A Concrete Representation of Observational Equivalence for PCF

Martin Churchill, Jim Laird and Guy McCusker
University of Bath

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Overview

- Observational equivalence for PCF terms
- This talk describes some work to give a concrete representation of (a superset of) the equivalence classes
- This goes via the game semantics model of the mid-nineties by Hyland, Ong, Abramsky et al
- We define a mapping obs into sets of finite sets which equates equivalent PCF terms.
Types and Terms of PCF

- Prototypical functional programming language introduced by Plotkin
- Based on Scott’s LCF.

Types are of the form:

\[ T = \text{nat} \mid T_1 \to T_2 \]

Terms are of the form:

\[ M ::= x \mid \lambda x : A. M \mid M_1 M_2 \mid \text{succ} M \mid \text{pred} M \mid n \mid \text{ifzero} M_1 \text{ then } M_2 \text{ else } M_3 \mid Y_A M \]
Observational Equivalence 1

- We define a relation $\Downarrow$ between closed terms and values.
- $S$ is *refined by* $T$ if replacing $S$ by $T$ in any terminating program gives a terminating program.
- A *context* is a PCF term possibly with a placeholder/hole $\_$. 
- Given closed terms $M$ and $N$ of the same type, $M \leq_{\text{obs}} N$ iff for all valid contexts $C[\_]$, $C[M] \Downarrow$ implies $C[N] \Downarrow$.
- Write $S \equiv_{\text{obs}} T$ if $S \leq_{\text{obs}} T$ and $T \leq_{\text{obs}} S$.
=_{obs} involves a large quantification over all contexts.

Undecidable for finite types (Loader).

Denotational (games) models:

In the mid-nineties, Hyland/Ong, Abramsky/Jagadeesan/Malacaria, Nickau provided a model of PCF based on game semantics.

Gives an intrinsic account of PCF terms as *innocent strategies* + definability/quotienting.
Games and Plays

- A *play* is a sequence of moves where most moves are equipped with a pointer to some previous move.
  
  \[
  a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g
  \]

- A *game* is a set of moves + further data defining which plays are legal.

- Moves are divided into *player* moves and *opponent* moves; plays must be *alternating* \(^1\)

Example: Game \(\mathbb{N}\) — O-move \(q\) + P-response for each \(n \in \mathbb{N}\).

Example legal play: \(q 5 \ x 6 \ x 7 \ x 42\).

\(^1\) We also require visibility and well-bracketing.
Function Space

- If $A$ and $B$ are games, we can define $A \Rightarrow B$ and $A \times B$
- Plays in these games consists of a play in $A$ interleaved with a play in $B$
- In the case of $A \Rightarrow B$, the roles of $P$ and $O$ are reversed in the subgame $A$

Example of a play in $(\mathbb{N} \times \mathbb{N}) \Rightarrow \mathbb{N}$:

$$(\mathbb{N} \times \mathbb{N}) \Rightarrow \mathbb{N}$$
Strategies

- A P-strategy on a game is a set of even-length plays that are even-prefixed closed and even-branching.
- Represents a partial function from odd-lengthed plays to the next P-move.

Example of a strategy on \((\mathbb{N} \times \mathbb{N}) \Rightarrow \mathbb{N}\):

\[
\begin{align*}
(N \times N) & \Rightarrow N \\
q & \quad O \\
q & \quad P \\
m & \quad P \\
n & \quad P \\
m + n & \quad P
\end{align*}
\]

We can compose strategies.
Composition

Let $\sigma : N \to N$ and $\tau : N \to N$ have maximal plays

\[
\begin{array}{cccc}
N & \Rightarrow & N \\
q & \downarrow & q \\
n & \uparrow & n + 1 \\
\end{array} \quad \begin{array}{cccc}
N & \Rightarrow & N \\
q & \downarrow & q \\
O & \downarrow & P \\
O & \uparrow & n \\
\end{array}
\]

For $\sigma; \tau$ we use “parallel composition plus hiding”

\[
\begin{array}{cccc}
N & N & N & \Rightarrow & N \\
q & \downarrow & q \\
n & \uparrow & n + 1 \\
q & \downarrow & q \\
n & \uparrow & 2(n + 1) \\
\end{array}
\]
Views

The P-view of a play $s$ is the subsequence of $s$ removing moves between an opponent move and its justifier.

- $\langle \epsilon \rangle = \epsilon$
- $\langle sp \rangle = \langle s \rangle p$ where $p$ is a P-move
- $\langle si \rangle = i$ where $i$ is an initial move
- $\langle spto \rangle = \langle s \rangle po$, where P-move $p$ is the justifier of O-move $o$

Can also define O-view of $s$:

- $\langle \epsilon \rangle = \epsilon$
- $\langle so \rangle = \langle s \rangle o$ where $o$ is an O-move
- $\langle sotp \rangle = \langle s \rangle op$, where O-move $o$ justifies P-move $p$
Innocent Strategies

- An innocent strategy $\sigma$ over a game is a strategy where the next P-move depends only on the P-view.
- We can give the denotation of each PCF term as an innocent strategy.
- Soundness + definability — all compact innocent strategies represent some PCF term.
- This allows us to give a semantic definition of observational equivalence; and via quotienting a fully abstract model of PCF.
Innocent Equivalence

We define $\leq_{ib}$ on innocent strategies giving a semantic definition of the $\leq_{obs}$

Let $\Sigma$ denote the game with one initial O-move $q$ and it's P-response $a$ enabled by $q$. Let $\top$ denote the strategy $\{\epsilon, qa\}$ on $\Sigma$.

Let $\sigma$ and $\tau$ be innocent strategies over a game $A$. $\sigma \leq_{ib} \tau$ if for any innocent strategy $\alpha : A \Rightarrow \Sigma$ if $\sigma;\alpha = \top$ then $\tau;\alpha = \top$.

Theorem

Given two PCF terms $M, N : A$ we have $M \leq_{obs} N$ iff $\llbracket M \rrbracket \leq_{ib} \llbracket N \rrbracket$
Innocent Tests

- Given a strategy $\sigma : A$ we consider innocent tests passed by $\sigma$, i.e. functions from P-views of plays in $A \Rightarrow \Sigma$ to the next move.
- But P-views of plays in $A \Rightarrow \Sigma \cong$ O-views of plays in $A$.
- Thus an innocent test on $A$ corresponds to an O-view function on $A$. We can represent this as a set of O-views.

**Definition**

Let $s$ be a play over some game. Define $\operatorname{ovw}(s) = \{ \bot t \bot : t \sqsubseteq s \}$. 
The obs Construction

**Definition**

A play $s$ is *O-innocent* if for $s_1 o_1, s_2 o_2 \sqsubseteq s$ with $\ll s_1 \gg = \ll s_2 \gg$ we must have $o_1 = o_2$.

**Definition**

Let $\sigma$ be an innocent strategy over some game. Define $\bar{\sigma}$ to be the subset of $\sigma$ consisting of only the O-innocent, single-threaded, complete plays.

**Definition**

Let $\sigma$ be an innocent strategy. Define $\text{obs}(\sigma) = \{\text{ovw}(s) : s \in \bar{\sigma}\}$
Example 1

We describe an innocent strategy succ

\[
\begin{align*}
N & \Rightarrow N \\
q & \quad q \\
q_n & \quad O \\
n+1 & \quad P
\end{align*}
\]

Then \( \text{obs}(\text{succ}) = \{\{\epsilon, q_2, q_2q_1, q_2q_1n_1, q_2(n+1)\} : n \in \mathbb{N}\} \)

(Maximal O-views: \( \{\{q_2q_1n_1, q_2(n+1)\} : n \in \mathbb{N}\} \).)
Example 2

We also consider a strategy $\text{succ}_2$ with maximal plays

$$
\begin{align*}
\mathbb{N} & \Rightarrow \mathbb{N} \\
q & \quad O \\
\_n & \quad P \\
\_m & \quad P \\
q & \quad O \\
m + 1 & \quad P
\end{align*}
$$

O-innocence implies $m = n$. Thus

$$
\text{obs}(\text{succ}_2) = \{q_2 q_1 m_1, q_2(m + 1)_2 : m \in \mathbb{N}\} \text{ (maximal O-views only.)}
$$
Forgetfulness

- We see $\text{succ} =_{ib} \text{succ}_2$ and $\text{obs}(\text{succ}) = \text{obs}(\text{succ}_2)$.
- $\text{obs}$ forgets the order and number of times the arguments are interrogated (and O-innocence guarantees the same each time.)
- Similarly, strategies for left-strict and right-strict addition ($\not\equiv$ but $=_{ib}$) both $\text{obs}$ to
  \[\{|q_3q_1m_1, q_3q_2n_2, q_3(m + n)\}_{m, n \in \mathbb{N}}\].

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Concrete Representation of PCF

We can show

**Theorem**  
Let $\sigma$ and $\tau$ be innocent strategies over a game $A$. Then $\sigma \equiv_{ib} \tau$ iff $\text{obs}(\sigma) = \text{obs}(\tau)$.

Thus, combining this with the full abstraction results for PCF of the mid nineties, we have:

**Corollary**  
If $S$ and $T$ are terms of PCF then $S \equiv_{obs} T$ iff $\text{obs}(\llbracket S \rrbracket) = \text{obs}(\llbracket T \rrbracket)$. 
Observational Preorder

We can also give a characterisation of $\leq_{\text{obs}}$ in this setting.

**Definition**

Suppose $\sigma$ and $\tau$ are sets of O-view sets over an arena $A$. Write $\sigma \leq_{os} \tau$ if $\forall S \in \sigma \exists T \in \tau$ with $T \subseteq S$.

▶ We can show that $\text{obs}(\sigma) \leq_{os} \text{obs}(\tau)$ iff $\sigma \leq_{ib} \tau$ (so corresponds to $\leq_{\text{obs}}$.)
Definability

- No concrete representation of the image of obs (not effectively presentable, Loader.)
- We could describe a category where objects are games and arrows are sets of the form obs(σ) for an innocent strategy σ; this would be a fully abstract model.
- Can we define composition in terms of the O-view sets directly?
- Loader’s result places some restrictions on this.
Possible definition of composition:

**Definition**

Given sets of O-view sets $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ we define

$$\sigma; \tau = \{ \text{ovw}(s|_{A,C}) : \begin{align*} s &\in \text{int}(A, B, C) \land \\
\text{singlethreaded}(s) &\land \\
\text{complete}(s) &\land \\
\text{Oinnocent}(s|_{A,C}) &\land \\
\text{ovw}(s|_{B,C}) &\in \tau \land \\
(\forall q \in \text{init}(s|_{A,B}))(\text{ovw}(s|_{A,B} \upharpoonright q) \in \sigma) \end{align*} \}$$

But it is not yet clear which conditions on these sets are needed for associativity to work (and such that composition preserves such conditions.)
Questions?