Pebble Games for Logics with Counting and Rank

Anuj Dawar
University of Cambridge

Visiting ENS, Cachan

GaLoP, 28 March 2009

Expressive Power of Logics

We are interested in the *expressive power* of logics on finite structures.

We consider finite structures in a *relational vocabulary*.

A finite set A, with relations $R_1, \ldots, R_m$ and constants $c_1, \ldots, c_n$.

A *property* of finite structures is any *isomorphism-closed* class of structures.

For a logic (i.e., a *description* or *query* language) $\mathcal{L}$, we ask for which properties P, there is a sentence φ of the language such that

$$\mathbb{A} \in P \quad \text{if, and only if,} \quad \mathbb{A} \models \varphi.$$

In our examples, we will confine ourselves to vocabularies with just one binary relation E.

First-Order Logic

terms – c, x

atomic formulae – $R(t_1, \ldots, t_a), t_1 = t_2$

boolean operations – $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$

first-order quantifiers – $\exists x\varphi$, $\forall x\varphi$

Graphs which contain a triangle:

$\exists x\exists y\exists z(x \neq y \wedge y \neq z \wedge x \neq y \wedge E(x, y) \wedge E(y, z) \wedge E(x, z))$

Unions of cycles: $\forall x(\exists! yE(x, y) \wedge \exists! zE(z, y))$

Can we define the class of *connected graphs*? No, but how do we prove it?

Quantifier Rank

The *quantifier rank* of a formula φ, written $\mathrm{qr}(\varphi)$ is defined inductively as follows:

1. if φ is atomic then $\mathrm{qr}(\varphi) = 0$,
2. if $\varphi = \neg\psi$ then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi)$,
3. if $\varphi = \psi_1 \vee \psi_2$ or $\varphi = \psi_1 \wedge \psi_2$ then
   $\mathrm{qr}(\varphi) = \max(\mathrm{qr}(\psi_1), \mathrm{qr}(\psi_2))$.
4. if $\varphi = \exists x\psi$ or $\varphi = \forall x\psi$ then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi) + 1$

In a finite relational vocabulary, it is easily proved that in a finite vocabulary, for each q, there are (up to logical equivalence) only finitely many sentences φ with $\mathrm{qr}(\varphi) \leq q$.

Finitary Elementary Equivalence

For two structures $\mathbb{A}$ and $\mathbb{B}$, we say $\mathbb{A} \equiv_p \mathbb{B}$ if for any sentence φ with $\mathrm{qr}(\varphi) \leq p$,

$$\mathbb{A} \models \varphi \text{ if, and only if, } \mathbb{B} \models \varphi.$$

Key fact:

a class of structures S is definable by a first order sentence if, and only if, S is closed under the relation $\equiv_p$ for some p.

In a finite relational vocabulary, for any structure $\mathbb{A}$ there is a sentence $\theta^p_{\mathbb{A}}$ such that

$$\mathbb{B} \models \theta^p_{\mathbb{A}} \quad \text{if, and only if,} \quad \mathbb{A} \equiv_p \mathbb{B}$$

Ehrenfeucht-Fraïssé Game

The p-round Ehrenfeucht game on structures $\mathbb{A}$ and $\mathbb{B}$ proceeds as follows:

- There are two players called Spoiler and Duplicator.
- At the ith round, Spoiler chooses one of the structures (say $\mathbb{B}$) and one of the elements of that structure (say b_i).
- Duplicator must respond with an element of the other structure (say a_i).
- If, after p rounds, the map $a_i \mapsto b_i$ is a partial isomorphism, then Duplicator has won the game, otherwise Spoiler has won.

Theorem (Fraïssé 1954; Ehrenfeucht 1961)

Duplicator has a strategy for winning the p-round Ehrenfeucht game on $\mathbb{A}$ and $\mathbb{B}$ if, and only if, $\mathbb{A} \equiv_p \mathbb{B}$.

Proof by Example

Suppose $\mathbb{A} \not\equiv_3 \mathbb{B}$, in particular, suppose $\theta(x, y, z)$ is quantifier free, such that:

$$\mathbb{A} \models \exists x \forall y \exists z \theta \quad \text{and} \quad \mathbb{B} \models \forall x \exists y \forall z \neg\theta$$

round 1: Spoiler chooses $a_1 \in A$ such that $\mathbb{A} \models \forall y \exists z \theta[a_1]$.
Duplicator responds with $b_1 \in B$.

round 2: Spoiler chooses $b_2 \in B$ such that $\mathbb{B} \models \forall z \neg\theta[b_1, b_2]$.
Duplicator responds with $a_2 \in A$.

round 3: Spoiler chooses $a_3 \in A$ such that $\mathbb{A} \models \theta[a_1, a_2, a_3]$.
Duplicator responds with $b_3 \in B$.

Spoiler wins, since $\mathbb{B} \not\models \theta[b_1, b_2, b_3]$.

Using Games

To show that a class of structures S is not definable in FO, we find, for every p, a pair of structures $\mathbb{A}_p$ and $\mathbb{B}_p$ such that

- $\mathbb{A}_p \in S$, $\mathbb{B}_p \in \overline{S}$; and
- *Duplicator* wins a p round game on $\mathbb{A}_p$ and $\mathbb{B}_p$.

Example:

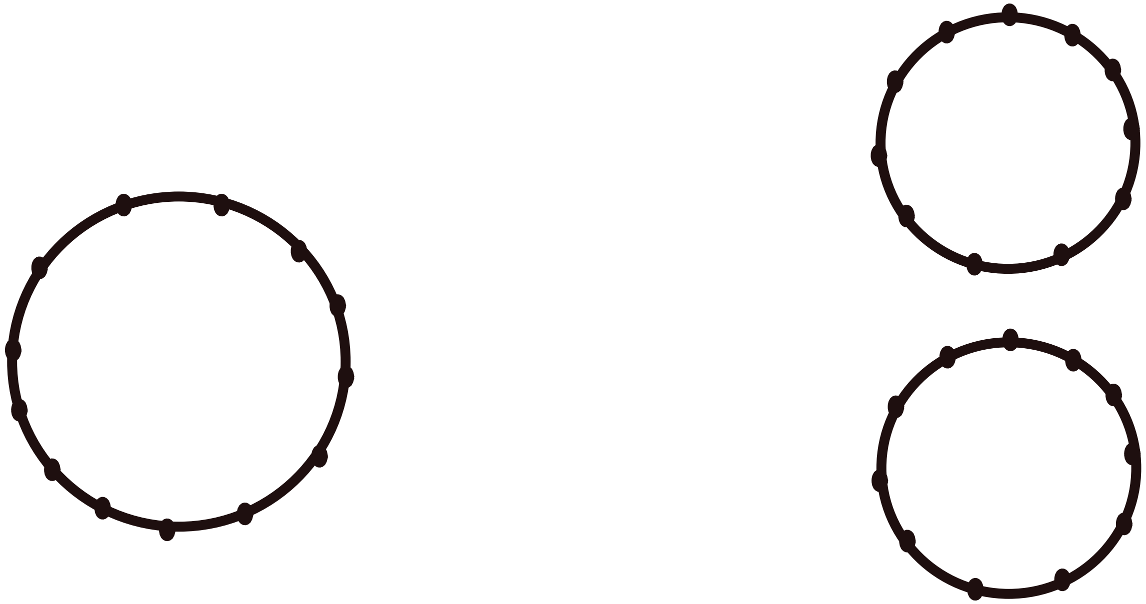
C_n—a cycle of length n.

Duplicator wins the p round game on $C_{2^p} \oplus C_{2^p}$ and C_{2^p+1}.

- 2-Colourability is not definable in FO.
- Even cardinality is not definable in FO.
- Connectivity is not definable in FO.

Using Games

An illustration of the game for undefinability of *connectivity* and *2-colourability*.



Duplicator's strategy is to ensure that after r moves, the distance between corresponding pairs of pebbles is either *equal* or $\geq 2^{p-r}$.

Inductive Definitions

Let $\varphi(R, x_1, \ldots, x_k)$ be a first-order formula in the vocabulary $\sigma \cup \{R\}$

Associate an operator Φ on a given structure $\mathbb{A}$:

$$\Phi(R^{\mathbb{A}}) = \{\mathbf{a} \mid (\mathbb{A}, R^{\mathbb{A}}, \mathbf{a}) \models \varphi(R, \mathbf{x})\}$$

We define the *increasing* sequence of relations on $\mathbb{A}$:

$$\Phi^0 = \emptyset$$

$$\Phi^{m+1} = \Phi^m \cup \Phi(\Phi^m)$$

The *inflationary fixed point* of Φ is the limit of this sequence.

On a structure with n elements, the limit is reached after at most n^k stages.

IFP

The logic IFP is formed by closing first-order logic under the rule:

If φ is a formula of vocabulary $\sigma \cup \{R\}$ then $[\textbf{ifp}_{R,\mathbf{x}}\varphi](\mathbf{t})$ is a formula of vocabulary σ.

The formula is read as:

the tuple $\mathbf{t}$ is in the inflationary fixed point of the operator defined by φ

LFP is the similar logic obtained using *least fixed points* of *monotone* operators defined by *positive* formulas.

LFP and IFP have the same expressive power **(Gurevich-Shelah; Kreutzer)**.

Transitive Closure

The formula

$$[\textbf{ifp}_{T,xy}(x = y \vee \exists z(E(x,z) \wedge T(z,y)))](u,v)$$

defines the *reflexive and transitive closure* of the relation E

The expressive power of IFP properly extends that of first-order logic.

On structures which come equipped with a linear order IFP expresses exactly the properties that are in P.

(Immerman; Vardi)

Open Question: Is there a logic that expresses exactly the properties for *unordered* structures?

Finite Variable Logic

We write L^k for the first order formulas using only the variables $x_1, \ldots, x_k$.

$$\mathbb{A} \equiv^k \mathbb{B}$$

denotes that $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k.

For any k, $\quad \mathbb{A} \equiv^k \mathbb{B} \quad \Rightarrow \quad \mathbb{A} \equiv_k \mathbb{B}$

However, for any q, there are $\mathbb{A}$ and $\mathbb{B}$ such that

$$\mathbb{A} \equiv_q \mathbb{B} \quad \text{and} \quad \mathbb{A} \not\equiv^2 \mathbb{B}.$$

Axiomatisability

Any class of finite structures closed under isomorphisms is *axiomatised* by a first-order theory.

A class of finite structures is closed under $\equiv_q$ (for some q) if, and only if, it is *finitely axiomatised*, i.e. defined by a single FO sentence.

A class of finite structures is closed under $\equiv^k$ if, and only if, it is axiomatisable in L^k (possibly by an infinite collection of sentences).

Every sentence of IFP is equivalent, *on finite structures*, to an L^k theory, for some k.

$$\varphi(R, x_1, \ldots, x_l) \in L^k$$

Each stage of the induction φ^m can be written as a formula in L^{k+l}.

Pebble Games

The k-pebble game is played on two structures $\mathbb{A}$ and $\mathbb{B}$, by two players—*Spoiler* and *Duplicator*—using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

Spoiler moves by picking a pebble and placing it on an element (a_i on an element of $\mathbb{A}$ or b_i on an element of $\mathbb{B}$).

Duplicator responds by picking the matching pebble and placing it on an element of the other structure

Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ defined by the pebble pairs is not a partial isomorphism

If *Duplicator* has a winning strategy for q moves, then $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k of quantifier rank at most q. **(Barwise)**

Using Pebble Games

To show that a class of structures S is not definable in first-order logic:

$$\forall k\ \forall q\ \exists \mathbb{A}, \mathbb{B}\ (\mathbb{A} \in S \wedge \mathbb{B} \notin S \wedge \mathbb{A} \equiv_q^k \mathbb{B})$$

To show that S is not axiomatisable with a finite number of variables:

$$\forall k\ \exists \mathbb{A}, \mathbb{B}\ \forall q\ (\mathbb{A} \in S \wedge \mathbb{B} \notin S \wedge \mathbb{A} \equiv_q^k \mathbb{B})$$

Evenness

To show that *Evenness* is not definable in IFP, it suffices to show that:

for every k, there are structures $\mathbb{A}_k$ and $\mathbb{B}_k$ such that $\mathbb{A}_k$ has an even number of elements, $\mathbb{B}_k$ has an odd number of elements and

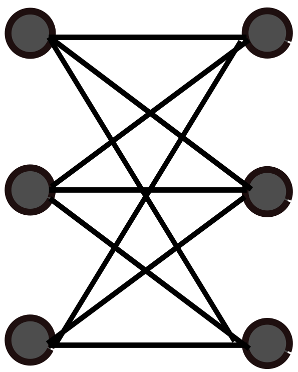
$$\mathbb{A} \equiv^k \mathbb{B}.$$

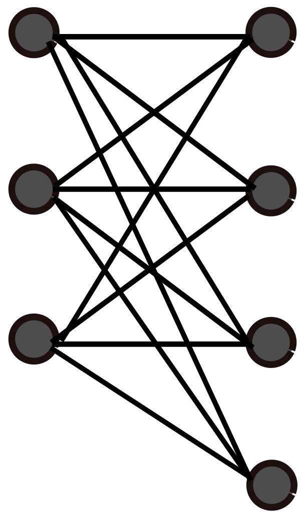
It is easily seen that *Duplicator* has a strategy to play forever when one structure is a set containing k elements (and no other relations) and the other structure has $k+1$ elements.

Hamiltonicity

Take $K_{k,k}$—the complete bipartite graph on two sets of k vertices.

and $K_{k,k+1}$—the complete bipartite graph on two sets, one of k vertices, the other of $k+1$.





These two graphs are $\equiv^k$ equivalent, yet one has a Hamiltonian cycle, and the other does not.

Fixed-point Logic with Counting

Immerman proposed IFP $+$ C—the extension of IFP with a mechanism for *counting*

Two sorts of variables:

- $x_1, x_2, \ldots$ range over $|A|$—the domain of the structure;
- $\nu_1, \nu_2, \ldots$ which range over *numbers* in the range $0, \ldots, |A|$

If $\varphi(x)$ is a formula with free variable x, then $\nu = \#x\varphi$ denotes that ν is the number of elements of A that satisfy the formula φ.

We also have the order $\nu_1 < \nu_2$, which allows us (using recursion) to define arithmetic operations.

Counting Quantifiers

C^k is the logic obtained from *first-order logic* by allowing:

- allowing *counting quantifiers*: $\exists^i x\, \varphi$; and
- only the variables $x_1, \ldots .x_k$.

Every formula of C^k is equivalent to a formula of first-order logic, albeit one with more variables.

For every sentence φ of IFP $+$ C, there is a k such that if $\mathbb{A} \equiv^{C^k} \mathbb{B}$, then

$$\mathbb{A} \models \varphi \quad \text{if, and only if,} \quad \mathbb{B} \models \varphi.$$

Counting Game

Immerman and Lander (1990) defined a *pebble game* for C^k.

This is again played by *Spoiler* and *Duplicator* using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

At each move, *Spoiler* picks a subset of the universe (say $X \subseteq B$)

Duplicator responds with a subset of the other structure (say $Y \subseteq A$) of the same *size*.

Spoiler then places a b_i pebble on an element of Y and *Duplicator* must place a_i on an element of X.

Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ defined by the pebble pairs is not a partial isomorphism

If *Duplicator* has a winning strategy for q moves, then $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of C^k of quantifier rank at most q.

Cai-Fürer-Immerman Graphs

There are polynomial-time decidable properties of graphs that are not definable in IFP $+$ C. **(Cai, Fürer, Immerman, 1992)**

More precisely, we can construct a sequence of pairs of graphs $G_k, H_k (k \in \omega)$ such that:

- $G_k \equiv^{C^k} H_k$ for all k.
- There is a polynomial time decidable class of graphs that includes all G_k and excludes all H_k.

Still, IFP $+$ C is a *natural* level of expressiveness within P.

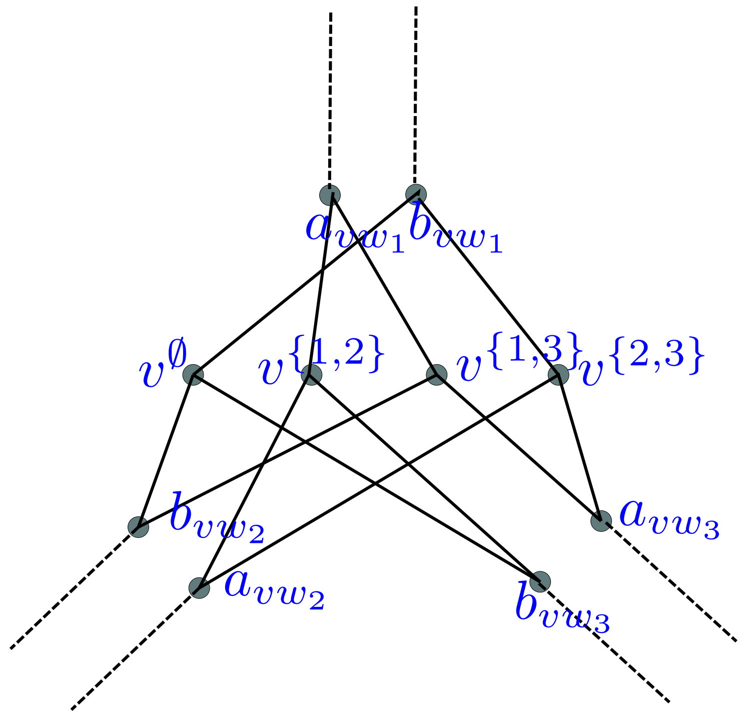
Constructing G_k and H_k

Given any graph G, we can define a graph X_G by replacing every edge with a pair of edges, and every vertex with a gadget.

The picture shows the gadget for a vertex v that is adjacent in G to vertices w_1, w_2 and w_3.

The vertex v^S is adjacent to $a_{vw_i} (i \in S)$ and $b_{vw_i} (i \notin S)$ and there is one vertex for all *even size* S.

The graph $\tilde{X}_G$ is like X_G except that at *one vertex* v, we include V^S for *odd size* S.



Properties

If G is *connected* and has *treewidth* at least k, then:

1. $X_G \not\cong \tilde{X}_G$; and
2. $X_G \equiv^{C^k} \tilde{X}_G$.

(1) allows us to construct a polynomial time property separating X_G and $\tilde{X}_G$.

(2) is proved by a game argument.

The original proof of **(Cai, Fürer, Immerman)** relied on the existence of balanced separators in G. The characterisation in terms of treewidth is from **(D., Richerby 07)**.

Bijection Games

$\equiv^{C^k}$ is characterised by a k-pebble *bijection game*. **(Hella 96)**.

The game is played on structures $\mathbb{A}$ and $\mathbb{B}$ with pebbles $a_1, \ldots, a_k$ on $\mathbb{A}$ and $b_1, \ldots, b_k$ on $\mathbb{B}$.

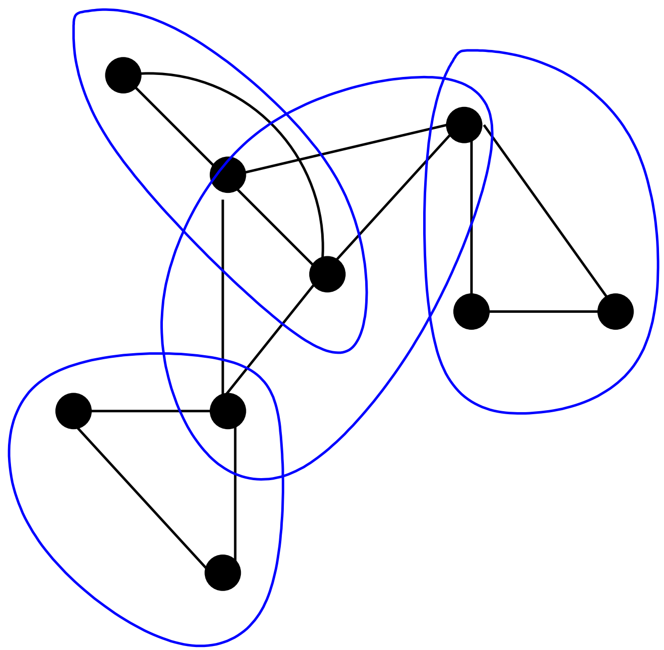
- *Spoiler* chooses a pair of pebbles a_i and b_i;
- *Duplicator* chooses a bijection $h : A \to B$ such that for pebbles a_j and $b_j (j \neq i)$, $h(a_j) = b_j$;
- *Spoiler* chooses $a \in A$ and places a_i on a and b_i on $h(a)$.

Duplicator loses if the partial map $a_i \mapsto b_i$ is not a partial isomorphism.
Duplicator has a strategy to play forever if, and only if, $\mathbb{A} \equiv^{C^k} \mathbb{B}$.

TreeWidth

The *treewidth* of a graph is a measure of its interconnectedness.

A graph has treewidth k if it can be covered by subgraphs of at most $k+1$ nodes in a tree-like fashion.



TreeWidth

Formal Definition:

For a graph $G = (V, E)$, a *tree decomposition* of G is a relation $D \subset V \times T$ with a tree T such that:

- for each $v \in V$, the set $\{t \mid (v, t) \in D\}$ forms a connected subtree of T; and
- for each edge $(u, v) \in E$, there is a $t \in T$ such that $(u, t), (v, t) \in D$.

The *treewidth* of G is the least k such that there is a tree T and a tree-decomposition $D \subset V \times T$ such that for each $t \in T$,

$$|\{v \in V \mid (v, t) \in D\}| \leq k + 1.$$

Cops and Robbers

A game played on an undirected graph $G = (V, E)$ between a player controlling k *cops* and another player in charge of a *robber*.

At any point, the cops are sitting on a set $X \subseteq V$ of the nodes and the robber on a node $r \in V$.

A move consists in the cop player removing some cops from $X' \subseteq X$ nodes and announcing a new position Y for them. The robber responds by moving along a path from r to some node s such that the path does not go through $X \setminus X'$.

The new position is $(X \setminus X') \cup Y$ and s. If a cop and the robber are on the same node, the robber is caught and the game ends.

Strategies and Decompositions

Theorem (Seymour and Thomas 93):
There is a winning strategy for the *cop player* with k cops on a graph G if, and only if, the tree-width of G is at most $k-1$.

It is not difficult to construct, from a tree decomposition of width k, a winning strategy for $k+1$ cops.

Somewhat more involved to show that a winning strategy yields a decomposition.

Cops, Robbers and Bijections

If G has treewidth k or more, than the *robber* has a winning strategy in the k-*cops and robbers* game played on G.

We use this to construct a winning strategy for Duplicator in the k-pebble bijection game on X_G and $\tilde{X}_G$.

- A bijection $h : X_G \to \tilde{X}_G$ is *good bar* v if it is an isomorphism everywhere except at the vertices v^S.
- If h is good bar v and there is a path from v to u, then there is a bijection h' that is good bar u such that h and h' differ only at vertices corresponding to the path from v to u.
- Duplicator plays bijections that are good bar v, where v is the robber position in G when the cop position is given by the currently pebbled elements.

Solvability of Linear Equations

A natural P problem that has been shown to be undefinable in IFP $+$ C is the problem of solving linear equations over the two element field $\mathbb{Z}_2$.

(Atserias, Bulatov, D. 07)

The question arose in the context of classification of *Constraint Satisfaction Problems*.

The problem is clearly solvable in polynomial time by means of Gaussian elimination.

We see how to represent systems of linear equations as *unordered* relational structures.

Systems of Linear Equations

Consider structures over the domain $\{x_1, \ldots, x_n, e_1, \ldots, e_m\}$, (where $e_1, \ldots, e_m$ are the equations) with relations:

- unary E_0 for those equations e whose r.h.s. is 0.
- unary E_1 for those equations e whose r.h.s. is 1.
- binary M with $M(x, e)$ if x occurs on the l.h.s. of e.

Solv$(\mathbb{Z}_2)$ is the class of structures representing solvable systems.

Undefinability in IFP + C

Take $\mathcal{G}$ a 3-regular, connected graph with treewidth $> k$.

Define equations $\mathbf{E}_\mathcal{G}$ with two variables x_0^e, x_1^e for each edge e.

For each vertex v with edges e_1, e_2, e_3 incident on it, we have eight equations:

$$E_v : \qquad x_i^{e_1} + x_j^{e_2} + x_k^{e_3} \equiv i + j + k \pmod 2$$

$\tilde{\mathbf{E}}_\mathcal{G}$ is obtained from $\mathbf{E}_\mathcal{G}$ by replacing, for exactly one vertex v, E_v by:

$$E'_v : \qquad x_i^{e_1} + x_j^{e_2} + x_k^{e_3} \equiv i + j + k + 1 \pmod 2$$

We can show: $\mathbf{E}_\mathcal{G}$ is satisfiable; $\tilde{\mathbf{E}}_\mathcal{G}$ is unsatisfiable; $\mathbf{E}_\mathcal{G} \equiv^{C^k} \tilde{\mathbf{E}}_\mathcal{G}$

Computational Problems from Linear Algebra

Linear Algebra is a testing ground for exploring the boundary of the expressive power of IFP $+$ C.

It may also be a possible source of new operators to extend the logic.

For a set I, and binary relation $A \subseteq I \times I$, take the matrix M over the two element field $\mathbb{Z}_2$:

$$M_{ij} = 1 \quad \Leftrightarrow \quad (i, j) \in A.$$

Most interesting properties of M are invariant under permutations of I.

Representing Finite Fields

We can represent matrices M over a finite field $\mathbb{F}_q$ by taking, for each $a \in \mathbb{F}_q$ a binary relation $A_a \subseteq I \times I$ with

$$M_{ij} = a \quad \Leftrightarrow \quad (i, j) \in A_a.$$

Alternatively, we could have the elements of $\mathbb{F}_q$ (along with the field operations) as a *separate sort* and include a ternary relation R

$$M_{ij} = a \quad \Leftrightarrow \quad (i, j, a) \in R.$$

These two representations are inter-definable.

IFP $+$ C over Finite Fields

Over $\mathbb{F}_q$, *matrix multiplication*; *non-singularity* of matrices; the *inverse* of a matrix; are all definable in IFP $+$ C.

determinants and more generally, the coefficients of the *characteristic polynomial* can be expressed IFP $+$ C.

(D., Grohe, Holm, Laubner, 2009)

solvability of systems of equations is *undefinable*.

Rank Operators

We introduce an operator for *matrix rank* into the logic.

> $\mathsf{rk}_{x,y}\varphi$ is a *term* denoting the number that is the rank of the matrix defined by $\varphi(x, y)$.

More generally, we could have, for each finite field $\mathbb{F}_q$, an operator rk^q.

(D., Grohe, Holm, Laubner, 2009)

Adding rank operators to IFP, we obtain a proper extension of IFP + C.

$$\#x\varphi \quad = \quad \mathsf{rk}_{x,y}[x = y \wedge \varphi(x)]$$

In IFP + rank we can express the solvability of linear systems of equations, as well as the Cai-Fürer-Immerman graphs and the order on multipedes.

Games for Logics with Rank

What might a pebble game for IFP + rank look like?

We could, as in the *Immerman-Lander* game, let *Spoiler* pick a relation and have *Duplicator* respond with one of equal rank.

> This works if we restrict the players to playing *definable* relations. A rather unsatisfactory solution.

Is there a game to be obtained by modifying the Hella game, replacing bijections with *invertible linear maps*?

Open Questions

With a suitable notion of game, we could try and tackle problems like:

- Are there any problems in P that are not definable in IFP + rank?
- Show for any concrete problem (say an NP-complete one) that it is not definable in IFP + rank.
- Are rk^p and rk^q interdefinable for $p \neq q$?
- *etc.*