Geometric Foundations for Game Semantics?

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Game Semantics

- Quite successful in modelling logics and programming languages
  - full abstraction/full completeness
- lots of *different* models, with similar underlying ideas
- perhaps it's worth revisiting the foundations...

Combinatoric Foundations

- Harmer, Hyland and Mellies (LICS 2007) give a new, combinatoric/algebraic development of the key ideas.
- Central result: an algebraic explanation of why innocent strategies compose.

Categorical Combinatorics for Innocent Strategies, LICS 2007

A geometric foundation?

- We present baby steps in giving a foundation based on geometry.
- Why?
  - In practice, “we” draw pictures when working with games.
  - Geometry makes certain things obvious.

Distinguished history

- Compare with the development of monoidal categories, braided, tortile and traced monoidal categories
- In that work, geometry took over as the foundation:
  - definitions were *given* geometrically;
  - combinatoric/algebraic analogues took a back seat.

Game semantics basics

- A game describes a tree of valid plays — certain sequences.
- A strategy is a subtree:
  - its root is the root of the game
  - it has at most one branch at odd-depth nodes

Maps in game semantics

- A map $A \longrightarrow B$ is a strategy on $A \multimap B$, whose plays are given by
  - a play of A
  - a play of B
  - an interleaving between them.

A picture (already!)

$A \multimap$	B
	b_1
	b_2
	b_3
a_1	
a_2	
	b_4
	b_5
a_3	

A picture (already!)

$A \multimap B$

A	B	
	b_1	R
	b_2	R
	b_3	R
a_1		L
a_2		L
	b_4	R
	b_5	R
a_3		L

Schedules

- A play of $A \multimap B$ is given by a pair of plays of A and B and a *schedule*.
- Schedule: a sequence $s \in \{L,R\}^*$ such that
    - $s_0 = R$
    - $s_{2i} = s_{2i-1}$ for all $i \geq 1$.

Composing schedules

- $s : m \longrightarrow n$ means s is a schedule with m L's and n R's.
- Given $s: m \longrightarrow n$ and $t: n \longrightarrow p$ we can compose to get $s;t: m \longrightarrow p$ as follows.
  - Relabel the R's in s and the L's in t as M ("middle")
  - $\exists! u \in \{L,M,R\}^*$ such that $u|_{L,M} = s$, $u|_{M,R} = t$.
  - Define $s;t$ to be $u|_{L,R}$.

Composition

m	n	p
		R
	M	
	M	
		R
		R
	M	
L		
L		
	M	
	M	
L		

Composition

m

p

R

R

R

L

L

L

The Zipping Lemma

- The existence and uniqueness of this *u* is tedious to prove.
- It's a dull induction on length.
- But once you see what's going on, it's something that ought to be obvious.
- Note that we need this lemma just to have a complete definition of composition.

Identities

- Identities are *copycat schedules*
- *R L L R R L L R R L L R R L ...*

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...which also does not contain the proof.

Category of games

- Objects: games
- Maps: (certain) sets of tuples

$$(p_A, p_B, s)$$

where:

- p_A is a play of A, p_B is a play of B, and

  $s \colon |p_A| \longrightarrow |p_B|$ is a schedule

Composition of maps

Let $\sigma : A \longrightarrow B$, $\tau : B \longrightarrow C$. Define

$$\sigma ; \tau = \{(p_A, p_C, u) \mid \exists\, p_B, s, t.$$

$$(p_A, p_B, s) \in \sigma,$$

$$(p_B, p_C, t) \in \tau,$$

$$u = s; t \,\}$$

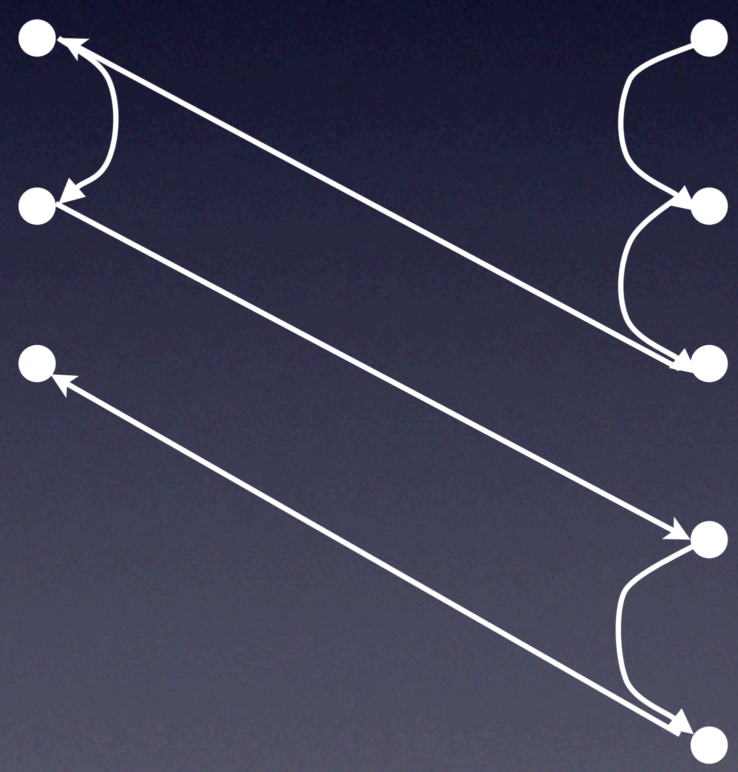
So the category of games is category of "scheduled relations".

Aside

- For deterministic strategies, the schedules are redundant: they can be recovered from the sets of plays.
- So the category of games is a subcategory of **Rel**.
- But we need schedules to describe the closed structure.

Schedules Geometrically

- A schedule is a path in the plane: here's a schedule from 3 to 5.



More precisely...

- Work in an oriented plane: we can tell up from down and left from right.
- Each natural number comes with an embedding L_n of {0, 1, ..., n-1} in the real line .
- A schedule $s: m \longrightarrow n$ consists of
  - embeddings of L_m and L_n in the plane, vertically, with *m* to the left of *n*
  - a path through the vertices of these lines, lying entirely in the strip between them.

Switching condition

- We also insist that schedules satisfy the *switching condition:*
  - the path can only cross from L_m to L_n (or the other way) after visiting an odd number of vertices in total.

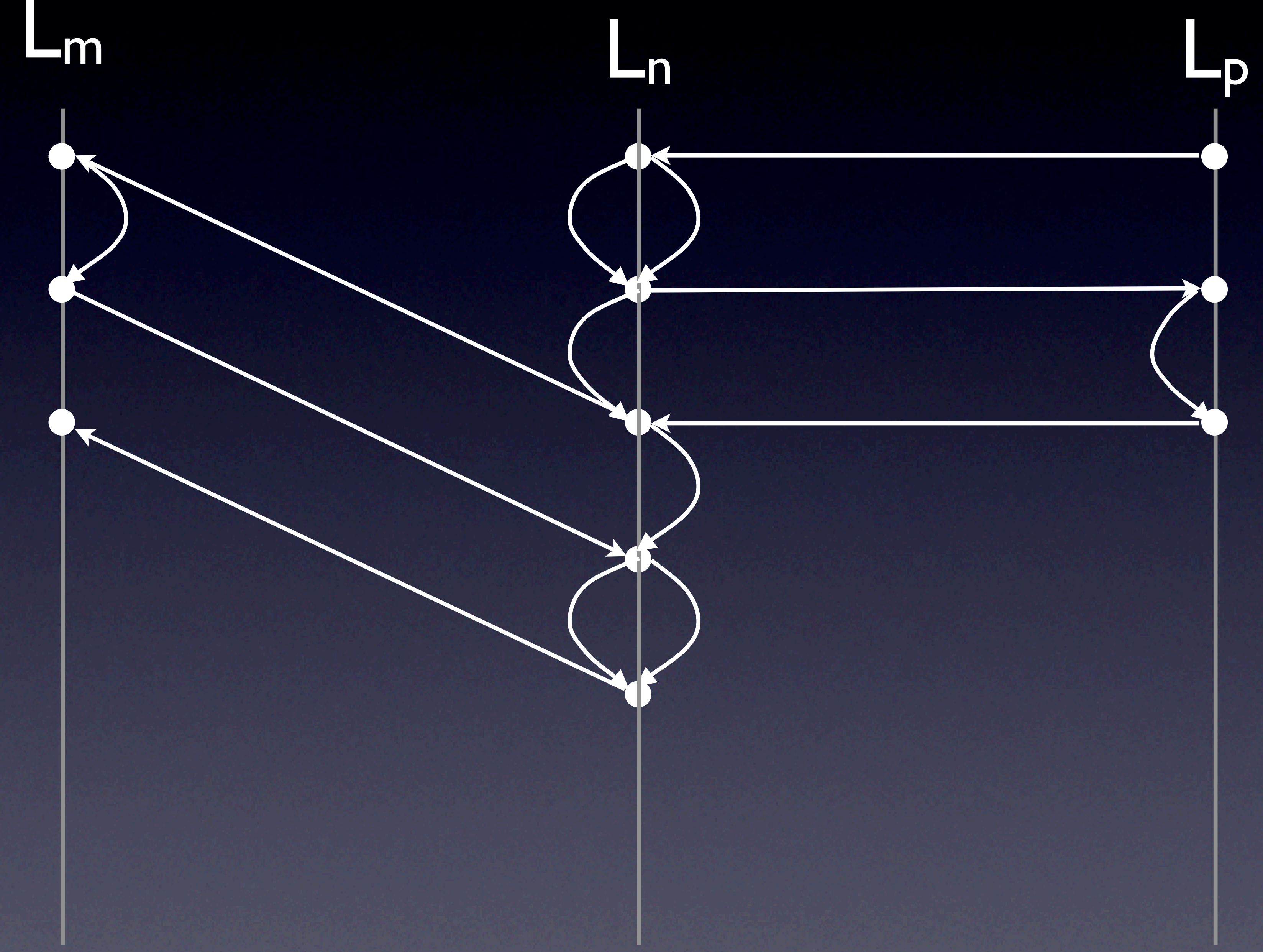
Isotopy

- We identify these paths up to *what they look like*.
- That is, really we're working with the underlying graphs, which might as well be sequences RLLLLRR...
- The extra geometric data lets us reason pictorially.

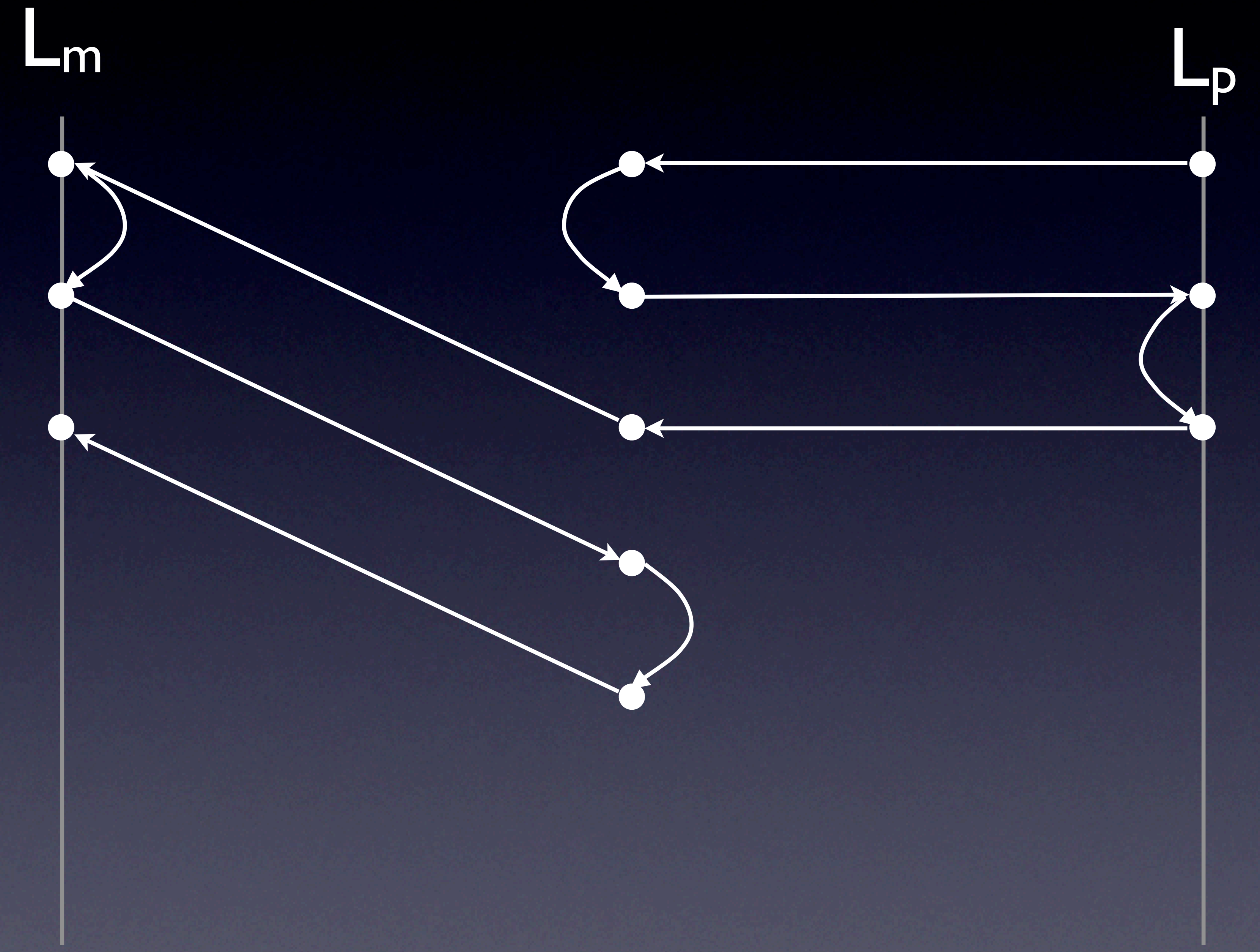
Composition

- Given *s*: $m \longrightarrow n$ and *t*: $u \longrightarrow p$, compose as follows:
  - draw them next to each other
  - follow the path from the top-right vertex, which is p_1.
  - when you reach L_n, you face a choice...
    - *always cut L_n when you reach it.*

Composition



Composition



Associativity

- Define the obvious 3-schedule (or arbitrary *n*-schedule) version of composition.
- Only need show that the subpath on any three consecutive lines is a composition as above.
- Suffices to show that choices are resolved the same way.
- So it's enough to show that vertices are always entered from the same side.

Right, left, right, left

- Every topmost vertex is entered from the right

  because the path starts at top right, and cannot cross a line without passing the top vertex.

- The next level vertices are entered from the left

  because the path entered the part of the plan to their left at the previous level, and cannot cross the line elsewhere.

- And so on.
- Done!

Stating the obvious

- This proof makes use of "obvious geometric facts" — the intermediate value theorem, for instance.
- Cultural shift: when working geometrically, this sort of reasoning is acceptably precise; not so combinatorically.

Conclusions?

- None really.
- The geometric setup makes things *formally* obvious.
- It's closer to intuition and to practice than combinatoric formalisms.

Where to?

- Can we give useful geometric accounts of views, visibility, innocence?
- Can we use geometry to make other generalizations "thinkable"?
  - e.g. plays or views which are DAGs?