\mathbb{A}$ such that $f_{a,b}\big|_{\{a,b\}}$ is a majority operation. Edges $(a, b), (b, a)$ exist and are coloured *blue* if none of them can be coloured red or yellow, and there is a term operation $f_{a,b}$ of $\mathbb{A}$ such that $f_{a,b}\big|_{\{a,b\}}$ is an affine operation. Thus, for each pair $a, b \in A$, either (a, b) or (b, a) is an edge of $\mathcal{G}(\mathbb{A})$; if (a, b) is a yellow or blue edge then (b, a) is also an edge of the same colour; while if (a, b) is red then the edge (b, a) may do not exist. First we show that all operations of the form $f_{a,b}$ can be considerably unified. Recall that an operation f on A is called a *projection* if there is $i \in \{1, \ldots, n\}$ such that $f(x_1, \ldots, x_n) = x_i$ for any $x_1, \ldots, x_n \in A$.

Proposition 5 *Let $\mathcal{A} = \{\mathbb{A}_1, \dots, \mathbb{A}_n\}$, $\mathbb{A}_i = (A_i; F^{\mathbb{A}_i})$ be a collection of similar conservative algebras that satisfies the conditions of Theorem 8. There are term operations $f(x,y), g(x,y,z)$, $h(x,y,z)$ of $\mathcal{A}$ such that, for every $i \in \underline{n}$ and every two-element subset $B \subseteq \mathbb{A}_i$,*

- *$f^{\mathbb{A}_i}\big|_B$ is a semilattice operation whenever B is red, and $f^{\mathbb{A}_i}\big|_B(x,y) = x$ otherwise;*
- *$g^{\mathbb{A}_i}\big|_B$ is the majority operation if B is yellow, $g^{\mathbb{A}_i}\big|_B(x,y,z) = x$ if B is blue, and $g^{\mathbb{A}_i}\big|_B(x,y,z) = f^{\mathbb{A}_i}\big|_B(f^{\mathbb{A}_i}\big|_B(x,y),z)$ if B is red;*
- *$h^{\mathbb{A}_i}\big|_B$ is the affine operation if B is blue, $h^{\mathbb{A}_i}\big|_B(x,y,z) = x$ if B is yellow, and $h^{\mathbb{A}_i}\big|_B(x,y,z) = f^{\mathbb{A}_i}\big|_B(f^{\mathbb{A}_i}\big|_B(x,y),z)$ if B is red.*

Proof: Show first that there is an operation f that is semilattice on each red edge. Let $B_1, \dots, B_n$ be the list of all red edges in the graphs $\mathcal{G}(\mathbb{A}_j)$. To avoid clumsy notation we shall denote the operation $f^{\mathbb{A}_i}\big|_{B_j}$, $B_j \subseteq \mathbb{A}_i$ simply by $f\big|_{B_j}$. Let also $f_1, \dots, f_n$ be the list of term operations of the algebras such that $f_i\big|_{B_i}$ is a semilattice operation. Notice that every binary idempotent operation on a 2-element set is either a projection or a semilattice operation. Since each f is idempotent, for any i, j, $f_i\big|_{B_j}$ is either a projection, or a semilattice operation. We prove by induction, that the operation f^i constructed via the following rules is a semilattice operation on $B_1, \dots, B_i$:

- $f^1 = f_1$;
- $f^i(x,y) = f_i(f^{i-1}(x,y), f^{i-1}(y,x))$.

The base case of induction, $i = 1$ holds by the choice of f_1. Suppose that f^{i-1} satisfies the required conditions. If $f^{i-1}\big|_{B_i}$ is a projection, say, $f^{i-1}\big|_{B_i}(x,y) = x$, then

$$f^i(x,y) = f_i(f^{i-1}(x,y), f^{i-1}(y,x)) = f_i(x,y),$$

that is a semilattice operation on B_i. Let $B_i = \{a,b\}$, and f^{i-1} a semilattice operation such that $f^{i-1}(a,b) = f^{i-1}(b,a) = a$. Then

$$\begin{aligned} f^i(a,b) &= f_i(f^{i-1}(a,b), f^{i-1}(b,a)) = f_i(a,a) = a, \\ f^i(b,a) &= f_i(f^{i-1}(b,a), f^{i-1}(a,b)) = f_i(a,a) = a, \end{aligned}$$

hence, f^i is again a semilattice operation.

Thus, for each edge B, $f^n\big|_B$ is a semilattice operation if B is red and either a semilattice operation or a projection otherwise. However, if B is not red, then the subalgebra with the universe B has no semilattice operation, therefore, $f^n\big|_B$ is a projection whenever B is yellow or blue. Finally, it is easy to check that $f(x,y) = f^n(f^n(x,y),x)$ satisfies the conditions of the proposition.

Now let $B_1,\dots,B_k$, $C_1,\dots,C_l$ be the lists of all yellow and all blue edges respectively, and $g_1,\dots,g_k$, $h_1,\dots,h_l$ the lists of term operations of the algebras such that $g_i\big|_{B_i}$ is the majority operation, and $h_i\big|_{C_i}$ is the minority operation. Notice first, that since neither $\mathbb{B}_i=(B_i;F\big|_{B_i})$ nor $\mathbb{C}_i=(C_i;F\big|_{C_i})$ has a term semilattice operation, every their binary term operation is a projection. Therefore, for any i,j, $g_i\big|_{B_j}(x,y,y), g_i\big|_{B_j}(y,x,y), g_i\big|_{B_j}(y,y,x)$, $h_i\big|_{B_j}(x,y,y), h_i\big|_{B_j}(y,x,y), h_i\big|_{B_j}(y,y,x)$,$g_i\big|_{C_j}(x,y,y), g_i\big|_{C_j}(y,x,y), g_i\big|_{C_j}(y,y,x)$,$h_i\big|_{C_j}(x,y,y)$, $h_i\big|_{C_j}(y,x,y), h_i\big|_{C_j}(y,y,x)\in\{x,y\}$. This means that the operations $g_i\big|_{B_j}, h_i\big|_{B_j}, g_i\big|_{C_j}, h_i\big|_{C_j}$ are of one of the following types: a projection, the minority operation, the majority operation, a 2/3-*minority operation*, that is an operation satisfying the equalities $m(x,y,y)=y$, $m(y,x,y)=m(y,y,x)=x$ or similar. Moreover, it is well known [43], that if a 2-element algebra has a 2/3-minority term operation then it also has a majority term operation. Hence, $g_i\big|_{C_j}, h_i\big|_{C_j}$ are either minority operations or projections.

First we prove by induction that for every $1\le j\le k$ there is an operation $g^j(x,y,z)$ which is majority on B_i for $i\le j$. The operation $g^1=g_1$ gives the base case of induction. Let us assume that g^{j-1} is already found. If $g^{j-1}\big|_{B_j}$ is the majority operation, set $g^j=g^{j-1}$. Otherwise, it is either a projection, or a 2/3-minority operation, or the minority operation. In all these case its variables can be permuted such that $g^{j-1}\big|_{B_j}(x,y,y)=x$. Then the operation $p(x,y)=g^{j-1}(x,y,y)$ satisfies the conditions $p\big|_{B_j}(x,y)=x$, and $p\big|_{B_i}(x,y)=y$ for all $i<j$. It is not hard to see that the operation

$$g^j(x,y,z)=p(g_j(x,y,z),g^{j-1}(x,y,z))$$

satisfies the required conditions.

Further, consider the operation g^k. Its restriction $g^k\big|_{C_j}$, $1\le j\le l$, is either a projection, or the minority operation. If $g^k\big|_{C_j}$ is the minority operation, then as above we can derive an operation $p(x,y)$ such that $p\big|_{B_i}(x,y)=y$ for all $1\le i\le k$, and $p\big|_{C_j}(x,y)=x$. The operation $g'(x,y,z)=p(x,g^k(x,y,z))$ is majority on B_i, $1\le i\le k$, the first projection on C_j, and a projection on each C_i such that $g^k\big|_{C_j}$ is a projection. Therefore, $g^k\big|_{C_i}$ can be assumed to be a projection for all $1\le i\le l$. Then for the operation

$$g''(x,y,z)=g^k(x,g^k(y,x,y),g^k(z,z,x))$$

we have

$$\begin{aligned}
g''\big|_{B_i}(x,y,z) &= g^k\big|_{B_i}(x,g^k\big|_{B_i}(y,x,y),g^k\big|_{B_i}(z,z,x))=g^k\big|_{B_i}(x,y,z), \quad \text{for any } 1\le i\le k;\\
g''\big|_{C_i}(x,y,z) &= g^k\big|_{C_i}(x,g^k\big|_{C_i}(y,x,y),g^k\big|_{C_i}(z,z,x))=x, \quad \text{for any } 1\le i\le l\\
&\qquad \text{such that } g^k\big|_{C_i}(x,y,z)=x;\\
g''\big|_{C_i}(x,y,z) &= g^k\big|_{C_i}(x,g^k\big|_{C_i}(y,x,y),g^k\big|_{C_i}(z,z,x))=g^k\big|_{C_i}(y,x,y)=x,\\
&\qquad \text{for any } 1\le i\le l \text{ such that } g^k\big|_{C_i}(x,y,z)=y;
\end{aligned}$$

$$g''\big|_{C_i}(x,y,z) \;=\; g^k\big|_{C_i}(x, g^k\big|_{C_i}(y,x,y), g^k\big|_{C_i}(z,z,x)) = g^k\big|_{C_i}(z,z,x) = x,$$
$$\text{for any } 1 \le i \le l \text{ such that } g^k\big|_{C_i}(x,y,z) = z.$$

Finally, to make g'' acting correctly on red edges we set

$$g(x,y,z) = g''(f(x, f(y,z)), f(y, f(z,x)), f(z, f(x,y))).$$

The operation g is as required.

Next we show that for any $1 \le j \le l$ there is h^j such that $h_j\big|_{C_i}$ is the minority operation for $i \le j$. As usual, $h^1 = h_1$ gives the base case of induction. If h^{j-1} is obtained, then if $h^{j-1}\big|_{C_j}$ is the minority operation then set $h^j = h^{j-1}$. Otherwise, we may assume $h^{j-1}\big|_{C_j}(x,x,y) = x$, and h^j can be chosen to be $r(h_j(x,y,z), h^{j-1}(x,y,z))$ where $r(x,y) = h^{j-1}(x,x,y)$. Finally, set $p(x,y) = g(x,y,y)$,

$$h''(x,y,z) = p(h^l(x,y,z), x).$$

and

$$h(x,y,z) = h''(f(x, f(y,z)), f(y, f(z,x)), f(z, f(x,y))).$$

As is easily seen h satisfies the conditions required. □

Example 5 We now are able to show that the problem LIST H_2-COLORING, where H_2 is the graph from Fig. 1, is NP-complete. If this problem were tractable then there would exist polymorphisms f, g, h of $\varrho = \varrho_{H_2}$ specified in Proposition 5. We show that the restrictions of these operations onto $B = \{d, k\}$ cannot be a semilattice, majority, or affine operation. Notice first that if $f\big|_B$ is a semilattice operation with $f(k,d) = f(d,k) = d$ $[f(k,d) = f(d,k) = k]$ then so are $f\big|_{\{a,c\}}, f\big|_{\{b,c\}}, f\big|_{\{k,l\}}, f\big|_{\{k,m\}}$ and $f(a,c) = c, f(b,c) = c, f(k,l) = l, f(k,m) = m$ $[f(a,c) = a, f(b,c) = b, f(k,l) = k, f(k,m) = k]$. Indeed, for example, since f is a polymorphism of ϱ and $f(k,d) = f(d,k) = d$,

$$f\left(\begin{pmatrix} a \\ k \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right), \quad f\left(\begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} a \\ k \end{pmatrix}\right)$$

must be equal to $\begin{pmatrix} d \\ c \end{pmatrix}$, which means $f(a,c) = f(c,a) = c$. Analogously, if $g\big|_B$ is a majority operation or $h\big|_B$ is an affine operation then so are $g\big|_{\{a,c\}}, g\big|_{\{b,c\}}, g\big|_{\{k,l\}}, g\big|_{\{k,m\}}$ or $h\big|_{\{a,c\}}, h\big|_{\{b,c\}}, h\big|_{\{k,l\}}, h\big|_{\{k,m\}}$.

Suppose that $f\big|_B$ is a semilattice operation with $f(d,k) = d$. Then we have

$$f\left(\begin{pmatrix} a \\ k \end{pmatrix}, \begin{pmatrix} b \\ m \end{pmatrix}\right) = \begin{pmatrix} e \\ m \end{pmatrix}, \quad f\left(\begin{pmatrix} a \\ l \end{pmatrix}, \begin{pmatrix} b \\ k \end{pmatrix}\right) = \begin{pmatrix} e \\ l \end{pmatrix}$$

where $e = f(a,b)$. The first equality implies $f(a,b) = b$ while the second implies $f(a,b) = a$, a contradiction.

Then, let $g\big|_B$ is a majority operation, and e denote $g(a,b,c)$. Then the pairs

$$\begin{array}{rcl}
g\left(\begin{pmatrix}a\\k\end{pmatrix},\begin{pmatrix}b\\k\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}\right) & = & \begin{pmatrix}e\\k\end{pmatrix},\\
g\left(\begin{pmatrix}a\\l\end{pmatrix},\begin{pmatrix}b\\k\end{pmatrix},\begin{pmatrix}c\\l\end{pmatrix}\right) & = & \begin{pmatrix}e\\l\end{pmatrix},\\
g\left(\begin{pmatrix}a\\k\end{pmatrix},\begin{pmatrix}b\\m\end{pmatrix},\begin{pmatrix}c\\m\end{pmatrix}\right) & = & \begin{pmatrix}e\\m\end{pmatrix}
\end{array}$$

belong to ϱ, and hence, $e \in \{a,b\}$ follows from the first equality, $e \in \{a,c\}$, $e \in \{b,c\}$ from the second and the third equalities respectively, a contradiction.

Finally, it can be shown in a similar way that h also cannot be properly defined. $\square$

Along with operations f,g,h we will often use the operations $p(x,y) = g(x,y,y)$, $r(x,y) = h(x,x,y)$. The operation p is the first projection on every blue edge, the second projection on every yellow edge, and a semilattice operation on every red edge; while the operation r is the first projection on every yellow edge, the second projection on every blue edge, and a semilattice operation on every red edge.

By simple algebraic calculation it can be shown that the collection $\mathcal{A}' = \{\mathbb{A}'_1,\dots,\mathbb{A}'_n\}$ where $\mathbb{A}'_i = (A_i; f^{\mathbb{A}_i}, g^{\mathbb{A}_i}, h^{\mathbb{A}_i})$ satisfies the conditions of Theorem 8. Since $\mathrm{CSP}(\mathcal{A}) \subseteq \mathrm{CSP}(\mathcal{A}')$, the tractability of $\mathcal{A}$ will be proved if we show that $\mathcal{A}'$ is tractable. So, from now on we assume that $\mathcal{A}$ has only 3 basic operations of the form specified in Proposition 5. Notice that in this case $\mathcal{G}(\mathbb{A}'_i)$, $i \in \underline{n}$, has no pair of mutually inverse red edges, because f determine the direction of the red edges uniquely.

To conclude this section we survey several particular cases when the colouring of $\mathcal{G}(\mathbb{A}_1)$ is restricted.

CASE 1. All edges of all $\mathcal{G}(\mathbb{A}_i)$ are blue.
In this case h is the affine operation on every two-element set, hence it satisfies the conditions $h(x,y,y) = h(y,y,x) = x$. This means that h is a Mal'tsev operation and every $\mathbb{A}_i$ is a Mal'tsev algebra. The tractability of $\mathcal{A}$ follows from the results of [4].

CASE 2. All edges of all $\mathcal{G}(\mathbb{A}_i)$ are yellow.
In this case g is the majority operation on every two-element set, hence it satisfies the conditions $g(x,x,y) = g(x,y,x) = g(y,x,x) = x$. This means that g is a majority operation. Then the tractability of $\mathcal{A}$ follows from the results of [33, 34, 8].

CASE 3. All edges of all $\mathcal{G}(\mathbb{A}_i)$ are red.
In this case g,h can be neglected as they do not affect the set of term operations of $\mathcal{A}$. The operation f is idempotent, that is $f(x,x) = x$, commutative, that is $f(x,y) = f(y,x)$, and satisfies the condition $f(a,b) \in \{a,b\}$. Therefore, $(A_i; f)$ is a commutative conservative groupoid. As is proved in [7, 2], any collection of such algebras is tractable.

4.2 Connectedness

Let $\mathbb{A} = (A; F)$ be a conservative algebra. We use several forms of connectedness in $\mathcal{G}(\mathbb{A})$ defined as existing a path (directed or not) consisting of edges of some restricted set of colours. Vertices a, b are said to be *(strongly) r- [r+y-, r+b-]connected* if there is a (directed) path from a to b consisting of red [red and yellow, red and blue] edges. We then define *r- [r+y-, r+b-] connected components*, and *strongly r- [r+y-, r+b-]connected components* in a natural way.

It also will be convenient for us to introduce 3 binary relations on A. Let α, β, γ be reflexive binary relations defined as follows: α consists of red edges, β — of yellow edges, and γ — of blue ones.

Generally, subalgebras of direct products of conservative algebras are not conservative anymore, but still posses many of properties of conservative algebras. We call a subdirect products of conservative algebras a *semiconservative* algebras.

In particular, every semiconservative algebra can be treated as an edge-coloured graph. Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \ldots, \mathbb{D}_n$. The vertex set of the graph $\mathcal{G}(\mathbb{D})$ is $\mathbb{D}$; a pair $(\mathbf{a}, \mathbf{b})$, $\mathbf{a}, \mathbf{b} \in \mathbb{D}$, is a red [yellow, blue] edge of $\mathcal{G}(\mathbb{D})$ if and only if, for each $i \in \underline{n}$, $(\mathbf{a}[i], \mathbf{b}[i])$ is a red [yellow, blue] edge of $\mathcal{G}(\mathbb{D}_i)$. (Strongly) connectedness in $\mathcal{G}(\mathbb{D})$ is defined in the same way as for conservative algebras. We also introduce relations analogous to α, β, γ as follows: for $\mathbf{a}, \mathbf{b} \in \mathbb{D}$, $\langle \mathbf{a}, \mathbf{b} \rangle \in \alpha$ $[\beta, \gamma]$ iff $(\mathbf{a}[i], \mathbf{b}[i]) \in \alpha$ $[\beta, \gamma]$, for all $i \in \underline{n}$. To simplify the notation we do not specify the set on which relations α, β, γ are defined. Since this set is always clear from the context, this does not lead to a confusion.

If (a, b) is a red edge and $f(a, b) = b$ then we write $a \leq b$. For semiconservative algebras we use the same notation. We shall also use $\prec$ for the transitive closure of $\leq$.

We complete this section with several simple properties of operations f, p, r.

Lemma 2 *Let $\mathbb{D}$ be a semiconservative algebras. Then for any $\mathbf{x}, \mathbf{y} \in \mathbb{D}$*
(1) $\mathbf{x} \leq f(\mathbf{x}, \mathbf{y})$;
(2) $\mathbf{x}$ is strongly r+b-connected to $p(\mathbf{y}, \mathbf{x})$, $r(\mathbf{x}, \mathbf{y})$;
(3) $\mathbf{x}$ is strongly r+y-connected to $p(\mathbf{x}, \mathbf{y})$, $r(\mathbf{y}, \mathbf{x})$.

Proof: (1) is straightforward.

(2) Let $\mathbb{D}$ be a subdirect product of conservative algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$. We set $\mathbf{a} = f(\mathbf{x}, \mathbf{y})$ and $\mathbf{b} = p(\mathbf{y}, \mathbf{a})$. By (1), $\mathbf{x} \leq \mathbf{a}$, moreover, $\mathbf{a}[i] = \mathbf{x}[i]$ for all $i \in \underline{n}$ such that $\mathbf{x}[i] \not\leq \mathbf{y}[i]$. Therefore, $\mathbf{b}[i] = \mathbf{a}[i]$ whenever $\mathbf{y}[i] \leq \mathbf{x}[i] = \mathbf{a}[i]$, or $\mathbf{x}[i] \leq \mathbf{y}[i] = \mathbf{a}[i]$, or $\langle \mathbf{y}[i], \mathbf{x}[i] \rangle \in \beta$. Hence $\langle \mathbf{a}, \mathbf{b} \rangle \in \beta$, and $\mathbf{x}$ is strongly r+b-connected to $\mathbf{b}$. Finally, noticing that $\mathbf{b} = p(\mathbf{y}, \mathbf{x})$ we get the result.

The proof of (3) and for the operation r is quite similar. □

4.3 Orders, paths and maximal elements

Let $\mathbb{D}$ be a semiconservative algebra. For strongly r- [r+y-, r+b] connected components B, C of $\mathbb{D}$ we write $A \leq_r B$ $[A \leq_y B,\ A \leq_b B]$ if there are $b \in B, c \in C$ such that $b \leq c$. Clearly, the family of strongly r- [r+y-, r+b-] connected component endowed with the transitive closure $\prec_r$ [$\prec_y$, $\prec_b$] of $\leq_r$ [$\leq_y$, $\leq_b$] is a poset. The maximal elements of this poset will be called *maximal strongly r- [r+y-, r+b-] connected* components. For every $a \in \mathbb{D}$ there is a maximal strongly r- [r+y- r+b-] connected component C such that if a belongs to a strongly r- [r+y-, r+b-] connected component B then $B \prec_r C$ [$B \prec_y C$, $B \prec_b C$].

For short we shall call maximal strongly r-connected components *maximal components* and maximal strongly r+b-connected components *rb-maximal components*. Elements from a maximal [rb-maximal] component of a conservative algebra are called *maximal [rb-maximal] elements*. An element $\mathbf{a}$ of a semiconservative algebra $\mathbb{D}$ is called *maximal [rb-maximal]* if, for any $i \in \underline{n}$, $\mathbf{a}[i]$ is maximal [rb-maximal]. In Section 6 we show that every maximal [rb-maximal] element of $\mathbb{D}$ belongs to a maximal [rb-maximal] component of $\mathbb{D}$.

A sequence $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathbb{D}$ is called a *path* if $\mathbf{a}_i \leq \mathbf{a}_{i+1}$ for every i. It is called an *rb-path* if, for any i, either $\mathbf{a}_i \leq \mathbf{a}_{i+1}$ or $\langle \mathbf{a}_i, \mathbf{a}_{i+1} \rangle \in \gamma$.

Lemma 3 *Let $\mathbb{D}$ be a subdirect product of semiconservative algebras $\mathbb{D}_1, \mathbb{D}_2$, $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{D}$, $a = a_1, a_2, , \ldots, a_k \in \mathbb{D}_1$ a path. Then there are $b = b_1, b_2, \ldots, b_k \in \mathbb{D}_2$ such that $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_k \\ b_k \end{pmatrix}$ is a path in $\mathbb{D}$. Moreover, if b is maximal then all $b_1, \ldots, b_k$ are also maximal and belong to the same maximal component.*

Proof: Let $c_1, \ldots, c_k \in \mathbb{D}_2$ be such that $\begin{pmatrix} a_1 \\ c_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_k \\ c_k \end{pmatrix} \in \mathbb{D}$, and $c_1 = b$. Then set $b_1 = c_1$, and $\begin{pmatrix} a_i \\ b_i \end{pmatrix} = f\left(\begin{pmatrix} a_{i-1} \\ b_{i-1} \end{pmatrix}, \begin{pmatrix} a_i \\ c_i \end{pmatrix}\right)$. It is straightforward that $\begin{pmatrix} a_{i-1} \\ b_{i-1} \end{pmatrix} \leq \begin{pmatrix} a_i \\ b_i \end{pmatrix}$. Finally, if b is maximal, then, as $b_1, \ldots, b_k \in \widetilde{\mathcal{F}}(b)$, they all belong to the same maximal component. $\Box$

Lemma 4 *Let $\mathbb{D}$ be a subdirect product of semiconservative algebras $\mathbb{D}_1, \mathbb{D}_2$, $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{D}$, $a = a_1, a_2, , \ldots, a_k \in \mathbb{D}_1$ an rb-path. Then there are $b = b_1, b_2, \ldots, b_{2k} \in \mathbb{D}_2$ such that $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_3 \end{pmatrix}, \ldots, \begin{pmatrix} a_k \\ b_{2k} \end{pmatrix}$ is an rb-path in $\mathbb{D}$. Moreover, if b is rb-maximal then all $b_1, \ldots, b_k$ are also rb-maximal and belong to the same rb-maximal component.*

Proof: Let $c_1, \ldots, c_k \in \mathbb{D}_2$ be such that $\begin{pmatrix} a_1 \\ c_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_k \\ c_k \end{pmatrix} \in \mathbb{D}$, and $c_1 = b$. Then set $b_1 = c_1$. If $a_i \leq a_{i+1}$ then we set $\begin{pmatrix} a_i \\ b_{2i} \end{pmatrix} = \begin{pmatrix} a_i \\ b_{2i-1} \end{pmatrix}$ and $\begin{pmatrix} a_i \\ b_{2i+1} \end{pmatrix} = f\left(\begin{pmatrix} a_i \\ b_{2i-1} \end{pmatrix}, \begin{pmatrix} a_{i+1} \\ c_{i+1} \end{pmatrix}\right)$. If $\langle a_i, a_{i+1} \rangle \in \gamma$, then set $\begin{pmatrix} a_i \\ b_{2i} \end{pmatrix} = f\left(\begin{pmatrix} a_i \\ b_{i-1} \end{pmatrix}, \begin{pmatrix} a_{i+1} \\ c_{i+1} \end{pmatrix}\right)$ and $\begin{pmatrix} a_i \\ b_{2i+1} \end{pmatrix} = f\left(\begin{pmatrix} a_i \\ b_{2i} \end{pmatrix}, \begin{pmatrix} a_{i+1} \\ c_{i+1} \end{pmatrix}\right)$. Making use of Lemma 2(2), it is straightforward that the obtained tuples form an rb-path. Finally, if b is rb-maximal, then, as $b_1, \ldots, b_{2k}$ are strongly r+b-connected with b, they all belong to the same rb-maximal component. $\square$

Lemma 5 *Let $\mathbb{D}$ be a subdirect product of conservative $\mathbb{D}_1, \ldots, \mathbb{D}_n$.*

(1) For any maximal a, there is a maximal $\mathbf{a}$ such that $\mathbf{a}[1] = a$.

(2) For any $\mathbf{a} \in \mathbb{D}$, there is a maximal $\mathbf{b}$ such that $\mathbf{a} \prec \mathbf{b}$.

(3) For any rb-maximal a, there is an rb-maximal $\mathbf{a}$ such that $\mathbf{a}[1] = a$.

(4) For any $\mathbf{a} \in \mathbb{D}$, there is an rb-maximal $\mathbf{b}$ such that $\mathbf{a}$ is strongly r+b-connected to $\mathbf{b}$.

Proof:
(2) We prove by induction on $j \leq n$ that there is $\mathbf{b}_j$ such that $\mathbf{a} \prec \mathbf{b}_j$ and $\mathbf{b}_j[1], \ldots, \mathbf{b}_j[j]$ are maximal. In the base case of induction, $j = 0$, one may set $\mathbf{b}_0 = \mathbf{a}$.

Suppose that the claim is proved for $j - 1$. Then there is a path $\mathbf{b}_{j-1}[j] = c_1 \leq c_2 \leq \ldots \leq c_k$ such that c_k is maximal. By Lemma 3, this path can be extended to a path $\mathbf{b}_{j-1} = \mathbf{c}_1 \leq \mathbf{c}_2 \leq \ldots \leq \mathbf{c}_k$. Since, for any $i \leq j - 1$, the element $\mathbf{b}_{j-1}[i]$ is maximal and $\mathbf{b}_{j-1}[i] = \mathbf{c}_1[i] \leq \mathbf{c}_2[i] \leq \ldots \leq \mathbf{c}_k[i]$, the element $\mathbf{c}_k[i]$ is also maximal, and $\mathbf{b}_j$ can be chosen to be $\mathbf{c}_k$.

Finally, $\mathbf{b}_n$ is the required maximal tuple.

(1) Take $\mathbf{b} \in \mathbb{D}$ such that $\mathbf{b}[1] = a$. By part (2), there exists a maximal element $\mathbf{c} \in \mathbb{D}$ such that $\mathbf{b} \prec \mathbf{c}$. Therefore, these two elements are connected by a path $\mathbf{b} = \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k = \mathbf{c}$. Since $a = \mathbf{b}_1[1] \leq \mathbf{b}_2[1] \leq \ldots \leq \mathbf{b}_k[1]$, they all lie in the same maximal component. Therefore, there is a path $\mathbf{c}[1] = c_1 \leq c_2 \leq \ldots \leq c_l = a$. By Lemma 3, it can be extended to a path $\mathbf{c} = \mathbf{c}_1 \leq \mathbf{c}_2 \leq \ldots \leq \mathbf{c}_l$. Since $\mathbf{c}$ is maximal and $\mathbf{c} \prec \mathbf{c}_1, \ldots, \mathbf{c}_l$, the tuple $\mathbf{c}_l$ is as required.

The proof for (3),(4) is quite similar. $\square$

Lemma 6 *Let $\mathbb{D}$ be a subdirect product of semiconservative algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$, and $\mathbb{D}' = \mathrm{pr}_{\{2,\ldots,n\}}\mathbb{D}$.*

(1) If $\mathbb{D}'$ is strongly r-connected, then, for any maximal strongly r-connected component B of $\mathbb{D}_1$ and any $\mathbf{a} \in \mathbb{D}'$, there is $a \in B$ such that $\begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} \in \mathbb{D}$.

(2) If $\mathbb{D}'$ is strongly r+b-connected, then, for any maximal strongly r+b-connected component B of $\mathbb{D}_1$ and any $\mathbf{a} \in \mathbb{D}'$, there is $a \in B$ such that $\begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} \in \mathbb{D}$.

(3) If $\mathbb{D}'$ is strongly r+y-connected, then, for any maximal strongly r+y-connected component B of $\mathbb{D}_1$ and any $\mathbf{a} \in \mathbb{D}'$, there is $a \in B$ such that $\begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} \in \mathbb{D}$.

Proof: (1) Let $B \subseteq \mathbb{D}_1$ be a maximal strongly r-connected component and $D_B = \{\mathbf{a} \in \mathbb{D}' \mid \binom{a}{\mathbf{a}} \in \mathbb{D}$ for a certain $a \in B\}$. Suppose the contrary, $D_B \neq \mathbb{D}'$. Since $\mathbb{D}'$ is strongly r-connected, there are $\mathbf{a} \in D_B$ and $\mathbf{b} \in \mathbb{D}' - D_B$ such that $\mathbf{a} \leq \mathbf{b}$. There exists also $d, c \in \mathbb{D}_1$ with $\binom{c}{\mathbf{b}}, \binom{d}{\mathbf{a}} \in \mathbb{D}$ and $d \in B$, $c \in \mathbb{D}_1 - B$. By Lemma 2(1), $d' \in B$, and we get

$$\binom{d'}{\mathbf{b}} = f\left(\binom{d}{\mathbf{a}}, \binom{c}{\mathbf{b}}\right) \in \mathbb{D}.$$

Hence $\mathbf{b} \in D_B$, a contradiction.

(2) Let $B \subseteq \mathbb{D}_1$ be a maximal strongly r+b-connected component. Suppose that $D_B \neq \mathbb{D}'$. Since $\mathbb{D}'$ is strongly r+b-connected, there are $\mathbf{a} \in D_B$ and $\mathbf{b} \in \mathbb{D}' - D_B$ such that $\mathbf{a} \leq \mathbf{b}$ or $\langle \mathbf{a}, \mathbf{b} \rangle \in \gamma$ where γ is a relation on $\mathbb{D}'$. There exists also $d, c \in \mathbb{D}_1$ with $\binom{c}{\mathbf{b}}, \binom{d}{\mathbf{a}} \in \mathbb{D}$ and $d \in B$, $c \in \mathbb{D}_1 - B$. This means, in particular that $c \leq d$ or $\langle c, d \rangle \in \beta$. If $\mathbf{a} \leq \mathbf{b}$, then $d' = f(d, c) \in B$, and we get

$$\binom{d'}{\mathbf{b}} = f\left(\binom{d}{\mathbf{a}}, \binom{c}{\mathbf{b}}\right) \in \mathbb{D}.$$

Hence $\mathbf{b} \in D_B$, a contradiction. If $\langle \mathbf{a}, \mathbf{b} \rangle \in \gamma$ then, by Lemma 2(2), $d' = p(c, d) \in B$, and

$$\binom{d'}{\mathbf{b}} = p\left(\binom{c}{\mathbf{b}}, \binom{d}{\mathbf{a}}\right) \in \mathbb{D},$$

a contradiction.

(3) Let $B \subseteq \mathbb{D}_1$ be a maximal strongly r+y-connected component Suppose that $D_B \neq \mathbb{D}'$. As before, since $\mathbb{D}'$ is strongly r+y-connected, there are $\mathbf{a} \in D_B$ and $\mathbf{b} \in \mathbb{D}' - D_B$ such that $\mathbf{a} \leq \mathbf{b}$ or $\langle \mathbf{a}, \mathbf{b} \rangle \in \beta$ where β is a relation on $\mathbb{D}'$. There exists also $d, c \in \mathbb{D}_1$ with $\binom{c}{\mathbf{b}}, \binom{d}{\mathbf{a}} \in \mathbb{D}$ and $d \in B$, $c \in \mathbb{D}_1 - B$. If $\mathbf{a} \leq \mathbf{b}$, then we argue as in case (2). If $\langle \mathbf{a}, \mathbf{b} \rangle \in \beta$ then, by Lemma 2(3), $d' = p(d, c) \in B$, and we have

$$\binom{d'}{\mathbf{b}} = p\left(\binom{d}{\mathbf{a}}, \binom{c}{\mathbf{b}}\right) \in \mathbb{D},$$

a contradiction. □

5 Double-connected algebras

In this section we consider subdirect products of algebras that both strongly r+y-connected and strongly r+b-connected. We show that such products satisfy some properties that can be used when solving corresponding constraint satisfaction problems. Some statements we will prove for strongly r+y- and r+b-connected semiconservative algebras, some for strongly r+b-connected algebras.

5.1 Preliminary results

Lemma 7 *Let $\mathbb{D}$ be a subdirect product of strongly r+b-connected semiconservative algebras $\mathbb{D}_1, \mathbb{D}_2$. If there exists an element $a \in \mathbb{D}_1$ such that $\{a\} \times \mathbb{D}_2 \subseteq \mathbb{D}$, then $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2$.*

Proof: We prove by induction that $\{c\} \times \mathbb{D}_2 \subseteq \mathbb{D}$ for every $c \in \mathbb{D}_1$. The inclusion $\{a\} \times \mathbb{D}_2 \subseteq \mathbb{D}$ forms the induction base. Furthermore, suppose that the property is proved for all $d \in C \subseteq \mathbb{D}_1$. Since $\mathbb{D}_1$ is strongly r+b-connected, there are $d \in C$, $c \in \mathbb{D}_1 - C$ such that $d \leq c$ or $\langle d, c\rangle \in \gamma$. Set $B = \{b \in \mathbb{D}_2 \mid (c, b) \in \mathbb{D}\}$. Again by the strongly r+b-connectedness of $\mathbb{D}_2$, there are $b \in B$, $a \in \mathbb{D}_2 - B$ such that $b \leq a$ or $\langle a, b\rangle \in \gamma$. If $d \leq c$ or $b \leq a$ then we get a contradiction, because

$$\begin{pmatrix} c \\ a \end{pmatrix} = f\left(\begin{pmatrix} d \\ a \end{pmatrix}, \begin{pmatrix} c \\ b \end{pmatrix}\right) \in \mathbb{D} \qquad \text{if } d \leq c,$$
$$\begin{pmatrix} c \\ a \end{pmatrix} = f\left(\begin{pmatrix} c \\ b \end{pmatrix}, \begin{pmatrix} d \\ a \end{pmatrix}\right) \in \mathbb{D} \qquad \text{if } b \leq a.$$

If $\langle d, c\rangle, \langle b, a\rangle \in \gamma$ then

$$\begin{pmatrix} c \\ a \end{pmatrix} = h\left(\begin{pmatrix} c \\ b \end{pmatrix}, \begin{pmatrix} d \\ b \end{pmatrix}, \begin{pmatrix} d \\ a \end{pmatrix}\right) \in \mathbb{D},$$

a contradiction again. □

Let $\mathbb{D}$ be a semiconservative algebra. By a *strongly r+y/r+b-connected* component of $\mathbb{D}$ we mean the first set in the following sequence which is both strongly r+y- and strongly r+b-connected. Set B_1 to be a maximal strongly r+y-connected component of $\mathbb{D}$ if it is not strongly r+y-connected; set C_1 to be a maximal strongly r+b-connected component of $\mathbb{D}$ if it is strongly r+y-connected, but not strongly r+b-connected. Then set B_i to be a maximal strongly r+y-connected component of C_{i-1}, and C_i to be a maximal strongly r+b-connected component of B_i.

Lemma 8 *Let $\mathbb{D}$ be a subdirect product of semiconservative algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$, and $\mathbb{D}' = \mathrm{pr}_{\{2,\ldots,n\}}\mathbb{D}$. If $\mathbb{D}'$ is strongly r+y/r+b-connected, then, for any maximal strongly r+y/r+b-connected component B of $\mathbb{D}_1$ and any $\mathbf{a} \in \mathbb{D}'$, there is $a \in B$ such that $\begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} \in \mathbb{D}$.*

Proof: Let B be a strongly r+y/r+b-connected component resulted from the sequence $C_0 = B_0 = \mathbb{D}_1, B_1, C_1, \ldots, B_k, C_k = B$ where B_i is a maximal strongly r+y-connected component of C_{i-1}, and C_i is a strongly r+b-connected component of B_i. We prove by induction that for any $j \in \underline{k}$, $D_{B_j} = D_{C_J} = \mathbb{D}' = \mathrm{pr}_{\{2,\ldots,n\}}\mathbb{D}$, where D_A for $A \subseteq \mathbb{D}_1$ denotes the set $\{\mathbf{a} \in \mathbb{D}' \mid (a, \mathbf{a}) \in \mathbb{D}$ for a certain $a \in A\}$. The base case of induction, $j = 0$, is obvious.

Suppose that the required condition is proved for $j-1$. Applying Lemma 6(2),(3) to $\mathbb{D} \cap (C_{j-1} \times D_{C_{j-1}})$ and $\mathbb{D} \cap (B_j \times D_{B_j})$, we get $D_{B_j} = \mathbb{D}'$ or $D_{C_j} = \mathbb{D}'$, as required. □

Lemma 9 *Let $\mathbb{D}$ be a semiconservative algebra.*

(1) For every $a, b \in \mathbb{D}$ such that they belong to strongly r-connected components B, C, respectively, and $B \prec C$, there exists $n \in \mathbb{N}$ such that the equation

$$f(x_n, f(x_{n-1}, \ldots f(x_2, f(x_1, a)) \ldots)) = b$$

is solvable.

(2) For every $a, b \in \mathbb{D}$ such that they belong to strongly r+y-connected components B, C, respectively, and $B \prec_y C$, there exists $n \in \mathbb{N}$ such that the equation

$$r(x_n, r(x_{n-1}, \ldots r(x_2, r(x_1, a)) \ldots)) = b$$

is solvable.

(3) For every $a, b \in \mathbb{D}$ such that they belong to strongly r+b-connected components B, C, respectively, and $B \prec_b C$, there exists $n \in \mathbb{N}$ such that the equation

$$p(x_n, p(x_{n-1}, \ldots p(x_2, p(x_1, a)) \ldots)) = b$$

is solvable.

Proof: (1) By the definition of strongly r-connectedness, there are $a = a_0, a_1, \ldots, a_n = b$ such that $a_i \le a_{i+1}$. This implies $f(a_{i+1}, a_i) = a_{i+1}$, and the result follows.

(2) By the definition of strongly r+y-connectedness, there are $a = a_0, a_1, \ldots, a_n = b$ such that either $a_i \le a_{i+1}$ or $\langle a_i, a_{i+1} \rangle \in \beta$. This implies $r(a_{i+1}, a_i) = a_{i+1}$, and the result follows.

For (3) the proof is quite analogous. $\square$

Lemma 10 *Let $\mathbb{D}$ be a subdirect product of algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$ and $I_1, \ldots, I_k \subseteq \{1, \ldots, n\}$. Let also $\mathbb{D}_{I_t}$ denote the union of strongly r+y/r+b-connected components of $\mathrm{pr}_{I_t}\mathbb{D}$ [maximal r-, r+y-, r+b-connected components of $\mathrm{pr}_{I_t}\mathbb{D}$]. Then for any tuple $\mathbf{a}' = (a[i])_{i \in I_1} \in \mathbb{D}_{I_1}$ there exist $a[j] \in \mathbb{D}_j$, $j \in \{1, \ldots, n\} - I_1$, such that $\mathbf{a} = (a[1], \ldots, a[n]) \in \mathbb{D}$ and $\mathrm{pr}_{I_t}\mathbf{a} \in \mathbb{D}_{I_t}$ for each $t \in \{2, \ldots, k\}$.*

Proof: Let us consider first the case of r-, r+y-, and r+b-connected components. We prove by induction that for any $j \le k$ there is $\mathbf{a}_j \in \mathbb{D}$ such that $\mathrm{pr}_{I_1}\mathbf{a}_j$ is in the same maximal strongly r- [r+y-, r+b-] connected component as $\mathbf{a}'$, and for any $i \le j$, $\mathrm{pr}_{I_i}\mathbf{a}_j$ belongs to a maximal strongly r- [r+y-, r+b-]connected component. As $\mathbf{a}' \in \mathrm{pr}_{I_1}\mathbb{D}$, the base case of induction, $j = 1$ is trivial: $\mathbf{a}_1$ can be chosen to be any tuple from $\mathbb{D}$ with $\mathbf{a}' = \mathrm{pr}_{I_1}\mathbf{a}_1$.

Notice that, for any $\mathbf{b} \in \mathbb{D}$ and for any $i \le j$, the tuple $\mathrm{pr}_{I_i} f(\mathbf{a}_j, \mathbf{b})$ [$\mathrm{pr}_{I_i} p(f(\mathbf{a}_j, \mathbf{b}), \mathbf{a}_j)$, $\mathrm{pr}_{I_i} p(\mathbf{b}, f(\mathbf{a}_j, \mathbf{b}))$] belongs to the same strongly r- [r+y-, r+b-] connected component of $\mathrm{pr}_{I_i}\mathbb{D}$ as $\mathrm{pr}_{I_i}\mathbf{a}_j$. Then let $\mathbf{b}_1, \ldots, \mathbf{b}_l \in \mathbb{D}$ be such that

$$\begin{array}{l} f(\mathrm{pr}_{I_{j+1}}\mathbf{b}_l, f(\ldots f(\mathrm{pr}_{I_{j+1}}\mathbf{b}_2, f(\mathrm{pr}_{I_{j+1}}\mathbf{b}_1, \mathrm{pr}_{I_{j+1}}\mathbf{a}_j) \ldots)) \\ {[r(\mathrm{pr}_{I_{j+1}}\mathbf{b}_l, r(\ldots r(\mathrm{pr}_{I_{j+1}}\mathbf{b}_2, r(\mathrm{pr}_{I_{j+1}}\mathbf{b}_1, \mathrm{pr}_{I_{j+1}}\mathbf{a}_j) \ldots)),} \\ p(\mathrm{pr}_{I_{j+1}}\mathbf{b}_l, p(\ldots p(\mathrm{pr}_{I_{j+1}}\mathbf{b}_2, p(\mathrm{pr}_{I_{j+1}}\mathbf{b}_1, \mathrm{pr}_{I_{j+1}}\mathbf{a}_j) \ldots))] \end{array}$$

belongs to a maximal strongly r- [r+y-, r+b-] connected component. Obviously, $\mathbf{a}_{j+1} = f(\mathbf{b}_l, f(\dots f(\mathbf{b}_2, f(\mathbf{b}_1, \mathbf{a}_j)\dots))$ $[\mathbf{a}_{j+1} = r(\mathbf{b}_l, r(\dots r(\mathbf{b}_2, r(\mathbf{b}_1, \mathbf{a}_j)\dots))$, $\mathbf{a}_{j+1} = p(\mathbf{b}_l, p(\dots p(\mathbf{b}_2, p(\mathbf{b}_1, \mathbf{a}_j)\dots))]$ satisfies the required conditions.

Finally, by strongly r- [r+y-, r+b-] connectedness, there are $\mathbf{c}_1, \dots, \mathbf{c}_m \in \mathbb{D}$ such that

$$\begin{array}{rcl} f(\mathrm{pr}_{I_1}\mathbf{c}_m, f(\dots f(\mathrm{pr}_{I_1}\mathbf{c}_2, \dots, f(\mathrm{pr}_{I_1}\mathbf{c}_1, \mathrm{pr}_{I_1}\mathbf{a}_k))) &=& \mathbf{a}' \\ [r(\mathrm{pr}_{I_1}\mathbf{c}_m, r(\dots r(\mathrm{pr}_{I_1}\mathbf{c}_2, \dots, r(\mathrm{pr}_{I_1}\mathbf{c}_1, \mathrm{pr}_{I_1}\mathbf{a}_k))) &=& \mathbf{a}', \\ p(\mathrm{pr}_{I_1}\mathbf{c}_m, p(\dots p(\mathrm{pr}_{I_1}\mathbf{c}_2, \dots, p(\mathrm{pr}_{I_1}\mathbf{c}_1, \mathrm{pr}_{I_1}\mathbf{a}_k))) &=& \mathbf{a}']. \end{array}$$

Therefore, the tuple $f(\mathbf{c}_m, f(\dots f(\mathbf{c}_2, f(\mathbf{c}_1, \mathbf{a}_k)\dots))$ $[r(\mathbf{c}_m, r(\dots r(\mathbf{c}_2, r(\mathbf{c}_1, \mathbf{a}_k)\dots))$, $p(\mathbf{c}_m, p(\dots p(\mathbf{c}_2, p(\mathbf{c}_1, \mathbf{a}_k)\dots))]$ is as required.

In the case of strongly r+y/r+b-connected components, we set $\mathbb{D}^0 = \mathbb{D}$. Suppose that $\mathbb{D}^{i-1}$ is already defined. Then, if i is odd then there is $\mathbf{a} \in \mathbb{D}^{i-1}$ such that $\mathrm{pr}_{I_1}\mathbf{a} = \mathbf{a}'$ and $\mathrm{pr}_{I_t}\mathbf{a}$ belongs to a maximal strongly r+y-connected component B_t of $\mathrm{pr}_{I_t}\mathbb{D}^{i-1}$. Analogously, if i is even then there is $\mathbf{a} \in \mathbb{D}^{i-1}$ such that $\mathrm{pr}_{I_1}\mathbf{a} = \mathbf{a}'$ and $\mathrm{pr}_{I_t}\mathbf{a}$ belongs to a maximal strongly r+b-connected component C_t of $\mathrm{pr}_{I_t}\mathbb{D}^{i-1}$. We set

$$\mathbb{D}^i = \begin{cases} \{\mathbf{b} \in \mathbb{D}^{i-1} \mid \mathrm{pr}_{I_t}\mathbf{b} \in B_t, 2 \le t \le k\}, & \text{if } i \text{ is odd}, \\ \{\mathbf{b} \in \mathbb{D}^{i-1} \mid \mathrm{pr}_{I_t}\mathbf{b} \in C_t, 2 \le t \le k\}, & \text{if } i \text{ is even} \end{cases}$$

By what was proved above $\mathrm{pr}_{I_t}\mathbb{D}^i = B_t$ if i is odd and $\mathrm{pr}_{I_t}\mathbb{D}^i = C_t$ if i is even.

For a certain s we get $\mathbb{D}^s = \mathbb{D}^{s+1}$. This means that, for any $t \ge 2$, $\mathrm{pr}_{I_t}\mathbb{D}^s$ is both strongly r+y- and strongly r+b-connected. Moreover, $\mathrm{pr}_{I_t}\mathbb{D}^s$ is a strongly r+y/r+b-connected component of $\mathrm{pr}_{I_t}\mathbb{D}$. Since $\mathbf{a}' \in \mathrm{pr}_{I_1}\mathbb{D}^s$, the lemma is proved. □

5.2 Simple red-connected algebras

Recall that an algebra $\mathbb{A}$ is said to be *simple* if its only congruences are the equality relation and the total relation. A *polynomial* of $\mathbb{A}$ is an operation $f(x_1, \dots, x_k, a_1, \dots, a_l)$ where f is a term operation and $a_1, \dots, a_l \in \mathbb{A}$. It is well known that, if $\mathbb{A}$ is simple, then for any pair $a, b \in \mathbb{A}$ such that $a \neq b$ the reflexive-transitive closure of the binary relation $\{(f(a), f(b)) \mid f \text{ is a unary polynomial of } \mathbb{A}\}$ is the total relation.

Our first purpose is to show that a subdirect product of simple strongly r- [r+b-, r+y/r+b-] connected semiconservative algebras is of a very restricted form.

Proposition 6 *Let $\mathbb{D}$ be a subdirect product of simple strongly r- [r+b-, r+y/r+b-] connected semiconservative algebras $\mathbb{D}_1, \mathbb{D}_2$. Then $\mathbb{D}$ is either the graph of a bijective mapping from $\mathbb{D}_1$ to $\mathbb{D}_2$, or $\mathbb{D}_1 \times \mathbb{D}_2$.*

Proof: Clearly, it is enough to prove the lemma only for strongly r+b-connected algebras. Notice first, that if $\mathbb{D}$ is the graph of a mapping $\pi : \mathbb{D}_1 \to \mathbb{D}_2$, then the relation θ: $(a, b) \in \theta \iff \pi(a) = \pi(b)$ is a congruence of $\mathbb{D}_1$ and, since $\mathbb{D}_1$ is simple, π is a bijection. The same holds if $\mathbb{D}$ is the graph of a mapping from $\mathbb{D}_2$ onto $\mathbb{D}_1$.

Suppose that $\mathbb{D}$ is neither $\mathbb{D}_1 \times \mathbb{D}_2$ nor the graph of a bijective mapping, and that $|\mathbb{D}_1| + |\mathbb{D}_2|$ is the smallest number such that there exists a subdirect product of simple strongly r+b-connected semiconservative algebras with this property. We show that there is $b \in \mathbb{D}_1$ $[b \in \mathbb{D}_2]$ such that $\{b\} \times \mathbb{D}_2 \subseteq \mathbb{D}$ $[\mathbb{D}_1 \times \{b\} \subseteq \mathbb{D}]$.

For $a \in \mathbb{D}_1, b \in \mathbb{D}_2$ by B_a, C_b we denote the sets $\{c \mid (a,c) \in \mathbb{D}\}$, $\{c \mid (c,b) \in \mathbb{D}\}$ respectively.

Claim 1. For any $A \subset \mathbb{D}_1$ $[A \subset \mathbb{D}_2]$, there is $a \in \mathbb{D}_2$ $[a \in \mathbb{D}_1]$ and $b \in A$, $c \in \mathbb{D}_1 - A$ $[b \in A,\ c \in \mathbb{D}_2 - A]$ such that $(b,a),(c,a) \in \mathbb{D}$ $[(a,b),(a,c) \in \mathbb{D}]$.

Since $\mathbb{D}$ is not the graph of a mapping, there are $(d_1,e),(d_2,e) \in \mathbb{D}$ with $d_1 \neq d_2$. Consider the graph $(\mathbb{D}_1, F)$ where $(b,c) \in F$ if and only if there are unary polynomials f_1, f_2 of $\mathbb{D}$ such that $f_1\begin{pmatrix} d_1 \\ e \end{pmatrix} = \begin{pmatrix} b \\ e' \end{pmatrix}$, $f_2\begin{pmatrix} d_2 \\ e \end{pmatrix} = \begin{pmatrix} c \\ e' \end{pmatrix}$. Notice that, for any $(b,c) \in F$, the pair (b,c) belongs to the congruence generated by (d_1,d_2) and there is $e' \in \mathbb{D}_2$ such that $(b,e'),(c,e') \in \mathbb{D}$. Finally, as $\mathbb{D}_1$ is simple, $(\mathbb{D}_1, F)$ is connected, that implies the required statement.

Take an arbitrary $a \in \mathbb{D}_1$ and set $E_1 = \{a\}$, and for each $i > 0$

$$E_{i+1} = \begin{cases} \bigcup_{b \in E_i} B_b & \text{if } i \text{ is odd} \\ \bigcup_{b \in E_i} C_b & \text{if } i \text{ is even.} \end{cases}$$

By Claim 1, for each $i > 0$, $E_i \subset E_{i+2}$ unless $E_i = \mathbb{D}_1$ or $E_i = \mathbb{D}_2$. Therefore, for some $l > 1$, $E_l = \mathbb{D}_1$ or $E_l = \mathbb{D}_2$. Without loss of generality let $E_l = \mathbb{D}_2$, and $E_{l-1} \neq \mathbb{D}_1$, $E_{l-2} \neq \mathbb{D}_2$.

Claim 2. For each i, $1 \le i \le l$, E_i is a subalgebra of $\mathbb{D}_1$ or $\mathbb{D}_2$.

We prove the claim by induction. In the base case of induction $E_1 = \{a\}$ is a subalgebra, because $\mathbb{D}_1$ is idempotent. If E_i is a subalgebra, and $E_i \subseteq \mathbb{D}_1$, then for any $a_2, b_2, c_2 \in E_{i+1}$ there are $a_1, b_1, c_1 \in E_i$ such that $(a_1,a_2),(b_1,b_2),(c_1,c_2) \in \mathbb{D}$. Then

$$\begin{aligned}
\begin{pmatrix} f(a_1,b_1) \\ f(a_2,b_2) \end{pmatrix} &= f\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in \mathbb{D}, \\
\begin{pmatrix} g(a_1,b_1,c_1) \\ g(a_2,b_2,c_2) \end{pmatrix} &= g\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right) \in \mathbb{D}, \\
\begin{pmatrix} h(a_1,b_1,c_1) \\ h(a_2,b_2,c_2) \end{pmatrix} &= h\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right) \in \mathbb{D}.
\end{aligned}$$

Since $f(a_1,b_1), g(a_1,b_1,c_1), h(a_1,b_1,c_1) \in E_i$, we get $f(a_2,b_2), g(a_2,b_2,c_2), h(a_2,b_2,c_2) \in E_{i+1}$. The proof in the case $E_i \subseteq \mathbb{D}_2$ is analogous.

Thus, E_{l-1} is a proper subalgebra of $\mathbb{D}_1$ such that $\bigcup_{b \in E_{l-1}} B_b = \mathbb{D}_2$.

Define a sequence $\mathbb{B}_0, \mathbb{B}_1, \ldots, \mathbb{B}_k$ of algebras and a sequence of congruences $\theta_0, \theta_1, \ldots, \theta_k$ where θ_i is a congruence of $\mathbb{B}_i$ through the following rules.
1) $\mathbb{B}_0$ is a maximal strongly r+b-connected component of E_{l-1}.
2) Suppose that $\mathbb{B}_i$ is already defined. Let θ_i be its maximal congruence or the identity

relation if $\mathbb{B}_i$ is simple.
3) If $\mathbb{B}_i$ is a singleton, then $k = i$ and the process stops. Otherwise set $\mathbb{B}_{i+1}$ to be a maximal strongly r+b-connected component of a class of θ_i containing an element b with $|B_b| > 1$ (as we shall prove later, such an element exists).

Further, set $\mathbb{B}'_i = \mathbb{B}_i/_{\theta_i}$ and

$$\mathbb{D}^{(i)} = \{(a,b) \mid a = a'/_{\theta_i} \text{ where } a' \in \mathbb{B}_i,\ (a',b) \in \mathbb{D}\} \subseteq \mathbb{B}'_i \times \mathbb{A}_2.$$

We prove that, for every i, (i) for any $b \in \mathbb{D}_2$ there exists $a \in \mathbb{B}_i$ such that $(a,b) \in \mathbb{D}$, (ii) there exists $b \in \mathbb{B}_i$ such that $|B_b| > 1$, and (iii) $\mathbb{D}^{(i)} = \mathbb{B}'_i \times \mathbb{D}_2$.

If $i = 0$, then (i) holds by the choice of E_{l-1}, and Lemma 6. Furthermore, there is $a \in E_{l-1}$ such that $|B_a| > 1$. If a belongs to a maximal strongly r+b-connected component of E_{l-1} then (ii) can be satisfied. Otherwise, let $b, c \in B_a$. If $b \not\leq c$ and $b \not\leq f(c,b)$, then replace b with $b' = p(f(b,c),c)$; furthermore, if $\langle b', c\rangle \notin \gamma$ then $c \leq b'' = p(c,b')$. So, b, c can be assumed to be such that either $b \leq c$ or $\langle b,c\rangle \in \gamma$. By Lemma 6, there is d from a maximal strongly r+b-connected component A of E_{l-1} such that $\begin{pmatrix} d \\ b \end{pmatrix} \in \mathbb{D}$. If $b \leq c$ then

$$\begin{aligned}
\begin{pmatrix} d' \\ b \end{pmatrix} &= f\left(\begin{pmatrix} d \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) \in \mathbb{D} \\
\begin{pmatrix} d' \\ c \end{pmatrix} &= f\left(\begin{pmatrix} d \\ b \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix}\right) \in \mathbb{D}
\end{aligned}$$

where $d' \in A$. If $\langle b,c\rangle \in \gamma$ then

$$\begin{aligned}
\begin{pmatrix} d' \\ b \end{pmatrix} &= p\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} d \\ b \end{pmatrix}\right) \in \mathbb{D}, \\
\begin{pmatrix} d' \\ c \end{pmatrix} &= p\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} d \\ b \end{pmatrix}\right) \in \mathbb{D}
\end{aligned}$$

where $d' \in A$ by Lemma 2(2). In both cases $|B_{d'}| > 1$. Therefore, $\mathbb{D}^{(0)}$ is not the graph of a mapping, and, since $\mathbb{B}'_0$ is a simple algebra and $|\mathbb{B}'_0| + |\mathbb{D}_2| \leq |\mathbb{D}_1| + |\mathbb{D}_2|$, we get $\mathbb{D}^{(0)} = \mathbb{B}'_0 \times \mathbb{D}_2$.

Suppose that for $i-1$ the properties (i), (ii), (iii) hold. Then, for any $a' \in \mathbb{B}'_{i-1}$ we have $\{a'\} \times \mathbb{D}_2 \subseteq \mathbb{D}^{(i-1)}$, that is, by Lemma 6, for every $b \in \mathbb{D}_2$ there exists $a \in \mathbb{B}_i$ such that $(a,b) \in \mathbb{D}$, that proves (i) for i. By (ii) for $i-1$ the θ_{i-1}-class A containing $\mathbb{B}_i$ contains an element b such that $|B_b| > 1$. Arguing as in the previous paragraph, b can be chosen to be in $\mathbb{B}_i$. Finally, as $\mathbb{B}'_i$ is simple we have $\mathbb{D}^{(i)} = \mathbb{B}'_i \times \mathbb{D}_2$.

We have proved $\mathbb{D}^{(k)} = \mathbb{B}'_k \times \mathbb{D}_2$. Since $\mathbb{B}_k$ is a singleton, say, $\mathbb{B}_k = \{b\}$ this implies $\mathbb{B}_k = \mathbb{B}'_k$, that is $\{a\} \times \mathbb{D}_2 \subseteq \mathbb{D}$.

To complete the proof we just have to apply Lemma 7. □

Note that the second part of the proof is valid for a subdirect product not only simple strongly r+b-connected algebras.

Corollary 10 *Let $\mathbb{D}$ be a subdirect product of semiconservative algebras $\mathbb{D}_1, \mathbb{D}_2$ where $\mathbb{D}_2$ is simple strongly r+b-connected and $\mathbb{D}$ is not the graph of any mapping $\pi: \mathbb{D}_1 \to \mathbb{D}_2$. Then there exists $a \in \mathbb{D}_1$ such that $\{a\} \times \mathbb{D}_2 \subseteq \mathbb{D}$.*

To prove this we should put $\mathbb{B}_0$ equal to a maximal strongly connected component of $\mathbb{D}_1$.

Lemma 11 *Let $\mathbb{D}$ be a subdirect product of simple semiconservative strongly r+b-connected algebras $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ such that each of them contains a red or yellow edge. If $\mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{i,j}\mathbb{D}$ for every $i, j \in \{1, 2, 3\}$, then $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3$.*

Proof: Suppose without lost of generality that $|\mathbb{D}_1| \geq |\mathbb{D}_2| \geq |\mathbb{D}_3|$. For $a \in \mathbb{D}_1$ set

$$\mathbb{D}_a = \{(b_2, b_3) \mid (a, b_2, b_3) \in \mathbb{D}\}.$$

Notice that, for every $a \in \mathbb{D}_1$, $\mathbb{D}_a$ is a subalgebra of $\mathrm{pr}_{2,3}\mathbb{D}$, and, since $\mathrm{pr}_{1,2}\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2$, $\mathrm{pr}_{1,3}\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_3$, the algebra $\mathbb{D}_a$ is a subdirect product of $\mathbb{D}_2, \mathbb{D}_3$. By Lemma 6, $\mathbb{D}_a$ is either the graph of a bijective mapping or $\mathbb{D}_2 \times \mathbb{D}_3$.

Let us assume $\mathbb{D}_a = \mathbb{D}_2 \times \mathbb{D}_3$ for a certain $a \in \mathbb{D}_1$. Since $\mathbb{D}_2 \times \mathbb{D}_3$ and $\mathbb{D}_1$ are strongly r+b-connected, by Lemma 7, $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3$.

Now suppose that, for every $a \in \mathbb{D}_1$, the set $\mathbb{D}_a$ is the graph of a bijective mapping $\pi_a: \mathbb{D}_2 \to \mathbb{D}_3$. This immediately implies $|\mathbb{D}_2| = |\mathbb{D}_3|$, let us denote this number by k, and as $\mathrm{pr}_{2,3}\mathbb{D} = \mathbb{D}_2 \times \mathbb{D}_3$, there are at least k different relations of the form $\mathbb{D}_a$. Therefore, $|\mathbb{D}_1| = k$ and $|D_a| = k$ for any $a \in \mathbb{D}_1$. Moreover, $|\mathrm{pr}_{2,3}\mathbb{D}| = k^2$, which means $D_a \cap D_{a'} = \varnothing$ whenever $a \neq a'$, $a, a' \in \mathbb{D}_1$. The equivalence relation $\sim$ on $\mathrm{pr}_{2,3}\mathbb{D}$ where $(a, b) \sim (c, d)$ iff $(a, b), (c, d) \in \mathbb{D}_e$ for some $e \in \mathbb{D}_1$, is a congruence of $\mathrm{pr}_{2,3}\mathbb{D} = \mathbb{D}_2 \times \mathbb{D}_3$.

Since $\mathbb{D}_2$ is isomorphic to $\mathbb{D}_3$, $\mathbb{D}_2 \times \mathbb{D}_3$ can be treated as a subalgebra of $\mathbb{D}_2^2$. Recall that an element a of an algebra $\mathbb{A}$ is said to be *absorbing* if whenever $t(x, y_1, \ldots, y_n)$ is an $(n+1)$-ary term operation of $\mathbb{A}$ such that t depends on x and $(b_1, \ldots, b_n) \in A^n$, then $t(a, b_1, \ldots, b_n) = a$. A congruence θ of $\mathbb{A}^2$ is said to be *skew* if it is the kernel of no projection mapping of $\mathbb{A}^2$ onto its factors. $\mathbb{D}_2$ is a simple idempotent algebra, therefore, by results of [35] one of the following holds: (a) $\mathbb{D}_2$ is term equivalent to a module; (b) $\mathbb{D}_2$ has an absorbing element; or (c) $\mathbb{D}_2^2$ has no skew congruence. Case (a) is impossible, because $\mathbb{D}_2$ has a 2-element subalgebra term equivalent to a semilattice, but no module has such a subalgebra. If in case (b) a is an absorbing element, then $f(a, b) = a$ for any $b \in \mathbb{D}_2$ that contradicts the strong r/r+b-connectedness. Finally, case (c) is also impossible, because $\sim$ is skew congruence. □

Lemma 12 *Let $\mathbb{D}$ be a subdirect product of simple semiconservative strongly r+b-connected algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$ such that each of them contains a red or yellow edge. If $D_i \times D_j \subseteq \mathrm{pr}_{i,j}\mathbb{D}$ for every $i, j \in \underline{n}$, then $\mathbb{D} = \mathbb{D}_1 \times \ldots \times \mathbb{D}_n$.*

Proof: We prove the lemma by induction. The induction base $n = 2, 3$ was proved in Lemmas 6 and 11. Suppose that the lemma holds for each number less than n. Take $a \in \mathbb{D}_1$ and denote by $\mathbb{D}_a$ the set $\{(b_2, \ldots, b_n) \mid (a, b_2, \ldots, b_n) \in \mathbb{D}\}$. By Lemma 11 $\mathbb{D}_1 \times \mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{1,i,j}\mathbb{D}$ for every $2 \leq i, j \leq n$. Then $\mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{i,j}\mathbb{D}_a$ and by induction hypothesis $\mathbb{D}_a = \mathbb{D}_2 \times \ldots \times \mathbb{D}_n$ and the lemma is proved. $\Box$

Definition 15 *A relation* $\mathbb{D} \subseteq \mathbb{D}_1 \times \ldots \times \mathbb{D}_n$ *is said to be* almost trivial *if there exists an equivalence relation* θ *on the set* $\{1, \ldots, n\}$ *with classes* $I_1, \ldots, I_k$*, such that*

$$\mathbb{D} = \mathrm{pr}_{I_1}\mathbb{D} \times \ldots \times \mathrm{pr}_{I_k}\mathbb{D}$$

where $\mathrm{pr}_{I_j}\mathbb{D} = \{(a_{i_1}, \pi_{i_2}(a_{i_1}), \ldots, \pi_{i_l}(a_{i_1})) \mid a_{i_1} \in \mathbb{D}_{i_1}\}$, $I_j = \{i_1, \ldots, i_l\}$, *for certain bijective mappings* $\pi_{i_2}: \mathbb{D}_{i_1} \to \mathbb{D}_{i_2}, \ldots, \pi_{i_l}: \mathbb{D}_{i_1} \to \mathbb{D}_{i_l}$.

Proposition 7 *A subdirect product of simple semiconservative strongly r+b-connected algebras each of which contains a red or yellow edge is an almost trivial relation.*

Proof: Let $\mathbb{D}$ be a subdirect product of simple semiconservative r+b-connected algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$. We prove the proposition by induction on n. When $n = 1$ the result holds trivially.

We now prove the induction step. By Lemma 6, for any pair $i, j \in \underline{n}$ the projection $\mathrm{pr}_{i,j}\mathbb{D}$ is either $\mathbb{D}_i \times \mathbb{D}_j$, or the graph of a bijective mapping. Assume that there exist i, j such that $\mathrm{pr}_{i,j}\mathbb{D}$ is the graph of a mapping $\pi: \mathbb{D}_i \to \mathbb{D}_j$. By the inductive hypothesis $\mathrm{pr}_{\underline{n}-\{j\}}\mathbb{D}$ is almost trivial, and therefore can be represented in the form

$$\mathrm{pr}_{\underline{n}-\{j\}}\mathbb{D} = \mathrm{pr}_{I_1}\mathbb{D} \times \ldots \times \mathrm{pr}_{I_k}\mathbb{D}$$

where $I_1 \cup \ldots \cup I_k = \underline{n} - \{j\}$. Suppose, for simplicity, that i is the coordinate position in I_1, that is,

$$\begin{array}{rcl} \mathrm{pr}_{I_1}\mathbb{D} & = & \{(a_{i_1}, \ldots, a_{i_k}, a_i) \mid a_{i_1} \in \mathbb{D}_{i_1},\ a_{i_s} = \pi_{s1}(a_{i_1}) \\ & & \text{for } s \in \{2, \ldots, k\},\ a_i = \pi_i(a_{i_1})\}. \end{array}$$

Then

$$\begin{array}{rcl} \mathrm{pr}_{I_1 \cup \{j\}}\mathbb{D} & = & \{(a_{i_1}, \ldots, a_{i_k}, a_i, a_j) \mid a_{i_1} \in \mathbb{D}_{i_1},\ a_{i_s} = \pi_{s1}(a_{i_1}) \\ & & \text{for } s \in \{2, \ldots, k\},\ a_i = \pi_i(a_{i_1}),\ a_j = \pi\pi_i(a_{i_1})\}, \end{array}$$

and we have $\mathbb{D} = \mathrm{pr}_{I_1 \cup \{j\}}\mathbb{D} \times \ldots \times \mathrm{pr}_{I_k}\mathbb{D}$, as required.

Finally, if $\mathrm{pr}_{i,j}\mathbb{D} = \mathbb{D}_i \times \mathbb{D}_j$ for all $i, j \in \underline{n}$, then the result follows by Lemma 12. $\Box$

Corollary 11 *A subdirect product of simple semiconservative strongly r- [strongly r+y/r+b-] connected algebras is an almost trivial relation.*

Proof: To prove the corollary, it is enough to observe that every strongly r- or r+y/r+b-connected algebra is also strongly r+b-connected and contains a red or yellow edge. $\Box$

5.3 General double-connected algebras

Lemma 13 *Let $\mathbb{D}$ be a subdirect product of strongly connected r+y/r+b-connected algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$ where $\mathbb{D}_1$ is simple, $\mathrm{pr}_{2,\dots,n}\mathbb{D}$ is strongly r+y/r+b-connected, and $\mathrm{pr}_{1,i}\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_i$ for $i \in \{2, \dots, n\}$, Then $\mathbb{D} = \mathbb{D}_1 \times \mathrm{pr}_{2,\dots,n}\mathbb{D}$.*

Proof: We prove the lemma by induction on n. The case $n = 2$ is obvious. Consider the case $n = 3$. We proceed by induction on $|\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3|$. The trivial case $|\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3| = 3$ gives the base case of induction. Take a maximal congruence θ of $\mathbb{D}_3$, and a θ-block $C = d/_\theta$. Then set $\mathbb{D}'_3 = \mathbb{D}_3/_\theta$, and consider $\mathbb{D}' \subseteq \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3/_\theta$, $\mathbb{D}'' \subseteq \mathbb{D}$ such that

$$\begin{aligned}
\mathbb{D}' &= \{(a[1], a[2], a[3]/_\theta) \mid (a[1], a[2], a[3]) \in \mathbb{D}\}\\
\mathbb{D}'' &= \{(a[1], a[2], a[3]) \mid (a[1], a[2], a[3]) \in \mathbb{D}, a[3] \in C\}.
\end{aligned}$$

Set also $\mathbb{D}'''$ to be a strongly r+y/r+b-connected component of $\mathbb{D}''$, and $\overline{C} = \mathrm{pr}_3\mathbb{D}'''$. The induction hypothesis implies that $\mathbb{D}' = \mathbb{D}_1 \times \mathrm{pr}_{2,3}\mathbb{D}'$. By Lemma 10, two cases are possible: $\mathrm{pr}_{2,3}\mathbb{D}' = \mathbb{D}_2 \times \mathbb{D}_3/_\theta$, or there is a mapping $\pi : \mathbb{D}_2 \to \mathbb{D}'_3$ such that $\mathrm{pr}_{2,3}\mathbb{D}' = \{\begin{pmatrix} a \\ \pi(a) \end{pmatrix} \mid a \in \mathbb{D}_2\}$.

Case 1. $\mathrm{pr}_{2,3}\mathbb{D}'$ is the graph of a mapping $\pi: \mathbb{D}_2 \to \mathbb{D}_3/_\theta$.

By Lemma 10, $\overline{B} = \mathrm{pr}_2\mathbb{D}'''$ is a strongly r+y/r+b-connected component of $B = \pi^{-1}(d)$. Since for each $(a, b) \in \mathbb{D}_1 \times B \subseteq \mathrm{pr}_{1,2}\mathbb{D}$ there is $c \in C$ with $(a, b, c) \in \mathbb{D}$, we have $\mathrm{pr}_{1,2}\mathbb{D}'' = \mathbb{D}_1 \times B$. Furthermore, again by Lemma 10, $\mathbb{D}_1 \times \overline{B}$ is a strongly r+y/r+b-connected component of $\mathbb{D}_1 \times B$, hence, $\mathrm{pr}_{1,2}\mathbb{D}''' = \mathbb{D}_1 \times \overline{B}$.

Since $|\mathbb{D}_1| + |\overline{B}| + |\overline{C}| < |\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3|$, and $\mathrm{pr}_{2,3}\mathbb{D}'''$ is strongly r+y/r+b-connected, inductive hypothesis implies $\mathbb{D}_1 \times \mathrm{pr}_{2,3}\mathbb{D}''' \subseteq \mathbb{D}'''$. In particular, there is $(a, b) \in \mathrm{pr}_{2,3}\mathbb{D}''' \subseteq \mathrm{pr}_{2,3}\mathbb{D}$ such that $\mathbb{D}_1 \times \{(a, b)\} \subseteq \mathbb{D}$. To finish the proof we just apply Lemma 7.

Case 2. $\mathrm{pr}_{2,3}\mathbb{D}' = \mathbb{D}_2 \times \mathbb{D}_3/_\theta$.

Since $|\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3/_\theta| < |\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3|$, $\mathbb{D}_3/_\theta$ is simple, and $\mathrm{pr}_{1,2}\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2$, by inductive hypothesis, $\mathbb{D}' = \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3/_\theta$. Therefore, $\mathrm{pr}_{1,2}\mathbb{D}'' = \mathbb{D}_1 \times \mathbb{D}_2$. By Lemma 6, since $\mathbb{D}_1 \times \mathbb{D}_2$ is strongly r+y/r+b-connected, $\mathrm{pr}_{1,2}\mathbb{D}''' = \mathbb{D}_1 \times \mathbb{D}_2$. Then we argue as in Case 1.

Let us assume that the lemma is proved for $n-1$. Then $\mathrm{pr}_1\mathbb{D} \times \mathrm{pr}_{3,\dots,n}\mathbb{D} \subseteq \mathrm{pr}_{1,3,\dots,n}\mathbb{D}$. Denoting $\mathrm{pr}_{3,\dots,n}\mathbb{D}$ by $\mathbb{D}'$ we have $\mathbb{D} \subseteq \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}'$, and the conditions of the lemma hold for this subdirect product. Thus $\mathbb{D} = \mathbb{D}_1 \times \mathrm{pr}_{2,\dots,n}\mathbb{D}$ as required. $\square$

6 Connectedness, rectangularity and decomposition

6.1 Connectedness for semiconservative algebras

Proposition 8 *(1) A subdirect product of strongly r-connected conservative algebras is strongly r-connected.*

(2) A subdirect product of strongly r+b-connected conservative algebras is strongly r+b-connected.

(3) A subdirect product of strongly r+y-connected conservative algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$ such that, for any $i \in \underline{n}$, each r-connected component of $\mathbb{D}_i$ is strongly r-connected, is strongly r+y-connected.

Proof: We start with (2).

Let $\mathbb{D}$ be a subdirect product of simple conservative strongly r+b-connected algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$. Without loss of generality, let us assume that $\mathbb{D}_1, \dots, \mathbb{D}_m$ contain a red or yellow edge while $\mathbb{D}_{m+1}, \dots, \mathbb{D}_n$ do not. We use $\mathbb{D}^1$ to denote $\mathrm{pr}_{1,\dots,m}\mathbb{D}$ and $\mathbb{D}^2$ to denote $\mathrm{pr}_{m+1,\dots,n}\mathbb{D}$. It is not hard to see that Proposition 7 implies that $\mathbb{D}^1$ is strongly r+b-connected.

Take $\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}, \begin{pmatrix}\mathbf{c}\\ \mathbf{d}\end{pmatrix} \in \mathbb{D} \subseteq \mathbb{D}^1 \times \mathbb{D}^2$. There is an rb-path $\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k = \mathbf{c}$ and tuples $\mathbf{b} = \mathbf{b}_1\mathbf{b}_2, \dots, \mathbf{b}_k \in \mathbb{D}^2$ such that $\begin{pmatrix}\mathbf{a}_1\\ \mathbf{b}_1\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ \mathbf{b}_2\end{pmatrix}, \dots, \begin{pmatrix}\mathbf{a}_k\\ \mathbf{b}_k\end{pmatrix} \in \mathbb{D}$ Clearly, any two elements of $\mathbb{D}^2$ constitute a blue edge. We build an rb-path from $\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$ to $\begin{pmatrix}\mathbf{c}\\ \mathbf{d}\end{pmatrix}$ as follows. Set $\begin{pmatrix}\mathbf{a}_1\\ \mathbf{d}_1\end{pmatrix} = \begin{pmatrix}\mathbf{a}_1\\ \mathbf{b}_1\end{pmatrix} = \begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$. Then, if $\begin{pmatrix}\mathbf{a}_i\\ \mathbf{d}_{2i-1}\end{pmatrix}$ is already constructed and $\langle \mathbf{a}_i, \mathbf{a}_{i+1}\rangle \in \gamma$ then set $\begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{d}_{2i}\end{pmatrix} = \begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{c}_{i+1}\end{pmatrix}$ and $\begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{d}_{2i+1}\end{pmatrix} = \begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{d}_{2i}\end{pmatrix}$. If $\mathbf{a}_i \le \mathbf{a}_{i+1}$, then set $\begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{d}_{2i}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_i\\ \mathbf{d}_{2i-1}\end{pmatrix}, \begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{c}_{i+1}\end{pmatrix}\right)$ and $\begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{d}_{2i}\end{pmatrix} = \begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{c}_{i+1}\end{pmatrix}$. As is easily seen the sequence $\begin{pmatrix}\mathbf{a}_1\\ \mathbf{d}_1\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ \mathbf{d}_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ \mathbf{d}_3\end{pmatrix}, \dots, \begin{pmatrix}\mathbf{a}_k\\ \mathbf{c}_{2k-1}\end{pmatrix} = \begin{pmatrix}\mathbf{c}\\ \mathbf{d}_{2k-1}\end{pmatrix}$ is an rb-path. Finally, as $\begin{pmatrix}\mathbf{c}\\ \mathbf{d}_{2k-1}\end{pmatrix}$ and $\begin{pmatrix}\mathbf{c}\\ \mathbf{d}\end{pmatrix}$ constitute a blue edge, $\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}, \begin{pmatrix}\mathbf{c}\\ \mathbf{d}\end{pmatrix}$ are strongly r+b-connected.

We prove the proposition by induction on the number of non-simple factors.

Notice first that if $\mathbb{D}$ is a conservative algebra, and θ is its congruence, then, for any θ-blocks $a/_\theta$, $b/_\theta$ and any $c \in a/_\theta, d \in b/_\theta$, the pairs $(a, b), (c, d)$ are both edges of the same colour, and moreover, if $a \le b$ then $c \le d$. Indeed, if, for instance, $a \le b$ but $d \le c$ then $f(a, b) = b$ while $f(c, d) = c$ that destroys the condition $\langle f(a, b), f(c, d)\rangle \in \theta$.

Let $\mathbb{D}$ be a subdirect product of conservative strongly r+b-connected algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$, and $\mathbb{D}_n$ not simple. Take a maximal congruence θ of $\mathbb{D}_n$, and consider the algebra

$$\mathbb{D}' = \{(a[1], \dots, a[n-1], a[n]/_\theta) \mid (a[1], \dots, a[n-1], a[n]) \in \mathbb{D}\}.$$

The algebra $\mathbb{D}'$ is a subdirect product of $\mathbb{D}_1, \dots, \mathbb{D}_{n-1}, \mathbb{D}_n/_\theta$, all of these algebras are strongly r+b-connected, and as $\mathbb{D}/_\theta$ is simple. Hence, they have one non-simple algebra less than $\mathbb{D}_1, \dots, \mathbb{D}_n$. Therefore, by inductive hypothesis, $\mathbb{D}'$ is strongly r+b-connected.

For any $\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{b}\\ b\end{pmatrix} \in \mathbb{D}$ (here $\mathbf{a}, \mathbf{b} \in \mathrm{pr}_{1,\dots,n-1}\mathbb{D}$) the corresponding tuples $\begin{pmatrix}\mathbf{a}\\ a/_\theta\end{pmatrix}$, $\begin{pmatrix}\mathbf{b}\\ b/_\theta\end{pmatrix}$ are r+b-connected. Suppose first that $a/_\theta \neq b/_\theta$. Then there is a sequence $\begin{pmatrix}\mathbf{a}\\ a/_\theta\end{pmatrix} = \begin{pmatrix}\mathbf{a}_0\\ a_0/_\theta\end{pmatrix}, \begin{pmatrix}\mathbf{a}_1\\ a_1/_\theta\end{pmatrix}, \dots, \begin{pmatrix}\mathbf{a}_k\\ a_k/_\theta\end{pmatrix} = \begin{pmatrix}\mathbf{b}\\ b/_\theta\end{pmatrix}$ where $\begin{pmatrix}\mathbf{a}_i\\ a_i/_\theta\end{pmatrix} \leq \begin{pmatrix}\mathbf{a}_{i+1}\\ a_{i+1}/_\theta\end{pmatrix}$ or $\left\langle \begin{pmatrix}\mathbf{a}_{i+1}\\ a_{i+1}/_\theta\end{pmatrix}, \begin{pmatrix}\mathbf{a}_i\\ a_i/_\theta\end{pmatrix} \right\rangle \in \gamma$. The observation above implies that $a_i \leq a_{i+1}$, $\langle a_i, a_{i+1}\rangle \in \gamma$ if and only if the similar connection is valid for $a_i/_\theta, a_{i-1}/_\theta$, and also for $\begin{pmatrix}\mathbf{a}_{i+1}\\ a_{i+1}\end{pmatrix}, \begin{pmatrix}\mathbf{a}_i\\ a_i\end{pmatrix}$. Therefore, $\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix} = \begin{pmatrix}\mathbf{a}_0\\ a_0\end{pmatrix}$ and $\begin{pmatrix}\mathbf{a}_k\\ a_k\end{pmatrix} = \begin{pmatrix}\mathbf{b}\\ b\end{pmatrix}$ are and r+b-connected.

Now suppose that $a/_\theta = b/_\theta$. Since $|\mathbb{D}_n/_\theta| > 1$, there is $\begin{pmatrix}\mathbf{c}\\ c\end{pmatrix} \in \mathbb{D}$ with $a/_\theta \neq c/_\theta$. By what is already proved, $\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{c}\\ c\end{pmatrix}$ and $\begin{pmatrix}\mathbf{c}\\ c\end{pmatrix}, \begin{pmatrix}\mathbf{b}\\ b\end{pmatrix}$ are r+b-connected. Hence, $\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{b}\\ b\end{pmatrix}$ are also r+b-connected.

(1) can be proved in a very similar way, but the base case of induction follows straightforwardly from Proposition 7.

(3) We prove the lemma by induction on n. The base case of induction, $n = 1$, is obvious. So, suppose that the result is proved for all subdirect products of $n-1$ strongly r+y-connected algebras.

Claim 1. Let $B_1, \dots, B_n$ be strongly r-connected components of $\mathbb{D}_1, \dots, \mathbb{D}_n$ respectively. Then either $\mathbb{D}' = \mathbb{D} \cap (B_1 \times \dots \times B_n) = \varnothing$ or $\mathbb{D}'$ is strongly r-connected.

Let us suppose that $\mathbb{D}' \neq \varnothing$, say, $(\mathbf{a}, a) \in \mathbb{D}'$ where $\mathbf{a} \in \mathbb{D}'' = \mathrm{pr}_{\underline{n-1}}\mathbb{D} \cap (B_1 \times \dots \times B_{n-1})$, and $B'_n = \mathrm{pr}_n\mathbb{D}'$. Since, by induction hypothesis, $\mathbb{D}''$ is strongly r-connected, by Lemma 6(1), if B''_n is a maximal strongly r-connected component of B'_n, then for any $\mathbf{b} \in \mathbb{D}''$ there is $b \in B''_n$ such that $(\mathbf{b}, b) \in \mathbb{D}$.

Next we show that $B'_n = B_n$. Since B_n is strongly r-connected it is enough to prove that if $b, c \in B_n$ are such that $b \leq c$ and $(\mathbf{b}, b) \in \mathbb{D}$ for some $\mathbf{b} \in \mathbb{D}''$ then there exists $\mathbf{c} \in \mathbb{D}''$ with $(\mathbf{c}, c) \in \mathbb{D}$. Take $\mathbf{d} \in \mathrm{pr}_{\underline{n-1}}\mathbb{D}$ with $(\mathbf{d}, c) \in \mathbb{D}$. For

$$\begin{pmatrix}\mathbf{c}\\ c\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{b}\\ b\end{pmatrix}, \begin{pmatrix}\mathbf{d}\\ c\end{pmatrix}\right) \in \mathbb{D}$$

we have $\mathbf{b} \leq \mathbf{c}$, and therefore $\mathbf{c} \in \mathbb{D}''$.

We have proved that $\mathbb{D}'$ is a subdirect product of $B_1, \dots, B_n$. By Proposition 8(1), $\mathbb{D}'$ is strongly r-connected.

Claim 2. Let $\mathbb{D}'$ denote $\mathrm{pr}_{\underline{n-1}}\mathbb{D}$. Any tuples of the form $(\mathbf{a}, a), (\mathbf{a}, b) \in \mathbb{D}$ where $\mathbf{a} \in \mathbb{D}'$ are strongly r+y-connected.

Suppose the contrary, there are $(\mathbf{a}, a), (\mathbf{a}, b) \in \mathbb{D}$ which are not strongly r+y-connected. Notice first that if $a \le b$ or $\langle a, b\rangle \in \beta$ the tuples are strongly r+y-connected by definition, and if $b \le a$ then they are strongly r+y-connected by Claim 1. So, we may assume $\langle a, b\rangle \in \gamma$. Since $\mathbb{D}_n$ is strongly r+y-connected, there are $a = a_1, a_2, \dots, a_k = b$ such that $a_i < a_{i+1}$ or $\langle a_i, a_{i+1}\rangle \in \beta$. Without loss of generality, we may assume that a, b and k are such that k is the least possible. Let $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{D}'$ be such that $\mathbf{a}_1 = \mathbf{a}$ and $(\mathbf{a}_i, a_i) \in \mathbb{D}$.

Suppose first that $k > 3$. Then $a_2 \not< b$ and $\langle a_2, b\rangle \notin \beta$. If $b < a_2$ then a_2, b lie in the same strongly r-connected component. If also $a < a_2$, then a, b in the same strongly r-connected component, and we get the result by Claim 1. Therefore, $\langle a, a_2\rangle \in \beta$, and

$$\begin{pmatrix}\mathbf{a}'\\ a\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right), \begin{pmatrix}\mathbf{a}''\\ a_2\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}\right), \begin{pmatrix}\mathbf{a}'\\ b\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}, \begin{pmatrix}\mathbf{a}'\\ a\end{pmatrix}\right) \in \mathbb{D}.$$

Since $\mathbf{a} \le \mathbf{a}'$, if we prove that $(\mathbf{a}', a), (\mathbf{a}' b)$ are strongly r+y-connected, then by Claim 1, we get the result. As $\langle p(\mathbf{a}', \mathbf{a}''), \mathbf{a}'\rangle \in \beta$, we get

$$\begin{pmatrix}\mathbf{a}'\\ a_2\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}'\\ b\end{pmatrix}, p\left(\begin{pmatrix}\mathbf{a}'\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{a}''\\ a_2\end{pmatrix}\right)\right) \in \mathbb{D}.$$

By Claim 1, $(\mathbf{a}', a_2)$ and $(\mathbf{a}', b)$ are strongly r+y-connected. Hence, $(\mathbf{a}, a), (\mathbf{a}, b)$ are strongly r+y-connected, a contradiction.

If $\langle a_2, b\rangle \in \gamma$ then by replacing $\mathbf{a}_2$ with $p(\mathbf{a}_2, \mathbf{a})$ we may assume that $\langle \mathbf{a}[i], \mathbf{a}_2[i]\rangle \notin \gamma$ for all $i \in \underline{n-1}$. For the tuple $\begin{pmatrix}\mathbf{a}'\\ a'\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right)$ we have $\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix} \le \begin{pmatrix}\mathbf{a}'\\ a'\end{pmatrix}$ and $\left\langle \begin{pmatrix}\mathbf{a}'\\ a'\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right\rangle \in \beta$. This means that $\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}$ and $\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}$ are strongly r+y-connected. Furthermore,

$$\begin{pmatrix}\mathbf{a}_2\\ b\end{pmatrix} = r\left(\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}\right) \in \mathbb{D}.$$

By the assumption made, $\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}$ and $\begin{pmatrix}\mathbf{a}_2\\ b\end{pmatrix}$ are strongly r+y-connected, a contradiction.

Then, let $k = 3$. If $a \le a_2 \le b$ then the result follows from Claim 1. Therefore, either $\langle a, a_2\rangle \in \beta$ or $\langle a_2, b\rangle \in \beta$. Suppose that $\langle a, a_2\rangle \in \beta$. As before, we may assume that $\langle \mathbf{a}[i], \mathbf{a}_2[i]\rangle \in \gamma$ for no $i \in \underline{n-1}$. Then $\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix} \le \begin{pmatrix}\mathbf{a}'\\ a\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right)$ and $\left\langle \begin{pmatrix}\mathbf{a}'\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right\rangle \in \beta$. If $a_2 < b$ then $\begin{pmatrix}\mathbf{a}_2\\ b\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}\right) \in \mathbb{D}$. Since $\mathbf{a}'$ and $\mathbf{a}$ are strongly r+y-connected, by Claim 1, the result follows. If $\langle a_2, b\rangle \in \beta$ then set $\begin{pmatrix}\mathbf{a}''\\ a_2\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}\right)$. We have $\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix} \le \begin{pmatrix}\mathbf{a}''\\ a_2\end{pmatrix}$, $\left\langle \begin{pmatrix}\mathbf{a}''\\ a_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}\right\rangle \in \beta$. Thus, $(\mathbf{a}_2, a_2), (\mathbf{a}, b)$ are strongly r+y-connected, and so are $(\mathbf{a}, a), (\mathbf{a}, b)$.

Finally, if $\langle a_2, b\rangle \in \beta$ then the result can be proved by the same argument swapping a and b. The claim is proved.

Finally take $(\mathbf{a}, a), (\mathbf{b}, b) \in \mathbb{D}$ where $\mathbf{a}, \mathbf{b} \in \mathbb{D}'$. Since $\mathbb{D}'$ is strongly r+y-connected, there are $\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k = \mathbf{b}$ such that $\mathbf{a}_i \leq \mathbf{a}_{i+1}$ or $\langle \mathbf{a}_i, \mathbf{a}_{i+1} \rangle \in \beta$. These tuples can be extended to tuples from $\mathbb{D}$: there are $a_1, \ldots, a_k \in \mathbb{D}_n$ with $(\mathbf{a}_i, a_i) \in \mathbb{D}$, $i \in \underline{k}$. We prove that, for every i, $(\mathbf{a}_i, a_i)$ and $(\mathbf{a}_{i+1}, a_{i+1})$ are strongly r+y-connected.

First, suppose $\mathbf{a}_i \leq \mathbf{a}_{i+1}$. If $a_i \leq a_{i+1}$, then $\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix} \leq \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}$. If $a_{i+1} \leq a_i$ then the result follows by Claim 1. If $\langle a_i, a_{i+1} \rangle \in \beta$, then $\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix} \leq \begin{pmatrix} \mathbf{a}_{i+1} \\ a_i \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}\right)$ and $\left\langle \begin{pmatrix} \mathbf{a}_{i+1} \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix} \right\rangle \in \beta$. Finally, if $\langle a_i, a_{i+1} \rangle \in \gamma$ then $\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix} \leq \begin{pmatrix} \mathbf{a}_{i+1} \\ a_i \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}\right)$ and $\begin{pmatrix} \mathbf{a}_{i+1} \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}$ are strongly r+y-connected by Claim 2.

Then, let $\langle \mathbf{a}_i, \mathbf{a}_{i+1} \rangle \in \beta$. If $\langle a_i, a_{i+1} \rangle \in \beta$ then $\left\langle \begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix} \right\rangle \in \beta$. If $\langle a_i, a_{i+1} \rangle \in \gamma$ or $a_i \leq a_{i+1}$ then $\left\langle \begin{pmatrix} \mathbf{a}_i \\ a_{i+1} \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix} \right\rangle \in \beta$ where $\begin{pmatrix} \mathbf{a}_i \\ a_{i+1} \end{pmatrix} = r\left(\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}\right)$, and $\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_i \\ a_{i+1} \end{pmatrix}$ are strongly r+y-connected by Claim 2. Finally, if $a_{i+1} \leq a_i$ then $\left\langle \begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix} \right\rangle \in \beta$ where $\begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}, \begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}\right)$, and $\begin{pmatrix} \mathbf{a}_{i+1} \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}$ are strongly r+y-connected by Claim 1. $\square$

Obviously, if $\mathbb{D}$ is a subdirect product of conservative algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$ and is strongly r- [r+b-, r+y-] connected, then $\mathbb{D}_1, \ldots, \mathbb{D}_n$ are also r- [r+b-, r+y-] connected. Thus, Proposition 8 holds also for semiconservative algebras.

Corollary 12 *(1) A subdirect product of strongly r-connected conservative algebras is strongly r-connected.*

(2) A subdirect product of strongly r+b-connected conservative algebras is strongly r+b-connected.

(3) A subdirect product of strongly r+y-connected conservative algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$ such that, for any $i \in \underline{n}$, each r-connected component of $\mathbb{D}_i$ is strongly r-connected, is strongly r+y-connected.

Corollary 13 *(1) Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \ldots, \mathbb{D}_n$ such that, for any $i \in \underline{n}$, every r-connected component of $\mathbb{D}_i$ is strongly r-connected. Then, for any $B_1, \ldots, B_n$ where B_i is an r-connected component of $\mathbb{D}_i$, $\mathbb{D} \cap (B_1 \times \ldots \times B_n)$ is either empty or strongly r-connected. (2) Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \ldots, \mathbb{D}_n$ such that, for any $i \in \underline{n}$, every r+b-connected component of $\mathbb{D}_i$ is strongly r+b-connected. Then, for any $B_1, \ldots, B_n$ where B_i is an r+b-connected component of $\mathbb{D}_i$, $\mathbb{D} \cap (B_1 \times \ldots \times B_n)$ is either empty or strongly r+b-connected. (3) Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \ldots, \mathbb{D}_n$ such that, for any $i \in \underline{n}$, every r-connected component of $\mathbb{D}_i$ is strongly r-connected. Then, for* *any $B_1, \ldots, B_n$ where B_i is an r+y-connected component of $\mathbb{D}_i$, $\mathbb{D} \cap (B_1 \times \ldots \times B_n)$ is either empty or strongly r+y-connected.*

Proof: (2) Set $\mathbb{D}' = \mathbb{D} \cap (B_1 \times \ldots \times B_n)$. To prove the corollary it is enough to show that $\mathbb{D}'$, provided it is non-empty, is a subdirect of $B_1, \ldots, B_n$, because then the result follows from Proposition 8(2). Take $i \in \underline{n}$; without loss of generality we may assume $i = 1$. Since B_1 is strongly r+b-connected, it is enough to prove that if $a \in \mathrm{pr}_1 \mathbb{D}'$ and $b \in B_1$ is such that $a \leq b$ or $\langle a, b \rangle \in \gamma$ then $b \in \mathrm{pr}_1 \mathbb{D}'$. So, let $a, b \in B_1$ are of this type and $a = \mathbf{a}[1]$, $b = \mathbf{b}[1]$ for some $\mathbf{a} \in \mathbb{D}'$, $\mathbf{b} \in \mathbb{D}$. For $\mathbf{c} = p(\mathbf{b}, \mathbf{a})$ we have

$$\mathbf{c}[i] = \begin{cases} \mathbf{a}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i] \rangle \in \beta \text{ or } \mathbf{b}[i] \leq \mathbf{a}[i], \\ \mathbf{b}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i] \rangle \in \gamma \text{ or } \mathbf{a}[i] \leq \mathbf{b}[i]; \end{cases}$$

in particular, $\mathbf{c}[1] = \mathbf{b}[1] = b$. As is easily seen, $\mathbf{c}[i] \in B_i$ for all $i \in \underline{n}$, that is $\mathbf{c} \in \mathbb{D}'$, as required.

(1) The proof is quite similar.

(3) The proof in this case is quite similar, we just have to use $r(\mathbf{b}, \mathbf{a})$ instead of $p(\mathbf{b}, \mathbf{a})$. $\square$

Lemma 14 *Let $\mathbb{D}$ be a subdirect product of conservative algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$ and $\mathbb{D}_i$ denote the union of maximal components of $\mathbb{D}_i$.*

(1) Let C_i be a strongly r+y-connected component of $\mathbb{D}'_i$, $i \in \underline{n}$. Then if $\mathbb{D}' = \mathbb{D} \cap (C_1 \times \ldots \times C_n)$ is non-empty then it is a subdirect product of $C_1, \ldots, C_n$ and is strongly r+y-connected.

(2) Let C_i be a strongly r-connected component of $\mathbb{D}'_i$, $i \in \underline{n}$. Then if $\mathbb{D}' = \mathbb{D} \cap (C_1 \times \ldots \times C_n)$ is non-empty then it is a subdirect product of $C_1, \ldots, C_n$ and is strongly r-connected.

Proof: (1) Take $i \in \underline{n}$; without loss of generality we may assume $i = 1$. Suppose that $C = \mathrm{pr}_1 \mathbb{D}' \neq C_1$. Since C_1 is strongly r+y-connected, there are $a \in C$ and $b \in C_1 - C$ such that $a \leq b$ or $\langle a, b \rangle \in \beta$. Let $a = \mathbf{a}[1]$, $b = \mathbf{b}[1]$ for some $\mathbf{a} \in \mathbb{D}'$ and $\mathbf{b} \in \mathbb{D}$. By Lemma 5(1), we may assume $\mathbf{b}$ to be maximal. This means that if $\langle \mathbf{a}[i], \mathbf{b}[i] \rangle \in \beta$ or $\mathbf{a}[i] \leq \mathbf{b}[i]$ then $\mathbf{b}[i] \in C_i$. For $\mathbf{c} = p(\mathbf{a}, \mathbf{b})$ we have

$$\mathbf{c}[i] = \begin{cases} \mathbf{a}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i] \rangle \in \gamma \text{ or } \mathbf{b}[i] \leq \mathbf{a}[i], \\ \mathbf{b}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i] \rangle \in \beta \text{ or } \mathbf{a}[i] \leq \mathbf{b}[i]; \end{cases}$$

in particular, $\mathbf{c}[1] = \mathbf{b}[1] = b$. As is easily seen, $\mathbf{c} \in \mathbb{D}'$. Thus, $\mathbb{D}'$ is a subdirect product of $C_1, \ldots, C_n$.

Since every C_i is strongly r+y-connected and every its r-conected component is strongly r-connected, $\mathbb{D}'$ satisfies the conditions of Proposition 8(3). Thus it is strongly r+y-connected.

(2) The proof is quite similar. $\square$

Corollary 14 *Let $\mathbb{D}$ be a subdirect product of conservative algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$. Let also B_i be an rb-maximal component of $\mathbb{D}_i$, $i \in \underline{n}$. Then if $\mathbb{D}' = \mathbb{D} \cap (B_1 \times \ldots \times B_n)$ is non-empty then it is a subdirect product of $B_1, \ldots, B_n$ and is strongly r+b-connected.*

Proof: Take $i \in \underline{n}$; without loss of generality we may assume $i = 1$. Suppose that $B = \mathrm{pr}_1 \mathbb{D}' \neq B_1$. Since B_1 is strongly r+b-connected, there are $a \in B$ and $b \in B_1 - B$ such that $a \leq b$ or $\langle a, b\rangle \in \gamma$. Let $a = \mathbf{a}[1]$, $b = \mathbf{b}[1]$ for some $\mathbf{a} \in \mathbb{D}'$ and $\mathbf{b} \in \mathbb{D}$. By Lemma 5(4), we may assume $\mathbf{b}$ to be rb-maximal. This means that if $\langle \mathbf{a}[i], \mathbf{b}[i]\rangle \in \gamma$ or $\mathbf{a}[i] \leq \mathbf{b}[i]$ then $\mathbf{b}[i] \in B_i$. For $\mathbf{c} = r(\mathbf{a}, \mathbf{b})$ we have

$$\mathbf{c}[i] = \begin{cases} \mathbf{a}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i]\rangle \in \beta \text{ or } \mathbf{b}[i] \leq \mathbf{a}[i], \\ \mathbf{b}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i]\rangle \in \gamma \text{ or } \mathbf{a}[i] \leq \mathbf{b}[i]; \end{cases}$$

in particular, $\mathbf{c}[1] = \mathbf{b}[1] = b$. As is easily seen, $\mathbf{c} \in \mathbb{D}'$. Thus, $\mathbb{D}'$ is a subdirect product of $C_1, \ldots, C_n$.

Since every B_i is strongly r+b-connected, $\mathbb{D}'$ satisfies the conditions of Proposition 8(2). Thus it is strongly r+b-connected. □

6.2 Rectangularity

Let x be one of the symbols r+y,r+b, $\mathbb{D} \in \Gamma'$ a subdirect product of $\mathbb{D}_1, \ldots, \mathbb{D}_n$, $j \in \underline{n}$, and $B \subseteq \mathbb{D}_j$ a strongly x-connected component of $\mathbb{D}_j$. We define the set $J(\mathrm{x}, j, B) \subseteq \underline{n}$ as follows:

$$\begin{aligned} J(\mathrm{x}, j, B) \quad = \quad & \{i \in \underline{n} \mid \text{for any } \mathbf{a}, \mathbf{b} \in \mathbb{D}, \text{ such that } \mathbf{a}[j] \in \\ & B, \text{ the element } \mathbf{b}[j] \text{ belongs to } B \text{ if and only if } \mathbf{a}[i], \mathbf{b}[i] \\ & \text{are (strongly) x-connected in } \mathbb{D}_i \end{aligned}$$

Definition 16 *The relation $\mathbb{D}$ is said to be x-*rectangular *if for any $\mathbf{a} \in \mathbb{D}$, any $j \in \underline{n}$, and any $\mathbf{b} \in \mathrm{pr}_{J(\mathrm{x},j,B)} \mathbb{D}$, where B is the strongly x-connected component containing $\mathbf{a}[j]$, and $\mathbf{b}[j] \in B$, the tuple $\mathbf{c}$ belongs to $\mathbb{D}$:*

$$\mathbf{c}[i] = \begin{cases} \mathbf{b}[i], & \text{if } i \in J(\mathrm{x}, j, B), \\ \mathbf{a}[i], & \text{otherwise.} \end{cases}$$

Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \ldots, \mathbb{D}_n$, $j \in \underline{n}$, and B is a strongly r+b-connected component of $, \ldots, _i$. For any $i \in J(\mathrm{r+b}, j, B)$, we use B_i to denote the strongly r+b-connected component of $\mathbb{D}_i$ such that, for any $\mathbf{a} \in \mathbb{D}$, $\mathbf{a}[i] \in B_i$ whenever $\mathbf{a}[j] \in B$; in particular, $B_j = B$.

Lemma 15 *If B is an rb-maximal component of $\mathbb{D}_j$ then, for any $i \in J_\varrho(\mathrm{r+b}, j, B)$, B_i is a maximal strongly r+b-connected component of $\mathbb{D}_i$.*

Proof: If, say B_i, is not rb-maximal, then take $\mathbf{a} \in \mathbb{D}$ such that $\mathbf{a}[j] \in B_j$. By Lemma 5(3), there is $\mathbf{b} \in \mathbb{D}$ such that $\mathbf{b}[j] = \mathbf{a}[j]$, but $\mathbf{b}[i]$ belongs to an rb-maximal component, i.e. $\mathbf{b}[i] \notin B_i$. $\square$

In this section we prove two rectangularity results which allow us to isolate r+y- and r+b-connected components.

Lemma 16 *Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \ldots, \mathbb{D}_n$. If for any i, every r-component of $\mathbb{D}_i$ is strongly r-connected then $\mathbb{D}$ is r+y-rectangular.*

Proof: Take $\mathbf{a} \in \mathbb{D}$, $q \in \underline{n}$, set $I = J(\text{r+y}, q, B)$ where B is the r+y-connected component of $\mathbb{D}_q$ containing $\mathbf{a}[q]$, and take $\mathbf{b} \in \text{pr}_I\mathbb{D}$ such that $\mathbf{b}[i], \mathbf{a}[i]$ are strongly r+y-connected for $i \in I$. By Corollary 13(2), $\text{pr}_I\mathbf{a}$ and $\mathbf{b}$ are strongly r+y-connected. We prove the proposition by induction on n. If $I = \underline{n}$, in particular, if $n = 1$, then the result follows trivially. So, suppose that $I \neq \underline{n}$.

As $I = J(\text{r+y}, q, B)$, there is $\mathbf{c} \in \mathbb{D}$ and $t \in \underline{n}$ such that either $\mathbf{c}[q], \mathbf{a}[q]$ are strongly r+y-connected while $\mathbf{c}[t], \mathbf{a}[t]$ are not, or $\mathbf{c}[q], \mathbf{a}[q]$ are not strongly r+y-connected while $\mathbf{c}[t], \mathbf{a}[t]$ are. Without loss of generality, suppose that the former case holds. Let J_1, J_2 be the partition of $\underline{n}$ such that $\mathbf{c}[i], \mathbf{a}[i]$ are strongly r+y-connected for $i \in J_1$, $\mathbf{c}[i], \mathbf{a}[i]$ are not strongly r+y-connected (and therefore $\langle \mathbf{c}[i], \mathbf{a}[i] \rangle \in \gamma$ as every r-connected component of $\mathbb{D}_i$ is strongly r-connected) for $i \in J_2$. Since $I \subseteq J_1$, $t \in J_2$, they are nonempty. By Corollary 13(2), $\text{pr}_{J_1}\mathbf{a}$ and $\text{pr}_{J_1}\mathbf{c}$ are r+y-connected, moreover, it is not hard to see that $\mathbf{a}, \mathbf{c}$ can be assumed to be such that $\text{pr}_{J_1}\mathbf{a} \leq \text{pr}_{J_1}\mathbf{c}$ or $\langle \text{pr}_{J_1}\mathbf{a}, \text{pr}_{J_1}\mathbf{c} \rangle \in \text{pr}_{J_1}\beta$. Set $\text{pr}_{J_1}\mathbf{a} = \mathbf{a}_1$, $\text{pr}_{J_1}\mathbf{c} = \mathbf{c}_1$, $\text{pr}_{J_2}\mathbf{a} = \mathbf{a}_2$, $\text{pr}_{J_2}\mathbf{c} = \mathbf{c}_2$.

Let A_i denote the r+y-connected component of $\mathbb{D}_i$ containing $\mathbf{a}[i]$, and C_i the r+y-connected component containing $\mathbf{c}[i]$. By the choice of J_1, J_2, $A_i = C_i$ for $i \in J_1$, and $A_i \neq C_i$ for $i \in J_2$. Let us denote

$$\mathbb{D}' = \text{pr}_{J_1}\mathbb{D} \cap \prod_{i \in J_1} A_i, \quad \mathbb{D}'' = \text{pr}_{J_2}\mathbb{D} \cap \prod_{i \in J_2} A_i, \quad \mathbb{D}''' = \text{pr}_{J_2}\mathbb{D} \cap \prod_{i \in J_2} C_i.$$

By Corollary 13(2), $\mathbb{D}', \mathbb{D}'', \mathbb{D}'''$ are strongly r+y-connected.

Claim 1. For every $\mathbf{b}_1 \in \mathbb{D}'$ we have $\begin{pmatrix}\mathbf{b}_1\\ \mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{b}_1\\ \mathbf{c}_2\end{pmatrix} \in \mathbb{D}$.

Prove first that the claim holds for one of $\mathbf{a}_1, \mathbf{c}_1$. If, say, $\mathbf{a}_1 \leq \mathbf{c}_1$, then we have

$$f\left(\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{pmatrix}\right) = \begin{pmatrix}\mathbf{c}_1\\ \mathbf{a}_2\end{pmatrix} \in \mathbb{D}.$$

If $\mathbf{c}_1 \leq \mathbf{a}_1$ we get $\begin{pmatrix}\mathbf{a}_1\\ \mathbf{c}_2\end{pmatrix} \in \mathbb{D}$. If $\langle \mathbf{a}_1, \mathbf{c}_1 \rangle \in \beta$ then

$$\begin{pmatrix}\mathbf{a}_1\\ \mathbf{c}_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}\right), \quad \begin{pmatrix}\mathbf{c}_1\\ \mathbf{a}_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{pmatrix}\right) \in \mathbb{D}.$$

Now let B be the set of tuples from $\mathbb{D}'$ for which the claim holds. If $B \neq \mathbb{D}'$ then there is $\mathbf{b}_1 \in B$ and $\mathbf{d}_1 \in \mathbb{D}' - B$ such that $\mathbf{b}_1 \leq \mathbf{d}_1$ or $\langle \mathbf{b}_1, \mathbf{d}_1 \rangle \in \beta$. There is $\mathbf{d}_2 \in \text{pr}_{J_2}\mathbb{D}$

with $\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix} \in \mathbb{D}$. By Lemma 2(3), in the tuple

$$\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}'_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{b}_1\\ \mathbf{c}_2\end{pmatrix}, \begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix}\right)$$

$\mathbf{c}_2[i] \le \mathbf{d}'_2[i]$ or $\langle \mathbf{c}_2[i], \mathbf{d}'_2[i]\rangle \in \beta$ for any $i \in J_2$, and therefore $\langle \mathbf{d}'_2[i], \mathbf{a}[i]\rangle \in \gamma$. Then

$$\begin{aligned}\begin{pmatrix}\mathbf{d}_1\\ \mathbf{a}_2\end{pmatrix} &= p\left(\begin{pmatrix}\mathbf{b}_1\\ \mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}'_2\end{pmatrix}\right) \in \mathbb{D} \quad \text{and}\\ \begin{pmatrix}\mathbf{d}_1\\ \mathbf{c}_2\end{pmatrix} &= p\left(\begin{pmatrix}\mathbf{b}_1\\ \mathbf{c}_2\end{pmatrix}, \begin{pmatrix}\mathbf{d}_1\\ \mathbf{a}_2\end{pmatrix}\right) \in \mathbb{D},\end{aligned}$$

a contradiction. The claim is proved.

Claim 2. For any $\mathbf{a}'_2 \in \mathrm{pr}_{J_2}\mathbb{D}''$ the tuple $\mathbf{d}$ with

$$\mathbf{d}[i] = \begin{cases} \mathbf{a}_1[i], & \text{if } i \in J_1,\\ \mathbf{a}'_2[i], & \text{if } i \in J_2,\end{cases}$$

belongs to $\mathbb{D}$.

Since $\mathbf{a}'_2 \in \mathbb{D}''$ and $\mathbb{D}''$ is strongly r+y-connected, it is enough to prove the claim under assumption $\mathbf{a}_2 \le \mathbf{a}'_2$ or $\langle \mathbf{a}_2, \mathbf{a}'_2\rangle \in \beta$. There is $\mathbf{a}'_1 \in \mathrm{pr}_{J_1}\mathbb{D}$ such that $\begin{pmatrix}\mathbf{a}'_1\\ \mathbf{a}'_2\end{pmatrix} \in \mathbb{D}$. Set

$$\begin{pmatrix}\mathbf{a}''_1\\ \mathbf{a}'_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}'_1\\ \mathbf{a}'_2\end{pmatrix}\right)$$

where $\mathbf{a}_1[i] \le \mathbf{a}''_1[i]$ or $\langle \mathbf{a}_1[i], \mathbf{a}''_1[i]\rangle \in \beta$ for any $i \in J_1$. Since $\langle \mathbf{a}'_2, \mathbf{c}_2\rangle \in \gamma$ and

$$\begin{pmatrix}\mathbf{a}''_1\\ \mathbf{c}_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{a}_1\\ \mathbf{c}_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}''_1\\ \mathbf{a}'_2\end{pmatrix}\right) \in \mathbb{D},$$

applying Claim 1 to the pair $\mathbf{a}'_2, \mathbf{c}_2$, we get what is required.

By the induction hypothesis, there is the tuple $\mathbf{d} \in \mathrm{pr}_{J_1}\mathbb{D}''$ such that

$$\mathbf{d}[i] = \begin{cases} \mathbf{b}[i], & \text{if } i \in I,\\ \mathbf{a}[i], & \text{if } i \in J_1 - I.\end{cases}$$

By Claims 1,2 $\begin{pmatrix}\mathbf{d}\\ \mathbf{a}_2\end{pmatrix} \in \mathbb{D}$, and the lemma is proved. □

Lemma 17 *Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \ldots, \mathbb{D}_n$. If, for any $\in \underline{n}$, every r+b-connected component is strongly r+b-connected then $\mathbb{D}$ is r+b-rectangular.*

Proof: Take $\mathbf{a} \in \mathbb{D}$, $q \in \underline{n}$, set $I = J(\text{r+b}, q, B)$ where B is the strongly r+b-connected component of $\mathbb{D}_q$ containing $\mathbf{a}[q]$, and take $\mathbf{b} \in \mathrm{pr}_I \mathbb{D}$ such that $\mathbf{b}[i], \mathbf{a}[i]$ are strongly r+b-connected for $i \in I$. We prove the lemma by induction on n. If $I = \underline{n}$, in particular if $n = 1$, then the result follows trivially. So, suppose that $I \neq \underline{n}$.

As $I = J(\text{r+b}, q, B)$, there is $\mathbf{c} \in \mathbb{D}$ and $t \in \underline{n}$ such that either $\mathbf{a}[q], \mathbf{c}[q]$ are strongly r+b-connected while $\mathbf{a}[t], \mathbf{c}[t]$ are not, or $\mathbf{a}[q], \mathbf{c}[q]$ are not strongly r+b-connected while $\mathbf{a}[t], \mathbf{c}[t]$ are. Without loss of generality suppose that the former case holds. Let J_1, J_2 be the partition of $\underline{n}$ such that $\mathbf{a}[i], \mathbf{c}[i]$ are strongly r+b-connected for $i \in J_1$, and $\mathbf{a}[i], \mathbf{c}[i]$ are not strongly r+b-connected for $i \in J_2$. By Corollary 13(1), $\mathrm{pr}_{J_1}\mathbf{a}, \mathrm{pr}_{J_1}\mathbf{c}$ are in the same strongly r+b-connected component, and $\langle \mathrm{pr}_{J_2}\mathbf{c}, \mathrm{pr}_{J_2}\mathbf{a}\rangle \in \beta$. As $I \subseteq J_1$, $t \in J_2$, they are nonempty. As $\mathrm{pr}_{J_1}\mathbf{a}, \mathrm{pr}_{J_1}\mathbf{c}$ are strongly r+b-connected, it is not hard to see that $\mathbf{a}, \mathbf{c}$ can be chosen such that $\mathrm{pr}_{J_1}\mathbf{a} \le \mathrm{pr}_{J_1}\mathbf{c}$ or $\langle \mathrm{pr}_{J_1}\mathbf{a}, \mathrm{pr}_{J_1}\mathbf{c}\rangle \in \gamma$. Set $\mathrm{pr}_{J_1}\mathbf{a} = \mathbf{a}_1$, $\mathrm{pr}_{J_1}\mathbf{c} = \mathbf{c}_1$, $\mathrm{pr}_{J_2}\mathbf{a} = \mathbf{a}_2$, $\mathrm{pr}_{J_2}\mathbf{c} = \mathbf{c}_2$.

Let A_i denote the r+b-connected component of $\mathbb{D}_i$ containing $\mathbf{a}[i]$, and C_i the r+b-connected component containing $\mathbf{c}[i]$. By the choice of J_1, J_2, $A_i = C_i$ for $i \in J_1$, and $A_i \neq C_i$ for $i \in J_2$. Let us denote

$$\mathbb{D}' = \mathrm{pr}_{J_1}\mathbb{D} \cap \prod_{i \in J_1} A_i, \quad \mathbb{D}'' = \mathrm{pr}_{J_2}\mathbb{D} \cap \prod_{i \in J_2} A_i, \quad \mathbb{D}''' = \mathrm{pr}_{J_2}\mathbb{D} \cap \prod_{i \in J_2} C_i.$$

By Corollary 13(1), $\mathbb{D}', \mathbb{D}'', \mathbb{D}'''$ are strongly r+b-connected.

Claim 1. For every $\mathbf{b}_1 \in \mathbb{D}'$ we have $\begin{pmatrix}\mathbf{b}_1\\ \mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{b}_1\\ \mathbf{c}_2\end{pmatrix} \in \mathbb{D}$.

Prove first that the claim holds for one of $\mathbf{a}_1, \mathbf{c}_1$. If, say, $\mathbf{a}_1 \le \mathbf{c}_1$, then

$$f\left(\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{pmatrix}\right) = \begin{pmatrix}\mathbf{c}_1\\ \mathbf{a}_2\end{pmatrix} \in \mathbb{D}.$$

If $\mathbf{c}_1 \le \mathbf{a}_1$ we get $\begin{pmatrix}\mathbf{a}_1\\ \mathbf{c}_2\end{pmatrix} \in \mathbb{D}$. If $\langle \mathbf{a}_1, \mathbf{c}_1\rangle \in \gamma$, then

$$r\left(\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{pmatrix}\right) = \begin{pmatrix}\mathbf{c}_1\\ \mathbf{a}_2\end{pmatrix} \in \mathbb{D}.$$

Now let B be the set of tuples from $\mathbb{D}'$ for which the claim holds. If $B \neq \mathbb{D}'$ then there is $\mathbf{b}_1 \in B$ and $\mathbf{d}_1 \in \mathbb{D}' - B$ such that $\mathbf{b}_1 \le \mathbf{d}_1$ or $\langle \mathbf{b}_1, \mathbf{d}_1\rangle \in \gamma$. There is $\mathbf{d}_2 \in \mathrm{pr}_{J_2}\mathbb{D}$ with $\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix} \in \mathbb{D}$. For the tuple

$$\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}'_2\end{pmatrix} = r\left(\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix}, \begin{pmatrix}\mathbf{b}_1\\ \mathbf{a}_2\end{pmatrix}\right)$$

we have $\mathbf{d}'_2[i] = \mathbf{a}_2[i]$ whenever $\mathbf{d}_2[i] \le \mathbf{a}_2[i]$ or $\langle \mathbf{d}'_2[i], \mathbf{a}_2[i]\rangle \in \beta$. Therefore, $\mathbf{d}'_2, \mathbf{a}_2$ are strongly r+b-connected, and $\langle \mathbf{d}'_2, \mathbf{c}_2\rangle \in \beta$. Then if $\mathbf{b}_1 \le \mathbf{d}_1$ then

$$f\left(\begin{pmatrix}\mathbf{b}_1\\ \mathbf{c}_2\end{pmatrix}, \begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix}\right) = \begin{pmatrix}\mathbf{d}_1\\ \mathbf{c}_2\end{pmatrix} \in \mathbb{D} \quad \text{and}$$

$$f\left(\begin{pmatrix}\mathbf{b}_1\\ \mathbf{a}_2\end{pmatrix},\begin{pmatrix}\mathbf{d}_1\\ \mathbf{c}_2\end{pmatrix}\right) = \begin{pmatrix}\mathbf{d}_1\\ \mathbf{a}_2\end{pmatrix}\in\mathbb{D},$$

and if $\langle\mathbf{b}_1,\mathbf{d}_1\rangle\in\mathrm{pr}_{J_1}\gamma$ then

$$p\left(\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix},\begin{pmatrix}\mathbf{b}_1\\ \mathbf{c}_2\end{pmatrix}\right) = \begin{pmatrix}\mathbf{d}_1\\ \mathbf{c}_2\end{pmatrix}\in\mathbb{D}\quad\text{and}$$
$$p\left(\begin{pmatrix}\mathbf{d}_1\\ \mathbf{c}_2\end{pmatrix},\begin{pmatrix}\mathbf{b}_1\\ \mathbf{a}_2\end{pmatrix}\right) = \begin{pmatrix}\mathbf{d}_1\\ \mathbf{a}_2\end{pmatrix}\in\mathbb{D},$$

a contradiction. The claim is proved.

Claim 2. For any $\mathbf{a}'_2\in\mathbb{D}''$ the tuple $\mathbf{d}$ with

$$\mathbf{d}[i]=\begin{cases}\mathbf{a}_1[i], & \text{if } i\in J_1,\\ \mathbf{a}'_2[i], & \text{if } i\in J_2,\end{cases}$$

belongs to $\mathbb{D}$.

As $\mathbb{D}''$ is strongly r+b-connected, it is enough to prove the claim under assumption $\mathbf{a}_2\le\mathbf{a}'_2$ or $\langle\mathbf{a}_2,\mathbf{a}'_2\rangle\in\gamma$. There is $\mathbf{a}'_1\in\mathrm{pr}_{J_1}\mathbb{D}$ such that $\begin{pmatrix}\mathbf{a}'_1\\ \mathbf{a}'_2\end{pmatrix}\in\mathbb{D}$. If $\mathbf{a}_1,\mathbf{a}'_1$ are not strongly r+b-connected, then for

$$\begin{pmatrix}\mathbf{a}''_1\\ \mathbf{a}'_2\end{pmatrix}=f\left(\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix},\begin{pmatrix}\mathbf{a}'_1\\ \mathbf{a}'_2\end{pmatrix}\right),$$

if $\mathbf{a}_2\le\mathbf{a}'_2$, and

$$\begin{pmatrix}\mathbf{a}''_1\\ \mathbf{a}'_2\end{pmatrix}=p\left(\begin{pmatrix}\mathbf{a}'_1\\ \mathbf{a}'_2\end{pmatrix},\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}\right),$$

if $\langle\mathbf{a}_2,\mathbf{a}'_2\rangle\in\gamma$, we have $\mathbf{a}''_1[i]=\mathbf{a}_1[i]$ whenever $\mathbf{a}''_1[i]\le\mathbf{a}_1[i]$ or $\langle\mathbf{a}''_1[i],\mathbf{a}_1[i]\rangle\in\beta$. Thus, we may assume $\mathbf{a}_1[i]\le\mathbf{a}'_1[i]$ or $\langle\mathbf{a}_1[i],\mathbf{a}'_1[i]\rangle\in\gamma$ for $i\in J_1$. Since

$$\begin{pmatrix}\mathbf{a}'_1\\ \mathbf{c}_2\end{pmatrix}=p\left(\begin{pmatrix}\mathbf{a}'_1\\ \mathbf{a}'_2\end{pmatrix},\begin{pmatrix}\mathbf{a}_1\\ \mathbf{c}_2\end{pmatrix}\right)\in\mathbb{D},$$

applying Claim 1 to the pair $\mathbf{a}'_2,\mathbf{c}_2$, we get what is required.

By the induction hypothesis, there is the tuple $\mathbf{d}\in\mathrm{pr}_{J_1}\mathbb{D}''$ such that

$$\mathbf{d}[i]=\begin{cases}\mathbf{b}[i], & \text{if } i\in I,\\ \mathbf{a}[i], & \text{if } i\in J_1-I.\end{cases}$$

By Claims 1,2, $\begin{pmatrix}\mathbf{d}\\ \mathbf{a}_2\end{pmatrix}\in\mathbb{D}$, and the proposition is proved. $\square$

6.3 Decomposition

In [32], it was proved that relations invariant with respect to a so-called *near-unanimity operation* (a majority operation is a ternary near-unanimity operation) are *decomposable*, that is are completely defined by their projections of bounded arity. We show that a similar property holds also for some types of semiconservative algebras.

Lemma 18 *Let $\mathbb{D}$ be a subdirect product of conservative algebras $\mathbb{D}_1,\dots,\mathbb{D}_n$, and $\mathbf{a}\in\mathbb{D}_1\times\dots\times\mathbb{D}_n$. If, for any $i,j\in\underline{n}$, $\begin{pmatrix}\mathbf{a}[i]\\ \mathbf{a}[j]\end{pmatrix}\in\mathrm{pr}_{i,j}\mathbb{D}$ and, for any $i\in\underline{n}$, $\mathbf{a}[i]$ belongs to a maximal strongly r+b-connected component B_i of $\mathbb{D}_i$ and $\mathrm{pr}_{J(\mathrm{r+b},i,B_i)}\mathbf{a}\in\mathrm{pr}_{J(\mathrm{r+b},i,B_i)}\mathbb{D}$, then $\mathbf{a}\in\mathbb{D}$.*

Proof: Let $I_1,\dots,I_k$ be the sets of the form $J(\mathrm{r+b},i,B_i)$. We use $\mathbf{a}_j$ to denote the tuple $\mathrm{pr}_{I_j}\mathbf{a}$. We prove by induction on $l\le k$ that, for any choice of $I_{j_1},\dots,I_{j_l}$, the tuple $\mathrm{pr}_{I_{j_1}\cup\dots\cup I_{j_l}}\mathbf{a}$ belongs to $\mathrm{pr}_{I_{j_1}\cup\dots\cup I_{j_l}}\mathbb{D}$. The base case of induction is provided by the conditions of the lemma. Indeed, if $l=1$ then the required property follows from the condition $\mathrm{pr}_{J(\mathrm{r+b},i,B_i)}\mathbf{a}\in\mathrm{pr}_{J(\mathrm{r+b},i,B_i)}\mathbb{D}$. If $l=2$ then take $i_1\in I_{j_1}$, $i_2\in I_{j_2}$. By the conditions, $\begin{pmatrix}\mathbf{a}[i_1]\\ \mathbf{a}[i_2]\end{pmatrix}\in\mathrm{pr}_{\{i_1,i_2\}}\mathbb{D}$ and, therefore, there is $\mathbf{b}\in\mathrm{pr}_{I_{j_1}\cup I_{j_2}}\mathbb{D}$ such that $\mathrm{pr}_{I_{j_1}}\mathbf{a}$, $\mathrm{pr}_{I_{j_1}}\mathbf{b}$ and $\mathrm{pr}_{I_{j_2}}\mathbf{a}$, $\mathrm{pr}_{I_{j_2}}\mathbf{b}$ are strongly r+b-connected. By Lemma 17, $\mathrm{pr}_{I_{j_1}\cup I_{j_2}}\mathbf{a}\in\mathrm{pr}_{I_{j_1}\cup I_{j_2}}\mathbb{D}$.

Suppose that for any subset $J\subseteq\{1,\dots,k\}$ containing less than $l>2$ element the tuple $\mathrm{pr}_{I_J}\mathbf{a}$, $I_J=\bigcup_{j\in J}I_j$, belongs to $\mathbb{D}$. Take $K\subseteq\{1,\dots,k\}$ with $|K|=l$, and $s,t,q\in K$. Without loss of generality we may assume $K=\{1,\dots,l\}$, $s=1,t=2,q=3$. We have to show that $\mathrm{pr}_{I_K}\mathbf{a}\in\mathrm{pr}_{I_K}\mathbb{D}$.

Since $\mathrm{pr}_{I_{K-\{s\}}}\mathbf{a},\mathrm{pr}_{I_{K-\{t\}}}\mathbf{a},\mathrm{pr}_{I_{K-\{q\}}}\mathbf{a}$ belong to $\mathrm{pr}_{I_{K-\{s\}}}\mathbb{D}$, $\mathrm{pr}_{I_{K-\{t\}}}\mathbb{D}$, $\mathrm{pr}_{I_{K-\{q\}}}\mathbb{D}$, respectively, the tuples $(\mathbf{a}_2,\mathbf{a}_3,\dots,\mathbf{a}_l),(\mathbf{a}_1,\mathbf{a}_3,\mathbf{a}_4,\dots,\mathbf{a}_l),(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_4,\dots,\mathbf{a}_l)$ belong to $\mathrm{pr}_{I_J-I_1}\mathbb{D},\mathrm{pr}_{I_J-I_2}\mathbb{D},\mathrm{pr}_{I_J-I_3}\mathbb{D}$ respectively. Therefore, there are $\mathbf{b}_1\in\mathrm{pr}_{I_1}\mathbb{D}$, $\mathbf{b}_2\in\mathrm{pr}_{I_2}\mathbb{D}$, $\mathbf{b}_3\in\mathrm{pr}_{I_3}\mathbb{D}$ such that $(\mathbf{b}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4,\dots,\mathbf{a}_l),(\mathbf{a}_1,\mathbf{b}_2,\mathbf{a}_3,\mathbf{a}_4,\dots,\mathbf{a}_l),(\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_3,\mathbf{a}_4,\dots,\mathbf{a}_l)\in\mathrm{pr}_{I_K}\mathbb{D}$. By Lemma 10, $\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3$ can be chosen from maximal components of $\mathrm{pr}_{I_1}\mathbb{D},\mathrm{pr}_{I_2}\mathbb{D}$, $\mathrm{pr}_{I_3}\mathbb{D}$. If, say, $\mathbf{b}_1$ is in the same strongly r+b-connected component as $\mathbf{a}_1$, then by Proposition 17, $(\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_l)\in\mathrm{pr}_{I_K}\mathbb{D}$, and we are done. So, we may assume that the pairs $\mathbf{a}_1,\mathbf{b}_1$, $\mathbf{a}_2,\mathbf{b}_2$, $\mathbf{a}_3,\mathbf{b}_3$ lie in different strongly r+b-connected components. Therefore, for every $j\in\{1,2,3\}$ and every $v\in I_j$, the pair $(\mathbf{a}_j[v],\mathbf{b}_j[v])$ is yellow. Hence,

$$\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\\ \mathbf{a}_4\\ \vdots\\ \mathbf{a}_l\end{pmatrix}=g\left(\begin{pmatrix}\mathbf{b}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\\ \mathbf{a}_4\\ \vdots\\ \mathbf{a}_l\end{pmatrix},\begin{pmatrix}\mathbf{a}_1\\ \mathbf{b}_2\\ \mathbf{a}_3\\ \mathbf{a}_4\\ \vdots\\ \mathbf{a}_l\end{pmatrix},\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{b}_3\\ \mathbf{a}_4\\ \vdots\\ \mathbf{a}_l\end{pmatrix}\right)\in\mathrm{pr}_K\mathbb{D}.$$

$\square$

7 Red/blue-connected algebras

If $\mathbb{D}$ is a subdirect product of doubly connected algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$ or all $\mathbb{D}_1, \dots, \mathbb{D}_n$ are strongly r+y-connected, then results of the previous section or Proposition 11 give enough information for developing an efficient algorithm, as it will be shown in Section 8. So, in this section, we consider the case when at least one of $\mathbb{D}_1, \dots, \mathbb{D}_n$ is not strongly r+y-connected. For short, a strongly r+y-connected component we will call *ry-component*.

7.1 Filters and paths

The following definitions and notation we make, first, for conservative algebras, and then generalise them for semiconservative. Let $\mathbb{A}$ be a conservative algebra. Recall that we write $a \le b$, for $a, b \in \mathbb{A}$, if $a = b$ or the edge (a, b) is red and $f(a, b) = b$; and for strongly r-connected components C, D, we write $C \le_r D$ if there are $a \in C, b \in D$ such that $a \le b$. The transitive reflexive closure $\prec$ of $\le$ and that $\prec_r$ of $\le_r$ are quasi-orders on $\mathbb{A}$ and the set of strongly r-connected components of $\mathbb{A}$.

For a set $B \subseteq \mathbb{A}$ we set

$$\widetilde{\mathcal{F}}(B) = \{a \in \mathbb{A} \mid b \prec a \text{ for a certain } b \in B\}.$$

The set $\widetilde{\mathcal{F}}(B)$ is called the *filter generated* by B. If B is a singleton, then the filter it generates is said to be *principal*. Notice that $\widetilde{\mathcal{F}}(B)$ consists of all $a \in \mathbb{A}$ such that there are $b_1, \dots, b_l \in \mathbb{A}$ with $b_1 \in B$, $b_l = a$, and $b_i \le b_{i+1}$. Recall that such a sequence is called a *path*.

Now, let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \dots, \mathbb{D}_n$. For $\mathbf{a}, \mathbf{b} \in \mathbb{D}$ we write $\mathbf{a} \le \mathbf{b}$ if and only if, for any $i \in \underline{n}$, $\mathbf{a}[i] \le \mathbf{b}[i]$; we denote $\prec$ the transitive closure of $\le$. Then filters and paths for the semiconservative algebra $\mathbb{D}$ can be defined in the same way as for a conservative algebra.

Let $a_1, \dots, a_k$ be a path in a conservative algebra $\mathbb{A}$. It is said to be *irreducible* if, for any $i \in \underline{n-2}$, if $a_i \le a_{i+2}$ then $a_{i+1} = \dots = a_k$.

Lemma 19 *Let $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{D}$ be a path. Then, for any $i, j \in \underline{n}$, there is a path $\mathbf{b}_1, \dots, \mathbf{b}_m$ such that $\mathbf{b}_1 = \mathbf{a}_1$, $\mathbf{b}_1[l], \dots, \mathbf{b}_m[l] \in \{\mathbf{a}_1[l], \dots, \mathbf{a}_k[l]\}$, for $l \in \underline{n}$, $\mathbf{b}_m[i] = \mathbf{a}_k[i], \mathbf{b}_m[j] = \mathbf{a}_k[j]$, and both $\mathbf{a}_1[i], \dots, \mathbf{a}_k[i]$ and $\mathbf{a}_1[j], \dots, \mathbf{a}_k[j]$ are irreducible.*

Proof: Without loss of generality we may assume that $i = 1, j = 2$. We prove the lemma by induction on the number of different elements amongst $\mathbf{a}_1[1], \dots, \mathbf{a}_k[1], \mathbf{a}_1[2], \dots, \mathbf{a}_k[2]$. In the base case of induction when this number equals 2, we have $k = 1$, and the result hold trivially.

We also may assume that either $\mathbf{a}_{k-1}[1] \ne \mathbf{a}_k[1]$ or $\mathbf{a}_{k-1}[2] \ne \mathbf{a}_k[2]$, and by induction hypothesis, that $\mathbf{a}_1[1], \dots, \mathbf{a}_{k-1}[1]$, $\mathbf{a}_1[2], \dots, \mathbf{a}_{k-1}[2]$ are irreducible. There are two cases to consider.

CASE 1. $\mathbf{a}_{k-2}[1] \ne \mathbf{a}_{k-1}[1]$ and $\mathbf{a}_{k-2}[2] \ne \mathbf{a}_{k-1}[2]$.

SUBCASE 1A. $\mathbf{a}_{k-2}[1] \not\le \mathbf{a}_k[1]$, $\mathbf{a}_{k-2}[2] \not\le \mathbf{a}_k[2]$.

In this case $\mathbf{a}_1[1],\ldots,\mathbf{a}_{k-1}[1],\mathbf{a}_k[1]$ and $\mathbf{a}_1[2],\ldots,\mathbf{a}_{k-1}[2],\mathbf{a}_k[2]$ are irreducible and we are done.

SUBCASE 1B. $\mathbf{a}_{k-2}[1]\not\leq\mathbf{a}_k[1]$ and $\mathbf{a}_{k-2}[2]\leq\mathbf{a}_k[2]$, or $\mathbf{a}_{k-2}[1]\leq\mathbf{a}_k[1]$ and $\mathbf{a}_{k-2}[2]\not\leq\mathbf{a}_k[2]$.

If $\mathbf{a}_{k-1}[2]=\mathbf{a}_k[2]$ then $\mathbf{a}_1[1],\ldots,\mathbf{a}_{k-1}[1],\mathbf{a}_k[1]$ and $\mathbf{a}_1[2],\ldots,\mathbf{a}_{k-1}[2],\mathbf{a}_k[2]$ are irreducible and we are done. Otherwise, by setting $\mathbf{a}'_{k-2}=f(\mathbf{a}_{k-2},\mathbf{a}_k)$ and $\mathbf{a}'_{k-1}=f(\mathbf{a}_{k-1},\mathbf{a}'_{k-2})$, we get

$$\mathbf{a}'_{k-2}[1]=\mathbf{a}_{k-2}[1],\quad \mathbf{a}'_{k-2}[2]=\mathbf{a}_k[2],\qquad \mathbf{a}'_{k-1}[1]=\mathbf{a}_{k-1}[1],\quad \mathbf{a}'_{k-1}[2]=\mathbf{a}_k[2].$$

In the path $\mathbf{a}_1,\ldots,\mathbf{a}_{k-2},\mathbf{a}'_{k-1},\mathbf{a}_k$ the number of different elements in $\mathbf{a}_1[1],\ldots,\mathbf{a}_{k-2}[1]$, $\mathbf{a}'_{k-1}[1],\mathbf{a}_k[1]$, $\mathbf{a}_1[2],\ldots,\mathbf{a}_{k-2}[2],\mathbf{a}'_{k-1}[2],\mathbf{a}_k[2]$ is less than that in $\mathbf{a}_1,\ldots,\mathbf{a}_{k-2},\mathbf{a}_{k-1},\mathbf{a}_k$, as $\mathbf{a}_{k-1}[2]$ has disappeared. Therefore we can apply induction hypothesis.

SUBCASE 1C. $\mathbf{a}_{k-2}[1]\leq\mathbf{a}_k[1]$ and $\mathbf{a}_{k-2}[2]\leq\mathbf{a}_k[2]$.

In this case it is enough to remove $\mathbf{a}_{k-1}$ and apply induction hypothesis.

CASE 2. $\mathbf{a}_{k-2}[1]\neq\mathbf{a}_{k-1}[1]$ and $\mathbf{a}_{k-l}[2]=\ldots=\mathbf{a}_{k-2}[2]=\mathbf{a}_{k-1}[2]$, or $\mathbf{a}_{k-l}[1]=\ldots=\mathbf{a}_{k-2}[1]=\mathbf{a}_{k-1}[1]$ and $\mathbf{a}_{k-2}[2]\neq\mathbf{a}_{k-1}[2]$.

We assume that $\mathbf{a}_{k-2}[2]\neq\mathbf{a}_{k-1}[2]$ and $\mathbf{a}_{k-l}[1]=\ldots=\mathbf{a}_{k-2}[1]=\mathbf{a}_{k-1}[1]$. Clearly, $\mathbf{a}_{k-l}[1]=\ldots=\mathbf{a}_{k-2}[1]=\mathbf{a}_{k-1}[1]\leq\mathbf{a}_k[1]$.

SUBCASE 2A. $\mathbf{a}_{k-l-1}[1]\not\leq\mathbf{a}_k[1]$, $\mathbf{a}_{k-2}[2]\not\leq\mathbf{a}_k[2]$.

By setting $\mathbf{a}'_{k-2}=f(\mathbf{a}_{k-2},\mathbf{a}_k)$, $\mathbf{a}'_{k-1}=f(\mathbf{a}'_{k-2},\mathbf{a}_{k-1})$, and then $\mathbf{a}'_j=f(\mathbf{a}_j,\mathbf{a}'_{j+2})$ for all $k-l+1\leq j\leq k-3$, we get

$$\mathbf{a}'_{k-l+1}[1]=\ldots=\mathbf{a}'_{k-1}=\mathbf{a}_k[1]$$

and

$$\mathbf{a}'_{k-l+1}[2]=\mathbf{a}_{k-l+1}[2],\ldots,\mathbf{a}'_{k-1}[2]=\mathbf{a}_{k-1}[2].$$

Then, for the path $\mathbf{a}_1,\ldots,\mathbf{a}_{n-l},\mathbf{a}'_{n-l+1},\ldots,\mathbf{a}'_{k-1},\mathbf{a}_k$, the paths $\mathbf{a}_1[1],\ldots,\mathbf{a}_{n-l}[1],\mathbf{a}'_{n-l+1}[1]$, $\ldots,\mathbf{a}'_{k-1}[1],\mathbf{a}_k[1]$ and $\mathbf{a}_1[2],\ldots,\mathbf{a}_{n-l}[2],\mathbf{a}'_{n-l+1}[2],\ldots,\mathbf{a}'_{k-1}[2],\mathbf{a}_k[2]$ are irreducible.

SUBCASE 2B. $\mathbf{a}_{k-l-1}[1]\leq\mathbf{a}_k[1]$, $\mathbf{a}_{k-2}[2]\not\leq\mathbf{a}_k[2]$.

If $\mathbf{a}_{k-1}[1]=\mathbf{a}_k[1]$ then $\mathbf{a}_1,\ldots,\mathbf{a}_k$ is of the required form. Otherwise, as in the previous case we may get $\mathbf{a}'_{k-l},\ldots,\mathbf{a}'_{k-1}$ such that $\mathbf{a}'_{k-l}[1]=\ldots=\mathbf{a}'_{k-1}=\mathbf{a}_k[1]$ and $\mathbf{a}'_{k-l}[2]=\mathbf{a}'_{k-l}[2],\ldots,\mathbf{a}'_{k-1}[2]=\mathbf{a}_{k-1}[2]$. Then in the path $\mathbf{a}_1,\ldots,\mathbf{a}_{k-l-1},\mathbf{a}'_{k-l},\ldots,\mathbf{a}'_{k-1},\mathbf{a}_k$ the number of different elements in $\mathbf{a}_1[1],\ldots,\mathbf{a}_{k-l-1}[1],\mathbf{a}'_{k-l}[1],\ldots,\mathbf{a}'_{k-1}[1],\mathbf{a}_k[1]$ and $\mathbf{a}_1[2],\ldots,$ $\mathbf{a}_{k-l-1}[2],\mathbf{a}'_{k-l}[2],\ldots,\mathbf{a}'_{k-1}[2],\mathbf{a}_k[2]$ is less than that for the path $\mathbf{a}_1,\ldots,\mathbf{a}_k$, as $\mathbf{a}_{k-l}[1]=\ldots\mathbf{a}_{k-1}[1]$ has disappeared. Therefore we may apply induction hypothesis.

SUBCASE 2C. $\mathbf{a}_{k-2}[2]\leq\mathbf{a}_k[2]$.

If $\mathbf{a}_{k-1}[2]\neq\mathbf{a}_k[2]$ then induction hypothesis can be applied to the path $\mathbf{a}_1,\ldots,\mathbf{a}_{k-2},\mathbf{a}_k$. Otherwise, replace the original path with $\mathbf{a}_1,\ldots,\mathbf{a}_{k-2},\mathbf{a}_k$. □

Lemma 20 *Let $\mathbf{a}_1,\dots,\mathbf{a}_k \in \mathbb{D}$ be a path such that $\mathbf{a}_1[1],\dots,\mathbf{a}_k[1]$, $\mathbf{a}_1[2],\dots,\mathbf{a}_k[2]$ are irreducible, and $\mathbf{a}_{k-1}[1] \neq \mathbf{a}_k[1]$. If there is $\mathbf{b} \in \mathbb{D}$ such that $\mathbf{b}[1] = \mathbf{a}_k[1]$ and $\mathbf{b}[2] = \mathbf{a}_{k-1}[2]$, then*

(1) there is $\mathbf{c} \in \mathbb{D}$ such that $\mathbf{c}[1] = \mathbf{a}_1[1]$ and $\mathbf{c}[2] = \mathbf{a}_{k-1}[2]$;

(2) if $k > 2$ then, for any $1 \le i,j \le k$ there is $\mathbf{c}_i^j \in \mathbb{D}$ such that $\mathbf{c}_i^j[1] = \mathbf{a}_i[1]$ and $\mathbf{c}_i^j[2] = \mathbf{a}_j[2]$.

If, in addition, for any $i \in \{3,\dots,n\}$, $\mathbf{b}[i] \in \{\mathbf{a}_1[i],\dots,\mathbf{a}_k[i]\}$, then the tuples $\mathbf{c}$, $\mathbf{c}_i^j$ can be chosen such that $\mathbf{c}[i], \mathbf{c}_i^j[l] \in \{\mathbf{a}_1[l],\dots,\mathbf{a}_k[l]\}$ for any $l \in \{3,\dots,n\}$.

Proof: (1) We begin with constructing a subsequence $j_1,\dots,j_l$ from $1,\dots,k$ such that $j_1 = k-1$; then if $\mathbf{a}_{k-l-1}[2] \neq \mathbf{a}_{k-l}[2] = \mathbf{a}_{k-l+1}[2] = \ldots = \mathbf{a}_{k-1}[2]$ then $j_2 = k-l-2$; and if j_i is defined then $j_{i+1} = j_i - 2$. Thus, $\langle \mathbf{a}_{j_{i+1}}[2], \mathbf{a}_{j_i}[2]\rangle \in \beta \cup \gamma$ or $\mathbf{a}_{j_i}[2] \le \mathbf{a}_{j_{i+1}}[2]$ for any $i < l$.

Claim 1. For any $r \in \underline{l-1}$, there are $\mathbf{b}^r_{j_{r-1}+1},\dots,\mathbf{b}^r_k \in \mathbb{D}$ such that $\mathbf{b}^r_s[1] = \mathbf{a}_s[1]$, $\mathbf{b}^r_s[2] = \mathbf{a}_{j_r}[2]$, and $\mathbf{b}^r_s[i] \in \{\mathbf{a}_1[i],\dots,\mathbf{a}_k[i]\}$ for any $i \in \{3,\dots,n\}$ and for any $s \in \{j_{r-1}+1,\dots,k\}$. There are $\mathbf{b}^l_1,\dots,\mathbf{b}^l_k \in \mathbb{D}$ such that $\mathbf{b}^l_s[1] = \mathbf{a}_s[1]$, $\mathbf{b}^l_s[2] = \mathbf{a}_{j_l}[2]$, and $\mathbf{b}^l_s[i] \in \{\mathbf{a}_1[i],\dots,\mathbf{a}_k[i]\}$ for any $i \in \{3,\dots,n\}$.

We prove the claim by induction on r. To avoid repetitions we start with proving of induction step, and then observe that the base case of induction also follows from our arguments. First we show the existence of $\mathbf{b}^r_s$ for $s \in \{j_r,\dots,k\}$. Clearly, we do not need to do this for $r = 1$.

The tuple $\mathbf{b}^r_{j_r}$ can be chosen to be $\mathbf{a}_{j_r}$. Furthermore, if $\mathbf{b}^r_s$ is already obtained, then set $\mathbf{b}^r_{s+1} = f(\mathbf{b}^r_s, \mathbf{b}^{r-1}_{s+1})$. Since $\mathbf{b}^r_s[1] = \mathbf{a}_s[1] \le \mathbf{a}_{s+1}[1] = \mathbf{b}^{r-1}_{s+1}[1]$ and $\langle \mathbf{b}^r_s[2], \mathbf{b}^{r-1}_{s+1}[2]\rangle \in \beta \cup \gamma$ or $\mathbf{b}^{r-1}_{s+1}[2] \le \mathbf{b}^r_s[2]$ where $\mathbf{b}^r_s[2] = \mathbf{a}_{j_r}[2]$, $\mathbf{b}^{r-1}_{s+1}[2] = \mathbf{a}_{j_{r-1}}[2]$, the obtained tuple satisfies the required conditions.

Notice that $\mathbf{b}^r_{j_r-1}$ can be chosen to be $f(\mathbf{a}_{j_r-1}, \mathbf{b}^r_{j_r+1})$. If $r \neq 1$ then we are done. Otherwise, for each $s = j_1-2,\dots,j_2+1$ we set $\mathbf{b}^1_s = f(\mathbf{a}_s, \mathbf{b}^1_{s+2})$. The claim is proved.

Claim 2. For any $j_l \le t < k$, there are $\mathbf{c}^t_1,\dots,\mathbf{c}^t_k \in \mathbb{D}$ such that $\mathbf{c}^t_1[2] = \ldots = \mathbf{c}^t_k[2] = \mathbf{a}_t[2]$ and $\mathbf{c}^t_s[1] = \mathbf{a}_s[1]$ for $s \in \underline{k}$.

We prove the claim by induction on t. By Claim 1, there exist $\mathbf{b}^l_1,\dots,\mathbf{b}^l_k$ such that $\mathbf{b}^l_1[2] = \ldots = \mathbf{b}^l_k[2] = \mathbf{a}_{j_l}[2]$ and $\mathbf{b}^l_s[1] = \mathbf{a}_s[1]$, which proves the base case of induction. So, suppose that $\mathbf{c}^t_1,\dots,\mathbf{c}^t_k$ are found. Then, $\mathbf{c}^{t+1}_{t+1} = \mathbf{a}_{t+1}$ and, for $j \in \{t+2,\dots,k\}$, we get $\mathbf{c}^{t+1}_j = f(\mathbf{c}^t_j, \mathbf{c}^{t+1}_{j+1})$. Furthermore, $\mathbf{c}^{t+1}_t = f(\mathbf{c}^t_t, \mathbf{c}^{t+1}_{t+1})$ if $t < k-2$, and $\mathbf{c}^{t+1}_t = \mathbf{b}^1_{k-2}$ if $t = k-2$. Finally, for $j \in \{t-1,\dots,1\}$, we set $\mathbf{c}^{t+1}_j = f(\mathbf{c}^t_j, \mathbf{c}^{t+1}_{j+2})$.

Claim 2, implies, in particular, that there is $\mathbf{c} = \mathbf{c}^{k-1}_1$ satisfying the conditions specified in part (1) of the lemma.

(2) If $k > 2$ then set $\mathbf{c}_k = \mathbf{a}_k$, $\mathbf{c}_{k-2} = f(\mathbf{c}^{k-1}_{k-2}, \mathbf{c}_k)$, and $\mathbf{c}_{k-1} = f(\mathbf{c}_{k-2}, \mathbf{a}_{k-1})$. Then, for any $j \in \{k-3,\dots,1\}$, $\mathbf{c}_j = f(\mathbf{c}^{k-1}_j, \mathbf{c}_{j+2})$. The tuples $\mathbf{c}_1,\dots,\mathbf{c}_k$ satisfies the conditions:

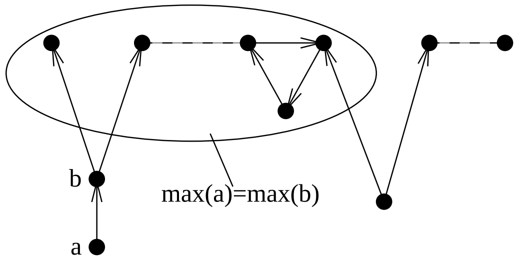

Figure 3: Maximal elements. Red edges are drawn solid, yellow edges are dotted, blue edges are omitted

$\mathbf{c}_j[1] = \mathbf{a}_j[1]$, $j \in \underline{k}$, and $\mathbf{c}_1[2] = \dots \mathbf{c}_k[2] = \mathbf{a}_k[2]$. By Claim 2, we just have to show the existence of $\mathbf{c}_1^1, \dots, \mathbf{c}_k^1$. As is easily seen, $\mathbf{c}_1^1$ can be chosen to be $\mathbf{a}_1$, and, for any $1 < i \leq k$, $\mathbf{c}_i^1 = f(\mathbf{c}_{i-1}^1, \mathbf{c}_i^3)$. $\square$

7.2 Maximal elements

We use $\mathbb{A}'$ to denote the union of all maximal strongly r-connected components from $\mathbb{A}$. Recall that maximal strongly r-connected are said to be maximal, and elements from $\mathbb{A}'$ are said to be *maximal*. For any $a \in \mathbb{A}$, we define the set of corresponding maximal elements as follows:

$$\begin{aligned} \max(a) \quad = \quad & \{b \in \mathbb{A}' \mid \text{there is } c \in \mathbb{A}' \cap \widetilde{\mathcal{F}}(a) \text{ such that} \\ & c, b \text{ are strongly r+y-connected in } \mathcal{G}(\mathbb{A}') \end{aligned}$$

(see Fig. 3).

If $\mathbb{D}$ is a subdirect product of conservative algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$, then, for any $\mathbf{a} \in \mathbb{D}$, we set

$$\max(\mathbf{a}) = \{\mathbf{b} \in \varrho \mid \mathbf{b}[i] \in \max(\mathbf{a}[i]) \text{ for any } i \in \underline{n}\}.$$

For $a, b \in \mathbb{A}$, we write $a \sqsubseteq b$ if and only if $\max(b) \subseteq \max(a)$. Clearly, $\prec$ refines $\sqsubseteq$, as $a \prec b$ implies $\max(b) \subseteq \max(a)$. If $a \sqsubseteq b$ and $b \sqsubseteq a$ then a, b are said to be *indistinguishable*. Obviously, if a, b lie in the same r-component then they are indistinguishable. The classes of the equivalence relation $\sqsubseteq \cap \sqsubseteq^{-1}$ are called the *i-components*.

We first make an easy observation.

Lemma 21 *If a, b are not indistinguishable, say $a \not\sqsubseteq b$, then there is $c \in \widetilde{\mathcal{F}}(b)$ such that $c \notin \widetilde{\mathcal{F}}(a)$.*

Lemma 22 *Let $\mathbb{D}$ be a subdirect product of conservative algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$. If $\mathbf{a}, \mathbf{b} \in \mathbb{D}$ are such that $\mathbf{a} \in \max(\mathbf{b})$, then there is $\mathbf{c} \in \widetilde{\mathcal{F}}(b) \cap \max(b)$ such that $\mathbf{c}$ is strongly r+y-connected to $\mathbf{a}$ in $\mathbb{D}'$.*

Proof: We should prove that every strongly r+y-connected component of $\max(\mathbf{b})$ contains a tuple $\mathbf{a}$ such that $\mathbf{a} \in \widetilde{\mathcal{F}}(\mathbf{b})$. By Lemma 14(1), the strongly r+y-connected components of $\max(\mathbf{b})$ are of the form $\mathbb{D}' = \mathbb{D} \cap (C_1 \times \ldots \times C_n)$ where $C_1, \ldots, C_n$ are strongly r+y-connected components of $\max(\mathbf{b}[1]), \ldots, \max(\mathbf{b}[n])$ respectively.

We prove this by induction on n. The base case of induction, $n = 1$, is obvious by definition of maximal elements. So, suppose the lemma is proved for all $m < n$.

By induction hypothesis, there is $\mathbf{c} \in \max(\mathbf{b})$ such that $\mathbf{c}[i] \in C_i$ for $i \in \underline{n-1}$ and $\mathrm{pr}_{\underline{n-1}}\mathbf{b} \prec \mathrm{pr}_{\underline{n-1}}\mathbf{c}$. Moreover, by Lemma 5(1), $\mathbf{c}$ can be chosen to be maximal. We also may assume that $\mathbf{b} \prec \mathbf{c}$. Indeed, let $\mathrm{pr}_{\underline{n-1}}\mathbf{b} = \mathbf{b}'_1 \leq \mathbf{b}'_2 \leq \ldots \leq \mathbf{b}'_k = \mathrm{pr}_{\underline{n-1}}\mathbf{c}$ be a path, and $\mathbf{b} = \mathbf{b}_1 \leq \mathbf{b}_2 \leq \ldots \leq \mathbf{b}_k \in \mathbb{D}$ its extension. Then the tuples $\mathbf{c}_1 = \mathbf{b}_1 = \mathbf{b}$, $\mathbf{c}_{i+1} = f(\mathbf{c}_i, \mathbf{c})$, for $2 \leq i \leq k$, form a path that ends up with $\mathbf{c}'$ such that $\mathrm{pr}_{\underline{n-1}}\mathbf{c}' = \mathrm{pr}_{\underline{n-1}}\mathbf{c}$. By Lemma 5(1),this path can be continued to a tuple $\mathbf{c}''$ such that $\mathbf{c}''[n]$ is maximal and $\mathrm{pr}_{\underline{n-1}}\mathbf{c}'' = \mathrm{pr}_{\underline{n-1}}\mathbf{c}$. Let $\mathbf{b} = \mathbf{b}_1 \leq \mathbf{b}_2 \leq \ldots \leq \mathbf{b}_k = \mathbf{c}$ be a path connecting $\mathbf{b}$ and $\mathbf{c}$.

If $\mathbf{c}[n] \in C_n$ then we are done. Otherwise, let $\mathbf{a} \in \mathbb{D}'$. By Lemma 16, $\max(\mathbf{b})$ is r+y-rectangular, which implies that the tuple $\mathbf{d}$:

$$\mathbf{d}[i] = \begin{cases} \mathbf{c}[i], & \text{if } i \neq n, \\ \mathbf{a}[i], & \text{if } i = n, \end{cases}$$

belongs to $\mathbb{D}'$. Thus we may assume that there is $\mathbf{a} \in \mathbb{D}' \subseteq \max(\mathbf{b})$ such that $\mathrm{pr}_{\underline{n-1}}\mathbf{a} = \mathrm{pr}_{\underline{n-1}}\mathbf{c}$, $\langle \mathbf{c}[n], \mathbf{a}[n] \rangle \in \gamma$. We shall show that $\mathbf{b} \prec \mathbf{a}$.

Notice that if $\langle \mathbf{a}[n], \mathbf{b}_j[n] \rangle \notin \beta \cup \gamma$ for a certain $j \in \underline{k}$ then $\mathbf{b} \prec \mathbf{a}$. Indeed, suppose first $\mathbf{a}[n] \leq \mathbf{b}_l[n]$ for a certain l. Consider the tuples $\mathbf{b}'_l = f(\mathbf{a}, \mathbf{b}_l)$, and if $\mathbf{b}'_j$ is obtained, $\mathbf{b}'_{j+1} = f(\mathbf{b}'_j, \mathbf{b}_{j+1})$. It is not hard to see that $\mathbf{b}'_j[i] \in \{\mathbf{b}_j[i], \mathbf{b}_k[j]\}$. Therefore, $\mathbf{b}'_k = \mathbf{b}_k$. Since $\mathbf{a} \leq \mathbf{b}'_j$, we have $\mathbf{a} \prec \mathbf{b}_k$. Since $\mathbf{a}$ is maximal and by Lemma 14(2), $\mathbf{b}_k \prec \mathbf{a}$. Furthermore, suppose that, for any $j \leq k$, $\mathbf{b}_j[n] \leq \mathbf{a}[n]$ or $\langle \mathbf{b}_j[n], \mathbf{a}[n] \rangle \in \beta \cup \gamma$; and $\mathbf{b}_l[n] \leq \mathbf{a}[n]$ for a certain $l \leq k$. Then the path $\mathbf{a}_l = f(\mathbf{b}_l, \mathbf{a})$, $\mathbf{a}_{j+1} = f(\mathbf{a}_j, \mathbf{b}_{j+1})$, for $j > l$, ends up with $\mathbf{a}$, which proves that $\mathbf{b} \prec \mathbf{a}$.

Then we prove by induction on $m < n$ that there exists $\mathbf{d} \in \widetilde{\mathcal{F}}(\mathbf{b})$ such that $\mathrm{pr}_{\underline{m}}\mathbf{d}, \mathrm{pr}_{\underline{m}}\mathbf{c}$ are strongly r-connected, $\mathbf{d}[n] = \mathbf{b}[n]$, and $\mathbf{d}[i] \in \{\mathbf{b}_1[i], \ldots, \mathbf{b}_k[i]\}$ for $i \in \{m+1, \ldots, n-1\}$. In the base case of induction, $m = 0$, we may choose $\mathbf{d} = \mathbf{b}$.

By Lemma 19, the paths $\mathbf{b}_1[m], \ldots, \mathbf{b}_k[m]$, $\mathbf{b}_1[n], \ldots, \mathbf{b}_k[n]$ can be assumed to be irreducible. Clearly, by truncating the sequence $\mathbf{b}_1, \ldots, \mathbf{b}_k$, either all $\mathbf{b}_1[m], \ldots, \mathbf{b}_k[m]$ or all $\mathbf{b}_1[n], \ldots, \mathbf{b}_k[n]$ can be assumed different.

There are two cases.

Case 1. All $\mathbf{b}_1[m], \ldots, \mathbf{b}_k[m]$ are different.

If $k \leq 2$ then for the tuple $\mathbf{a}' = f(\mathbf{b}_1, \mathbf{a})$, we have $\mathbf{a}'[m] = \mathbf{c}[m]$, $\mathbf{a}'[n] = \mathbf{b}_1[n]$ and $\mathbf{b} = \mathbf{b}_1 \leq \mathbf{a}'$.

If $k > 2$ then set $\mathbf{a}' = f(\mathbf{b}_{k-1}, \mathbf{a})$. By Lemma 20(2), we obtain tuples $\mathbf{c}^1_1, \ldots, \mathbf{c}^1_k \in \widetilde{\mathcal{F}}(\mathbf{b})$ such that $\mathbf{c}^1_1[n] = \ldots = \mathbf{c}^1_k[n] = \mathbf{b}_1[n]$, $\mathbf{c}^1_j[m] = \mathbf{b}_j[m]$ for $j \in \underline{k}$, and $\mathbf{c}^1_1[i], \ldots, \mathbf{c}^1_l[i] \in \{\mathbf{b}_1[i], \ldots, \mathbf{b}_k[i]\}$ for $i \in \underline{n} - \{m, n\}$.

Case 2. All $\mathbf{b}_1[n], \ldots, \mathbf{b}_k[n]$ are different

Let l be such that $\mathbf{b}_{k-l}[m] = \ldots = \mathbf{b}_k[m]$, but $\mathbf{b}_{k-l-1}[m] \neq \mathbf{b}_{k-l}[m]$. Consider the path $\mathbf{b}_1, \ldots, \mathbf{b}_{k-l}$. As in the previous case, there are $\mathbf{c}_1^1 \leq \ldots \leq \mathbf{c}_{k-l}^1$ with $\mathbf{c}_1^1[n] = \ldots = \mathbf{c}_{k-l}^1 = \mathbf{b}_1[n]$ and $\mathbf{c}_j^1[m] = \mathbf{b}_j[m]$ for $j \in \underline{k-l}$.

In both cases, there are $\mathbf{b} = \mathbf{c}_1^1 \leq \ldots \leq \mathbf{c}_k^1 = \mathbf{d}$ where in Case 3 $\mathbf{c}_{n-l+1}^1, \ldots, \mathbf{c}_k^1$ can be taken to be equal $\mathbf{c}_{k-l}^1$. Let also $\mathbf{d}'$ be the tuple existing by induction hypothesis, that is $\mathrm{pr}_{\underline{m-1}}\mathbf{d}'$ is strongly r-connected with $\mathrm{pr}_{\underline{m-1}}\mathbf{c}$ and $\mathbf{d}'[n] = \mathbf{b}[n]$. There is $j \in \underline{k}$ such that $\mathbf{d}'[m] = \mathbf{b}_j[m]$. Let us define tuples $\mathbf{d}_j, \ldots, \mathbf{d}_k$ as follows. Set $\mathbf{d}_j = \mathbf{d}'$, and if $\mathbf{d}_i$ is already defined then set $\mathbf{d}_{i+1} = f(\mathbf{d}_i, \mathbf{c}_{i+1}^1)$. Clearly,$\mathbf{d}_k[m] = \mathbf{c}[m]$ and $\mathrm{pr}_{\underline{m-1}}\mathbf{d}_k$ is strongly r-connected with $\mathrm{pr}_{\underline{m-1}}\mathbf{c}$. By Lemma 14(2), $\mathrm{pr}_{\underline{m}}\mathbf{d}_k$ is strongly r-connected with $\mathrm{pr}_{\underline{m}}\mathbf{c}$.

Thus, $\widetilde{\mathcal{F}}(\mathbf{b})$ contains a tuple $\mathbf{d}_k$ such that $\mathbf{d}_k[n] = \mathbf{b}[n]$ and $\mathrm{pr}_{\underline{n-1}}\mathbf{d}_k$ is strongly r-connected with $\mathrm{pr}_{\underline{n-1}}\mathbf{c}$. Take a path $\mathbf{b}[n] = e_1 \leq e_2 \leq \ldots \leq e_r = \mathbf{a}[n]$, and its extension to a path $\mathbf{b} = \mathbf{e}_1 \leq \mathbf{e}_2 \leq \ldots \leq \mathbf{e}_r$. Then set $\mathbf{e}_1' = f(\mathbf{d}_k, \mathbf{e}_1)$, and if $\mathbf{e}_j'$ is already obtained, then $\mathbf{e}_{j+1}' = f(\mathbf{e}_j', \mathbf{e}_{j+1})$. It is not hard to see, that $\mathbf{e}_1', \ldots, \mathbf{e}_r'$ form a path. Since $\mathrm{pr}_{\underline{n-1}}\mathbf{a} \prec \mathrm{pr}_{\underline{n-1}}\mathbf{d}_k \prec \mathrm{pr}_{\underline{n-1}}\mathbf{e}_r'$, $\mathbf{a}[i]$ is in a maximal strongly r-connected component and $\mathbf{e}_r'[n] = \mathbf{a}[n]$, by Lemma 14(2), $\mathbf{a}$ and $\mathbf{e}_r'$ are strongly r-connected. Finally, as $\mathbf{b} \prec \mathbf{d}_k \prec \mathbf{e}_r' \prec \mathbf{a}$, we get $\mathbf{a} \in \widetilde{\mathcal{F}}(\mathbf{b})$, as required. □

The next lemma follows straightforwardly from Lemma 5(1).

Lemma 23 *Let $\mathbb{D}$ be a subdirect product of conservative algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$, $B \subseteq \mathbb{D}$, and $I \subseteq \underline{n}$. Then* $\max(\mathrm{pr}_I B) = \mathrm{pr}_I \max(B)$.

Lemma 24 *Let $\mathbb{D}$ be a subdirect product of semiconservative algebras $\mathbb{D}^1, \mathbb{D}^2$, and $A \subseteq \mathbb{D}^1$, $B \subseteq \mathbb{D}^2$ such that* $\max(A) \times \max(B) \subseteq \mathbb{D}$. *If for $\mathbf{a} \in \mathbb{D}^1$ there is $\mathbf{b} \in \mathbb{D}^2$ with* $\max(\mathbf{b}) \subseteq \max(B)$, *then* $\max(\mathbf{a}) \times \max(B) \subseteq \mathbb{D}$.

Proof: Suppose that $\mathbb{D}^1, \mathbb{D}^2$ are subdirect products of conservative algebras $\mathbb{D}_1, \ldots, \mathbb{D}_m$ and $\mathbb{D}_{m+1}, \ldots, \mathbb{D}_{m+n}$ respectively. Take $\mathbf{a}' \in \max(\mathbf{a})$. We have to show that, for any $\mathbf{c} \in \max(B)$, the tuple $\begin{pmatrix}\mathbf{a}'\\ \mathbf{c}\end{pmatrix} \in \mathbb{D}$. By Lemma 23, $\max(\mathbf{a}) = \mathrm{pr}_{\underline{m}} \max \begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$, i.e. there is $\mathbf{b}' \in \max(B)$ with $\begin{pmatrix}\mathbf{a}'\\ \mathbf{b}'\end{pmatrix} \in \mathbb{D}$.

Take $\mathbf{d} \in \max(A)$ such that the set $I = \{i \in \underline{m} \mid \mathbf{a}'[i], \mathbf{d}[i]$ are in different ry-component$\}$ is minimal. If $\mathbf{a}'[i], \mathbf{d}[i] \in C_i$ for $i \in \underline{m} - I$ where C_i is a strongly r+y-connected component of $\max(\mathbb{D}_i)$ then by Lemma 23,

$$\mathrm{pr}_{\underline{m}-I} \max(\mathbf{a}) \cap \prod_{i \in \underline{m}-I} C_i = \mathrm{pr}_{\underline{m}-I} \max(A) \cap \prod_{i \in \underline{m}-I} C_i.$$

Therefore, $\mathbf{d}$ can be chosen such that $\mathrm{pr}_{\underline{m}-I}\mathbf{d} = \mathrm{pr}_{\underline{m}-I}\mathbf{a}'$. Since $\max(A) \times \max(B) \subseteq \mathbb{D}$, we have $\begin{pmatrix}\mathbf{d}\\ \mathbf{b}'\end{pmatrix}, \begin{pmatrix}\mathbf{d}\\ \mathbf{c}\end{pmatrix} \in \mathbb{D}$.

Consider the tuple $\begin{pmatrix}\mathbf{d}'\\ \mathbf{e}\end{pmatrix} = h\left(\begin{pmatrix}\mathbf{d}\\ \mathbf{c}\end{pmatrix}, \begin{pmatrix}\mathbf{d}\\ \mathbf{b}'\end{pmatrix}, \begin{pmatrix}\mathbf{a}'\\ \mathbf{b}'\end{pmatrix}\right)$. If $i \in \underline{m}$ is such that $\langle \mathbf{a}'[i], \mathbf{d}[i]\rangle \in \alpha \cup \beta$ then $\mathbf{a}'[i] = \mathbf{d}[i]$ and $\mathbf{d}'[i] = \mathbf{a}'[i]$. If $\langle \mathbf{a}'[i], \mathbf{d}[i]\rangle \in \gamma$ then $\mathbf{d}'[i] = \mathbf{a}'[i]$, because h is a Mal'tsev operation on $\{\mathbf{a}'[i], \mathbf{d}[i]\}$. Thus $\mathbf{d}' = \mathbf{a}'$. Furthermore, $\mathbf{e}[i] = \mathbf{c}[i]$ whenever $\langle \mathbf{b}'[i], \mathbf{c}[i]\rangle \in \beta \cup \gamma$ or $\mathbf{b}'[i] \le \mathbf{c}[i]$, $i \in K = \{m+1, \dots, m+n\}$. Therefore $\mathbf{c}[i] \le \mathbf{e}[i]$; and, since both $\mathbf{e}, \mathbf{c}$ are maximal and by Lemma 14, $\mathbf{e}, \mathbf{c}$ are in the same strongly r-connected component.

There is a path $\mathbf{e} = \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_q = \mathbf{c}$. We have $\begin{pmatrix}\mathbf{a}'\\ \mathbf{e}_1\end{pmatrix} \in \mathbb{D}$; and since $\begin{pmatrix}\mathbf{d}\\ \mathbf{e}_i\end{pmatrix} \in \mathbb{D}$ for all $\in \underline{q}$, if $\begin{pmatrix}\mathbf{a}'\\ \mathbf{e}_i\end{pmatrix} \in \mathbb{D}$ then $\begin{pmatrix}\mathbf{a}'\\ \mathbf{e}_{i+1}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}'\\ \mathbf{e}_i\end{pmatrix}, \begin{pmatrix}\mathbf{d}\\ \mathbf{e}_{i+1}\end{pmatrix}\right) \in \mathbb{D}$. Thus $\begin{pmatrix}\mathbf{a}'\\ \mathbf{c}\end{pmatrix} \in \mathbb{D}$, and the Lemma is proved. $\Box$

Corollary 15 *Let $\mathbb{D}$ be a subdirect product of semiconservative algebras $\mathbb{D}^1, \mathbb{D}^2$, $A \subseteq \mathbb{D}^1$, $B \subseteq \mathbb{D}^2$, and $\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix} \in \mathbb{D}$ is such that $\max(\mathbf{a}_1) \subseteq \max(A)$, $\max(\mathbf{a}_2) \subseteq \max(B)$. If $\max(\mathbf{a}_1) \times \max(B)$ and, for any $\mathbf{b}_1 \in \max(A)$, there is $\mathbf{b}_2 \in \max(\mathbf{a}_2)$ such that $\begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{pmatrix} \in \mathbb{D}$, then $\max(A) \times \max(B) \subseteq \mathbb{D}$.*

Proof: For any $\mathbf{b}_1 \in \max(A)$ there is $\mathbf{b}_2 \in \max(\mathbf{a}_2) \subseteq \max(B)$ such that $\begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{pmatrix} \in \mathbb{D}$; moreover, as $\max(\mathbf{a}_1) \times \max(B) \subseteq \mathbb{D}$, we are in the conditions of Lemma 24, and therefore $\max(\mathbf{b}_1) \times \max(B) \subseteq \mathbb{D}$. Since this holds for an arbitrary tuple $\mathbf{b}_1 \in \max(A)$, we get what is required. $\Box$

Now we are in a position to prove the main technical lemma.

Lemma 25 *Let $\mathbb{D}$ be a subdirect product of semiconservative algebras $\mathbb{D}^1, \mathbb{D}^2$, which are subdirect products of conservative algebras $\mathbb{D}_1, \dots, \mathbb{D}_m$ and $\mathbb{D}_{m+1}, \dots, \mathbb{D}_{m+n}$ respectively, and $\mathbf{o} \in \mathbb{D}^1$, $B \subseteq \mathbb{D}^2$ such that $\{\mathbf{o}\} \times B \subseteq \mathbb{D}$. There is $I \subseteq K = \{m+1, \dots, m+n\}$ such that*

$$\max(\{\mathbf{o}\} \times \mathrm{pr}_{K-I} B) \times \max(\mathrm{pr}_I B) \subseteq \mathbb{D},$$

and all members of $\mathrm{pr}_{K-I} B$ are indistinguishable.

Proof: We proceed by induction on 3 parameters: n, $|\widetilde{\mathcal{F}}(\mathbf{o})|$, $\sum_{\mathbf{b} \in B} |\widetilde{\mathcal{F}}(\mathbf{b})|$.

Since $\max(B) = \bigcup_{\mathbf{b} \in B} \max(\mathbf{b})$, it is enough to prove that there is $I \subseteq K$ such that, for any $\mathbf{b} \in B$, we have $\max\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-I}\mathbf{b}\end{pmatrix} \times \max(\mathrm{pr}_I \mathbf{b}) \subseteq \mathbb{D}$ and the members of $\mathrm{pr}_{K-I} B$ are indistinguishable. In fact, this condition can be weakened even further.

Claim 1. If there is $\mathbf{b}^0 \in B$ and $\varnothing \ne J \subseteq K$ such that $\max\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J}\mathbf{b}^0\end{pmatrix} \times \max(\mathrm{pr}_J \mathbf{b}^0) \subseteq \mathbb{D}$ then the result follows.

Consider $\mathrm{pr}_{\underline{m}\cup(K-J)}\mathbb{D}$, $\mathbf{o}\in\mathbb{D}^1$, and $\mathrm{pr}_{K-J}B\subseteq\mathrm{pr}_{K-J}\mathbb{D}^2$. Since $J\neq\varnothing$, by induction hypothesis, there is $I\subseteq K-J$ such that $\max(\{\mathbf{o}\}\times\mathrm{pr}_{K-J-I}B)\times\max(\mathrm{pr}_I B)\subseteq\mathbb{D}$, and the members of $\mathrm{pr}_{K-J-I}B$ are indistinguishable. In particular, $\max\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J-I}\mathbf{b}^0\end{pmatrix}\times\max(\mathrm{pr}_{J\cup I}\mathbf{b}^0)\subseteq\mathbb{D}$.

Since, for any $\mathbf{b}\in B$, $\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J-I}\mathbf{b}\end{pmatrix}$ and $\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J-I}\mathbf{b}^0\end{pmatrix}$ are indistinguishable, we get $\max\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J-I}\mathbf{b}\end{pmatrix}\times\max(\mathrm{pr}_{J\cup I}\mathbf{b}^0)\subseteq\mathbb{D}$. Finally, by Corollary 15, where

$$
\begin{aligned}
A &= \{\mathrm{pr}_{J\cup I}\mathbf{b}^0,\mathrm{pr}_{J\cup I}\mathbf{b}\},\\
B &= \{\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J-I}\mathbf{b}^0\end{pmatrix},\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J-I}\mathbf{b}\end{pmatrix}\},\\
\max\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J-I}\mathbf{b}\end{pmatrix}\times\max(\mathrm{pr}_{J\cup I}\mathbf{b}) &\subseteq \mathbb{D}.
\end{aligned}
$$

The claim is proved.

As is easily seen, by Claim 1, we may restrict ourselves with the case $|B|=2$. So, let $B=\{\mathbf{b}_1,\mathbf{b}_2\}$, $\mathbf{b}_1,\mathbf{b}_2$ are not indistinguishable. The base case of induction, n is any, $\mathbf{o}$ and $\mathbf{b}_1,\mathbf{b}_2$ are maximal, follows from the next statement.

Claim 2. Let $\mathbf{b}_1,\mathbf{b}_2$ be maximal elements of $\mathbb{D}^2$, B^1_i,B^2_i the ry-components of $\max(\mathbb{D}_i)$, $i\in K$, containing $\mathbf{b}_1[i],\mathbf{b}_2[i]$ respectively, and $I=\{i\in K\mid B^1_i\neq B^2_i\}$. If $I=\varnothing$ then $\mathbf{b}_1,\mathbf{b}_2$ are indistinguishable, otherwise,

$$\max\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-I}\mathbf{b}_1\end{pmatrix}\times\max(\mathrm{pr}_I\mathbf{b}_1)\subseteq\mathbb{D}.$$

By Lemma 14, the algebra $\mathbb{E}^i=\mathbb{D}^2\cap(B^i_{m+1}\times\ldots\times B^i_{m+n})$, $i=1,2$, is a subdirect product of $B^i_{m+1},\ldots,B^i_{m+n}$, strongly r+y-connected, and, in fact, $\mathbb{E}^i=\max(\mathbf{b}_i)$.

Notice that $\max(\mathbf{o})$, $\max(\mathbf{b}_1)$, $\max(\mathbf{b}_2)$ equal the ry-components of $\mathbb{D}^1,\mathbb{D}^2$ containing $\mathbf{o},\mathbf{b}_1,\mathbf{b}_2$ respectively. Therefore, if $I=\varnothing$ then $\max(\mathbf{b}_1)=\max(\mathbf{b}_2)$, and $\mathbf{b}_1,\mathbf{b}_2$ are indistinguishable. Otherwise, we apply Lemma 16 to the algebra

$$\mathbb{D}\cap\left(\max(\mathbf{o})\times(\mathbb{D}^2\cap((B^1_{m+1}\cup B^2_{m+1})\times\ldots\times(B^1_{m+n}\cup B^2_{m+n})))\right).$$

This algebra is r+y-rectangular, which means that for any $\mathbf{a}\in\max\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-I}\mathbf{b}_1\end{pmatrix}$ and any $\mathbf{b}\in\mathrm{pr}_I\mathbf{b}_1$, the tuple $\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$ belongs to $\mathbb{D}$. The claim is proved.

Notice that if, say, $\mathbf{b}_1$ is not maximal while $\mathbf{b}_2$ is maximal, then $\mathbf{b}_2\not\sqsubseteq\mathbf{b}_1$. Indeed, $\max(\mathbf{b}_2)$ is the ry-component of $\mathbb{D}^2$ containing $\mathbf{b}_2$, therefore, if $\max(\mathbf{b}_1)\subseteq\max(\mathbf{b}_2)$

then $\max(\mathbf{b}_1) = \max(\mathbf{b}_2)$, that is $\mathbf{b}_1, \mathbf{b}_2$ are indistinguishable. Suppose that there is $\mathbf{c} \in \widetilde{\mathcal{F}}(\mathbf{b}_1) - \widetilde{\mathcal{F}}(\mathbf{b}_2)$. Moreover, we may assume $\mathbf{c}$ to be maximal and $\mathbf{c}, \mathbf{b}_1$ are in different strongly r-connected components. In other words, $|\widetilde{\mathcal{F}}(\mathbf{c})| \leq |\widetilde{\mathcal{F}}(\mathbf{b}_1)|$.

There is a path $\mathbf{b}_1 = \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k = \mathbf{c}$, and by Lemma 21, $\mathbf{b}_2 \leq \mathbf{d}$ for no $\mathbf{d} \in \mathbb{D}^2$ indistinguishable with one of $\mathbf{c}_1, \ldots, \mathbf{c}_k$. We again prove by induction on k.

Claim 3. Either there is $\mathbf{o}' \in \widetilde{\mathcal{F}}(\mathbf{o})$ such that $\begin{pmatrix}\mathbf{o}'\\ \mathbf{c}_2\end{pmatrix}, \begin{pmatrix}\mathbf{o}'\\ \mathbf{b}_2\end{pmatrix} \in \mathbb{D}$, or there is $J \subseteq K$ with

$$\max\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J}\mathbf{b}_2\end{pmatrix} \times \max(\mathrm{pr}_J \mathbf{b}_2) \subseteq \mathbb{D}.$$

There is $\mathbf{o}' \in \mathbb{D}^1$ with $\begin{pmatrix}\mathbf{o}'\\ \mathbf{c}_2\end{pmatrix} \in \mathbb{D}$. Replacing $\begin{pmatrix}\mathbf{o}'\\ \mathbf{c}_2\end{pmatrix}$ with $f\left(\begin{pmatrix}\mathbf{o}\\ \mathbf{b}_1\end{pmatrix}, \begin{pmatrix}\mathbf{o}'\\ \mathbf{c}_2\end{pmatrix}\right)$, we may assume that $\mathbf{o} \leq \mathbf{o}'$.

Consider the tuple $\begin{pmatrix}\mathbf{o}'\\ \mathbf{d}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{o}\\ \mathbf{b}_2\end{pmatrix}, \begin{pmatrix}\mathbf{o}'\\ \mathbf{c}_2\end{pmatrix}\right)$. Clearly, $\mathbf{b}_2 \leq \mathbf{d}$. Let us denote by L the set $\{i \in K \mid \mathbf{d}[i] = \mathbf{b}_2[i]\}$. If $L = K$ then $\begin{pmatrix}\mathbf{o}'\\ \mathbf{c}_2\end{pmatrix}, \begin{pmatrix}\mathbf{o}'\\ \mathbf{b}_2\end{pmatrix} \in \mathbb{D}$. If $L = \varnothing$ then $\mathbf{d} = \mathbf{c}_2$ which means $\mathbf{b}_2 \leq \mathbf{c}_2$, a contradiction to the condition $\mathbf{c} \in \widetilde{\mathcal{F}}(\mathbf{b}_1) - \widetilde{\mathcal{F}}(\mathbf{b}_2)$. Otherwise, $\mathbf{d}[i] = \mathbf{c}_2[i]$ for any $i \in K - L$, hence, we may use induction hypothesis for the pair $\begin{pmatrix}\mathbf{a}\\ \mathrm{pr}_L\mathbf{b}_1\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ \mathrm{pr}_L\mathbf{c}_2\end{pmatrix}$ where $\mathbf{a} = \begin{pmatrix}\mathbf{o}'\\ \mathrm{pr}_{K-L}\mathbf{c}_2\end{pmatrix}$. Therefore there is $\varnothing \neq J \subseteq L$ satisfying the required conditions.

If the second possibility (a direct decomposition) from Claim 3 is valid then, by Claim 1, the result follows. Otherwise, if $k = 2$ then, since $|\widetilde{\mathcal{F}}(\mathbf{c}_2)| + |\widetilde{\mathcal{F}}(\mathbf{b}_2)| < |\widetilde{\mathcal{F}}(\mathbf{b}_1)| + |\widetilde{\mathcal{F}}(\mathbf{b}_2)|$ and $|\widetilde{\mathcal{F}}(\mathbf{o}')| \leq |\widetilde{\mathcal{F}}(\mathbf{o})|$, by induction hypothesis, there is $J \subseteq K$ such that

$$\max\begin{pmatrix}\mathbf{o}'\\ \mathrm{pr}_{K-J}\mathbf{b}_2\end{pmatrix} \times \max(\mathrm{pr}_J \mathbf{b}_2) \subseteq \mathbb{D}.$$

Moreover, as $\begin{pmatrix}\mathbf{a}\\ \mathrm{pr}_J\mathbf{b}_2\end{pmatrix} \in \mathbb{D}$ where $\mathbf{a} = \begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J}\mathbf{b}_2\end{pmatrix}$, by Lemma 24,

$$\max\begin{pmatrix}\mathbf{o}\\ \mathrm{pr}_{K-J}\mathbf{b}_2\end{pmatrix} \times \max(\mathrm{pr}_J \mathbf{b}_2) \subseteq \mathbb{D}.$$

Therefore, we get what is required.

In the case $k > 2$, the argument is the same but we use induction hypothesis of the second induction process, on k. This completes the proof. $\square$

7.3 Max-completeness

For a subdirect product $\mathbb{D}$ of conservative algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$, a coordinate position $j \in \underline{n}$, and an i-component B of $\mathbb{D}_j$, minimal with respect to $\sqsubseteq$, we define the set $J(\mathrm{m}, j, B)$ as follows

$$J(\mathrm{m}, j, B) \;=\; \{i \in \underline{n} \mid \text{for any } \mathbf{a}, \mathbf{b} \in \mathbb{D}, \text{ such that } \mathbf{a}[j] \in B, \text{ the element } \mathbf{b}[j] \text{ belongs to } B \text{ if and only if } \mathbf{a}[i], \mathbf{b}[i] \text{ are minimal and indistinguishable}\}$$

Definition 17 *The algebra $\mathbb{D}$ is said to be* max-complete *if, for any maximal $\mathbf{a} \in \mathbb{D}$, and any minimal i-components $B_1, \dots, B_n$ of $\mathbb{D}_1, \dots, \mathbb{D}_n$, respectively, such that $\mathbf{a}[i] \in \max(B_i)$, $i \in \underline{n}$; any $j \in \underline{n}$, and any maximal $\mathbf{b} \in \mathrm{pr}_{J(\mathrm{m},j,B_j)}\mathbb{D} \cap \left(\prod_{i \in J(\mathrm{m},j,B_j)} \max(B_j)\right)$ the tuple $\mathbf{c}$:*

$$\mathbf{c}[i] = \begin{cases} \mathbf{b}[i], & \text{if } i \in J(\mathrm{m}, j, B), \\ \mathbf{a}[i], & \text{otherwise,} \end{cases}$$

belongs to $\mathbb{D}$.

Lemma 26 *Let $\mathbb{D}$ be a subdirect product of conservative algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$, $j \in \underline{n}$, $l \in J(\mathrm{m}, j, B)$ and B, C minimal i-components of $\mathbb{D}_j, \mathbb{D}_l$, respectively, such that, for any $\mathbf{o} \in \mathbb{D}$ with $\mathbf{o}[j] \in B$, we have $\mathbf{o}[l] \in C$. For any $\mathbf{a} \in \mathbb{D}$ such that $\mathbf{a}[j] \in \widetilde{\mathcal{F}}(B)$ the element $\mathbf{a}[l]$ belongs to $\widetilde{\mathcal{F}}(C)$.*

Proof: Without loss of generality we may assume that $j = 1$, $l = 2$. Let us suppose that $\mathbf{a} \in \mathbb{D}$ is such that $\mathbf{a}[1] \in \widetilde{\mathcal{F}}(B)$ while $\mathbf{a}[2] \notin \widetilde{\mathcal{F}}(C)$. Notice that if $\mathbf{a}[1] \in B$ then $\mathbf{a}[2] \in C$, therefore, we may assume that $\mathbf{a}[1] \in \widetilde{\mathcal{F}}(B) - B$. There is $\mathbf{b} \in \mathbb{D}$ such that $\mathbf{b}[1] \in B$ and $\mathbf{a}[1] \in \widetilde{\mathcal{F}}(\mathbf{b}[1])$. This means that there is also a path $\mathbf{b}[1] = b_1, b_2, \dots, b_k = \mathbf{a}[1]$. This path can be extended to a path $\mathbf{b} = \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$ with $\mathbf{b}_i[1] = b_i$, $i \in \underline{k}$. By Lemma 19, $\mathbf{b}_1[1], \dots, \mathbf{b}_k[1]$ and $\mathbf{b}_1[2], \dots, \mathbf{b}_k[2]$ can be assumed to be irreducible. Since $\mathbf{b}_1[2], \dots, \mathbf{b}_k[2] \in \widetilde{\mathcal{F}}(\mathbf{b}[2])$, $\mathbf{b}_i[2] \le \mathbf{a}[2]$ for no $i \in \underline{k}$. Therefore, this path and the tuple $\mathbf{c} = f(\mathbf{b}_{k-1}, \mathbf{a})$ satisfies the conditions of Lemma 20. Hence, there is $\mathbf{d} \in \mathbb{D}$ with $\mathbf{d}[1] = \mathbf{b}[1]$, $\mathbf{d}[2] = \mathbf{b}_{k-1}[2]$. If $\mathbf{b}_{k-1}[2] \notin C$ then this contradicts the assumption $2 \in J(\mathrm{m}, 1, B)$. If $\mathbf{b}_{k-1}[2] \in C$ then as $\mathbf{c}[1] = \mathbf{a}[1] \notin B$, we again get a contradiction with the assumption $1 \in J(\mathrm{m}, 2, C)$. $\square$

Proposition 9 *Any subdirect product of conservative strongly r+b-connected algebras is max-complete.*

Proof: Let $\mathbb{D}$ be a subdirect product of conservative algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$, $\mathbf{a} \in \max(\mathbb{D})$ and $B_1, \dots, B_n$ minimal i-components of $\mathbb{D}_1, \dots, \mathbb{D}_n$, respectively, such that $\mathbf{a}[i] \in B_i$. Let also $I_1, \dots, I_k \subseteq \underline{n}$ be the sets of the form $J(\mathrm{m}, i, B_i)$, and $I_1 = J(\mathrm{m}, 1, B_1)$. Without loss of generality $I_1 = \{1, \dots, m\}$. We prove that, for any $\mathbf{a}' \in \mathrm{pr}_{\underline{m}}\mathbb{D} \cap (B_1 \times \ldots \times B_m)$, the tuple $\mathbf{a}''$:

$$\mathbf{a}''[i] = \begin{cases} \mathbf{a}'[i], & \text{if } i \in I_1; \\ \mathbf{a}[i], & \text{otherwise,} \end{cases}$$

belongs to $\mathbb{D}$.

CLAIM 1. If $\mathbf{b} \in \mathbb{D}$ is such that, for certain $j \in \underline{k}$ and $i \in I_j$, $\mathbf{b}[i] \in B_i$, then, for any $l \in I_j$, $\mathbf{b}[l] \in B_l$.

By Lemma 26, $\mathbf{b}[l] \in \widetilde{\mathcal{F}}(B_l)$. On the other hand, $\mathbf{b}[l]$ belongs to a minimal i-component B of $\mathbb{D}_l$. Since $B_l \sqsubseteq B$ and both are minimal, we get $B = B_l$.

Claim 1 implies that, for any $j \in \underline{k}$,

$$\mathrm{pr}_{I_j}\left(\mathbb{D} \cap (\prod_{i \in I_j} \max(B_i))\right) = \max(\mathrm{pr}_{I_j}\mathbf{b})$$

where $\mathbf{b} \in \mathbb{D}$ is an arbitrary tuple such that $\mathbf{b}[i] \in B_i$ for $i \in I_j$.

If B_i contains a maximal element, then $\max(B_i) \subseteq B_i$ and is strongly r+y-connected. Indeed, maximal elements from different ry-components of $\max(B_i)$ are not indistinguishable.

CLAIM 2. Let, for a certain $j \in \underline{n}$, B_j contain a maximal element. Then $J(\mathrm{r{+}y}, j, \max(B_j)) \subseteq J(\mathrm{m}, j, B_j)$.

We have to show that, for any $j \in \underline{n} - J(\mathrm{m}, i, B_i)$, $j \notin J(\mathrm{r{+}y}, i, C)$. Without loss of generality, let us assume that $i = 1$, $j = n$. Since $j \notin J(\mathrm{m}, 1, B_1)$, either

(1) there is $\mathbf{c} \in \mathbb{D}$ such that $\mathbf{c}[1] \in B_1$, $\mathbf{c}[n]$ is not in a minimal i-component; or

(2) there are $\mathbf{c}, \mathbf{d}$ such that $\mathbf{c}[1], \mathbf{d}[1] \in B_1$, $\mathbf{c}[n], \mathbf{d}[n]$ belong to different minimal i-components; or

(3) there is a minimal i-component B of $\mathbb{D}_n$ such that, for any $\mathbf{c} \in \mathbb{D}$ with $\mathbf{c}[1] \in B_1$, $\mathbf{c}[n] \in B$, but there is $\mathbf{d} \in \mathbb{D}$ with $\mathbf{d}[n] \in B$, $\mathbf{d}[1] \notin B_1$.

(1) Take $\mathbf{d} \in \mathbb{D}$ such that $\mathbf{d}[n]$ is minimal and $\mathbf{d}[n] \sqsubseteq \mathbf{c}[n]$. Since $\max(\mathbf{d}[n]) \not\subseteq \max(\mathbf{c}[n])$, there is $d \in \widetilde{\mathcal{F}}(\mathbf{d}[n]) - \max(\mathbf{c}[n])$. This element can be chosen maximal and can be extended to a maximal tuple $\mathbf{e} \in \max(\mathbf{d}) \cap \widetilde{\mathcal{F}}(\mathbf{d})$, in particular, there exists a path $\mathbf{d} = \mathbf{d}_1 \le \ldots \le \mathbf{d}_r = \mathbf{e}$.

Suppose first that $\mathbf{d}[1] \in B_1$, that is $\max(\mathbf{c}[1]) = \max(\mathbf{d}[1])$ and $\mathbf{e}[1] \in \max(\mathbf{c}[1])$. Therefore, $\widetilde{\mathcal{F}}(\mathbf{c}) \cap \max(\mathbf{c})$ contains a tuple $\mathbf{e}'$ such that $\mathbf{e}[1], \mathbf{e}'[1]$ are strongly r+y-connected. Clearly, $\mathbf{e}[n], \mathbf{e}'[n]$ are in different ry-components of $\max(\mathbb{D}_n)$. Thus, $\mathbf{e}, \mathbf{e}'$ witness that $n \notin J(\mathrm{r{+}y}, 1, \max(B_1))$.

Now, let $\mathbf{d}[1] \notin B_1$. Since B_1 is a minimal i-component, by Lemma 21, there is $d \in \widetilde{\mathcal{F}}(\mathbf{c}[1]) - \max(\mathbf{d}[1])$. The element d can be assumed to be maximal, and there is a maximal tuple $\mathbf{e} \in \widetilde{\mathcal{F}}(\mathbf{c})$ with $\mathbf{e}[1] = d$. Since $\max(\mathbf{c}[n]) \subseteq \max(\mathbf{d}[n])$, there is a maximal tuple $\mathbf{e}' \in \mathbb{D}$ such that $\mathbf{e}[n], \mathbf{e}'[n]$ are strongly r+y-connected in $\max(\mathbf{d}[n])$. Furthermore, as $\mathbf{e}[1] \notin \max(\mathbf{d}[1])$, $\mathbf{e}[1], \mathbf{e}'[1]$ are in different strongly r+y-components of $\max(\mathbb{D}_1)$, which implies $n \notin J(\mathrm{r{+}y}, 1, \max(B_1))$.

(2) Since $\mathbf{c}[n], \mathbf{d}[n]$ belong to different minimal i-components, there is $d \in \widetilde{\mathcal{F}}(\mathbf{c}[n]) - \max(\mathbf{d}[n])$. This element can be extended to a maximal tuple $\mathbf{e}$ from $\widetilde{\mathcal{F}}(\mathbf{c})$. Furthermore, as $\max(\mathbf{c}[1]) = \max(\mathbf{d}[1])$, there is a maximal $\mathbf{e}' \in \widetilde{\mathcal{F}}(\mathbf{d})$ such that $\mathbf{e}'[1]$ is in the same ry-component with $\mathbf{e}[1]$. The elements $\mathbf{e}[n], \mathbf{e}'[n]$ are not strongly r+y-connected, that implies $n \notin J(\mathrm{r{+}y}, 1, \max(B_1))$.

(3) There is $d \in \widetilde{\mathcal{F}}(\mathbf{c}[1]) - \max(\mathbf{d}[1])$ which can be extended to a maximal tuple $\mathbf{e} \in \widetilde{\mathcal{F}}(\mathbf{c})$. Since $\mathbf{e}[n] \in \max(\mathbf{d}[n])$, there is a maximal $\mathbf{e}' \in \widetilde{\mathcal{F}}(\mathbf{d})$ such that $\mathbf{e}[n], \mathbf{e}'[n]$ are strongly r+y-connected. As $\mathbf{e}[1], \mathbf{e}'[1]$ are in different ry-components, we get what is required. The claim is proved.

Claim 2 implies that $J(\mathrm{m}, 1, B_1)$ a inion of certain sets of the form $J(\mathrm{r{+}y}, i, \max(B_i))$.

We proceed by induction on n. If $n = 1$ then the result is obvious. So, let us assume that it is proved for any number less than n.

If B_1 contains a maximal element then, by Claim 2, I_1 is a union of $J(\mathrm{r{+}y}, i, \max(B_i))$, $i \in I_1$. By Lemma 16, for any $\mathbf{a}'$ strongly r+y-connected to $\mathrm{pr}_{I_1}\mathbf{a}$, that is, by Lemma 14, for any $\mathbf{a}' \in \mathrm{pr}_{\underline{m}}\mathbb{D} \cap (B_1 \times \ldots \times B_m)$ the corresponding tuple $\mathbf{a}''$ belongs to $\mathbb{D}$. So, suppose that B_1 contains no maximal elements.

First we show that there are $\mathbf{b} \in \mathbb{D}$ with $\mathbf{b}[1] \in B_1$ and $I_1 \subseteq J \subseteq \underline{n}$, $n \in K = \underline{n} - J$ such that

$$\max(\mathrm{pr}_J \mathbf{b}) \times \max(\mathrm{pr}_K \mathbf{b}) \subseteq \mathbb{D}.$$

There are three cases to consider.

Case 1. There is $\mathbf{b} \in \mathbb{D}$, $j \in \underline{n} - I_1$, such that $\mathbf{b}[1] \in B_1$ and $\mathbf{b}[j]$ does not belong to a minimal i-component.

Without loss of generality we may assume that $j = n$. Take $\mathbf{c} \in \mathbb{D}$ such that $\mathbf{c}[n]$ is minimal and $\mathbf{c}[n] \sqsubseteq \mathbf{b}[b]$.

Subcase 1.1. $\mathbf{c}[1] \in B_1$.

Since $\max(\mathbf{c}[n]) \not\subseteq \max(\mathbf{b}[n])$, there is $d \in \widetilde{\mathcal{F}}(\mathbf{c}[n]) - \max(\mathbf{b}[n])$. This element can be chosen maximal and can be extended to a maximal tuple $\mathbf{d} \in \max(\mathbf{c}) \cap \widetilde{\mathcal{F}}(\mathbf{c})$. As $\max(\mathbf{b}[1]) = \max(\mathbf{c}[1])$, $\mathbf{d}[1] \in \max(\mathbf{b}[1])$. Therefore, $\widetilde{\mathcal{F}}(\mathbf{b}) \cap \max(\mathbf{b})$ contains a tuple $\mathbf{e}$ such that $\mathbf{e}[1], \mathbf{d}[1]$ are strongly r+y-connected, in particular, there exists a path $\mathbf{b} = \mathbf{b}_1 \le \ldots \le \mathbf{b}_r = \mathbf{e}$ such that $\mathbf{b}_1[1], \ldots, \mathbf{b}_r[1]$ and $\mathbf{b}_1[n], \ldots, \mathbf{b}_r[n]$ are irreducible.

Clearly, $\mathbf{e}[n], \mathbf{d}[n]$ are in different strongly r+y-connected components of $\max(\mathbb{D}_n)$. Therefore, $n \notin J(\mathrm{r{+}y}, 1, C)$ where C is the ry-component containing $\mathbf{e}[1], \mathbf{d}[1]$, and, by Lemma 16, there exists $\mathbf{e}' \in \mathbb{D}$ such that $\mathbf{e}'[1] = \mathbf{e}[1]$, $\mathbf{e}'[n] = \mathbf{d}[n]$. The path $\mathbf{b}_1, \ldots, \mathbf{b}_r$ and the tuple $f(\mathbf{b}_{r-1}, \mathbf{e}')$ satisfy the conditions of Lemma 20, hence, there is $\mathbf{b}' \in \mathbb{D}$ with $\mathbf{b}'[1] = \mathbf{e}[1]$, $\mathbf{b}'[n] = \mathbf{b}[n]$. Set

$$\begin{aligned} J &= \{i \in \underline{n} \mid \mathbf{b}[i], \mathbf{b}'[i] \text{ are indistinguishable}\} \\ K &= \underline{n} - J. \end{aligned}$$

Clearly, both sets are non-empty, as $n \in J$ and $1 \in K$. By Lemma 25,

$$\max(\mathrm{pr}_J \mathbf{b}) \times \max(\mathrm{pr}_K \mathbf{b}) \subseteq \mathbb{D}.$$

By induction hypothesis, $\max(\mathrm{pr}_K \mathbf{b}) = \max(\mathrm{pr}_{I_1} \mathbf{b}) \times \max(\mathrm{pr}_{K-I_1} \mathbf{b})$; consequently,

$$\max(\mathrm{pr}_{I_1} \mathbf{b}) \times \max(\mathrm{pr}_{\underline{n}-I_1} \mathbf{b}) \subseteq \mathbb{D}.$$

SUBCASE 1.2. $\mathbf{c}[1] \notin B_1$.

Since B_1 is a minimal i-component, by Lemma 21, there is $d \in \widetilde{\mathcal{F}}(\mathbf{b}[1]) - \max(\mathbf{c}[1])$. The element d can be assumed to be maximal, and there is a maximal tuple $\mathbf{d} \in \widetilde{\mathcal{F}}(\mathbf{b})$ with $\mathbf{d}[1] = d$. Since $\max(\mathbf{b}[n]) \subseteq \max(\mathbf{c}[n])$, there is a maximal tuple $\mathbf{e} \in \widetilde{\mathcal{F}}(\mathbf{c})$ such that $\mathbf{d}[n], \mathbf{e}[n]$ are strongly r+y-connected in $\max(\mathbf{c}[n])$. Furthermore, as $\mathbf{d}[1] \notin \max(\mathbf{c}[1])$, $\mathbf{d}[1], \mathbf{e}[1]$ are in different strongly r+y-components of $\max(\mathbb{D}_1)$.

By Lemmas 16,14, there is a maximal $\mathbf{e}' \in \mathbb{D}$ such that $\mathbf{e}'[1] = \mathbf{e}[1]$, $\mathbf{e}'[n] = \mathbf{d}[n]$. Consider a path $\mathbf{b} = \mathbf{d}_1 \leq \mathbf{d}_2 \leq \ldots \leq \mathbf{d}_r = \mathbf{d}$ such that $\mathbf{d}_1[1], \ldots, \mathbf{d}_r[1]$ and $\mathbf{d}_1[n], \ldots, \mathbf{d}_r[n]$ are irreducible. In all remaining subcases we show that there exists $\mathbf{b}' \in \mathbb{D}$ or $\mathbf{b}'' \in \mathbb{D}$ such that $\mathbf{b}'[n] = \mathbf{b}[n]$ and $\mathbf{b}'[1]$ equals either $\mathbf{d}[1]$ or $\mathbf{e}[1]$, or $\mathbf{b}''[1] = \mathbf{b}[1]$ and $\mathbf{b}''[n] = \mathbf{d}[n]$.

SUBCASE 1.2.1. $\mathbf{d}_{r-1}[1] \not\leq \mathbf{e}[1]$.

The path $\mathbf{d}_1, \ldots, \mathbf{d}_r$ and the tuple $\mathbf{d}' = f(\mathbf{d}_{r-1}, \mathbf{e}')$ satisfy the conditions of Lemma 20. Therefore, if $r > 2$ then there is $\mathbf{b}' \in \mathbb{D}$ such that $\mathbf{b}'[n] = \mathbf{b}[n]$ and $\mathbf{b}'[1] = \mathbf{e}'[1] = \mathbf{e}[1]$. If $r = 2$ then the tuple $\mathbf{b}'' = \mathbf{d}'$ satisfies the required conditions.

Let l be the least number such that $\mathbf{d}_l[1] \leq \mathbf{e}[1]$.

SUBCASE 1.2.2. $\mathbf{d}_l[n] \not\leq \mathbf{e}[n]$.

Consider the path $\mathbf{d}_1, \ldots, \mathbf{d}_l, \mathbf{d}'$ where $\mathbf{d}' = f(\mathbf{d}_l, \mathbf{e})$. Since $\mathbf{d}'[1] = \mathbf{e}[1]$, $\mathbf{d}'[n] = \mathbf{d}_l[n]$, this path satisfies the conditions of Lemma 20 and there is $\mathbf{b}'$ with the required properties.

Let m be the least number such that $\mathbf{d}_m[n] \leq \mathbf{e}[n]$.

SUBCASE 1.2.3. $m < l$.

For $\mathbf{d}' = f(\mathbf{d}_m, \mathbf{e})$ we have $\mathbf{d}'[n] = \mathbf{e}[n]$, $\mathbf{d}'[1] = \mathbf{d}_m[1]$. If $m = 1$ then we may choose $\mathbf{b}'' = \mathbf{d}'$. Otherwise, the path $\mathbf{d}_1, \ldots, \mathbf{d}_m$ and $\mathbf{d}'$ satisfy the conditions of Lemma 20. Therefore, there exists a tuple $\mathbf{d}'' \in \mathbb{D}$ such that $\mathbf{d}''[1] = \mathbf{d}_{m-1}[1]$, $\mathbf{d}''[n] = \mathbf{d}_m[n]$.

By induction on $j \in \{m, \ldots, r\}$ we show that there is a tuple $\mathbf{d}''_j$ such that $\mathbf{d}''_j[1] = \mathbf{d}_{j-1}[1]$, $\mathbf{d}''_j[n] = \mathbf{d}_j[n]$. For the base case of induction we may set $\mathbf{d}''_m = \mathbf{d}''$. If the existence of $\mathbf{d}''_j$ is proved, then set $\mathbf{d}''_{j+1} = f(\mathbf{d}_j, f(\mathbf{d}''_j, \mathbf{d}_{j+1}))$. It is not hard to verify that it satisfies the required conditions.

The path $\mathbf{d}_1, \ldots, \mathbf{d}_r$ and the tuple $\mathbf{d}''_r$ satisfies the conditions of Lemma 20, which implies that there exists $\mathbf{b}' \in \mathbb{D}$ with $\mathbf{b}'[1] = \mathbf{d}_r[1]$, $\mathbf{b}'[n] = \mathbf{b}[n]$.

SUBCASE 1.2.4. $m = l$.

Let $\mathbf{c} = \mathbf{e}_1 \leq \mathbf{e}_2 \leq \ldots \leq \mathbf{e}_q = \mathbf{e}$ be a path connecting $\mathbf{c}$ and $\mathbf{e}$, and s the maximal number such that either $\mathbf{d}_l[1] \not\lesssim \mathbf{e}_s[1]$ or $\mathbf{d}_l[n] \not\lesssim \mathbf{e}_s[n]$. Since $\mathbf{b}[n] \not\sqsubseteq \mathbf{c}[n]$, $\mathbf{d}_l[n] \not\lesssim \mathbf{c}[n]$ and such an s exists. Analogously to $\mathbf{e}'$ there is a maximal $\mathbf{e}'' \in \mathbb{D}$ such that $\mathbf{e}''[1] = \mathbf{d}[1]$, $\mathbf{e}''[n] = \mathbf{e}[n]$. As $\mathbf{d}[1] \not\in \widetilde{\mathcal{F}}(\mathbf{c}[1])$, we have $\mathbf{e}_{q-1}[1] \not\lesssim \mathbf{d}[1]$. Therefore, the tuple $\mathbf{d}'' = f(\mathbf{e}_{q-1}, \mathbf{e}'')$ satisfies the conditions: $\mathbf{d}''[1] = \mathbf{e}_{q-1}[1]$, $\mathbf{d}''[n] = \mathbf{e}[n] = \mathbf{e}_q[n]$. Thus, Lemma 20 can be applied to $\mathbf{e}_1, \ldots, \mathbf{e}_r$ and $\mathbf{d}''$. There are $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{D}$ such that $\mathbf{a}_1[1] = \mathbf{e}_s[1]$, $\mathbf{a}_1[n] = \mathbf{e}_q[n]$ and $\mathbf{a}_2[1] = \mathbf{e}_q[1]$, $\mathbf{a}_2[n] = \mathbf{e}_s[n]$.

If $\mathbf{d}_l[1] \not\lesssim \mathbf{e}_s[1]$ then consider the path $\mathbf{d}_1, \ldots, \mathbf{d}_l, \mathbf{e}$ and the tuple $\mathbf{d}' = f(\mathbf{d}_l, \mathbf{a}_1)$. Since $\mathbf{d}'[1] = \mathbf{d}_l[1]$ and $\mathbf{d}'[n] = \mathbf{e}[n]$, by Lemma 20, there is $\mathbf{b}'$ with $\mathbf{b}'[1] = \mathbf{e}[1]$, $\mathbf{b}'[n] = \mathbf{b}[n]$.

If $\mathbf{d}_l[n] \not\lesssim \mathbf{e}_s[n]$ then consider the path $\mathbf{d}_1, \ldots, \mathbf{d}_l, \mathbf{e}$ and the tuple $\mathbf{d}' = f(\mathbf{d}_l, \mathbf{a}_2)$. Since $\mathbf{d}'[1] = \mathbf{e}[1]$ and $\mathbf{d}'[n] = \mathbf{d}_l[n]$, again by Lemma 20, there is $\mathbf{b}'$ with $\mathbf{b}'[1] = \mathbf{e}[1]$, $\mathbf{b}'[n] = \mathbf{b}[n]$.

Now, set

$$\begin{aligned} J &= \{i \in \underline{n} \mid \mathbf{b}[i], \mathbf{b}'[i] \text{ are indistinguishable}\} \\ K &= \underline{n} - J. \end{aligned}$$

Clearly, both sets are non-empty, as $n \in J$ and $1 \in K$. By Lemma 25 where $\mathbf{o} = \mathbf{b}[n]$ and $I = J - \{n\}$,

$$\max(\mathrm{pr}_J \mathbf{b}) \times \max(\mathrm{pr}_K \mathbf{b}) \subseteq \mathbb{D}.$$

If we have obtained $\mathbf{b}''$, then set

$$\begin{aligned} J &= \{i \in \underline{n} \mid \mathbf{b}''[i], \mathbf{d}[i] \text{ are indistinguishable}\} \\ K &= \underline{n} - J. \end{aligned}$$

Clearly, both sets are non-empty, as $n \in J$ and $1 \in K$. Again by Lemma 25 where $\mathbf{o} = \mathbf{d}[n]$ and $I = J - \{n\}$,

$$\max(\mathrm{pr}_J \mathbf{b}'') \times \max(\mathrm{pr}_K \mathbf{b}'') \subseteq \mathbb{D}.$$

By induction hypothesis, $\max(\mathrm{pr}_K \mathbf{b}'') = \max(\mathrm{pr}_{I_1} \mathbf{b}'') \times \max(\mathrm{pr}_{K-I_1} \mathbf{b}'')$; consequently,

$$\max(\mathrm{pr}_{I_1} \mathbf{b}'') \times \max(\mathrm{pr}_{\underline{n}-I_1} \mathbf{b}'') = \max(\mathrm{pr}_{I_1} \mathbf{b}) \times \max(\mathrm{pr}_{\underline{n}-I_1} \mathbf{b}'') \subseteq \mathbb{D}.$$

Finally, by Lemma 24,

$$\max(\mathrm{pr}_{I_1} \mathbf{b}) \times \max(\mathrm{pr}_{\underline{n}-I_1} \mathbf{b}) \subseteq \mathbb{D}.$$

CASE 2. For any $\mathbf{b} \in \mathbb{D}$ such that $\mathbf{b}[1] \in B_1$, and any $i \in \underline{n}$, $\mathbf{b}[i]$ belongs to a minimal i-component, but there are $\mathbf{b}, \mathbf{c} \in \mathbb{D}$ such that $\mathbf{b}[1], \mathbf{c}[1] \in B_1$ and $\mathbf{b}[n], \mathbf{c}[n]$ are not indistinguishable.

We shall not use the fact that $\mathbf{c}[n]$ is in a minimal i-component. However, $\mathbf{b}[n]$ is assumed to be a member of a minimal i-component. There is $d \in \widetilde{\mathcal{F}}(\mathbf{b}[n]) - \max(\mathbf{c}[n])$.

The element d can be extended to a maximal tuple $\mathbf{d} \in \widetilde{\mathcal{F}}(\mathbf{b})$ so that $\mathbf{d}[n] = d$. There is a path $\mathbf{c} = \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_l = \mathbf{e}$ such that $\mathbf{e}$ is maximal, $\mathbf{e}[1], \mathbf{d}[1]$ are strongly r+y-connected and $\mathbf{c}_1[1], \ldots, \mathbf{c}_l[1]$, $\mathbf{c}_1[n], \ldots, \mathbf{c}_l[n]$ are irreducible.

As in Case (1), it can be shown that there is a tuple $\mathbf{e}'$ such that $\mathbf{e}'[1] = \mathbf{d}[1]$, $\mathbf{e}'[n] = \mathbf{e}[n]$. Again we shall show that there exist non-empty $J, K = \underline{n} - J \subseteq \underline{n}$ such that

$$\max(\mathrm{pr}_J \mathbf{b}) \times \max(\mathrm{pr}_K \mathbf{b}) \subseteq \mathbb{D}.$$

Since $\mathbf{d}[n] \notin \max(\mathbf{c}[n])$, we have $\mathbf{c}_{l-1} \not\leq \mathbf{d}[n]$. The path $\mathbf{c}_1, \ldots, \mathbf{c}_l$ and the tuple $\mathbf{d}' = f(\mathbf{c}_{l-1}, \mathbf{e}')$ satisfy the conditions of Lemma 20; hence, there is $\mathbf{c}' \in \mathbb{D}$ such that $\mathbf{c}'[1] = \mathbf{e}[1]$ and $\mathbf{c}'[n] = \mathbf{c}[n]$.

We set

$$\begin{aligned} J &= \{i \in \underline{n} \mid \mathbf{c}[i], \mathbf{c}'[i] \text{ are indistinguishable}\} \\ K &= \underline{n} - J. \end{aligned}$$

Clearly, both sets are non-empty, as $n \in J$ and $1 \in K$. By Lemma 25 where $\mathbf{o} = \mathbf{c}[n]$ and $I = J - \{n\}$,

$$\max(\mathrm{pr}_J \mathbf{c}) \times \max(\mathrm{pr}_K \mathbf{c}) \subseteq \mathbb{D}.$$

Since $\mathbf{c}[1] \in B_1$, by induction hypothesis

$$\max(\mathrm{pr}_{K-I_1} \mathbf{c}) \times \max(\mathrm{pr}_{I_1} \mathbf{c}) \subseteq \mathbb{D}.$$

Therefore,

$$\max(\mathrm{pr}_{\underline{n}-I_1} \mathbf{c}) \times \max(\mathrm{pr}_{I_1} \mathbf{c}) = \max(\mathrm{pr}_{\underline{n}-I_1} \mathbf{c}) \times \max(\mathrm{pr}_{I_1} \mathbf{b}) \subseteq \mathbb{D}.$$

Finally, making use of Lemma 24, we get

$$\max(\mathrm{pr}_J \mathbf{b}) \times \max(\mathrm{pr}_K \mathbf{b}) \subseteq \mathbb{D}.$$

Case 3. For any $\mathbf{b} \in \mathbb{D}$ such that $\mathbf{b}[1] \in B_1$, and any $i \in \underline{n}$, $\mathbf{b}[i]$ belongs to a minimal i-component, but there are $\mathbf{b}, \mathbf{c} \in \mathbb{D}$ such that $\mathbf{b}[1] \in B_1$, $\mathbf{b}[n], \mathbf{c}[n]$ belong to the same minimal i-component B and $\mathbf{b}[1], \mathbf{c}[1]$ are not indistinguishable.

Subcase 3.1. *B contains maximal elements.*

Let $\mathbf{d} \in \widetilde{\mathcal{F}}(\mathbf{b})$ be a maximal tuple. Let us denote the set $J(\mathrm{m}, n, B)$ by I. By Claim 2, I is the union of the sets $J(\mathrm{r+y}, i, C_i)$ where $i \in I$ and C_i is the ry-component of $\max(\mathbb{D}_i)$ containing $\mathbf{d}[i]$. Since $\max(\mathrm{pr}_I \mathbf{d})$ is strongly r+y-connected, by Lemma 16,

$$\mathrm{pr}_{\underline{n}-I} \mathbf{d} \times \max(\mathrm{pr}_I \mathbf{d}) = \mathrm{pr}_{\underline{n}-I} \mathbf{d} \times \max(\mathrm{pr}_I \mathbf{b}) \subseteq \mathbb{D}.$$

Furthermore, by Lemma 24,

$$\max(\mathrm{pr}_{\underline{n}-I} \mathbf{b}) \times \max(\mathrm{pr}_I \mathbf{b}) \subseteq \mathbb{D}$$

and we may set $J = \underline{n} - I$, $K = I$.

SUBCASE 3.2. B contains no maximal elements.

In this case the proof is similar to that in Case 2.

Now, by induction hypothesis

$$\max(\mathrm{pr}_{K-I_1}\mathbf{b}) \times \max(\mathrm{pr}_{I_1}\mathbf{b}) \subseteq \mathbb{D}.$$

Therefore,

$$\max(\mathrm{pr}_{\underline{n}-I_1}\mathbf{b}) \times \max(\mathrm{pr}_{I_1}\mathbf{b}) = \max(\mathrm{pr}_{\underline{n}-I_1}\mathbf{b}) \times \left(\mathbb{D} \cap \prod_{i \in I_1} \max(B_i)\right) \subseteq \mathbb{D}.$$

Finally, since $\mathrm{pr}_{I_1}\mathbf{a}, \mathbf{a}' \in \left(\mathbb{D} \cap \prod_{i \in I_1} \max(B_i)\right)$, making use of Lemma 24, we get the required result. $\square$

8 Algorithm

Notice that the constraint relations of any 1-minimal problem are subdirect products of their domains. By applying the algorithm 1-MINIMALITY we may reduce an arbitrary problem instance to a 1-minimal one. Thus, from now on we shall assume all problem instances to be 1-minimal, and all constraint relations to be subdirect products of their domains.

8.1 Finding a solution of bounded relational width problems

In this section we show that the classes of constraint satisfaction problems over strongly r+y/r+b-connected algebras are of relational width 3.

Lemma 27 *Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a 3-minimal problem instance, where each constraint relation is a subdirect product of its domains and almost trivial. Then if none of the constraints of $\mathcal{P}$ is empty then $\mathcal{P}$ has a solution.*

Proof: For every constraint $C = \langle s, \varrho \rangle \in \mathcal{C}$, denote by θ_C the equivalence relation on s corresponding to the almost trivial relation ϱ, and by θ the least equivalence relation on V that contains θ_C for all constraints C. By the 3-minimality, for any $u, v \in V$ and any constraint $C = \langle s, \varrho \rangle$, if $u, v \in s$ then $\mathrm{pr}_{u,v}\varrho = \mathcal{S}_{u,v}$. Therefore

$$\theta = \bigcup_{C \in \mathcal{C}} \theta_C = \{(u, v) \mid \mathcal{S}_{u,v} \text{ is the graph of a bijective mapping } \pi_{u,v}\colon A_{\delta(u)} \to A_{\delta(v)}\}.$$

Moreover, if $(u, v), (v, w) \in \theta$ then there is $C = \langle s, \varrho \rangle \in \mathcal{C}$ such that $u, v, w \in s$. For the constraint C the projections $\mathrm{pr}_{u,v}\varrho = \mathcal{S}_{u,v}$, $\mathrm{pr}_{v,w}\varrho = \mathcal{S}_{v,w}$ are the graphs of $\pi_{u,v}, \pi_{v,w}$ respectively. Therefore, $\pi_{u,w} = \pi_{u,v} \circ \pi_{v,w}$. Denote the classes of θ by $V_1, \ldots, V_k$.

It is not hard to see that any mapping ψ constructed through the following rules is a solution to $\mathcal{P}$.

– Choose a representative v_j from each V_j.
– Assign to v_j an arbitrary value $\psi(v_j) \in P_{v_j}$.
– Set $\psi(v) = \pi_{v_j,v}(\psi(v_j))$ for all $v \in V_j$ and $j \in \{1,\dots,k\}$. $\square$

Proposition 10 *Let* $\mathbf{K}$ *be a class of 1-minimal constraint satisfaction problems in which all the domains are r+y/r+b-connected. Then* $\mathbf{K}$ *is of relational width 3.*

Proof: Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a 3-minimal problem instance without empty constraints. We prove by induction on the number of elements in $\mathbb{A}_{\delta(v)}$, $v \in V$, that $\mathcal{P}$ has a solution.

THE BASE CASE OF INDUCTION. If all $\mathbb{A}_{\delta(v)}$, $v \in V$, are simple or 1-element, then the required result follows from Corollary 11 and Lemma 27.

INDUCTION STEP. Suppose that the theorem holds for all problem instances $\mathcal{P}' = (V; \mathcal{A}'; \delta'; \mathcal{C}')$ where $|\mathbb{A}'_{\delta'(v)}| \le |\mathbb{A}_{\delta(v)}|$ for $v \in V$ and at least one inequality is strict.

Let us assume that, for a certain $u \in V$, $\mathbb{A}_{\delta(u)}$ is not simple and θ is a maximal congruence of $\mathbb{A}_{\delta(u)}$. By Corollary 10, for any $v \in V - \{u\}$, $\mathcal{S}_{u,v}/_\theta = \{(a/_\theta, b) \mid (a,b) \in \mathcal{S}_{u,v}\}$ is either the direct product $\mathbb{A}_{\delta(u)}/_\theta \times \mathbb{A}_{\delta(v)}$, or the graph of a surjective mapping $\pi_v \colon \mathbb{A}_{\delta(v)} \to \mathbb{A}_{\delta(u)}/_\theta$. Let W denote the set consisting of u and all $v \in V$ such that $\mathcal{S}_{u,v}/_\theta$ is the graph of π_v, and

$$\theta_v = \begin{cases} \theta, & \text{if } v = u, \\ \ker \pi_v & \text{if } v \in W - \{u\}, \\ =_v & \text{otherwise}, \end{cases}$$

for $v \in V$ where $=_v$ denotes the equality relation on $\mathbb{A}_{\delta(v)}$. Consider the *factor problem* $\overline{\mathcal{P}} = (V; \widetilde{\mathcal{A}}; \tilde{\delta}; \widetilde{\mathcal{C}})$ where $\widetilde{\mathbb{A}}_{\tilde{\delta}(v)} = \mathbb{A}_{\delta(v)}/_{\theta_v}$, $v \in V$, $\widetilde{\mathcal{A}}$ consists of $\widetilde{\mathbb{A}}_{\tilde{\delta}(v)}$, and for each $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$, $s = (v_1,\dots,v_k)$, there is $\widetilde{C} = \langle s, \widetilde{\mathbb{C}}\rangle \in \widetilde{\mathcal{C}}$ such that

$$\widetilde{\mathbb{C}} = \{(a[v_1]/_{\theta_{v_1}}, \dots, a[v_k]/_{\theta_{v_k}}) \mid (a[v_1],\dots,a[v_k]) \in \mathbb{C}\}.$$

As is easily seen, the factor problem is 3-minimal, therefore, by the induction hypothesis it has a solution. Let $(\mathfrak{a}[1],\dots,\mathfrak{a}[n])$ be a solution. Notice that if $v \in W$, then $\mathfrak{a}[v]$ is a congruence block of $\mathbb{A}_{\delta(v)}$, that is, $\mathfrak{a}[v]$ is a subset of $A_{\delta(v)}$ in this case. The 3-minimality of $\mathcal{P}$ implies that, for any constraint $\langle s, \mathbb{C}\rangle \in \mathcal{C}$, any $v, w \in s \cap W$, and any $\mathbf{a} \in \mathbb{C}$, if $a[v] \in \mathfrak{a}[v]$ then $a[w] \in \mathfrak{a}[w]$. Set $\mathcal{P}' = (V; \mathcal{A}'; \delta'; \mathcal{C}')$ where

$$\mathbb{A}'_{\delta'(v)} = \begin{cases} \mathfrak{a}[v] & \text{if } v \in W, \\ \mathbb{A}_{\delta(v)} & \text{otherwise}, \end{cases}$$

$\mathcal{A}' = \mathcal{A} \cup \{\mathfrak{a}[v] \mid v \in W\}$, and for each $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$ there is $C' = \langle s, \mathbb{C}'\rangle \in \mathcal{C}'$ with

$$\mathbf{a} \in \mathbb{C}' \quad \text{if and only if} \quad \mathbf{a} \in \mathbb{C} \text{ and } a[v] \in \mathfrak{a}[v] \text{ for all } v \in s \cap W.$$

INPUT: Problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.

OUTPUT: a solution to $\mathcal{P}$ if it has one, $\varnothing$ otherwise.

Step 1. **Invoke** 3-MINIMALITY($\mathcal{P}$)
Step 2. **If** the constraint relations of $\mathcal{P}$ are empty, **then output** $\varnothing$ **and stop**
Step 3. Choose a variable $v \in V$, let $\mathbb{A}_{\delta(v)=\{a_1,\dots,a_k\}}$, **set** $i := 0$
Step 4. **Do**
Step 4.1. **Set** $i := i+1$, $a := a_i$
Step 4.2. **Set** $\mathcal{P}_{v,a}$ to be the problem instance $(V'; \mathcal{A}; \delta'; \mathcal{C}')$ where
- $V' = V - \{v\}$,
- $\delta' = \delta\big|_{V'}$,
- $\mathcal{C}' = \{C' = \langle s \cap V', \mathbb{C}'\rangle \mid \langle s, \mathbb{C}\rangle \in \mathcal{C}\}$ where
  - $\mathbb{C}' = \mathbb{C}$ if $v \notin s$, and
    $\mathbb{C}' = \{\mathbf{a} \in \mathbb{C} \mid \mathbf{a}[v] = a\}$ otherwise.

Until 3-MINIMALITY($\mathcal{P}_{v,a}$) has no empty constraint relations
Step 5. **Return** 3-WIDTH($\mathcal{P}_{v,a}$)$\cup\{v = a\}$.

Figure 4: Algorithm 3-WIDTH

Since $|\mathfrak{a}[u]| < |\mathbb{A}_{\delta(u)}|$, to complete the proof we just have to show that $\mathcal{P}'$ is 3-minimal. For $U = \{u_1, u_2, u_3\} \subseteq V$ set $T_U = \mathcal{S}_U \cap (S_1 \times S_2 \times S_3)$ where $S_i = \mathbb{A}'_{\delta'(u_i)}$. Clearly, for any $C' = \langle s, \mathbb{C}'\rangle \in \mathcal{C}'$, we have $\mathrm{pr}_{U\cap s}\mathbb{C}' \subseteq \mathrm{pr}_{U\cap s} T_U$. Therefore, if we prove the reverse inclusion then we get both, the equality $\mathrm{pr}_{U\cap s}\mathbb{C}' = \mathrm{pr}_{U\cap s} T_U$, and the 3-minimality of $\mathcal{P}'$.

Take $\mathbf{b} = (a[u_1], a[u_2], a[u_3]) \in T_U$, $\langle s, \mathbb{C}\rangle \in \mathcal{C}$, and $\mathbf{a} \in \mathbb{C}$ such that $\mathrm{pr}_{U\cap s}\mathbf{a} = \mathrm{pr}_{U\cap s}\mathbf{b}$. If $U \cap W \cap s \neq \varnothing$ then, for any $v \in s \cap W$, $a[v] \in \mathfrak{a}[v]$, and therefore $\mathbf{a} \in \mathbb{C}'$. If $s \cap W = \varnothing$ then $\mathbb{C}' = \mathbb{C}$, and again $\mathbf{a} \in \mathbb{C}'$. Otherwise, consider the relation $\widetilde{\mathbb{C}}$. Choose $v \in s \cap W$ and set $\mathbb{D} = \mathrm{pr}_{(s-W)\cup\{v\}}\widetilde{\mathbb{C}}$. Then, by Lemma 13, $\mathbb{D} = \mathrm{pr}_v\mathbb{D} \times \mathrm{pr}_{s-W}\widetilde{\mathbb{C}} = \mathrm{pr}_v\mathbb{D} \times \mathrm{pr}_{s-W}\mathbb{C}$. This means that there is $\mathbf{c} \in \mathbb{C}$ such that $\mathrm{pr}_{s-W}\mathbf{c} = \mathrm{pr}_{s-W}\mathbf{a}$ and $c[v] \in \mathfrak{a}[v]$. Therefore, $c[w] \in \mathfrak{a}[w]$ for any $w \in s \cap W$, and hence $\mathbf{c} \in \mathbb{C}'$. Since $s \cap U \subseteq s - W$, we have $\mathrm{pr}_{s\cap U}\mathbf{c} = \mathrm{pr}_{s\cap U}\mathbf{b}$, as required. $\square$

For our main algorithm we also need to be able to find a solution to a problem instance from a class of relational width 3 (provided there exists one). As is easily seen, this can be done by employing algorithm 3-WIDTH, see Fig. 4.

8.2 Solving algorithm

The main idea of our algorithm is to split a problem into several smaller subproblems, usually selected in a sophisticated way, so that the existence of a solution to each subproblem ensures the existence of a solution to the whole problem, and if some subproblem has no solution then the problem can be tightened by excluding tuples from the constraint relations. Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a problem instance from CSP($\mathcal{A}$). Without loss of generality $\mathcal{P}$ can be assumed to be 3-minimal. We distinguish three types of subproblems depending on the structure of the graphs $\mathcal{G}(\mathcal{S}_v)$, $v \in V$.

First we extend the notation $J(\mathrm{x}, i, B)$ from relations to problem instances.

Let x be one of r+b, r+y, m. Take a variable $v \in V$, a strongly connected component B of $\mathcal{G}_v$, and set J_1 to be the union of $J_{\mathbb{C}}(\mathrm{x}, v, B)$ where $\mathbb{C}$ runs over all constraint relations whose scope contains v. Note that, since $\mathcal{P}$ is 3-minimal, if for a certain $w \in J_1$, v, w are contained in the scopes of constraints $\langle s_1, \mathbb{C}_1\rangle, \langle s_2, \mathbb{C}_2\rangle$ then, for any $\mathbf{a} \in \mathbb{C}_1$, $\mathbf{b} \in \mathbb{C}_2$ such that $\mathbf{a}[v], \mathbf{b}[v] \in B$, the elements $\mathbf{a}[w], \mathbf{b}[w]$ are also strongly r+b-connected [strongly r+y-connected, lie in the same minimal i-component]. This allows us to define B_w for every $w \in J_1$ in the same way as for relations. Suppose that J_k is already defined. Then we set J_{k+1} to be the union of all sets of the form $J_{\mathbb{C}}(\mathrm{x}, w, B_w)$ where $w \in J_k$ and $\langle s, \mathbb{C}\rangle$ is a constraint such that $w \in s$. Finally, $J(\mathrm{x}, v, B)$ is defined to be the largest set among the sets J_k. Note that if φ is a solution to $\mathcal{P}$ such that $\varphi(v) \in B$ then $\varphi(w) \in B_w$ for any $w \in J(\mathrm{x}, v, B)$.

Maximal strongly r+b-connected components We use this type of subproblems when there is $v \in V$ such that $\mathcal{S}_v$ is not strongly r+b-connected.

We now define the subproblem corresponding to $J(\mathrm{r{+}b}, v, B)$. The problem $\mathcal{P}_{\mathrm{r+b},v,B}$ is defined to be the problem instance $(W; \mathcal{A}; \delta'; \mathcal{C}')$ where $W = J(\mathrm{r{+}b}, v, B)$, $\delta' = \delta|_W$ and, for every $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$, there is $C' = \langle s \cap W; \mathbb{C}'\rangle \in \mathcal{C}$ such that

$$\mathbb{C}' = \{\mathrm{pr}_{s\cap W}\mathbf{a} \mid \mathbf{a} \in \mathbb{C},\ \mathbf{a}[w] \in B_w \text{ for any } w \in s \cap W\}.$$

Proposition 11 *If, for any $v \in V$, and any maximal strongly r+b-connected component B of $\mathcal{S}_v$, the restricted problem $\mathcal{P}_{\mathrm{r+b},v,B}$ has a solution, then $\mathcal{P}$ also has a solution.*

Proof: Notice first, that by Lemma 10, every strongly r+b-component of $\mathbb{A}_{\delta(v)}$ for every $v \in V$ can be assumed to be rb-maximal. The proof we give here is a modification of the proof of Theorem 3.5 from [32].

For any $W \subseteq V$ and any $v \in V - W$, we prove that if φ_W is a partial solution to $\mathcal{P}_W$ such that, for any $w \in W$, $\varphi_W(w)$ belongs to a maximal strongly r+b-component B_w and there is a solution ψ_I to $\mathcal{P}_{\mathrm{r+b},w,B_w}$ such that $\varphi_W = \psi_I|_{W\cup I}$ where $I = J(\mathrm{r{+}b}, w, B_w)$, then it can be extended to a solution $\varphi_{W\cup\{v\}}$ to $\mathcal{P}_{W\cup\{v\}}$ satisfying the same conditions.

In the base case of induction W is a set of the form $J(\mathrm{r{+}b}, w, B_w)$, B_w is a maximal strongly r+b-connected component of $\mathbb{A}_{\delta(w)}$, and φ_W is a solution to $\mathcal{P}_{\mathrm{r+b},w,B_w}$. Take $w \in W$ and $a \in \mathbb{A}_{\delta(v)}$ such that $\begin{pmatrix}\varphi_W(w)\\ a\end{pmatrix} \in \mathcal{S}_{w,v}$. There is a solution ψ to $\mathcal{P}_{\mathrm{r+b},v,B_v}$ where B_v is the strongly r+b-connected component containing a. Let $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$ be a constraint such that $v \in s$, and $w' \in W \cap s$. Since $\mathcal{P}$ is 3-minimal, the tuple $\begin{pmatrix}\varphi_W(w)\\ a\end{pmatrix} \in \mathcal{S}_{w,v}$ can be extended to a tuple $\begin{pmatrix}\varphi_W(w)\\ a\\ b\end{pmatrix} \in \mathcal{S}_{w,v,w'}$. Moreover, by Proposition 17 applied to $\mathcal{S}_{w,v,w'}$, we may choose $b = \varphi_W(w')$. Therefore, $\begin{pmatrix}\varphi_W(w')\\ a\end{pmatrix} \in \mathrm{pr}_{w',v}\mathbb{C}$ can be extended to a tuple $\mathbf{b} \in \mathbb{C}$. For any $u \in J(\mathrm{r+b}, v, B_v) \cap s$, the element $\mathbf{b}[u]$ is a member of the same strongly r+b-connected component as $\psi(u)$, and for any $u \in W \cap s$, the element $\mathbf{b}[u]$ is a member of the same strongly r+b-connected component as $\varphi_W(u)$. Thus, $\mathrm{pr}_{W\cap s}\mathbf{b}$ is strongly r+b-connected with $\varphi_W|_{W\cap s}$, and $\mathrm{pr}_{J(\mathrm{r+b},v,B_v)\cap s}\mathbf{b}$ is strongly r+b-connected with $\psi|_{J(\mathrm{r+b},v,B_v)\cap s}$. By Proposition 17,

$$\begin{pmatrix}\mathrm{pr}_{W\cap s}\mathbf{b}\\ \mathrm{pr}_{J(\mathrm{r+b},v,B_v)\cap s}\mathbf{b}\end{pmatrix} \in \mathrm{pr}_{(W\cup J(\mathrm{r+b},v,B_v))\cap s}\mathbb{C}.$$

Now suppose that the result is proved for every $U \subseteq V$ with $|U| < k$. Take $W \subseteq V$ with $W = \{w_1, \ldots, w_k\}$ and $v \in V - W$, and let φ_W be a partial solution satisfying the conditions of the lemma. Assume for contradiction that φ cannot be extended to a solution to $\mathcal{P}_{W\cup\{v\}}$.

Let the constraints $\mathcal{P}_{W\cup\{v\}}$ be $\langle s_1, \mathbb{C}_1\rangle, \ldots, \langle s_q, \mathbb{C}_q\rangle$. To obtain the desired contradiction we shall construct a problem $\mathcal{P}'$ which also has q constraints, with the same constraint relations, but with different constraint scopes.

We define the set of variables of $\mathcal{P}'$ to be the union on q disjoint copies $W_1, \ldots, W_q$ and $\{v'\}$, $W_i = \{w^i_1, \ldots, w^i_k\}$. The sort function is defined by the equalities $\delta'(w^i_j) = \delta(w_j)$, $\delta'(v') = \delta(v)$.

Now, for each $i \in \underline{q}$, we define a mapping $f_i\colon W \to W_i$ by setting $f_i(w_j) = w^i_j$, and extend each f_i to v by setting $f_i(v) = v'$. The set of constraints of $\mathcal{P}'$ is then defined as:

$$\{\langle f_1(s_1), \mathbb{C}_1\rangle, \ldots, \langle f_q(s_q), \mathbb{C}_q\rangle\}.$$

Then let the $q\cdot k$-ary relation $\mathbb{D}$ be defined as follows

$$\begin{aligned}\mathbb{D} \;=\; &\{(\sigma(f_1(w_1)), \ldots, \sigma(f_1(w_k)), \sigma(f_2(w_1)), \ldots, \sigma(f_2(w_k)), \ldots,\\ &\sigma(f_q(w_1)), \ldots, \sigma(f_q(w_k))) \mid \sigma \text{ is a solution to } \mathcal{P}'\}.\end{aligned}$$

Note that $\mathbb{D}$ is a subdirect product of $\mathbb{A}_{\delta(w_1)}, \ldots, \mathbb{A}_{\delta(w_k)}, \mathbb{A}_{\delta(w_1)}, \ldots, \mathbb{A}_{\delta(w_k)}, \ldots, \mathbb{A}_{\delta(w_1)}, \ldots, \mathbb{A}_{\delta(w_k)}$.

The tuple

$$\mathbf{a} = (\varphi_W(w_1), \ldots, (\varphi_W(w_k)), \varphi_W(w_1), \ldots, (\varphi_W(w_k)), \ldots, \varphi_W(w_1), \ldots, (\varphi_W(w_k)))$$

does not belong to $\mathbb{D}$, since φ_W cannot be extended to a solution to $\mathcal{P}_{W\cup\{v\}}$. However, we shall show that $\mathbf{a}$ satisfies the conditions of Lemma 18 for $\mathbb{D}$, and thus derive a contradiction.

It is not hard to see that, for any w^i_j,

$$J_{\mathbb{D}}(\mathrm{r+b}, w^i_j, B_{w_j}) \subseteq \{w^1_t, \ldots, w^q_t \mid t \in J(\mathrm{r+b}, w_j, B_{w_j}) \cap W\}.$$

By the choice of φ_W, $\varphi_W|_{J(\mathrm{r+b},w_j,B_{w_j})}$ is a solution to $\mathcal{P}_{\mathrm{r+b},w_j,B_{w_j}}$. Arguing as in the base case of induction we can show that $\mathbf{a}|_{J_{\mathbb{D}}(\mathrm{r+b},w^i_j,B_{w_j})}$ belongs to $\mathrm{pr}_{J_{\mathbb{D}}(\mathrm{r+b},w^i_j,B_{w_j})}\mathbb{D}$.

INPUT: Problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.

OUTPUT: a solution to $\mathcal{P}$ if it has one, $\varnothing$ otherwise.

Step 1. **Invoke** 3-MINIMALITY($\mathcal{P}$)

Step 2. **If** the constraint relations of $\mathcal{P}$ are empty, **then output** $\varnothing$ **and stop**

Step 3. Choose a variable $v \in V$, let $a_1, \ldots, a_k$ be the elements of $\mathbb{A}_{\delta(v)}$ that can be extended to a solution of a problem of the form $\mathcal{P}_{\mathrm{r+b},v,B}$,
set $i := 0$

Step 4. **Do**

Step 4.1. **Set** $i := i + 1$, $a := a_i$

Step 4.2. **Set** $\mathcal{P}_{v,a}$ to be the problem instance $(V'; \mathcal{A}; \delta'; \mathcal{C}')$ where
- $V' = V - \{v\}$,
- $\delta' = \delta|_{V'}$,
- $\mathcal{C}' = \{C' = \langle s \cap V', \mathbb{C}' \rangle \mid \langle s, \mathbb{C} \rangle \in \mathcal{C}\}$ where
  - $\mathbb{C}' = \mathbb{C}$ if $v \notin s$, and
    $\mathbb{C}' = \{\mathbf{a} \in \mathbb{C} \mid \mathbf{a}[v] = a\}$ otherwise.

Until 3-MINIMALITY($\mathcal{P}_{v,a}$) has no empty constraint relations

Step 5. Let B the maximal r+b-connected component of $\mathcal{S}_v$, $W = J(\mathrm{r+b}, v, B)$ and ψ a solution to $\mathcal{P}_{v,B}$ such that $\psi(v) = a$

Step 6. **Set** $\mathcal{P}_{W,\psi}$ to be the problem instance $(V'; \mathcal{A}; \delta'; \mathcal{C}')$ where
- $V' = V - W$,
- $\delta' = \delta|_{V'}$,
- $\mathcal{C}' = \{C' = \langle s \cap V', \mathbb{C}' \rangle \mid \langle s, \mathbb{C} \rangle \in \mathcal{C}\}$ where
  - $\mathbb{C}' = \mathbb{C}$ if $W \cap s = \varnothing$, and
    $\mathbb{C}' = \{\mathbf{a} \in \mathbb{C} \mid \mathbf{a}[w] = \psi(w)$ for all $w \in W \cap s\}$ otherwise.

Step 7. **Return** BLOCK-3-WIDTH($\mathcal{P}_{W,\psi}$)$\cup\{w = \psi(w) \mid w \in W\}$.

Figure 5: Algorithm BLOCK-3-WIDTH

For any pair of indices $w_{j_1}^{i_1}, w_{j_2}^{i_2}$, we claim that $\mathrm{pr}_{w_{j_1}^{i_1}, w_{j_2}^{i_2}} \mathbf{a} \in \mathrm{pr}_{w_{j_1}^{i_1}, w_{j_2}^{i_2}} \mathbb{D}$. Since $\mathcal{P}$ is 3-minimal the tuple $\begin{pmatrix} \varphi_W(w_{j_1}) \\ \varphi_W(w_{j_2}) \end{pmatrix} \in \mathcal{S}_{w_{j_1}, w_{j_2}}$ can be extended to a solution $\begin{pmatrix} \varphi_W(w_{j_1}) \\ \varphi_W(w_{j_2}) \\ a \end{pmatrix} \in \mathcal{S}_{w_{j_1}, w_{j_2}, v}$ to $\mathcal{P}_{w_{j_1}, w_{j_2}, v}$. Furthermore, for this solution, we can construct a corresponding solution, σ, to $\mathcal{P}'$, such that $\sigma(f_{i_1}(w_{j_1}) = \varphi_W(w_{j_1})$, $\sigma(f_{i_2}(w_{j_2}) = \varphi_W(w_{j_2})$, by the construction of $\mathcal{P}'$. Hence $\mathbf{a}$ agrees with some element of $\mathbb{D}$ in $w_{j_1}^{i_1}, w_{j_2}^{i_2}$, which establishes the claim. □

Moreover, if the conditions of Proposition 11 hold then a solution to $\mathcal{P}$ can be found in polynomial time by the algorithm BLOCK-3-WIDTH (see Fig. 5.

Thus if all the subproblems have solutions then Lemma 11 allows one to solve the problem $\mathcal{P}$. Otherwise if $\mathcal{P}_{\mathrm{r+b},v,B}$ has no solution then $\mathcal{P}$ can be tightened by removing from every $\mathbb{C}$, $\langle s,\mathbb{C}\rangle\in\mathcal{C}$, such that $w\in s\cap J(\mathrm{r+b},v,B)$, every tuple $\mathbf{a}$ with $\mathbf{a}[w]\in B_w$.

Maximal strongly r-connected components. This type of subproblems is used if every $\mathcal{S}_v$ is strongly r+b-connected, but, for a certain $v\in V$, there is an r-connected component of $\mathcal{S}_v$ which is not strongly r-connected.

The union of all minimal i-components of $\mathcal{S}_v$ we denote by $\mathcal{S}_v^0$.

In this case the subproblems of $\mathcal{P}$ correspond to the sets of the form $J(\mathrm{m},v,B)$ and are defined as follows. By $\mathcal{P}^+_{v,B}$ we denote the constraint satisfaction problem $(W;\mathcal{A};\delta';\mathcal{C}')$ where $W=J(\mathrm{m},v,B)$, $\delta'=\delta|_W)$ and, for each constraint $\langle s,\mathbb{C}\rangle\in\mathcal{C}$, we include $C'=\langle s\cap W,\mathbb{C}'\rangle$ in $\mathcal{C}'$, with

$$\mathbb{C}'=\{\mathrm{pr}_{s\cap W}\mathbf{a}\mid \mathbf{a}\in\mathbb{C},\ \mathbf{a}[w]\in\mathbf{C}_w \text{ for } w\in s\cap W\},$$

where $C_w=\{a\in\mathcal{S}_w\mid B_w\sqsubseteq a\}$.

Lemma 28 *If every problem of the form $\mathcal{P}^+_{v,B}$ has a solution $\varphi_{v,B}$ such that every $\varphi_{v,B}(w)$, $w\in J(\mathrm{m},v,B)$, is maximal and $\varphi_{v,B}(w)\in\max(B_w)$, and $\mathcal{P}$ has a solution, then there is a solution ψ to $\mathcal{P}$ such that $\psi(w)$ is maximal for any $w\in V$.*

Proof: Let φ be a solution to $\mathcal{P}$ and B_v, $v\in V$, a minimal i-component such that $B_v\sqsubseteq\varphi(v)$. By the assumption, for any $v\in V$, the problem $\mathcal{P}^+_{v,B_v}$ has a solution φ_{v,B_v} such that $\varphi_{v,B_v}(w)\in\max(B_w)$ for any $w\in J(\mathrm{m},v,B_v)$. Moreover, for any v,w such that $w\in J(\mathrm{m},v,B_v)$, we have $\mathcal{P}^+_{v,B_v}=\mathcal{P}^+_{w,B_w}$ and, therefore, we may assume that $\varphi_{v,B_v}=\varphi_{w,B_w}$.

Let $W_1,\dots,W_k$ be the sets of the form $J(\mathrm{m},v,B_v)$ and $\psi_1,\dots,\psi_k$ the corresponding solutions. By Proposition 9, the mapping ψ such that $\psi(v)=\psi_i(v)$ whenever $v\in I_i$ is a solution to $\mathcal{P}$ with the required properties. $\square$

Having a formal description of the algorithm we shall be able to show that if a problem of the form $\mathcal{P}^+_{v,B}$ has no solution of the required type, then $\mathcal{P}$ can be tightened. Otherwise, Lemma 28 allows us to reduce $\mathcal{P}$ to the smaller problems $\mathcal{P}^+_{v,B}$, and the problem $\mathcal{P}^{\max}=(V;\mathcal{A};\delta;\mathcal{C}')$ in which every constraint relation $\langle s,\mathbb{C}\rangle$ is replaced with $\langle s,\mathbb{C}'\rangle$, $\mathbb{C}'=\{\mathbf{a}\in\varrho\mid$ for any $v\in s$, $\mathbf{a}[v]$ is maximal $\}$.

However, Lemma 28 cannot help us in the case when, for any $v\in V$, $\mathcal{S}_v=\mathcal{S}_v^0$. In this case we need another method.

Denote the set of maximal elements in $\mathcal{S}_v$ by $\mathcal{S}_v^{\max}$. By definition, $\mathcal{S}_v^{\max}$ is strongly r+y-connected. First suppose that, for any $v\in V$, it is also strongly r+b-connected. Since, by Lemma 10, $\mathcal{P}^{\max}$ is 3-minimal, it satisfies the conditions of Proposition 10 and can be solved by the algorithm 3-WIDTH.

Then, suppose that there is $v\in V$ such that $\mathcal{S}_v^{\max}$ is not strongly r+b-connected (and, therefore, is not r+b-connected). We consider the sets of the form $J_{\max}(\mathrm{r+b},v,B)$ defined as $J(\mathrm{r+b},v,B)$ for the problem $\mathcal{P}^{\max}$, where B is a strongly r+b-connected component of $\mathcal{S}_v^{\max}$. As usual, having fixed v and B we get a strongly r+b-connected component B_w for each $w\in J_{\max}(\mathrm{r+b},v,B)$.

We define the subproblems of $\mathcal{P}$ corresponding to the sets of the form $J_{\max}(\text{r+b}, v, B)$ as follows. By $\mathcal{P}^{\max}_{v,B}$ we denote the constraint satisfaction problem $(W; \mathcal{A}; \delta'; \mathcal{C}')$ where $W = J_{\max}(\text{r+b}, v, B)$, $\delta' = \delta_{|W}$ and, for each constraint $\langle s, \mathbb{C}\rangle \in \mathcal{C}$, we include $C' = \langle s \cap W, \mathbb{C}'\rangle$ in $\mathcal{C}'$, with

$$\mathbb{C}' = \{\mathrm{pr}_{s\cap W}\mathbf{a} \mid \mathbf{a} \in \mathbb{C},\ \mathbf{a}[w] \in \mathcal{S}_{w,B} \text{ for } w \in s \cap W\},$$

where $\mathcal{S}_{w,B}$ is the set of all $a \in \mathcal{S}_w$ such that $\max(a) \cap \widetilde{\mathcal{F}}(a) \subseteq B_w$. Note that $\max(\mathcal{S}_{w,B}) = B_w$.

Lemma 29 *Suppose that*

(1) $\mathcal{P}$ is 3-minimal;

(2) for any $v \in V$, $\mathcal{S}_v = \mathcal{S}^0_v$;

(3) there is $v \in V$ such that $\mathcal{S}^{\max}_v$ is not strongly r+b-connected;

(4) for any $v \in V$ and strongly r+b-connected components B of $\mathcal{S}^{\max}_v$, the problem $\mathcal{P}^{\max}_{v,B}$ has a solution $\psi_{v,B}$ such that $\psi_{v,B}(w) \in B_w$ for any $w \in J_{\max}(\text{r+b}, v, B)$.

Then $\mathcal{P}$ has a solution φ such that $\varphi(v)$ is maximal for any $v \in V$.

Proof: If conditions (1)–(4) hold, the problem $\mathcal{P}^{\max}$ satisfies the conditions of Proposition 11, and therefore can be solved in polynomial time. □

Strongly r+y-connected components. We use this type of subproblems if, for any $v \in V$, $\mathcal{S}_v$ is strongly r+b-connected and every its r-connected component is strongly r-connected.

Lemma 16 reduces the problem $\mathcal{P}$ to a collection of subproblems $\mathcal{P}_{\text{r+y},v,B}$, defined analogously to those for strongly r+b-connected components, and a *skeleton problem.* The latter problem is defined as follows. We may assume that each problem of the form $\mathcal{P}_{\text{r+y},v,B}$ has a solution $\mathcal{S}_{\text{r+y},v,B}$ found by the algorithm. Moreover, if $w \in J(\text{r+y}, v, B)$, then $J(\text{r+y}, v, B) = J(\text{r+y}, w, B_w)$ where B_w is the r+y-connected component corresponding to B, $\mathcal{P}_{\text{r+y},v,B} = \mathcal{P}_{\text{r+y},w,B_w}$, and $\mathcal{S}_{\text{r+y},v,B} = \mathcal{S}_{\text{r+y},w,B_w}$.

Then the skeleton problem is the problem $\mathcal{P}^s = (V; \mathcal{A}; \delta; \mathcal{C}^s)$ obtained by removing all tuples from the constraint relations but those consisting of the solutions $\mathcal{S}_{\text{r+y},v,B}$: for every $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$ there is $C^s = \langle s, \mathbb{C}^s\rangle \in \mathcal{C}^s$ where $\mathbf{a} \in \mathbb{C}^s$ if and only if $\mathbf{a} \in \mathbb{C}$ and, for any $v \in s$, any r+y-connected component B, we have $\mathrm{pr}_{s\cap J(\text{r+y},v,B)}\mathbf{a} = \mathrm{pr}_{s\cap J(\text{r+y},v,B)}\mathcal{S}_{\text{r+y},v,B}$.

Lemma 30 *If, for any $v \in V$ and any r+y-connected component, B, of $\mathcal{S}_v$, the problem $\mathcal{P}_{\text{r+y},v,B}$ has a solution, then the skeleton problem has a solution if and only if $\mathcal{P}$ has a solution. Moreover, a solution to $\mathcal{P}$ can be chosen to be a solution of the skeleton problem.*

Proof: The lemma follows straightforwardly from Lemma 16. □

Note that if $\mathcal{S}_v^s$ denotes the partial solution to the skeleton problem on $v \in V$, then every edge of $\mathcal{S}_v^s$ is blue. Therefore the skeleton problem can be solved by the algorithm MALTSEV described in [4]. If one of the subproblems $\mathcal{P}_{\mathrm{r+y},v,B}$ has no solution then $\mathcal{P}$ can be tightened by removing from every constraint relation $\mathbb{C}$, $\langle s, \mathbb{C}\rangle \in \mathcal{C}$, every tuple $\mathbf{a}$ such that $\mathbf{a}[w] \notin B_w$ for $w \in s \cap J(\mathrm{r+y}, v, B)$.

In the only case that is not yet considered, for any $v \in V$, $\mathcal{G}_v$ is strongly r+b- and strongly r+y-connected. In this case, Proposition 10 claims that $\mathcal{P}$ belongs to a problem class of width 3 and, therefore can be solved by algorithm 3-WIDTH.

Now we in are a position to give a formal description of the algorithm.

INPUT: Problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.

OUTPUT: Whether $\mathcal{P}$ has a solution.

Step 1. **Invoke** CONSERVATIVE($\mathcal{P}$)
Step 2. **If** $\mathcal{P} \neq \varnothing$ **output** "YES", **otherwise output** "NO".
Step 3. **Stop.**

Figure 6: Algorithm solving conservative constraint satisfaction problems

Algorithm CONSERVATIVE

INPUT: Problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.

OUTPUT: a solution to $\mathcal{P}$ and the tightened problem instance if $\mathcal{P}$ has a solution, $(\varnothing, \varnothing)$ otherwise.

Step 1. **invoke** 3-MINIMALITY($\mathcal{P}$)
Step 2. **find** (strongly) r+y-, r+b-, r- connected components of $\mathcal{S}_v$, $v \in V$.
Step 3. **ifcase:**
there exists $v \in V$ such that $\mathcal{G}_v$ is not strongly r+b-connected:
Step 3.1. **do**
Step 3.1.1. **for each** $v \in V$, **each** maximal strongly r+b-connected component $B \subseteq \mathcal{S}_v$, **do**
Step 3.1.1.1. $(\mathcal{S}_{v,B}, \mathcal{P}'_{\mathrm{r+b},v,B})$:=CONSERVATIVE($\mathcal{P}_{\mathrm{r+b},v,B}$)
Step 3.1.1.2. for every constraint $C = \langle s, \mathbb{C}\rangle$ of $\mathcal{P}$, **remove** from $\mathbb{C}$ every tuple $\mathbf{a}$ such that there is $v \in s$, $B \subseteq \mathcal{S}_v$, $W = J(\mathrm{r+b}, v, B)$ with $\mathrm{pr}_{W\cap s}\mathbf{a} \notin \mathbb{C}'$ where $C' = \langle W \cap s, \mathbb{C}'\rangle$ is the corresponding constraint of $\mathcal{P}'_{\mathrm{r+b},v,B}$
endfor
Step 3.1.2. **if** $\mathcal{P}$ changed **then return**(CONSERVATIVE($\mathcal{P}$))
Step 3.1.3. **otherwise return**(BLOCK-3-WIDTH($\mathcal{P}$))
enddo
there exists $v \in V$ and an r-connected component C of $\mathcal{G}_v$ such that C is not strongly r-connected and, for any $w \in V$, $\mathcal{S}_w = \mathcal{S}_w^0$:

Step 3.2. **do**
Step 3.2.1. **If** for any $v \in V$, $\mathcal{S}_v^{\max}$ is strongly r+b-connected, **then** **return**(3-Width($\mathcal{P}^{\max}$))
Step 3.2.2. **Otherwise for each** $v \in V$, **each** strongly r+b-connected component $B \subseteq \mathcal{S}_v^{\max}$, **do**
Step 3.2.2.1. $(\mathcal{S}_{v,B}, \mathcal{P}'_{v,B})$:=Conservative($\mathcal{P}^{\max}_{v,B}$)
Step 3.2.2.2. for every constraint $C = \langle s, \mathbb{C}\rangle$ of $\mathcal{P}$, **remove** from $\mathbb{C}$ every tuple **a** such that there is $v \in s$, $B \subseteq \mathcal{S}_v$, $W = J_{\max}(\text{r+b}, v, B)$ with $\text{pr}_{W\cap s}\mathbf{a} \notin \mathbb{C}'$ and $\mathbf{a}[v] \in B$ where $C' = \langle W \cap s, \mathbb{C}'\rangle$ is the corresponding constraint of $\mathcal{P}'_{v,B}$
endfor
Step 3.2.3. **if** $\mathcal{P}$ changed **then return**(Conservative($\mathcal{P}$))
Step 3.2.4. **otherwise return**(Block-3-Width($\mathcal{P}^{\max}$))
enddo
there exists $v \in V$ and an r-connected component C of $\mathcal{G}_v$ such that C is not strongly r-connected and $w \in V$ such that $\mathcal{S}_w \neq \mathcal{S}_w^0$:
Step 3.3. **do**
Step 3.3.1. **for each** $v \in V$, **each** minimal i-component $B \subseteq \mathcal{S}_v$, **do**
Step 3.3.1.1. $(\mathcal{S}_{v,B}, \mathcal{P}'_{v,B})$:=Conservative($\mathcal{P}^{+}_{v,B}$)
Step 3.3.1.2. for every constraint $C = \langle s, \mathbb{C}\rangle$ of $\mathcal{P}$, **remove** from $\mathbb{C}$ every tuple **a** such that there is $v \in s$, $B \subseteq \mathcal{S}_v$, $W = J(\text{m}, v, B)$ with $\text{pr}_{W\cap s}\mathbf{a} \notin \mathbb{C}'$ where $C' = \langle W \cap s, \mathbb{C}'\rangle$ is the corresponding constraint of $\mathcal{P}'_{v,B}$
endfor
Step 3.3.2. **if** $\mathcal{P}$ changed **then return**(Conservative($\mathcal{P}$))
Step 3.3.3. **otherwise return**(Conservative($\mathcal{P}^{\max}$))
enddo
there exists $v \in V$ such that $\mathcal{G}_v$ is not r+y-connected:
Step 3.4. **do**
Step 3.4.1. **for each** $v \in V$, each r+y-connected component B **do**
Step 3.4.1.1. $(\mathcal{S}_{v,B}, \mathcal{P}'_{\text{r+y},v,B})$:=Conservative($\mathcal{P}_{\text{r+y},v,B}$)
Step 3.4.1.2. for every constraint $C = \langle s, \mathbb{C}\rangle$ of $\mathcal{P}$, **remove** from $\mathbb{C}$ all tuples **a** such that there is $v \in s$, $B \subseteq \mathcal{S}_v$, $W = J(\text{r+y}, v, B)$ with $\text{pr}_{W\cap s}\mathbf{a} \notin \mathbb{C}'$ where $C' = \langle W \cap s, \mathbb{C}'\rangle$ is the corresponding constraint of $\mathcal{P}'_{\text{r+y},v,B}$
endfor
Step 3.4.2. **if** $\mathcal{P}$ changed **then return**(Conservative($\mathcal{P}$))
Step 3.4.3. **else return**(Maltsev($\mathcal{P}^s$))
enddo
$\mathcal{S}_v$ is strongly r+b- and r+y-connected for any $v \in V$:
Step 3.5. **return**(3-Width($\mathcal{P}$))
endifcase
Step 4. **invoke** 3-Minimality($\mathcal{P}$)

8.3 Soundness and complexity

To prove soundness of the algorithms from Section 8.2 we have to show that the algorithm CONSERVATIVE works correctly.

Proposition 12 *The algorithm* CONSERVATIVE *returns a solution of the given problem instance if and only if a solution exists.*

We prove Proposition 12 by induction on the depth of recursion. We shall also prove that CONSERVATIVE always either changes the problem or returns a solution from maximal strongly r+b- and r+y-connected components. Step 3.5 formsthe base case of induction. In this case the result follows from Proposition 10.

Let $\mathcal{P} = (V; \mathcal{A}; \delta, \mathcal{C})$ be a problem instance. Suppose that CONSERVATIVE solves correctly all problems whose size is less than $|\mathcal{P}|$. In particular, it solves subproblems of the form $\mathcal{P}_{\text{r+b},v,B}$, $\mathcal{P}^{\max}_{v,B}$, $\mathcal{P}^{+}_{v,B}$, $\mathcal{P}_{\text{r+y},v,B}$, derived problems $\mathcal{P}^{\max}$, $\mathcal{P}^{s}$ and also problems obtained from $\mathcal{P}$ after removing any tuples.

In Step 3.1 if every subproblem of the form $\mathcal{P}_{\text{r+b},v,B}$ have a solution, then $\mathcal{P}$ satisfies the conditions of Proposition 11. Thus, $\mathcal{P}$ can be solved by algorithm BLOCK-3-WIDTH. Maximality of a solution follows from Proposition 11.

In Step 3.2, if, for any $v \in V$, $\mathcal{S}^{\max}_{v}$ is strongly r+b-connected, then $\mathcal{P}^{\max}$ is 3-minimal and satisfies the conditions of Proposition 10. Hence, 3-WDTH is applicable. Analogously, if all the problems $\mathcal{P}^{\max}_{v,B}$ have solutions, then we are in the conditions of Lemma 29. We just have to show that if, for a certain $v \in V$ and a strongly r+b-connected component B of $\mathcal{S}^{\max}_{v}$, the problem $\mathcal{P}^{\max}_{v,B}$ has no solution ψ such that $\psi(v) \in B$, then, for any solution φ to $\mathcal{P}$, we have $\varphi(v) \notin B$.

Supposing the contrary, let φ be a solution to $\mathcal{P}$ such that $\varphi \in B$. Then there is $w \in J_{\max}(\text{r+b}, v, B)$ such that $\widetilde{\mathcal{F}}(\varphi(w)) \cap \max(\varphi(w)) \not\subseteq B_w$. Let $b \in \widetilde{\mathcal{F}}(\varphi(w)) \cap \max(\varphi(w))$ be such that $b \notin B_w$. Take a path in $\mathcal{S}_{v,w}$,

$$\begin{pmatrix}\varphi(v)\\ \varphi(w)\end{pmatrix} = \begin{pmatrix}a_1\\ b_1\end{pmatrix} \leq \begin{pmatrix}a_2\\ b_2\end{pmatrix} \leq \cdots \leq \begin{pmatrix}a_k\\ b_k\end{pmatrix} = \begin{pmatrix}a_k\\ b\end{pmatrix}.$$

Clearly, $a_k \in B$ while $b \notin B_w$, a contradiction to the assumption $w \in J_{\max}(\text{r+b}, v, B)$.

Finally, the maximality of a solution follows from induction hypothesis and construction of $\mathcal{P}^{\max}$.

For Step 3.3 the required result follows from Lemma 28, and for Step 3.4 Lemma 30 is applicable. □

Proposition 13 *If the size of the domains in $\mathcal{A}$ is bounded, then the algorithm* CONSERVATIVE *is polynomial time.*

Proof: We use $|\mathcal{P}|$ to denote the size of a problem instance $\mathcal{P}$. Suppose that CONSERVATIVE is applied to a problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$, and m is the total number of tuples in the constraint relations. We assume first, that the algorithm never changes the problem $\mathcal{P}$, that is Steps 3.1.2, 3.2.2, 3.3.2 and 3.4.2 are never performed. Let s denote the maximal size of domains in $\mathcal{A}$.

CLAIM 1. The depth of recursion when performing CONSERVATIVE($\mathcal{P}$) does not exceed $3s$.

Notice that, unless $\mathcal{S}_v$ or $\mathcal{S}_v^{\max}$ is both strongly r+y- and strongly r+b-connected, the size of $\mathcal{S}_v$ decreases as follows.

– If $\mathcal{S}_v$ is not strongly r+b-connected then its size decreases on Step 3.1.1.1.

– If there is an r-connected component of $\mathcal{S}_v$ which is not strongly r-connected and $\mathcal{S}_v^0 \neq \mathcal{S}_v$, then the size of $\mathcal{S}_v$ may stay unchanged when Step 3.1.1.1 is performed, but it decreases on the next round of recursion.

– If there is an r-connected component of $\mathcal{S}_v$ which is not strongly r-connected and $\mathcal{S}_v^0 = \mathcal{S}_v$, then the size of $\mathcal{S}_v$ may stay unchanged when Step 3.1.1.1 or 3.3.1.1 is performed. However, for every $w \in J(\mathrm{r{+}b}, v, \mathcal{S}_v)$, $\mathcal{S}_w$ is strongly r+b-connected and, for every $w \in J(\mathrm{m}, v, \mathcal{S}_v)$, $\mathcal{S}_w = \mathcal{S}_w^0$. This means that the size of $\mathcal{S}_v$ decreases on the next or on the third round of recursion.

– If $\mathcal{S}_v$ is strongly r+b-connected, is not r+y-connected and every r-connected component is strongly r-connected, then $\mathcal{S}_v$ contains more than one minimal i-components (every r+y-connected component is a minimal i-component). Therefore, the size of $\mathcal{S}_v$ may stay unchanged when Step 3.1.1.1 is performed, but v is not included into any set of the form $J(\mathrm{m}, w, B)$ or $J_{\max}(\mathrm{r{+}b}, w, B)$ on Step 3.2.1.1 or Step 3.3.1.1.

Therefore, for any variable $v \in V$, $|\mathcal{S}_v|$ decreases during at least each third iteration. Finally, algorithm CONSERVATIVE on Step 3.3.3 is applied to a problem in which, for any domain, every r-connected component is strongly r-connected, which means that for any variable $v \in V$, $|\mathcal{S}_v|$ decreases during at least each second iteration.

Let $p(n)$ be a polynomial that bounds the time complexity of 3-MINIMALITY, 3-WIDTH and MALTSEV, r its degree; and ζ the maximal number of sets of the form $\{a \in \mathcal{S}_v \mid B \sqsubseteq a\}$, where B is a minimal i-connected component of $\mathcal{S}_v$, sharing an element.

CLAIM 2. There is a constant D such that the number of operations performed by CONSERVATIVE($\mathcal{P}$) does not exceed $\xi^{lr} p(|\mathcal{P}|)$ where l is the depth of recursion and $\xi = \zeta + 3$.

We prove Claim 2 by induction on the depth l of recursion. The base case of induction, Step 3.5, is obvious.

The time complexity of each of Steps 1, 3.1.3, 3.2.1, 3.2.4, 3.4.3, 4 is bounded with $p(|\mathcal{P}|)$. Step 2 can be performed in linear time, so we neglect its contribution. By the assumption made, Steps 3.1.1.2, 3.1.2, 3.2.2.2, 3.2.3, 3.3.1.2, 3.3.2, 3.4.1.2, 3.4.2 are never performed. Therefore, it is enough to estimate the time complexity of Steps 3.1.1.1, 3.2.1.1, 3.3..1.1 and 3.4.1.1.

Suppose first, that the conditions for Step 3.4 are valid. Then $|\mathcal{P}_{v_1,B_1}|+\ldots+|\mathcal{P}_{v_k,B_k}| \leq |\mathcal{P}|$ where $B_1,\ldots,B_k$ are the strongly r+y-connected components of $\mathcal{S}_{v_1},\ldots,\mathcal{S}_{v_k}$. Therefore, the number of operations required on Step 3.4.1.1 equals $\xi^{r(l-1)}p(|\mathcal{P}_{v_1,B_1}|)+\ldots+\xi^{r(l-1)}p(|\mathcal{P}_{v_k,B_k}|)$. As is easily seen

$$\xi^{r(l-1)}p(|\mathcal{P}_{v_1,B_1}|)+\ldots+\xi^{r(l-1)}p(|\mathcal{P}_{v_k,B_k}|) \leq \xi^{r(l-1)}p(|\mathcal{P}|).$$

Since $|\mathcal{P}_{v_1,B_1}|+\ldots+|\mathcal{P}_{v_k,B_k}| \leq |\mathcal{P}|$ where $B_1,\ldots,B_k$ are the maximal r+b-connected components of $\mathcal{S}_{v_1},\ldots,\mathcal{S}_{v_k}$, the same bound is valid for Step 3.1.1.1.

The same arguments work for Step 3.2.1.1 and Step 3.4.1.1.

In Step 3..1.1, we have

$$\zeta|\mathcal{P}| \geq |\mathcal{P}^+_{v_1,B_1}|+\ldots+|\mathcal{P}^+_{v_k,B_k}|$$

where $B_1,\ldots,B_k$ are the minimal i-components of $,\ldots,\mathcal{S}k$. Therefore, the the number of operations required on Step 3.3.1.1 is bounded with

$$\xi^{r(l-1)}p(|\mathcal{P}^+_{v_1,B_1}|)+\ldots+\xi^{r(l-1)}p(|\mathcal{P}^+_{v_k,B_k}|) \leq$$
$$\xi^{r(l-1)}p(\zeta|\mathcal{P}|) \leq \xi^{r(l-1)}\zeta^r p(|\mathcal{P}|).$$

Thus, the overall time complexity is bounded with $\xi^{r(l-1)}\zeta^r p(|\mathcal{P}|)+3p(|\mathcal{P}|) \leq \xi^{rl}p(|\mathcal{P}|)$, as required.

To complete the proof, notice that in the general case, including performing Steps 3.1.1.2, 3.1.2, 3.2.2.2, 3.2.3, 3.3.1.2, 3.3.2, 3.4.1.2, 3.4.2, the problem $\mathcal{P}$ can be changed at most m times. Therefore, the time complexity of CONSERVATIVE($\mathcal{P}$) is bounded with $m\xi^{rl}p(|\mathcal{P}|)$. □

9 Conclusion

We have continued the research aiming to distinguish those constraint languages that give rise to tractable constraint satisfaction problems from those which do not. We achieve this goal in the case of conservative constraint languages which, besides its own importance, including applications, provides a possible prototype for the most general case of arbitrary constraint languages. The main result of the paper has become possible due to the algebraic approach to constraints. This idea being originally developed for unifying and expanding known tractable classes has grown up to one the most prolific methods of studying the constraint satisfaction problem. It allows one to tackle problems which can hardly be solved by other existing methods, and we strongly believe that eventually it will make it possible to completely solve the indicated research problem.

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