$\mathcal{EL}$-ifying Ontologies

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Abstract. The OWL 2 profiles are fragments of the ontology language OWL 2 for which standard reasoning tasks are feasible in polynomial time. Many OWL ontologies, however, contain a typically small number of out-of-profile axioms, which may have little or no influence on reasoning outcomes. We investigate techniques for rewriting axioms into the EL and RL profiles of OWL 2. We have tested our techniques on both classification and data reasoning tasks with encouraging results.

1 Introduction

Description Logics (DLs) are a family of knowledge representation formalisms underpinning the W3C standard ontology languages OWL and OWL 2. State-of-the-art DL reasoners such as Pellet [18], JFact, FaCT^{++} [21], RacerPro [9], and HermiT [15] are highly-optimised for classification (i.e., the problem of computing all subsumption relationships between atomic concepts in an ontology) and have been exploited successfully in many applications. In a recent large-scale evaluation campaign, these reasoners exhibited excellent performance on a corpus of more than $1,000$ ontologies, as they were able to classify 75%-85% of the corpus in less than 10 seconds when running on stock hardware [8,3].

However, notwithstanding extensive research into optimisation techniques, DL reasoning remains a challenge in practice. Indeed, the aforementioned evaluation also revealed that many ontologies are still hard for reasoners to classify. Furthermore, due to the high worst-case complexity of reasoning, systems are inherently not robust, and even minor changes to ontologies can have a significant effect on performance. Finally, the limitations of DL reasoners become even more apparent when reasoning with ontologies and large datasets.

These issues have motivated a growing interest in lightweight DLs: weaker logics that enjoy more favourable computational properties. OWL 2 specifies several profiles (language fragments) based on lightweight DLs [14]: OWL 2 EL (or just EL) is based on the $\mathcal{EL}$ family of DLs; OWL 2 RL (or just RL) is based on Datalog; and OWL 2 QL (or just QL) is based on DL-Lite. Standard reasoning tasks, including classification and fact entailment (checking whether an ontology and a dataset entail a given ground atom), are feasible in polynomial time for all profiles, and many highly scalable reasoners have been developed [22,11,2,4].

Unfortunately, many ontologies fall outside the OWL 2 profiles, and we are forced to resort to a fully-fledged reasoner if a completeness guarantee is required.

Even in such cases, the majority of axioms typically still fall within one of the profiles, and the out-of-profile axioms may have little or no influence on the results of classification or query answering. Effectively detecting cases where the additional expressivity is used in a "harmless" way is, however, challenging, since even a single axiom can have a dramatic effect on reasoning outcomes.

In this paper we investigate techniques for rewriting out-of-profile axioms so as to improve reasoner performance. All rewritings are polynomial and preserve classification and fact entailment reasoning outcomes. In Section 3, we consider rewritings that are applicable to $\mathcal{SHOIQ}$—a DL that covers OWL DL and most of OWL 2 [10]—and that can transform non-EL axioms into EL by elimination of inverse roles and universal restrictions. If all non-EL axioms can be rewritten, then we can provide completeness guarantees using only an EL reasoner. Otherwise, the rewritings can still improve the performance of fully-fledged reasoners (e.g., by enabling the use of optimisation techniques that are applicable only in the absence of certain constructs) and/or the effectiveness of modular reasoners that combine profile-specific with OWL 2 reasoners, such as MORe [1].

In Section 4, we focus on Horn ontologies and consider rewritings into OWL 2 RL. The RL profile is tightly connected to Datalog, and hence existential restrictions $\exists R.C$ occurring positively in axioms are disallowed, unless C is a singleton nominal $\{o\}$. We show that when R fulfills certain conditions, such concepts $\exists R.C$ can be rewritten into existential restrictions over nominals as accepted in OWL 2 RL; we call such roles R *reuse-safe*. In the limit case where all roles are reuse-safe, the ontology can be polynomially rewritten into RL; if, additionally, the ontology contains no cardinality constraints, it can also be rewritten into EL. Furthermore, if only some roles are reuse-safe, they can be treated by (hyper-)tableau reasoners in an optimised way, potentially reducing the size of the constructed pre-models and improving reasoning times.

We have implemented our rewriting techniques and evaluated their effect on reasoning times over a large repository of ontologies. Our experiments reveal that our EL-ification techniques can lead to substantial improvements in classification times for both standard and modular reasoners. Furthermore, we show that many ontologies contain only reuse-safe roles and hence can be rewritten into RL; thus, highly scalable RL triple stores can be exploited for large-scale data reasoning.

2 Preliminaries

A *signature* consists of disjoint countably infinite sets of *individuals* N_{I}, *atomic concepts* N_{C} and *atomic roles* N_{R}. A *role* is an element of $N_{\mathsf{R}} \cup \{R^- | R \in N_{\mathsf{R}}\}$. The function $\mathsf{Inv}(\cdot)$ is defined over the set of roles as follows, where $R \in N_{\mathsf{R}}$: $\mathsf{Inv}(R) = R^-$ and $\mathsf{Inv}(R^-) = R$. An *RBox* $\mathcal{R}$ is a finite set of *RIAs* $R \sqsubseteq R'$ and *transitivity axioms* $\mathsf{Tra}(R)$, with R and R' roles. We denote with $\sqsubseteq_{\mathcal{R}}$ the minimal relation over roles in $\mathcal{R}$ s.t. $R \sqsubseteq_{\mathcal{R}} S$ and $\mathsf{Inv}(R) \sqsubseteq_{\mathcal{R}} \mathsf{Inv}(S)$ hold if $R \sqsubseteq S \in \mathcal{R}$. We define $\sqsubseteq^*_{\mathcal{R}}$ as the reflexive-transitive closure of $\sqsubseteq_{\mathcal{R}}$. A role R is *transitive* in $\mathcal{R}$ if there is a role S such that $S \sqsubseteq^*_{\mathcal{R}} R$, $R \sqsubseteq^*_{\mathcal{R}} S$ and either $\mathsf{Tra}(S) \in \mathcal{R}$ or $\mathsf{Tra}(\mathsf{Inv}(S)) \in \mathcal{R}$. A role R is *simple* in $\mathcal{R}$ if no transitive role S exists s.t. $S \sqsubseteq^*_{\mathcal{R}} R$.

The set of $\mathcal{SHOIQ}$ *concepts* is the smallest set containing A (atomic concept), $\top$ (top), $\bot$ (bottom), $\{o\}$ (nominal), $\neg C$ (negation), $C \sqcap D$ (conjunction), $C \sqcup D$ (disjunction), $\exists R.C$ (existential restriction), $\forall R.C$ (universal restriction), $\leqslant nS.C$ (at-most restriction), and $\geqslant nR.C$ (at-least restriction), for $A \in N_\mathsf{C}$, C and D $\mathcal{SHOIQ}$ concepts, $o \in N_\mathsf{I}$, R a role and S a simple role, and n a nonnegative integer. A *literal concept* is either atomic or the negation of an atomic concept. A *TBox* $\mathcal{T}$ is a finite set of GCIs $C \sqsubseteq D$ with C, D concepts. An *ABox* $\mathcal{A}$ is a finite set of assertions $C(a)$ (concept assertion), $R(a, b)$ (role assertion), $a \approx b$ (equality assertion), and $a \not\approx b$ (inequality assertion), with C a concept, R a role and a, b individuals. A *fact* is either a concept assertion $A(a)$ with A atomic, a role assertion, an equality assertion, or an inequality assertion. A knowledge base is a triple $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$. The semantics is standard [10].

We assume familiarity with standard conventions for naming DLs, and we just provide here a definition of the OWL 2 profiles. A $\mathcal{SHOIQ}$ KB is:

- EL if *(i)* it does not contain inverse roles, negation, disjunction, at-most restrictions and at-least restrictions; and *(ii)* every universal restriction appears in a GCI of the form $\top \sqsubseteq \forall R.C$.
- RL if each GCI $C \sqsubseteq D$ satisfies *(i)* C does not contain negation as well as universal, at-least, and at-most restrictions; *(ii)* D does not contain negation (other than $\bot$), union, existential restrictions (other than of the form $\exists R.\{o\}$), at-least restrictions, and at-most restrictions with $n > 1$.
- QL if it does not contain transitivity and for each GCI $C \sqsubseteq D$ *(i)* C is either atomic or $\exists R.\top$; *(ii)* D is of the form $\bigsqcap_{i=1}^{n} B_i$ with each B_i either a literal concept, or $\bot$, or of the form $\exists R.A$ with R a role and A either atomic or $\top$.

Classification of $\mathcal{K}$ is the task of computing all subsumptions $\mathcal{K} \models A \sqsubseteq B$ with $A \in N_\mathsf{C} \cup \{\top\}$, and $B \in N_\mathsf{C} \cup \{\bot\}$. Fact entailment is to check whether $\mathcal{K} \models \alpha$, for α a fact. Both problems are reducible to knowledge base unsatisfiability.

3 $\mathcal{EL}$-ification of $\mathcal{SHOIQ}$ ontologies

In this section, we propose techniques for transforming non-EL axioms into EL. Whenever possible, inverse roles are replaced with fresh symbols and the knowledge base is extended with axioms simulating their possible effects. At the same time, we attempt to transform positive occurrences of universal restrictions into negative occurrences of existential restrictions while inverting the relevant role. Note that our techniques do not rewrite disjunctions and cardinality restrictions; thus, ontologies containing such constructs will not be fully rewritten into EL.

3.1 Preprocessing

Before attempting to rewrite a $\mathcal{SHOIQ}$ knowledge base $\mathcal{K}$ into EL, we first bring $\mathcal{K}$ into a suitable normal form. Normalisation facilitates further rewriting steps, and it allows us to identify axioms with a direct correspondence in EL. For example, $A \sqcup B \sqsubseteq \neg\forall R.\neg C$ is equivalent to the EL axioms $A \sqsubseteq \exists R.C$ and

$$
\begin{aligned}
\Theta(\mathcal{T}) &= \bigcup_{\alpha \in \mathcal{T}} \Theta(\alpha) \\
\Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \forall R.B) &= \Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \alpha_B) \cup \{\alpha_B \sqsubseteq \forall R.B\} \\
\Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \forall R.\neg B) &= \Theta(\mathbb{C} \sqcap \alpha_B \sqsubseteq \mathbb{D}) \cup \{\exists R.B \sqsubseteq \alpha_B\} \\
\Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \bowtie nR.B) &= \Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \alpha_B) \cup \{\alpha_B \sqsubseteq \bowtie nR.B\} \\
\Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \geqslant nR.\neg B) &= \Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \geqslant nR.\alpha_B) \cup \{\alpha_B \sqcap B \sqsubseteq \top\} \\
\Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \leqslant nR.\neg B) &= \Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \leqslant nR.\alpha_B) \cup \{\top \sqsubseteq \alpha_B \sqcup B\} \\
\Theta(\mathbb{C} \sqsubseteq \mathbb{D} \sqcup \neg B) &= \Theta(\mathbb{C} \sqcap B \sqsubseteq \mathbb{D}) \\
\Theta(\alpha) &= \alpha \text{ for any other axiom } \alpha.
\end{aligned}
$$

Fig. 1. $\mathbb{C}$ is a conjunction of atomic concepts or $\top$, $\mathbb{D}$ is a disjunction of concepts C, $\forall R.C$, $\bowtie nR.C$ ($\bowtie \in \{\leqslant, \geqslant\}$) or $\bot$, with C literal, B atomic, and α_B is fresh.

$B \sqsubseteq \exists R.C$. Furthermore, although $A \sqsubseteq \exists R.\neg B$ is not equivalent to an EL axiom, it can be trivially transformed into the EL axioms $A \sqsubseteq \exists R.X$ and $X \sqcap B \sqsubseteq \bot$ by introducing a fresh symbol X. We therefore introduce a normal form that makes explicit those axioms that are neither logically equivalent to EL axioms, nor can be transformed into EL by means of the trivial introduction of fresh symbols.

Definition 1. *A GCI is normalised if it is of either of the following forms, where each $A_{(i)}$ is atomic or $\top$, B is atomic, each $C_{(j)}$ is atomic, $\bot$, or a nominal, R is a role, $n \geqslant 2$, and $m \geqslant 1$:*

(**N1**) $\prod_{i=1}^{n} A_i \sqsubseteq \bigsqcup_{j=1}^{m} C_j$; (**N2**) $A \sqsubseteq \exists R.A_i$; (**N3**) $\exists R.A \sqsubseteq A_i$

(**N4**) $A \sqsubseteq \geqslant n\, R.A_i$; (**N5**) $A \sqsubseteq \forall R.B$; (**N6**) $A \sqsubseteq \leqslant m\, R.A_i$

A knowledge base $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ is normalised if $\mathcal{A}$ has only facts and each GCI in $\mathcal{T}$ is normalised. Finally, $\mathcal{K}$ is Horn *if $m = 1$ in each axiom* ***N1*** *or* ***N6***.

Note that axioms of type **N2** and **N3**, as well as Horn axioms of type **N1**, are EL. To normalise a knowledge base $\mathcal{K}$, we proceed in two steps. First, we translate $\mathcal{K}$ into the following disjunctive normal form [15].

Definition 2. *A GCI is in disjunctive normal form (DNF) if it is of the form $\top \sqsubseteq \bigsqcup_{i=1}^{n} C_i$, where each C_i is of the form B, $\{o\}$, $\exists R.B$, $\forall R.B$, $\geqslant n\, R.B$, or $\leqslant n\, R.B$, for B a literal concept, R a role, and n a nonnegative integer. A knowledge base $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ is in DNF if all roles in $\mathcal{A}$ are atomic, all concept assertions in $\mathcal{A}$ contain only a literal concept, and each GCI in $\mathcal{T}$ is in DNF.*

DNF normalisation can be seen as a variant of the structural transformation, in which all complex concepts are "flattened" and negations are made explicit (see [15] for details). Once $\mathcal{K}$ is in DNF, we can further normalise by replacing concepts $\neg B$ in restrictions $\forall R.\neg B$, $\exists R.\neg B$, $\geqslant n\, R.\neg B$ and $\leqslant n\, R.\neg B$ with

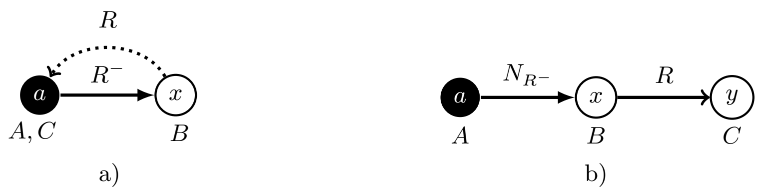

Fig. 2. A situation where rewriting away inverse roles leads to missing entailments

fresh symbols, bringing the remaining negated concepts to the left in GCIs, and introducing fresh symbols for all restrictions occurring in disjunctions.

Definition 3. *Let $\mathcal{K}$ be a KB. Then, $\Upsilon(\mathcal{K})$ is computed from $\mathcal{K}$ as follows:* (i) *apply the transformation in [15] to obtain $\mathcal{K}' = (\mathcal{R}', \mathcal{T}', \mathcal{A}')$ in DNF;* (ii) *replace each assertion $\alpha = \neg A(a)$ in $\mathcal{A}'$ with a fact $X_\alpha(a)$, with X_α fresh, and extend $\mathcal{T}'$ with $X_\alpha \sqcap A \sqsubseteq \bot$; and* (iii) *apply to $\mathcal{T}'$ the transformation Θ in Figure 1.*

The following proposition establishes the properties of normalisation.

Proposition 1. *Let $\mathcal{K}$ be a KB, then $\Upsilon(\mathcal{K})$ is normalised and can be computed in polynomial time in the size of $\mathcal{K}$. Furthermore, if $\mathcal{K}$ is EL, then so is $\Upsilon(\mathcal{K})$. Finally, $\mathcal{K}$ is satisfiable iff $\Upsilon(\mathcal{K})$ is satisfiable.*

3.2 Rewritable Inverse Roles

Satisfiability of $\mathcal{SHOIQ}$ KBs is NExpTime-complete, whereas for $\mathcal{SHOQ}$ it is ExpTime-complete; thus, in general, inverse roles cannot be faithfully eliminated from $\mathcal{SHOIQ}$ KBs by means of a polynomial transformation. The following example illustrates that an obstacle to rewritability is the interaction between inverses and at-most restrictions.

Example 1. Consider $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$, with $\mathcal{R} = \emptyset$, $\mathcal{A} = \{A(a)\}$, and $\mathcal{T}$ as follows:

$$\mathcal{T} = \{A \sqsubseteq \exists R^-.B; \quad B \sqsubseteq \exists R.C; \quad B \sqsubseteq \leqslant 1\, R.\top\}$$

Note that $\mathcal{K} \models C(a)$. In every model $(\Delta_\mathcal{I}, \cdot^\mathcal{I})$, object $a^\mathcal{I}$ must be R^--connected to some $x \in B^\mathcal{I}$ (due to the first axiom in $\mathcal{T}$); also, x must be R-connected to some $y \in C^\mathcal{I}$ (due to the second axiom). Then, for the last axiom to be satisfied, $a^\mathcal{I}$ and y must be identical; thus, $a^\mathcal{I} \in C^\mathcal{I}$. Figure 2 a) depicts such a model. Consider now $\mathcal{K}'$ obtained from $\mathcal{K}$ by replacing R^- with a fresh atomic role N_{R^-}. Then, $\mathcal{K}' \not\models C(a)$, and Figure 2 b) depicts a model of $\mathcal{K}'$ not satisfying $C(a)$. Extending $\mathcal{K}'$ with EL axioms to simulate the interaction between inverses and cardinality restrictions (and thus recover the missing entailment) seems infeasible. $\diamondsuit$

We next propose sufficient conditions for inverse roles to be rewritable in the presence of cardinality constraints. Our conditions ensure existence of a one-to-one correspondence between the *canonical* forest-shaped models of the original and rewritten KBs, and hence disallow cases such as Example 2.[3]

Definition 4. *Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be a normalised $\mathcal{SHOIQ}$ knowledge base. A (possibly inverse) role R is* generating *in $\mathcal{K}$ if there exists a role R' occurring in $\mathcal{T}$ in an axiom of type **N2** or **N4** such that $R' \sqsubseteq^*_{\mathcal{R}} R$.*

An inverse role S^- is rewritable *if for each $X \in \{S, S^-\}$ occurring in an axiom of type **N6** we have that $\mathsf{Inv}(X)$ is not generating in $\mathcal{K}$.*

Intuitively, roles R' in axioms **N2** or **N4** are those “inducing” the edges between individuals and their successors in a canonical model; then, a role R is generating if it is a super-role of one such R'. Our condition ensures that “backwards” edges in a canonical model of $\mathcal{K}$ (i.e., those induced by an inverse role) cannot invalidate an at-most cardinality restriction. In the limit case where all inverse roles in a $\mathcal{SHOIQ}$ KB are rewritable, we can faithfully eliminate inverses and rewrite the KB into $\mathcal{SHOQ}$ by means of a polynomial transformation.

Theorem 1. *Let $\mathcal{C}$ be the class of all normalised $\mathcal{SHOIQ}$ ontologies containing only rewritable inverse roles. Then, there exists a polynomial transformation mapping each $\mathcal{K} \in \mathcal{C}$ to an equisatisfiable $\mathcal{SHOQ}$ knowledge base.*[4]

Theorem 1 identifies a class of $\mathcal{SHOIQ}$ ontologies for which standard reasoning is feasible in ExpTime (in contrast to NExpTime). This result can also be exploited for optimisation: tableaux reasoners employ pairwise blocking techniques over $\mathcal{SHOIQ}$ ontologies, while they rely on more aggressive single blocking techniques for $\mathcal{SHOQ}$ inputs, which can reduce the size of pre-models.

3.3 The EL-ification Transformation

Before presenting our transformation formally, we provide two motivating examples. First, we show how a rewritable inverse role can be eliminated in the presence of cardinality constraints.

Example 2. Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be the following knowledge base:

$$\mathcal{R} = \{R \sqsubseteq T^-; \quad S \sqsubseteq T^-\}$$
$$\mathcal{T} = \{A \sqsubseteq \exists R.B; \quad A \sqsubseteq \exists S.C; \quad A \sqsubseteq \leqslant 1\, T^-.\top; \quad B \sqcap C \sqsubseteq D; \quad \exists R.D \sqsubseteq B\}$$
$$\mathcal{A} = \{A(a); \quad T(b, a)\}$$

[3] Roughly speaking, a forest-shaped model of a (normalised) knowledge base is canonical if every fact that holds in the model is “justified” by an axiom or assertion in the knowledge base. In particular, the result of unravelling a pre-model constructed by a (hyper-)tableau algorithm is a canonical forest-shaped model.

[4] Theorem 1 is given here for presentation purposes: it follows as a corollary of Theorem 3, which we state only after presenting our transformations.

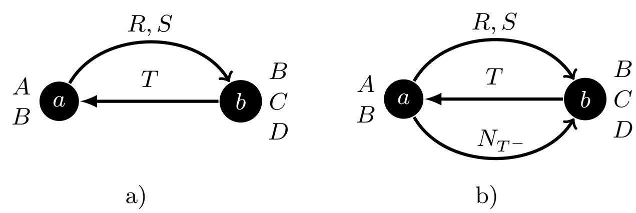

Fig. 3. Rewriting away inverse roles in a KB with cardinality constraints

Figure 3(a) depicts a canonical model for $\mathcal{K}$. The facts entailed by $\mathcal{K}$ are precisely those that hold in the canonical model. By Definition 4, T^- is rewritable since T is not generating; however, it does not suffice to replace T^- with a fresh N_{T^-} since the resulting KB will no longer entail the facts $R(a,b)$, $S(a,b)$, $B(b)$, $C(b)$, and $D(b)$. Instead, we can extend $\mathcal{A}$ with $T^-(a,b)$, and only then replace T^- with N_{T^-}. The canonical model of the resulting KB is given in Figure 3(b). $\diamondsuit$

Next, we show how axioms of type **N5**, which involve a universal restriction, can be replaced with EL axioms of type **N3** if the relevant roles are not generating.

Example 3. Consider $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ where $\mathcal{R} = \{R \sqsubseteq S^-\}$, $\mathcal{A} = \{A(a); S(a,b)\}$, and $\mathcal{T}$ is defined as follows:

$$\mathcal{T} = \{A \sqsubseteq \forall S.B; \quad B \sqsubseteq \exists R.C; \quad \exists S.B \sqsubseteq D; \quad C \sqcap D \sqsubseteq \bot\}$$

Clearly, $\mathcal{K}$ is unsatisfiable. Furthermore, it does not contain axioms **N6**, and hence S^- is rewritable. In a first step, we extend $\mathcal{K}$ with logically redundant axioms, which make explicit information that may be lost when replacing inverses with fresh symbols. Thus, we extend $\mathcal{T}$ with $\exists S^-.A \sqsubseteq B$, and $B \sqsubseteq \forall S^-.D$; furthermore, we extend $\mathcal{R}$ with $R^- \sqsubseteq S$; and finally, $\mathcal{A}$ with the assertion $S^-(b,a)$.

An important observation is that S is not generating. As a result, we can dispense with axiom $A \sqsubseteq \forall S.B$. Then we replace S^- with a fresh symbol N_{S^-} and R^- with N_{R^-}. The resulting $\mathcal{K}' = (\mathcal{R}', \mathcal{T}', \mathcal{A}')$ is as follows:

$$\mathcal{R}' = \{R \sqsubseteq N_{S^-}; \quad N_{R^-} \sqsubseteq S\}$$
$$\mathcal{T}' = \{\exists N_{S^-}.A \sqsubseteq B; \quad B \sqsubseteq \exists R.C; \quad \exists S.B \sqsubseteq D; \quad B \sqsubseteq \forall N_{S^-}.D; \quad C \sqcap D \sqsubseteq \bot\}$$
$$\mathcal{A}' = \{A(a); \quad S(a,b); \quad N_{S^-}(b,a)\}$$

$\mathcal{K}'$ is unsatisfiable; furthermore it is in EL except for axiom $B \sqsubseteq \forall N_{S^-}.D$. This axiom cannot be dispensed with since S^- is generating, and hence it is needed to propagate information along N_{S^-}-edges in a canonical model. $\diamondsuit$

We next present our transformation. For simplicity, we first restrict ourselves to $\mathcal{ALCHOIQ}$ KBs; later on, we discuss issues associated with transitivity axioms and show how our techniques extend to $\mathcal{SHOIQ}$.

Definition 5. *Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be a normalised $\mathcal{ALCHOIQ}$ knowledge base. The knowledge base $\Xi(\mathcal{K}) = (\mathcal{R}', \mathcal{T}', \mathcal{A}')$ is obtained as follows:*

1. Extension: *the knowledge base* $\mathcal{K}_e = (\mathcal{R}_e, \mathcal{T}_e, \mathcal{A}_e)$ *is defined as follows:*
   - $\mathcal{R}_e$ *extends* $\mathcal{R}$ *with an axiom* $\mathsf{Inv}(R) \sqsubseteq \mathsf{Inv}(S)$ *for each* $R \sqsubseteq S$ *in* $\mathcal{R}$*;*
   - $\mathcal{T}_e$ *extends* $\mathcal{T}$ *with the following axioms:*
     - *an axiom* $\exists \mathsf{Inv}(R).A \sqsubseteq B$ *for each axiom* $A \sqsubseteq \forall R.B$ *in* $\mathcal{T}$ *where either* $\mathsf{Inv}(R)$ *is generating, or* R *is not generating; and*
     - *an axiom* $A \sqsubseteq \forall \mathsf{Inv}(R).B$ *for each axiom* $\exists R.A \sqsubseteq B$ *in* $\mathcal{T}$ *where* $\mathsf{Inv}(R)$ *is generating;*
   - $\mathcal{A}_e$ *extends* $\mathcal{A}$ *with an assertion* $R^-(b, a)$ *for each* $R(a, b) \in \mathcal{A}$*.*
2. $\mathcal{EL}$-ification: $\Xi(\mathcal{K}) = (\mathcal{R}', \mathcal{T}', \mathcal{A}')$ *is obtained from* $\mathcal{K}_e$ *by first removing all axioms* $A \sqsubseteq \forall R.B$ *in* $\mathcal{T}_e$ *where* R *is not generating in* $\mathcal{T}$ *and then replacing each occurrence of an inverse role that is rewritable in* $\mathcal{K}_e$ *with a fresh role.*

The extension step only adds redundant information, and hence $\mathcal{K}$ and $\mathcal{K}_e$ are equivalent. Making such information explicit is crucial for the subsequent EL-ification step, where ineffectual axioms involving universal restrictions are removed, and rewritable inverse roles are replaced with fresh atomic roles. The following theorem extablishes the properties of the transformation.

Theorem 2. *Let* $\mathcal{K}' = \Xi(\mathcal{K})$*. The following conditions hold:*

1. $\mathcal{K}'$ *is satisfiable iff* $\mathcal{K}$ *is satisfiable;*
2. $\mathcal{K}'$ *is of size polynomial in the size of* $\mathcal{K}$*;*
3. *If* $\mathcal{K}$ *satisfies all of the following properties, then* $\mathcal{K}'$ *is EL:*
   - $\mathcal{K}$ *is Horn and does not contain axioms of type* ***N4*** *or* ***N6****;*
   - *each axiom* ***N5*** *satisfies either* $A = \top$*, or* R *is not generating.*
   - *each axiom* ***N3*** *satisfies either* $A = \top$*, or* $\mathsf{Inv}(R)$ *is not generating.*

Note that the third condition in the theorem establishes sufficient conditions on $\mathcal{K}$ for the transformed knowledge base $\mathcal{K}'$ to be in EL. A simple case is when $\mathcal{K}$ is in the QL profile of OWL 2, in which case the transformed KB is guaranteed to be in EL. An interesting consequence of this result is that highly optimised EL reasoners, such as ELK, can be exploited for classifying QL ontologies.

Corollary 1. *If* $\mathcal{K}$ *is a normalised QL knowledge base, then* $\Xi(\mathcal{K})$ *is in EL.*

In many cases our transformation may only succeed in partially rewriting a knowledge base into EL (c.f. Example 3). Even in these cases, our techniques can have substantial practical benefits (see Evaluation section). As discussed in Section 3.2, in the absence of inverse roles (hyper-)tableau reasoners may exploit more aggressive blocking techniques. Furthermore, modular reasoning systems such as MORe, which are designed to behave better for ontologies with a large EL subset, are substantially enhanced by our transformations.

3.4 Dealing with Transitivity Axioms

As shown by the following example, the transformation in Definition 5 is not applicable to knowledge bases containing transitivity axioms in the RBox.

Example 4. Consider $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ with $\mathcal{R} = \{R \sqsubseteq R^-; \mathsf{Tra}(R)\}$, $\mathcal{A} = \{A(a)\}$, and $\mathcal{T} = \{A \sqsubseteq \exists R.B; A \sqsubseteq C; \exists R^-.C \sqsubseteq D\}$. Let $\mathcal{K}' = \Xi(\mathcal{K})$, where we assume that the transitivity axiom $\mathsf{Tra}(R)$ stays unmodified in $\mathcal{K}'$. More precisely, $\mathcal{A}' = \mathcal{A}$, and $\mathcal{R}' = \{R \sqsubseteq N_{R^-}; N_{R^-} \sqsubseteq R; \mathsf{Tra}(R)\}$, and $\mathcal{T}' = \{A \sqsubseteq \exists R.B; A \sqsubseteq C; \exists N_{R^-}.C \sqsubseteq D; C \sqsubseteq \forall R.D\}$. It can be checked that $\mathcal{K} \models D(a)$, but $\mathcal{K}' \not\models D(a)$; thus, a relevant entailment is lost. An attempt to recover this entailment by making N_{R^-} transitive does not solve the problem. ♢

To address this issue, we eliminate transitivity before applying our transformation in Definition 5. Standard techniques for eliminating transitivity axioms in DLs (e.g., [15]) have the effect of introducing non-Horn axioms. As a result, a Horn knowledge base may not remain Horn after eliminating transitivity. Therefore, we propose a modification of the standard technique that preserves Horn axioms and which is compatible with our transformation in Definition 5.

Definition 6. *Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be a normalised $\mathcal{SHOIQ}$ knowledge base. For each axiom of the form $A \sqsubseteq \forall R.B$ in $\mathcal{T}$ and each transitive sub-role S of R in $\mathcal{R}$, let $X^S_{R,B}$ be an atomic concept uniquely associated to R, B, S. Furthermore, for each axiom $\exists R.A \sqsubseteq B$ in $\mathcal{T}$ and each transitive sub-role S of R in $\mathcal{R}$, let $Y^S_{R,B}$ be a fresh atomic concept uniquely associated to R, B, S.*

The knowledge base $\Omega(\mathcal{K}) = (\mathcal{R}', \mathcal{T}', \mathcal{A}')$ is defined as follows: (i) *$\mathcal{R}'$ is obtained from $\mathcal{R}$ by removing all transitivity axioms;* (ii) *$\mathcal{T}'$ is obtained from $\mathcal{T}$ by adding axioms $A \sqsubseteq \forall S.X^S_{R,B}$, $X^S_{R,B} \sqsubseteq \forall S.X^S_{R,B}$, and $X^S_{R,B} \sqsubseteq \forall S.B$ for each concept $X^S_{R,B}$, and axioms $\exists S.A \sqsubseteq Y^S_{R,B}$, $\exists S.Y^S_{R,B} \sqsubseteq Y^S_{R,B}$, and $\exists S.Y^S_{R,B} \sqsubseteq B$ for each concept $Y^S_{R,B}$; finally,* (iii) *$\mathcal{A}' = \mathcal{A}$.*

Lemma 1 establishes the properties of transitivity elimination, and Theorem 3 shows that our techniques extend to a $\mathcal{SHOIQ}$ knowledge base $\mathcal{K}$ by first applying Ω to $\mathcal{K}$ and then Ξ to the resulting KB.

Lemma 1. *Let $\mathcal{K}$ be a normalised $\mathcal{SHOIQ}$ KB. The following holds:*

1. *$\Omega(\mathcal{K})$ is satisfiable iff $\mathcal{K}$ is .*
2. *$\Omega(\mathcal{K})$ is a normalised $\mathcal{ALCHOIQ}$; furthermore, $\Omega(\mathcal{K})$ is Horn iff $\mathcal{K}$ is Horn.*
3. *$\Omega(\mathcal{K})$ can be computed in time polynomial in the size of $\mathcal{K}$.*
4. *if $\mathcal{K}$ is EL, then so is $\Omega(\mathcal{K})$.*
5. *If an inverse role R^- is rewritable in $\mathcal{K}$, then it is also rewritable in $\Omega(\mathcal{K})$.*

Theorem 3. *Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be a normalised $\mathcal{SHOIQ}$ knowledge base, and let $\mathcal{K}' = \Xi(\Omega(\mathcal{K}))$. Then, $\mathcal{K}'$ satisfies all properties $1-3$ in Theorem 2.*

4 Reuse-safe Roles

We next focus on Horn ontologies, and show how to further optimise reasoning by identifying roles that are "reuse-safe", and which can thus be treated by (hyper-)tableau reasoners in a more optimised way. Each application of an axiom **N2** or **N4** triggers the generation of fresh individuals in a (hyper-)tableau. If these

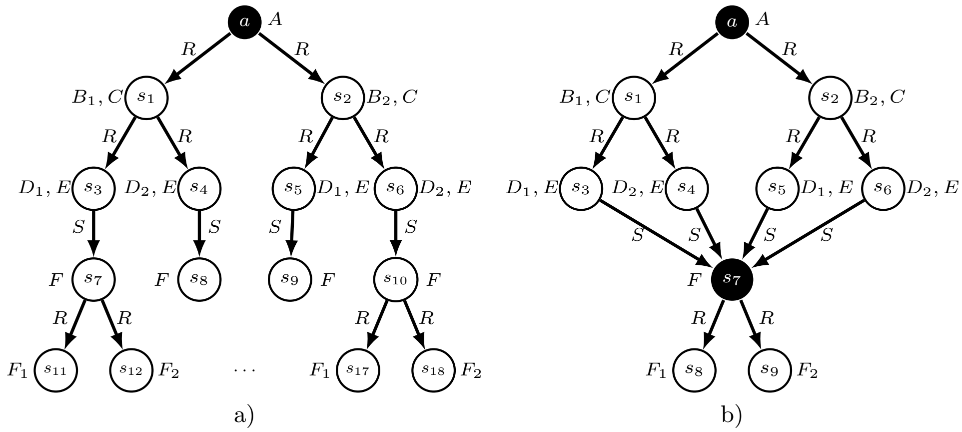

Fig. 4. Decreasing model size by reusing individuals

axioms involve a reuse-safe role, however, we show that reasoners can associate with each such axiom a single fresh nominal, which can be deterministically "reused" whenever the axiom is applied during construction of a pre-model. This may reduce the size of pre-models, and improve reasoning times. Our technique extends the results in [16], which show that for EL ontologies all roles admit reuse, and pre-model size can be bounded polynomially.

Example 5. Consider the following knowledge base $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ where $\mathcal{R} = \emptyset$, $\mathcal{A} = \{A(a)\}$, and $\mathcal{T}$ consists of the following axioms:

$$\begin{array}{llllll} A \sqsubseteq \exists R.B_1 & C \sqsubseteq \exists R.D_1 & A \sqsubseteq \forall R.C & E \sqsubseteq \exists S.F & F \sqsubseteq \exists R.F_2 & B_1 \sqcap B_2 \sqsubseteq \bot \\ A \sqsubseteq \exists R.B_2 & C \sqsubseteq \exists R.D_2 & C \sqsubseteq \forall R.E & F \sqsubseteq \exists R.F_1 & F_1 \sqcap F_2 \sqsubseteq \bot & D_1 \sqcap D_2 \sqsubseteq \bot \end{array}$$

Since R is generating and $\mathcal{K}$ has no inverses, we have $\Xi(\mathcal{K}) = \mathcal{K}$. Figure 4 a) depicts a canonical model of $\mathcal{K}$. Role S is reuse-safe since it is not "affected" by non-EL axioms involving universal restrictions. We can exploit this fact to "fold" the model by identifying all nodes with an S-predecessor to a single fresh nominal, as in Figure 4 b). In this way, we can reduce model size. ♢

Definition 7. *Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be a normalised Horn KB. A role R in $\mathcal{K}$ is* reuse-safe *if either R is not generating or the following conditions hold:*

- *Each axiom $A \sqsubseteq \leqslant 1\, S.B$ in $\mathcal{K}$ satisfies $R \not\sqsubseteq^*_{\mathcal{R}} S$ and $R \not\sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(S)$;*
- *Each axiom $A \sqsubseteq \forall S.B$ in $\mathcal{K}$ with $A \neq \top$ satisfies $R \not\sqsubseteq^*_{\mathcal{R}} S$; and*
- *Each axiom $\exists S.A \sqsubseteq B$ in $\mathcal{K}$ with $A \neq \top$ satisfies $R \not\sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(S)$.*

If a generating role R is reuse-safe, we can ensure that R-edges in a canonical model of $\mathcal{K}$ are irrelevant to the satisfaction of non-EL axioms in $\mathcal{K}$. To ensure that (hyper-)tableau algorithms will exploit reuse-safety, and construct succinct "folded" canonical models such as the one in Example 5, we provide the following transformation, which makes the relevant nominals explicit.

Definition 8. *Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be a normalised Horn knowledge base.*

For each each positive occurrence of a concept $\exists R.B$ (resp. $\geqslant nR.B$) in $\mathcal{K}$ with R reuse-safe, let $c_{R,B}$ (resp. $c_{i,R,B}$ for $1 \leqslant i \leqslant n$) be fresh individual(s). Then, $\Psi(\mathcal{K})$ is KB obtained from $\mathcal{K}$ by:

- *replacing each axiom in $\mathcal{T}$ of the form $A \sqsubseteq \exists R.B$, where R is safe, by $A \sqsubseteq \exists R.\{c_{R,B}\}$ and adding the fact $B(c_{R,B})$ to $\mathcal{A}$, and by*
- *replacing each axiom of the form $A \sqsubseteq \geqslant n\, R.B$, where R is safe, by all $\alpha \in \{A \sqsubseteq \exists R.\{c_{i,R,B}\}, \{c_{j,R,B}\} \sqcap \{c_{k,R,B}\} \sqsubseteq \bot \mid 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j < k \leqslant n\}$ and adding the facts $B(c_{i,R,B})$, for $1 \leqslant i \leqslant n$, to $\mathcal{A}$.*

The following theorem establishes the correctness of our transformation.

Theorem 4. *$\mathcal{K}$ is satisfiable iff $\Psi(\mathcal{K})$ is satisfiable.*

In practice, system developers can achieve the same goal as our transformation by making their implementations sensitive to reuse-safe roles: to satisfy an axiom involving existential or an at-least restrictions over such role, a system should reuse a suitable distinguished individual instead of generating a fresh one.

We next analyse the case where all roles in a Horn KB $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ are reuse-safe. In this case, we can show that $\Psi(\mathcal{K})$ is in RL. Furthermore, we can identify a new efficiently-recognisable class of DL knowledge bases that contains both EL and RL, and for which both classification and fact entailment are feasible in polynomial time.

Theorem 5. *Let $\mathcal{C}$ be the class of Horn knowledge bases $\mathcal{K}$ such that all roles in $\mathcal{K}$ are reuse-safe. Then, the following conditions hold:*

1. *Checking whether a $\mathcal{SHOIQ}$ KB $\mathcal{K}$ is in $\mathcal{C}$ is feasible in polynomial time;*
2. *Every EL and RL knowledge base is contained in $\mathcal{C}$;*
3. *$\Psi(\mathcal{K})$ is an RL knowledge base for each $\mathcal{K} \in \mathcal{C}$; and*
4. *Classification and fact entailment in $\mathcal{C}$ are feasible in polynomial time.*

Finally, it is worth emphasising that, although the transformations Ψ in Definition 8 and Ξ in Section 3 are very different and serve rather orthogonal purposes, they are connected in the limit case where all roles are reuse-safe and the ontology does not contain cardinality restrictions.

Proposition 2. *Let $\mathcal{K}$ be a normalised Horn KB that does not contain axioms* ***N4*** *or* ***N6****. Then, $\Xi(\Omega(\mathcal{K}))$ is EL iff all roles in $\mathcal{K}$ are reuse-safe.*

5 Evaluation

We have implemented the transformations described in Sections 3 and 4, and we have performed a range of classification and data reasoning experiments over both realistic ontologies and standard benchmarks.

5.1 Classification Experiments

For our input data, we used the OWL 2 ontologies in the Oxford Ontology Repository,[5] which contains 793 realistic ontologies, as well as a "hard" version

[5] http://www.cs.ox.ac.uk/isg/ontologies/

Ontology ID	00018	00352	00448	00461	00463	00470	00660	Fly
Original (HermiT)	76.787	18.679	68.545	2.260	t-out	286.89	102.80	840.014
Normalised (HermiT)	30.730	7.235	41.529	11.768	t-out	318.60	123.71	807.167
EL-ified (HermiT)	9.006	7.953	21.395	1.801	651.884	54.40	17.62	17.361
Original (MORe)	42.292	15.095	5.949	2.515	t-out	258.53	99.93	844.639
Normalised (MORe)	10.521	3.195	5.061	11.442	t-out	293.55	85.42	819.640
EL-ified (MORe)	3.0792	2.650	5.019	1.310	694.046	3.48	17.58	17.409

Table 1. Classification times for representative ontologies (in seconds).

of the FlyAnatomy ontology, which is not yet in the repository. Several of the test ontologies contain a small number of axioms exploiting constructs (such as complex RIAs) not available in $\mathcal{SHOIQ}$; in these cases we tested filtered versions of the ontologies where such axioms have been removed.

We tested classification times for the latest versions of HermiT (v.1.3.8) and MORe (v.0.1.5) using their standard settings. All experiments were performed on a laptop with 16 GB RAM and Intel Core 2.9 GHz processor running Java v.1.7.0_21, with a timeout set to 3,000s.

EL-ification Experiments. Out of the 793 ontologies in the corpus, we selected those 70 that contain inverse roles, and which HermiT takes at least 1s to classify. For each test ontology $\mathcal{K}$ we have computed a normalised version $\Upsilon(\mathcal{K})$ and an EL-ified version $\mathcal{K}'$ (see Section 3), and have compared classification times for HermiT and MORe on each version.

We found that 50 out of the 70 test ontologies contained only rewritable inverse roles, which could be successfully eliminated using our transformations, and 4 of these ontologies could be fully rewritten into EL. Of these 50 ontologies, 6 could not be classified by HermiT even after EL-ification; however, HermiT succeeded on 2 EL-ified ontologies that could not be classified in their original form. For the remaining 42 ontologies, normalisation alone leads to a slight deterioration in average performance due to the introduction of new class names (which HermiT must classify); however, EL-ification improves HermiT's performance by an average factor of approximately 3. We believe that this improvement is due to HermiT being able to use single blocking instead of pairwise blocking.

Like HermiT, MORe failed on 8 of the original ontologies, but succeeded on two of these after EL-ification. With the remaining 42, as for HermiT, normalisation alone leads to a slight deterioration in performance, but EL-ification improves performance by an average factor of approximately 6. The larger improvement can be explained by the fact that many axioms are rewritten into EL, and hence MORe can delegate a greater part of the computational work to ELK. Table 1 presents results for some representative cases.

Finally, as already mentioned, our test corpus contains 20 ontologies with non-rewritable inverse roles. As expected, in these cases we obtained no consistent improvement since the presence of inverses forces HermiT to use pairwise blocking; furthermore, in some cases the transformation negatively impacts performance, as it adds a substantial number of axioms to simulate the effect of inverse roles. Hence, it seems that our techniques are clearly beneficial only when all inverse roles are rewritable.

Reuse Safety. From the 793 ontologies in the corpus, we identified 174 Horn ontologies that do not fall within any of the OWL 2 profiles. We have applied our transformation in Definition 8 to these ontologies and found that 53 do not contain unsafe roles and hence are rewritten into RL. Furthermore, we found that in the remaining ontologies 89% of the roles were reuse-safe, on average. We have tested classification times with HermiT over the transformed ontologies, but found that the transformation had a negative impact on performance. This is explained by the fact that our transformation introduces nominals. In the presence of nominals, HermiT disables *anywhere blocking*—a powerful technique that makes nodes blockable by any other node in the tableau (and not just by its ancestors). As mentioned in Section 4, it would be more effective to implement safe reuse as a modification of HermiT's calculus; this, however, implies non-trivial modifications to the core of the reasoner, which is left for future work.

5.2 Data Reasoning Experiments

We have used the standard LUBM benchmark, which comes with an ontology about academic departments and a dataset generator parameterised by the number of universities for which data is generated (LUBM(n) denotes the dataset for n universities). The LUBM ontology is not in RL, as it contains axioms of type **N2**; however, all roles in LUBM are reuse-safe and hence we rewrote it into RL using the transformation in Definition 8. For each dataset, we recorded the times needed to compute the instances of all atomic concepts in the ontology. We compared HermiT over the original ontology and the RL reasoner RDFox[6] over the transformed ontology. HermiT took 3.7s for LUBM(1), and timed out for LUBM(5). In contrast RDFox only required 0.2s for LUBM(1), 1.5s for LUBM(10), and 7.4s for LUBM(20). These results suggest the clear benefits of transforming an ontology to RL and exploiting highly scalable reasoners such as RDFox.

6 Related Work

The observation that many ontologies consist of a large EL "backbone" and a relatively small number of non-EL axioms is exploited by the modular reasoner MORe [1] to delegate the bulk of the classification work to EL reasoner ELK [11]. Modular reasoning techniques, however, are sensitive to syntax and all non-EL axioms (as well as those "depending" on them) must be processed by a fully-fledged OWL reasoner. Ren et al. propose a technique for approximating an OWL ontology into EL [17]; this approximation, however, is incomplete for classification and hence valid subsumptions might be lost.

Several techniques for inverse role elimination in DL ontologies have been developed. Ding et al. [7] propose a polynomial reduction from $\mathcal{ALCI}$ into $\mathcal{ALC}$,

[6] http://www.cs.ox.ac.uk/isg/tools/RDFox/

which is then extended in [6] to $\mathcal{SHOI}$. Similarly, Song et al. [19] propose a polynomial reduction from $\mathcal{ALCHI}$ to $\mathcal{ALCH}$ KBs to optimise classification. In all of these approaches inverse roles are replaced with fresh symbols and new axioms are introduced to compensate for the loss of implicit inferences. These approaches, however, are not applicable to KBs with cardinality restrictions; furthermore, inverse role elimination heavily relies on the introduction of universal restrictions, and hence they are not well-suited for EL-ification. Calvanese et al. [5] propose a transformation from $\mathcal{ALCFI}$ knowledge bases to $\mathcal{ALC}$ which is sound and complete for classification; this technique exhaustively introduces universal restrictions to simulate at-most cardinality restrictions and inverse roles, and hence it is also not well-suited for EL-ification; furthermore, this technique is not applicable to knowledge bases with transitive roles or nominals. Finally, Lutz et al. study rewritability of first-order formulas into EL as a decision problem [13]; the rewritings studied in [13], however, require preservation of logical equivalence, whereas ours preserve satisfiability.

The techniques described in Section 4 extend the so-called combined approach to query answering in EL [12,20]. They are also related to are strongly related to individual reuse optimisations [16], where to satisfy existential restrictions a (hyper-)tableau reasoner tries to reuse an individual from the model constructed thus far. Individual reuse, however, may introduce non-determinism in exchange for a smaller model: if the reuse fails (i.e., a contradiction is derived), the reasoner must backtrack and introduce a fresh individual. In contrast, in the case of reuse-safe roles reuse can be done *deterministically* and hence model size is reduced without the need of backtracking.

Finally, Zhou et. al use a very similar transformation as ours to strengthen ontologies and overestimate query answers [23]. It follows from Theorem 5 that the technique in [23] leads to exact answers to atomic queries for Horn ontologies where all roles are reuse-safe.

7 Conclusions and Future Work

In this paper, we have proposed novel techniques for rewriting ontologies into the OWL 2 profiles. Our techniques are easily implementable as preprocessing steps in DL reasoners, and can lead to substantial improvements in reasoning times. Furthermore, we have established sufficient conditions for ontologies to be polynomially rewritable into the EL and RL profiles. Thus, for the class of ontologies satisfying our conditions reasoning becomes feasible in polynomial time. There are many avenues to explore for future work. For example, we will investigate extensions of our EL-ification techniques that are capable of rewriting away disjunctive axioms. Furthermore, we are planning to implement safe reuse in HermiT and evaluate the impact of this optimisation on classification.

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A Appendix

Theorem 2. 1.: $\mathcal{K}' = \Xi(\mathcal{K})$ is satisfiable iff $\mathcal{K}$ is satisfiable:
Proof. We show a one-to-one correspondence between (canonical forest) models of $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ and $\mathcal{K}' = (\mathcal{R}', \mathcal{T}', \mathcal{A}')$.
Let f be a function which maps roles in $\mathcal{K}'$ to roles in $\mathcal{K}$ as follows:

$$f(R) = \begin{cases} Q^-, & \text{when } R = NQ \\ R, & \text{otherwise .} \end{cases}$$

Also, let g be the inverse of f (f is bijective): g maps roles in $\mathcal{K}$ to roles in $\mathcal{K}'$ as follows:

$$g(R) = \begin{cases} NQ, & \text{when } R = Q^- \text{ and } Q^- \text{ is rewritable} \\ R, & \text{otherwise .} \end{cases}$$

$\Rightarrow$: Let $\mathcal{I}_1 = (\Delta^{\mathcal{I}_1}, \cdot^{\mathcal{I}_1})$ be a model for $\mathcal{K}$ (not necessarily forest-shaped). We show how to construct from $\mathcal{I}_1$ a model $\mathcal{I}_2 = (\Delta^{\mathcal{I}_2}, \cdot^{\mathcal{I}_2})$ for $\mathcal{K}'$.
Let $\Delta^{\mathcal{I}_2} = \Delta^{\mathcal{I}_1}$ and $\cdot^{\mathcal{I}_2}$ be defined as follows:

- $s^{\mathcal{I}_2} = s^{\mathcal{I}_1}$, for every $s \in N_\mathsf{I}$
- $A^{\mathcal{I}_2} = A^{\mathcal{I}_1}$, for every $A \in N_\mathsf{C}$
- $R^{\mathcal{I}_2} = R^{\mathcal{I}_1}$, for every $R \in N_\mathsf{R}$
- $NR^{\mathcal{I}_2} = (R^-)^{\mathcal{I}_1}$, for every NR in the signature of $\mathcal{K}'$.

Then, the following hold:

1. $(x, y) \in R^{\mathcal{I}_2}$ implies $(x, y) \in (f(R))^{\mathcal{I}_1}$, for every R in the signature of $\mathcal{K}'$, and
2. $(x, y) \in R^{\mathcal{I}_1}$ implies $(x, y) \in (g(R))^{\mathcal{I}_2}$, for every R in the signature of $\mathcal{K}$.

We show that $\mathcal{I}_2 = (\Delta^{\mathcal{I}_2}, \cdot^{\mathcal{I}_2})$ satisfies every axiom from $\mathcal{K}'$:

- Assume $R \sqsubseteq S \in \mathcal{R}'$, and $(x, y) \in R^{\mathcal{I}_2}$. Then, $f(R) \sqsubseteq^*_{\mathcal{R}} f(S) \in \mathcal{R}$ and $(x, y) \in (f(R))^{\mathcal{I}_1}$. Thus, $(x, y) \in (f(S))^{\mathcal{I}_1}$, which implies $(x, y) \in g(f(S))^{\mathcal{I}_2} = S^{\mathcal{I}_2}$.
- Assume $\exists R.A \sqsubseteq B \in \mathcal{T}'$, $(x, y) \in R^{\mathcal{I}_2}$, and $y \in A^{\mathcal{I}_2}$. Then, either:
  - $\exists f(R).A \sqsubseteq B \in \mathcal{T}$. As $(x, y) \in (f(R))^{\mathcal{I}_1}$ and $y \in A^{\mathcal{I}_1}$, it follows that: $x \in B^{\mathcal{I}_1} \Rightarrow x \in B^{\mathcal{I}_2}$, or
  - $A \sqsubseteq \forall \mathsf{Inv}(f(R)).B \in \mathcal{T}$. As $(y, x) \in (\mathsf{Inv}((f(R)))^{\mathcal{I}_1}$ and $y \in A^{\mathcal{I}_1}$, it follows that: $x \in B^{\mathcal{I}_1} \Rightarrow x \in B^{\mathcal{I}_2}$.
- Assume $A \sqsubseteq \forall R.B \in \mathcal{T}'$, $x \in A^{\mathcal{I}_2}$, and $(x, y) \in R^{\mathcal{I}_2}$. Then, either:
  - $\exists \mathsf{Inv}(f(R)).A \sqsubseteq B \in \mathcal{T}$. As $(y, x) \in (\mathsf{Inv}((f(R)))^{\mathcal{I}_1}$ and $y \in A^{\mathcal{I}_1}$, it follows that: $x \in B^{\mathcal{I}_1} \Rightarrow x \in B^{\mathcal{I}_2}$, or
  - $A \sqsubseteq \forall f(R).B \in \mathcal{T}$. As $(y, x) \in (f(R))^{\mathcal{I}_1}$ and $y \in A^{\mathcal{I}_1}$, it follows that: $x \in B^{\mathcal{I}_1} \Rightarrow x \in B^{\mathcal{I}_2}$.

- Assume $A \sqsubseteq \geqslant nR.B \in \mathcal{T}'$ and $x \in A^{\mathcal{I}_2}$. Then, $A \sqsubseteq \geqslant n(f(R)).B \in \mathcal{T}$ and $x \in A^{\mathcal{I}_1}$. As $(x,y) \in (f(R))^{\mathcal{I}_1}$ implies $(x,y) \in R^{\mathcal{I}_2}$ and $y \in B^{\mathcal{I}_1}$ implies $y \in B^{\mathcal{I}_2}$, it follows that $x \in (\geqslant nR.B)^{\mathcal{I}_2}$.
- Assume $A \sqsubseteq \leqslant nR.B \in \mathcal{T}'$ and $x \in A^{\mathcal{I}_2}$. Then, $A \sqsubseteq \leqslant n(f(R)).B \in \mathcal{T}$, $x \in A^{\mathcal{I}_1}$, and consequently $x \in (\leqslant n(f(R)))^{\mathcal{I}_1}$. We have that $(x,y) \in R^{\mathcal{I}_2}$ implies $(x,y) \in (f(R))^{\mathcal{I}_1}$ and $y \in B^{\mathcal{I}_2}$ implies $y \in B^{\mathcal{I}_1}$. Then, $x \notin (\geqslant nR.B)^{\mathcal{I}_2}$ implies $x \notin (\geqslant n(f(R)).B)^{\mathcal{I}_1}$ – contradiction. Thus, $x \in (\geqslant nR.B)^{\mathcal{I}_2}$.
- Assume $\bigsqcap A_i \sqsubseteq \bigsqcup B_j \in \mathcal{T}'$ and $x \in (\bigsqcap A_i)^{\mathcal{I}_2}$. Then, $\bigsqcap A_i \sqsubseteq \bigsqcup B_j \in \mathcal{T}$, $x \in (\bigsqcap A_i)^{\mathcal{I}_1}$, and thus $x \in (\bigsqcup B_j)^{\mathcal{I}_1}$, which implies $x \in (\bigsqcup B_j)^{\mathcal{I}_2}$.

$\Leftarrow$: Let $\mathcal{I}_2 = (\Delta^{\mathcal{I}_2}, \cdot^{\mathcal{I}_2})$ be a canonical forest model for $\mathcal{K}'$. We show how to construct from $\mathcal{I}_1$ a model $\mathcal{I}_1 = (\Delta^{\mathcal{I}_1}, \cdot^{\mathcal{I}_1})$ for $\mathcal{K}$.
Let $\Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2}$ and $\cdot^{\mathcal{I}_1}$ be defined as follows:

- $s^{\mathcal{I}_1} = s^{\mathcal{I}_2}$, for every $s \in N_\mathsf{I}$
- $A^{\mathcal{I}_1} = A^{\mathcal{I}_2}$, for every $A \in N_\mathsf{C}$
- $R^{\mathcal{I}_1} = \begin{cases} R^{\mathcal{I}_2}, & \text{for every } R \in N_\mathsf{R} \text{ s.t. } R^- \text{ is not rewritable} \\ & \text{or does not occur in } \mathcal{K} \\ R^{\mathcal{I}_2} \cup (NR^-)^{\mathcal{I}_2}, & \text{otherwise.} \end{cases}$

Before moving to show that $\mathcal{I}_1 = (\Delta^{\mathcal{I}_1}, \cdot^{\mathcal{I}_1})$ is a model of $\mathcal{K}$, we introduce a few helping lemmas:

Lemma 2. *Let R be a role which occurs in $\mathcal{K}'$. Then $(x,y) \in R^{\mathcal{I}_2}$ iff there exists a role S in $\mathcal{K}$ such that one of the following holds:*

1. *$g(S) \sqsubseteq^*_{\mathcal{R}'} R$, $g(S)(a,b) \in \mathcal{A}'$, $a^{\mathcal{I}_2} = x$, and $b^{\mathcal{I}_2} = y$,*
2. *$g(S) \sqsubseteq^*_{\mathcal{R}'} \mathsf{Inv}(R)$, $g(S)(b,a) \in \mathcal{A}'$, $a^{\mathcal{I}_2} = x$, and $b^{\mathcal{I}_2} = y$,*
3. *$g(S) \sqsubseteq^*_{\mathcal{R}'} R$, $g(S)$ occurs in an axiom of type **N2** or **N4** in $\mathcal{K}$', $x \in A^{\mathcal{I}_2}$ and $(x,y) \in g(S)^{\mathcal{I}_2}$,*
4. *$g(S) \sqsubseteq^*_{\mathcal{R}'} \mathsf{Inv}(R)$, $g(S)$ occurs in an axiom of type **N2** or **N4** in $\mathcal{K}$', $y \in A^{\mathcal{I}_2}$ and $(y,x) \in g(S)^{\mathcal{I}_2}$.*

Proof. It follows from the fact that: $\mathcal{I}_2 = (\Delta^{\mathcal{I}_2}, \cdot^{\mathcal{I}_2})$ is a canonical forest model. Intuitively, in each of the cases the S-role is the reason for the existence of $R(x^{\mathcal{I}_2}, y^{\mathcal{I}_2})$ in the interpretation. □

Lemma 3. *Let R and S be two roles in $\mathcal{K}$. Then, the following hold:*

i) $g(R) \sqsubseteq^*_{\mathcal{R}'} g(S)$ implies $R \sqsubseteq^*_{\mathcal{R}} S$, and
ii) $g(R) \sqsubseteq^*_{\mathcal{R}'} \mathsf{Inv}(g(S))$ implies $R \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(S)$.

Proof. First, from Definition 5 we observe that: $g(R) \sqsubseteq g(S) \in \mathcal{R}'$ implies $R \sqsubseteq S \in \mathcal{R}$ or $\mathsf{Inv}(R) \sqsubseteq \mathsf{Inv}(S) \in \mathcal{R}$: $R \sqsubseteq^*_{\mathcal{R}} S$.

Also, $g(R) \sqsubseteq \mathsf{Inv}(g(S)) \in \mathcal{R}'$ implies that: $f(g(R)) \sqsubseteq f(\mathsf{Inv}(g(S))) \in \mathcal{R}$ or $f(\mathsf{Inv}(g(R))) \sqsubseteq f(g(S)) \in \mathcal{R}$. From the definition of f it can be seen that $f(\mathsf{Inv}(R)) = \mathsf{Inv}(R) = \mathsf{Inv}(f(R))$ (note that R cannot be of the form NQ as f is not defined on inputs of type NQ^-), thus: $g(R) \sqsubseteq \mathsf{Inv}(g(S)) \in \mathcal{R}'$ implies that: $R \sqsubseteq \mathsf{Inv}(S) \in \mathcal{R}$ or $\mathsf{Inv}(R) \sqsubseteq S \in \mathcal{R}$: $R \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(S)$.

From the two base cases above it is easy to see how one can construct an induction argument over the role hierarchy to prove the original claims. □

Lemma 4. *Let R be a role in $\mathcal{K}$. Then, the following hold:*

- $(x, y) \in (g(R))^{\mathcal{I}_2}$ *and R is not generating in $\mathcal{K}$ implies* $(x, y) \in \mathsf{Inv}(g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$,
- $(x, y) \in \mathsf{Inv}(g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$ *and $\mathsf{Inv}(R)$ is not generating in $\mathcal{K}$ implies* $(x, y) \in (g(R))^{\mathcal{I}_2}$.

Proof.
We first show that the claims of the current lemma hold in the cases where the reason for $(x, y) \in (g(R))^{\mathcal{I}_2}$ and $(x, y) \in \mathsf{Inv}(g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$ is one of those enunciated at points 1. and 2. in Lemma 2. We distinguish between:

- $(x, y) \in (g(R))^{\mathcal{I}_2}$ and case 1. in Lemma 2 holds for $g(R)$. Then: $g(S) \sqsubseteq^*_{\mathcal{R}'} g(R)$ implies $g(\mathsf{Inv}(S)) \sqsubseteq^*_{\mathcal{R}'} g(\mathsf{Inv}(R))$ and $g(S)(a, b) \in \mathcal{A}'$ implies $g(\mathsf{Inv}(S))(b, a) \in \mathcal{A}'$. Together with $a^{\mathcal{I}_2} = x$, and $b^{\mathcal{I}_2} = y$, it follows that $(y, x) \in g(\mathsf{Inv}(R))^{\mathcal{I}_2}$, thus $(x, y) \in \mathsf{Inv}(g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$.
- $(x, y) \in (g(R))^{\mathcal{I}_2}$ and case 2. in Lemma 2 holds for $g(R)$ – similar to above.
- $(x, y) \in \mathsf{Inv}(g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$ and case 1. holds for $\mathsf{Inv}(g(\mathsf{Inv}(R)))$. Then: $(y, x) \in (g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$, $g(S) \sqsubseteq^*_{\mathcal{R}'} g(\mathsf{Inv}(R))$, and $g(S)(a, b) \in \mathcal{A}'$ with $a^{\mathcal{I}_2} = y$ and $b^{\mathcal{I}_2} = x$. But then: $g(\mathsf{Inv}(S)) \sqsubseteq^*_{\mathcal{R}'} g(R)$ and $g(\mathsf{Inv}(S))(b, a) \in \mathcal{A}'$). It follows that $(x, y) \in (g(R))^{\mathcal{I}_2}$.
- $(x, y) \in \mathsf{Inv}(g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$ and case 2. holds for $\mathsf{Inv}(g(\mathsf{Inv}(R)))$ – similar to above.

We next show that the claims of the current lemma hold even when the conditions at point 1. and 2. of Lemma 2 do not hold. We distinguish between:

- $(x, y) \in (g(R))^{\mathcal{I}_2}$, R is not generating in $\mathcal{K}$, and neither the conditions at point 1. nor the conditions at point 2. of Lemma 2 do not hold. Then the conditions at point 4. of Lemma 2 must be fulfilled w.r.t. $g(R)$: there exists a role S s.t. $g(S) \sqsubseteq^*_{\mathcal{R}'} \mathsf{Inv}(R)$ and $(y, x) \in g(S)^{\mathcal{I}_2}$. From Lemma 3 it follows that: $S \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(R)$ and thus $g(S) \sqsubseteq^*_{\mathcal{R}'} g(\mathsf{Inv}(R))$. But then, $(y, x) \in g(\mathsf{Inv}(R)^{\mathcal{I}_2}$ or $(x, y) \in \mathsf{Inv}(g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$.
- $(x, y) \in \mathsf{Inv}(g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$, $\mathsf{Inv}(R)$ is not generating in $\mathcal{K}$ and neither the conditions at point 1. nor the conditions at point 2. of Lemma 2 do not hold. Then, the conditions at point 3. of Lemma 2 must be fulfilled w.r.t. $\mathsf{Inv}(g(\mathsf{Inv}(R)))$: there exists a role S s.t. $g(S) \sqsubseteq^*_{\mathcal{R}'} \mathsf{Inv}(g(\mathsf{Inv}(R))$ and $(x, y) \in g(S)^{\mathcal{I}_2}$. But then, from Lemma 3 it follows that $S \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(\mathsf{Inv}(R)) = R$. Thus, $g(S) \sqsubseteq^*_{\mathcal{R}'} g(R)$ and $(x, y) \in (g(R))^{\mathcal{I}_2}$.

We finally proceed to show that $\mathcal{I}_1 = (\Delta^{\mathcal{I}_1}, \cdot^{\mathcal{I}_1})$ is a model of $\mathcal{K}$. As for the proof in the other direction it holds that:

– $(x,y) \in R^{\mathcal{I}_2}$ implies $(x,y) \in (f(R))^{\mathcal{I}_1}$, for every R in the signature of $\mathcal{K}'$, and
– $(x,y) \in R^{\mathcal{I}_1}$ implies $(x,y) \in (g(R))^{\mathcal{I}_2} \cup (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$, for every R in the signature of $\mathcal{K}$.

As we will see $\mathcal{I}_1$ satisfies every axiom from $\mathcal{K}$:

– Assume $R \sqsubseteq S \in \mathcal{R}$, and $(x,y) \in R^{\mathcal{I}_1}$. Then $g(R) \sqsubseteq g(S) \in \mathcal{R}'$, $g(\mathsf{Inv}(R)) \sqsubseteq g(\mathsf{Inv}(S)) \in \mathcal{R}'$ and $(x,y) \in (g(R))^{\mathcal{I}_2} \cup (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$. Thus, $(x,y) \in S^{\mathcal{I}_2}$ and $(x,y) \in S^{\mathcal{I}_1}$.
– Assume $\exists R.A \sqsubseteq B \in \mathcal{T}$ and $x \in \exists R.A^{\mathcal{I}_1}$: there exists $y \in \Delta^{\mathcal{I}_1}$ s.t. $(x,y) \in R^{\mathcal{I}_1}$ and $y \in A^{\mathcal{I}_1}$. Then, $\exists g(R).A \sqsubseteq B \in \mathcal{T}'$ and if $\mathsf{Inv}(R)$ is generating: $A \sqsubseteq \forall g(\mathsf{Inv}(R)).B \in \mathcal{T}'$. Also $(x,y) \in (g(R))^{\mathcal{I}_2} \cup (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$ and $y \in A^{\mathcal{I}_2}$.
When $(x,y) \in (g(R))^{\mathcal{I}_2}$, from $\exists g(R).A \sqsubseteq B \in \mathcal{T}'$ it follows that $x \in B^{\mathcal{I}_2}$, and thus $x \in B^{\mathcal{I}_1}$.
When $(x,y) \in (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$ and $\mathsf{Inv}(R)$ is generating: $(y,x) \in (g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$ and from $A \sqsubseteq \forall g(\mathsf{Inv}(R)).B \in \mathcal{T}'$ it follows that $x \in B^{\mathcal{I}_2}$, and thus $x \in B^{\mathcal{I}_1}$.
When $(x,y) \in (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$ and $\mathsf{Inv}(R)$ is not generating from Lemma 4 it follows that $(x,y) \in g(R)^{\mathcal{I}_2}$ and together with $\exists g(R).A \sqsubseteq B \in \mathcal{T}'$ it leads to $x \in B^{\mathcal{I}_2}$, and $x \in B^{\mathcal{I}_1}$.
– Assume $A \sqsubseteq \forall R.B \in \mathcal{T}$, $x \in A^{\mathcal{I}_1}$, and $(x,y) \in R^{\mathcal{I}_1}$: $x \in A^{\mathcal{I}_2}$, and $(x,y) \in (g(R))^{\mathcal{I}_2} \cup (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$. Then: $\exists g(\mathsf{Inv}(R)).A \sqsubseteq B \in \mathcal{T}'$ and if R is not generating or $\mathsf{Inv}(R)$ is generating, and $A \sqsubseteq \forall g(R).B \in \mathcal{T}'$ if R is generating.
When $(x,y) \in (g(R))^{\mathcal{I}_2}$ and R is generating from $A \sqsubseteq \forall g(R).B \in \mathcal{T}'$ it follows that $x \in B^{\mathcal{I}_2}$, and thus $x \in B^{\mathcal{I}_1}$.
When $(x,y) \in (g(R))^{\mathcal{I}_2}$ and R is not generating, from Lemma 4 it follows that $(x,y) \in (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$: $(y,x) \in (g(\mathsf{Inv}(R)))^{\mathcal{I}_2}$ and together with $\exists g(\mathsf{Inv}(R)).A \sqsubseteq B \in \mathcal{T}'$ it follows that $x \in B^{\mathcal{I}_2}$, and thus $x \in B^{\mathcal{I}_1}$.
When $(x,y) \in (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$ and $\mathsf{Inv}(R)$ is not generating from Lemma 4 it follows that $(x,y) \in g(R)^{\mathcal{I}_2}$ and together with $A \sqsubseteq \forall g(R).B \in \mathcal{T}'$ it leads to $x \in B^{\mathcal{I}_2}$, and $x \in B^{\mathcal{I}_1}$.
When $(x,y) \in (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$ and $\mathsf{Inv}(R)$ is generating from Lemma 4, from $\exists g(\mathsf{Inv}(R)).A \sqsubseteq B \in \mathcal{T}'$ it follows that $x \in B^{\mathcal{I}_2}$, and thus $x \in B^{\mathcal{I}_1}$.
– Assume $A \sqsubseteq \geqslant nR.B \in \mathcal{T}$ and $x \in A^{\mathcal{I}_1}$. Then, $A \sqsubseteq \geqslant n(g(R)).B \in \mathcal{T}'$ and $x \in A^{\mathcal{I}_2}$. As $(x,y) \in (g(R))^{\mathcal{I}_2}$ implies $(x,y) \in R^{\mathcal{I}_1}$ and $y \in B^{\mathcal{I}_2}$ implies $y \in B^{\mathcal{I}_1}$, it follows that $x \in (\geqslant nR.B)^{\mathcal{I}_1}$.
– Assume $A \sqsubseteq \leqslant nR.B \in \mathcal{T}$ and $x \in A^{\mathcal{I}_1}$. Then, $A \sqsubseteq \leqslant n(g(R)).B \in \mathcal{T}'$, $x \in A^{\mathcal{I}_2}$, and consequently $x \in (\leqslant n(g(R)))^{\mathcal{I}_2}$. We have that $(x,y) \in R^{\mathcal{I}_1}$ implies $(x,y) \in (g(R))^{\mathcal{I}_2} \cup (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$ and $y \in B^{\mathcal{I}_1}$ implies $y \in B^{\mathcal{I}_2}$.
Assume $(x,y) \in (\mathsf{Inv}(g(\mathsf{Inv}(R))))^{\mathcal{I}_2}$, but $(x,y) \notin (g(R))^{\mathcal{I}_2}$. Then R is not generating and $\mathsf{Inv}(R)$ is generating (otherwise there is a contradiction with Lemma 4). Also, it is either the case that $R = Q$ or $R = Q^-$ with Q^- being rewritable. But this is in contradiction with the notion of rewritable roles. Thus, in this case $(x,y) \in R^{\mathcal{I}_1}$ implies $(x,y) \in (g(R))^{\mathcal{I}_2}$ and $x \notin (\geqslant nR.B)^{\mathcal{I}_1}$ implies $x \notin (\geqslant n(g(R)).B)^{\mathcal{I}_2}$ – contradiction. Thus, $x \in (\geqslant nR.B)^{\mathcal{I}_1}$.

– Assume $\bigsqcap A_i \sqsubseteq \bigsqcup B_j \in \mathcal{T}$ and $x \in (\bigsqcap A_i)^{\mathcal{I}_1}$. Then, $\bigsqcap A_i \sqsubseteq \bigsqcup B_j \in \mathcal{T}'$, $x \in (\bigsqcap A_i)^{\mathcal{I}_2}$, and thus $x \in (\bigsqcup B_j)^{\mathcal{I}_2}$, which implies $x \in (\bigsqcup B_j)^{\mathcal{I}_1}$.

A.1 Reuse-safe Roles

In this section we verify the correctness of the rewriting presented in Definition 8 as stated in Theorem 5. To do so, we first proof correctness of a modified version of the hypertableau algorithm introduced in [15]. From this result it is straightforward to verify correctness of the aforementioned transformation.

We proceed with some preliminary notions necessary for the definition of the hypertableau algorithm and the reuse safe strategy. Note that some of these are slightly modified as compared to the ones appearing [15]. To improve readability, we have simplified the algorithm to only cover the cases for Horn-$\mathcal{SHOIQ}$ ontologies. Also, the $\geqslant$-rule is modified to automatically reuse auxiliary individuals to satisfy at least restrictions that feature existential roles.

The algorithm requires that the DL axioms are preprocessed into *DL-clauses* – universally quantified implications containing DL concepts and roles as predicates. More specifically, DL axioms are preprocessed into *HT-clauses* – syntactically restricted DL-clauses on which the hypertableau calculus is guaranteed to terminate. HT-clauses are formally defined in Definition 5 of [15].

Instead of reusing the lengthy preprocessing procedure introduced in Section 4.1 of [15] we directly map the set of Horn-$\mathcal{SHOIQ}$ axioms into DL-clauses as defined in Figure 5.[7] Since we are only considering axioms in a very succinct normal form this mapping is a much more straightforward procedure. By definition all clauses produced by the mappings described in Figure 5 are HT-clauses.

Note that, mapping (8) of Figure 5 contains a special type of atom, namely $y \approx z@^x_{\leqslant 1R.D}$. The *annotation* $@^x_{\leqslant 1R.D}$ does not affect the meaning and $y \approx z@^x_{\leqslant 1R.D}$ is semantically equivalent to $y \approx z$. These annotations appear as part of the premise of the *NI*-rule (described in Figure 6).
As in [15], the following lemma holds:

Lemma 5. *Let K be a Horn-$\mathcal{SHOIQ}$ knowledge base. Then K is equisatisfiable with $\Omega(K) = (\Omega_{\mathcal{TR}}(K), \Omega_{\mathcal{A}}(K))$, where $\Omega_{\mathcal{TR}}(K)$ is a set of HT-clauses and $\Omega_{\mathcal{A}}(K)$ is an ABox, both obtained as shown in Figure 5, where $\Omega(K)$ is to be interpreted with standard first order logic standard semantics.*

Lemma 5 follows from Lemma 3 in [15]. This result is completely straightforward since, given a normalized Horn-$\mathcal{SHOIQ}$ ontology K, the transformation defined in Figure 5 outputs the same set of HT-clauses as the more involved preprocessing process defined in Section 4.1 of the previously cited paper.
We proceed now with a simplified definition of the hypertableau algorithm.

Definition 9. (Hypertableau with Individual Reuse.)

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$$\Omega_{\mathcal{TR}}(K) = \{\Omega_{\mathcal{T}}(\alpha) | \alpha \in \mathcal{T}\} \cup \{\Omega_{\mathcal{R}}(\beta) | \beta \in \mathcal{R}\} \quad (1)$$
$$\Omega_{\mathcal{A}}(K) = \mathcal{A} \quad (2)$$

$$\Omega_{\mathcal{T}}(C_1 \sqcap \ldots \sqcap C_n \sqsubseteq D) \mapsto C_1(x) \wedge \ldots \wedge C_n(x) \rightarrow D(x) \quad (3)$$
$$\Omega_{\mathcal{T}}(\exists R.C \sqsubseteq D) \mapsto R(x,y) \wedge C(y) \rightarrow D(x) \quad (4)$$
$$\Omega_{\mathcal{T}}(C \sqsubseteq \forall R.D) \mapsto C(x) \wedge R(x,y) \rightarrow D(y) \quad (5)$$
$$\Omega_{\mathcal{T}}(C \sqsubseteq \exists R.D) \mapsto C(x) \rightarrow \geqslant 1R.D(x) \quad (6)$$
$$\Omega_{\mathcal{T}}(C \sqsubseteq \geqslant nR.D) \mapsto C(x) \rightarrow \geqslant nR.D(x) \quad (7)$$
$$\Omega_{\mathcal{T}}(C \sqsubseteq \leqslant 1R.D) \mapsto C(x) \wedge R(x,y) \wedge \; D(y) \wedge R(x,z) \wedge D(z) \rightarrow y \approx z @^{x}_{\leqslant 1R.D} \quad (8)$$
$$\Omega_{\mathcal{T}}(C \sqsubseteq \{a\}) \mapsto C(x) \wedge O_a(z) \rightarrow x \approx z \quad (9)$$

$$\Omega_{\mathcal{R}}(R \sqsubseteq S) \mapsto R(x,y) \rightarrow S(x,y) \quad (10)$$
$$\Omega_{\mathcal{R}}(R \sqsubseteq S^-) \mapsto R(x,y) \rightarrow S(y,x) \quad (11)$$

where $K = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ and every predicate of the form $C(x)$ appearing in the body of a resulting DL clause such that $C = \top$ or in the head such that $C = \bot$ is erased

Fig. 5. Horn-$\mathcal{SHOIQ}$ Clauses

Let $K = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ be a Horn-$\mathcal{SHOIQ}$ ontology defined over the signature N_I, N_C and N_R. Let $\mathcal{C}$ be a set of HT-clauses such that $\mathcal{C} = \Omega_{\mathcal{TR}}(K)$.

Individuals. *The set of* auxiliary individuals N_X *is the smallest set such that $\alpha_{iRD} \in N_X$ for every role R, atomic concept D, and positive integer i. The set of* root individuals N_O *is the smallest set such that $N_I \cup N_X \subseteq N_O$, and if $x \in N_O$ then $x.\langle R, D\rangle \in N_O$, for each role R and named concept D. The set of all individuals N_A is the smallest set such that $N_O \subseteq N_A$ and if $x \in N_A$ then $x.(i, R, D) \in N_A$ for every role R, atomic concept D, and positive integer i. The individuals in $N_A \setminus N_O$ are* blockable individuals. *An individual $x.(i, R, D)$ is a* successor *of x, and x is a* predecessor *of $x.(i, R, D)$.* Descendant *and* ancestor *are the respective transitive closures of successor and predecessor.*

ABoxes. *An ABox that contains only named individuals and no at-most equalities is called an* input ABox. *The hypertableau algorithm works with* generalized ABoxes, *which can contain assertions using the individuals from N_A, a special assertion $\bot$ that is false in all interpretations, and an acyclic confluent relation $\mapsto$ on root individuals called* renaming. *The* canonical name *of a root individual $a \in N_O$ with respect to $\mathcal{A}$, written $||a||_{\mathcal{A}}$, is the normal form of a with respect to $\mapsto$ in $\mathcal{A}$. If a occurs in $\mathcal{A}$, then the relation $\mapsto$ must be such that $||a||_{\mathcal{A}} = a$.*

Pairwise Anywhere Blocking. *The label of an individual is defined as $\mathcal{L}_{\mathcal{A}}(s) = \{C \mid C(s) \in \mathcal{A}$ and C is of the form A or $\geqslant nR.A\}$ and of an individual pair as $\mathcal{L}_{\mathcal{A}}(s,t) = \{R \mid R(s,t) \in \mathcal{A}\}$. Let $\prec$ be a transitive and irreflexive relation on N_A such that, if s' is an ancestor of s, then $s' \prec s$. By induction on $\prec$, we assign to each individual in $\mathcal{A}$ a status as follows: a blockable individual s with a predecessor s' is* directly blocked *by a blockable individual t with a predecessor t' if and only if t is not blocked, $t \prec s$, $\mathcal{L}_{\mathcal{A}}(s) = \mathcal{L}_{\mathcal{A}}(t)$, $\mathcal{L}_{\mathcal{A}}(s') = \mathcal{L}_{\mathcal{A}}(t')$,*

$\mathcal{L}_{\mathcal{A}}(s, s') = \mathcal{L}_{\mathcal{A}}(t, t')$ and $\mathcal{L}_{\mathcal{A}}(s', s) = \mathcal{L}_{\mathcal{A}}(t, t')$; s is indirectly blocked if and only if it has a predecessor that is blocked; and s is blocked if and only if it is directly or indirectly blocked.

Pruning. *The Abox $\mathsf{prune}_{\mathcal{A}}(s)$ is obtained from $\mathcal{A}$ by removing all assertions containing a descendent of s.*

Merging. *The ABox $\mathsf{merge}_{\mathcal{A}}(s \mapsto t)$ is obtained from $\mathsf{prune}_{\mathcal{A}}(s)$ by replacing the individual s with the individual t in all assertions (but not in the renaming relation $\mapsto$) and, if both t and s are root individuals, adding the renaming $s \mapsto t$.*

Derivation Rules. *Figure 6 specifies rules that, for $\mathcal{A}$ an ABox and $\mathcal{C}$ a set of HT-clauses, derive the ABoxes $\mathcal{A}_1, \ldots, \mathcal{A}_l$.*

Rule Precedence. *The $\approx$-rule can be applied to a (possibly annotated) equality $s \approx t$ in an ABox $\mathcal{A}$ only if $\mathcal{A}$ does not contain an equality of the form $s \approx t@^{u}_{\leqslant nR.D}$ to which the NI-rule is applicable.*

Clash. *An ABox $\mathcal{A}$ contains a* clash *if and only if $\bot \in \mathcal{A}$; otherwise, $\mathcal{A}$ is clash free.*

Derivation. *A* derivation *$D = (T, \lambda)$ for a set of HT-clauses $\mathcal{C}$ and an ABox $\mathcal{A}$ consists of a finitely branching tree T where every node has at most one child and a function λ labeling the nodes of T with ABoxes such that (i) $\lambda(\epsilon) = \mathcal{A}$ for ϵ the root of T, (ii) $t \in T$ is a leaf of T if $\bot \in \lambda(t)$ or no derivation rule is applicable to $\lambda(t)$ and $\mathcal{C}$, and (iii) otherwise, $t \in T$ has child s such that $\lambda(s)$ is exactly the result of applying on (arbitrarily chosen, but respecting the precedence) applicable derivation rule to $\lambda(t)$ and $\mathcal{C}$. The derivation D is successful if it contains a leaf node labeled with a clash-free ABox.*

The main result of this section is conveyed in the following theorem:

Theorem 6. *Let $\mathcal{C}$ be a set of HT-clauses and $\mathcal{A}$ an input ABox. Then, (1) each derivation produced by the hypertableau algorithm for $\mathcal{C}$ and $\mathcal{A}$ is finite, (2) if a derivation is successful then $(\mathcal{C}, \mathcal{A})$ is satisfiable and (3) if $(\mathcal{C}, \mathcal{A})$ is satisfiable there exists a successful derivation.*

Proof. [Theorem 6] Statement (1) is immediate from the proof of Lemma 7 in [15]. It is also trivial to see that (2) also holds: if there exist a successful derivation then a model can be constructed for for $(\mathcal{C}, \mathcal{A})$ as shown in the proof of Lemma 6 in [15]. Showing claim (3) requires a more elaborate argument.

We start this argument introducing function ar which maps a role R and two individuals or variables s, t to a binary predicate. Function ar will be used across the rest of the argument to formally define further claims and is formally defined as follows:

$$\mathsf{ar}(R, s, t) = \begin{cases} R(s, t) & R \in N_{\mathsf{R}} \\ \mathsf{Inv}(R)(t, s) & R \notin N_{\mathsf{R}} \end{cases}$$

We proceed with the definition of the precedence relation over individuals $\rightsquigarrow^{R}_{\mathcal{A}_n}$ and function $[\cdot]$ which maps an individual to a concept expression. These two will be used to properly formalize further claims.

Let R be a role and $\mathcal{A}_n$ an HT-ABox produced by the hypertableau algorithm during the reasoning process. Then $\rightsquigarrow^{R}_{\mathcal{A}_n} \subseteq N_{\mathsf{A}} \times N_{\mathsf{A}}$ is the minimal relation such

Fig. 6. Derivation Rules of the Hypertableau Calculus

Hyp-rule	if 1. $r \in \mathcal{C}$, where $r = U_1 \wedge \ldots \wedge U_n \rightarrow V \in \mathcal{C}$, and 2. a mapping σ from variables in r to the individuals of $\mathcal{A}$ exists such that 2.1 there is no $x \in N_\mathsf{V}$ such that $\sigma(x)$ is indirectly blocked, 2.2 $\sigma(U_i) \in \mathcal{A}$ for each $1 \leqslant i \leqslant m$, and 2.3 $\sigma(V) \notin \mathcal{A}$, then $\mathcal{A}_1 = \mathcal{A} \cup \{\bot\}$ if V is empty; $\mathcal{A}_1 = \mathcal{A} \cup \{\sigma(V)\}$ otherwise.
$\geqslant$-rule	if 1. $\geqslant nR.D(s) \in \mathcal{A}$, 2. s is not blocked, and 3. $\mathcal{A}$ does not contain individuals $u_1, \ldots, u_n$ such that 3.1 $\{R(s, u_i), D(u_i) \mid 1 \leqslant i \leqslant n\} \cup \{u_i \not\approx u_j \mid 1 \leqslant i < j \leqslant n\} \subseteq A$, and 3.2 for each $1 \leqslant i \leqslant n$, either u_i is a successor of s or u_i is not blocked, then $\mathcal{A}_1 := \mathcal{A} \cup \{R(s, u_i), C(u_i) \mid 1 \leqslant i \leqslant n\} \cup \{u_i \not\approx u_j \mid 1 < i < j \leqslant n\}$ where each $u_i = \alpha_{iRD}$ if R is a safe role; otherwise every u_i is a freshly introduced successors of s
$\approx$-rule	if 1. $s \approx t \in \mathcal{A}_n$, 2. $s \neq t$, and 3. neither s nor t is indirectly blocked then $\mathcal{A}_1 := \mathsf{merge}_\mathcal{A}(s \rightarrow t)$ if $t \in N_\mathsf{I}$, $t \in N_\mathsf{O}$ and $s \notin N_\mathsf{I}$, or s is a descendant of t; $\mathcal{A}_1 := \mathsf{merge}_\mathcal{A}(t \rightarrow s)$ otherwise
$\bot$-rule	if $s \not\approx s \in \mathcal{A}$ the $\mathcal{A}_1 := \mathcal{A} \cup \{\bot\}$
NI-rule	if 1. $s \approx t@^u_{\leqslant 1R.D} \in \mathcal{A}$ (the symmetry of $\approx$ applies as usual), 2. u is a root individual, 3. s is a blockable individual and it is not a successor of u, and 4. t is a blockable individual then $\mathcal{A}_1 := \mathsf{merge}_\mathcal{A}(s \rightarrow \|\|u.\langle R, D\rangle\|\|_\mathcal{A})$

that $\rightsquigarrow^R_{\mathcal{A}_0} = \{(s,t) \mid \mathsf{ar}(R,s,t) \in \mathcal{A}_0\}$ and depending on the expansion rule applied to derive $\mathcal{A}_n$ the relation $\rightsquigarrow^R_{\mathcal{A}_n}$ is defined from the relation $\rightsquigarrow^{n-1}_{\mathcal{A}_n}$ as follows:

- (Hyp-rule)
  - $\mathsf{ar}(S,x,y) \rightarrow \mathsf{ar}(R,x,y)$, $\mathsf{ar}(S,s,t) \in \mathcal{A}_{n-1}$ and $s \rightsquigarrow^S_{\mathcal{A}_{n-1}} t$: $\rightsquigarrow^R_{\mathcal{A}_n} = \rightsquigarrow^R_{\mathcal{A}_{n-1}} \cup \{(s,t)\}$.
  - All other cases: $\rightsquigarrow^R_{\mathcal{A}_n} = \rightsquigarrow^R_{\mathcal{A}_{n-1}}$.
- ($\geqslant$-rule)
  - $\geqslant nR.D(s) \in \mathcal{A}_{n-1}$ where R is safe: $\rightsquigarrow^R_{\mathcal{A}_n} = \rightsquigarrow^R_{\mathcal{A}_{n-1}} \cup \{(s, \alpha_{iRD})\}$.
  - $\geqslant nR.D(s) \in \mathcal{A}_{n-1}$ where R is unsafe: $\rightsquigarrow^R_{\mathcal{A}_n} = \rightsquigarrow^R_{\mathcal{A}_{n-1}} \cup \{(s, s.(i,R,D))\}$.
  - $\geqslant n\mathsf{Inv}(R).D(s) \in \mathcal{A}_{m-1}$ where R is unsafe: $\rightsquigarrow^R_{\mathcal{A}_n} = \rightsquigarrow^R_{\mathcal{A}_{n-1}} \cup \{(s.(i,R,D), s)\}$.
- (($\approx$-rule) or (NI-rule)) $\mathcal{A}_n = \mathsf{merge}_{\mathcal{A}_{n-1}}(s \mapsto t)$: $\rightsquigarrow^R_{\mathcal{A}_n} = \mathsf{merge}_{\rightsquigarrow^R_{\mathcal{A}_{n-1}}}(s \mapsto t)$.

Let $[\cdot]$ a function on N_A defined as follows:

$$[s] = \begin{cases} \{s\} & s \in N_\mathsf{I} \\ \exists \mathsf{Inv}(R).\top \sqcap D \sqcap X_i & s = \alpha_{iRD} \\ \exists \mathsf{Inv}(R).[t] \sqcap D \sqcap X_i^t & s = t.(i,R,D) \\ \exists \mathsf{Inv}(R).[t] \sqcap D & s = t.\langle R, D\rangle \end{cases}$$

where X_i are freshly introduced classes that allow us to differentiate the class of all ith successors created due to some application of the $\geqslant$-rule.

We show inductively that, if $(\mathcal{C}, \mathcal{A})$ is satisfiable, then the following properties $(1-9)$ hold for each HT-ABox in the derivation and each model $\mathcal{I}$ of $(\mathcal{C}, \mathcal{A})$:

1. $s \approx t@^u_{\leqslant 1R.B} \in \mathcal{A}_n$ implies $[s]^\mathcal{I} = [u.\langle R, B\rangle]^\mathcal{I}$.
2. $\mathsf{ar}(S, s, t) \in \mathcal{A}_n$ and $A(x) \wedge V(x, y) \wedge B(y) \wedge V(x, z) \wedge B(z) \rightarrow y \approx z@^x_{\leqslant 1R.D} \in \mathcal{C}$ such that $S \prec^*_\mathcal{R} V$ imply $[s]^\mathcal{I} \subseteq (\exists S.[t])^\mathcal{I}$.
3. $s \approx t \in \mathcal{A}_n$, where $s \approx t$ may be annotated, implies $[s]^\mathcal{I} = [t]^\mathcal{I}$.
4. $s \rightsquigarrow^S_{\mathcal{A}_n} t$ implies $[s]^\mathcal{I} \subseteq (\exists S.[t])^\mathcal{I}$.
5. $\mathsf{ar}(S, s, t) \in \mathcal{A}_n$ and $s \not\rightsquigarrow_{\mathcal{A}_n} t$ implies $V \sqsubseteq^*_\mathcal{R} \mathsf{Inv}(S)$ for some safe role V.
6. $\mathsf{ar}(S, s, t) \in \mathcal{A}_n$ implies $[s]^\mathcal{I} \subseteq (\exists S.\top)^\mathcal{I}$.
7. $D(s) \in \mathcal{A}_n$ implies $[s]^\mathcal{I} \subseteq D^\mathcal{I}$.
8. $s \not\approx t \in \mathcal{A}_n$ implies $[s]^\mathcal{I} \neq [t]^\mathcal{I}$.
9. s occurs syntactically in $\mathcal{A}_n$ implies $[s]^\mathcal{I} \neq \emptyset$.
10. $\bot \neq \mathcal{A}_n$.

Note that, to verify statement (3), it suffices to show IH (A.1) and (A.1). The other claims are explicitly stated to properly structure the proof. The base case of the induction, namely the ABox $\mathcal{A}_0$, trivially satisfies (1) - (A.1). For the inductive step, the IH states that (1 - A.1) hold for every assertion $\alpha \in \mathcal{A}_{n-1}$ when a rule from Figure 6 is applied to produce a new derivation $\mathcal{A}_n$.

IH (1): $s \approx t@^u_{\leqslant nR.D} \in \mathcal{A}_n$ only if

1. (Hyp-rule) $C(x) \wedge R(x, y) \wedge D(y) \wedge R(x, z) \wedge D(z) \rightarrow z \approx y \in \mathcal{C}$ and $\{C(u), R(u, s), D(s), R(u, t), B(t)\} \subseteq \mathcal{A}_{n-1}$.
   We apply (2) to $R(u, s)$ to obtain $[u]^\mathcal{I} = \exists R.[s]^\mathcal{I}$. We apply (7) and $D(s)$ to obtain $[u]^\mathcal{I} \subseteq C^\mathcal{I}$ and $s^\mathcal{I} \subseteq D^\mathcal{I}$. Since $C(x) \wedge R(x, y) \wedge D(y) \wedge R(x, z) \wedge D(z) \rightarrow z \approx y \in \mathcal{C}$ we can conclude $[s]^\mathcal{I} = [u.\langle R, D\rangle]^\mathcal{I}$. Note that $[u.\langle i, R, D\rangle] = \exists \mathsf{Inv}(R).[u] \sqcap D$.
2. ($\approx$-rule) Two possible cases arise:
   (a) $\{s \approx v@^u_{\leqslant nR.D}, v \approx t\} \subseteq \mathcal{A}_{n-1}$ such that $t \in N_\mathsf{I}$, v is a descendant of t, or $t \in N_\mathsf{O}$ and $v \notin N_\mathsf{I}$. We apply IH (1) to $s \approx v@^u_{\leqslant nR.D}$ and obtain $[s]^\mathcal{I} = [u.\langle i, R, D\rangle]^\mathcal{I}$.
   (b) $\{s \approx v, v \approx t@^u_{\leqslant nR.D}\} \subseteq \mathcal{A}_{n-1}$ such that $s \in N_\mathsf{I}$, u is a descendant of s, or $s \in N_\mathsf{O}$ and $u \notin N_\mathsf{I}$. We apply IH (1) to $v \approx t@^u_{\leqslant nR.D}$ to obtain $[v]^\mathcal{I} = [u.\langle i, R, D\rangle]^\mathcal{I}$. We apply IH (3) to $s \approx v$ to obtain $[s]^\mathcal{I} \approx [v]^\mathcal{I}$. Hence $[s]^\mathcal{I} = [u.\langle i, R, D\rangle]^\mathcal{I}$.
3. (NI-rule) Two cases arise:
   (a) $v \approx t@^u_{\leqslant 1R.D} \in \mathcal{A}_{n-1}$ such that $s = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, t is a blockable individual and neither v nor t is indirectly blocked. Then $[s]^\mathcal{I} = [u.(R, D)]^\mathcal{I}$ since $s = u.(R, D)$.
   (b) $v \approx s@^u_{\leqslant 1R.D} \in \mathcal{A}_{n-1}$ such that $t = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, s is a blockable individual and neither v nor s is indirectly blocked. We can apply IH (1) to $v \approx s@^u_{\leqslant 1R.D} \in \mathcal{A}_{n-1}$ to derive $[s]^\mathcal{I} = [u.(R, D)]^\mathcal{I}$ since $s = u.(R, D)$.

IH (2): $\mathsf{ar}(S,s,t) \in \mathcal{A}_n$ only if

1. (Hyp-rule) $\mathsf{ar}(R,x,y) \to \mathsf{ar}(S,x,y) \in \mathcal{C}$ and $\mathsf{ar}(R,s,t) \in \mathcal{A}_{n-1}$. Note that since $S \prec^*_{\mathcal{R}} V$ we also have that $R \prec^*_{\mathcal{R}} V$. We apply IH (2) to $\mathsf{ar}(R,s,t)$ to obtain $[s]^{\mathcal{I}} \subseteq (\exists R.[t])^{\mathcal{I}}$. The claim holds since $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$ $\mathsf{Inv}(R)^{\mathcal{I}} \subseteq \mathsf{Inv}(S)^{\mathcal{I}}$.
2. ($\exists$-rule) Two possible cases arise:
   (a) $\exists S.D(s) \in \mathcal{A}_{n-1}$. We apply IH (7) to $\exists S.D(s)$ to obtain $[s]^{\mathcal{I}} \subseteq (\exists S.D)^{\mathcal{I}}$. The claim holds since $t = s.(S,D)$ and $[t] = \exists \mathsf{Inv}(S).[s] \sqcap D$.
   (b) $\exists \mathsf{Inv}(S).D(t) \in \mathcal{A}_{n-1}$. The claim holds since $s = t.(S,D)$ and $[s] = \exists S.[t] \sqcap D$.
3. ($\approx$-rule) Two possible cases arise:
   (a) $\{\mathsf{ar}(S,s,u), u \approx t\} \subseteq \mathcal{A}_{n-1}$ such that $t \in N_{\mathsf{I}}$, u is a descendant of t, or $t \in N_{\mathsf{O}}$ and $u \notin N_{\mathsf{I}}$. We apply IH (2) to $\mathsf{ar}(S,s,u)$ to obtain $[s]^{\mathcal{I}} \subseteq (\exists S.[u])^{\mathcal{I}}$. We apply IH (3) to $u \approx t$ to obtain $[u]^{\mathcal{I}} = [t]^{\mathcal{I}}$. Hence $[s]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$.
   (b) $\{s \approx u, \mathsf{ar}(S,u,t)\} \subseteq \mathcal{A}_{n-1}$ such that $s \in N_{\mathsf{I}}$, u is a descendant of s, or $s \in N_{\mathsf{O}}$ and $u \notin N_{\mathsf{I}}$. Analogous to the previous case.
4. (NI-rule) Two cases arise:
   (a) $\{\mathsf{ar}(S,v,t), v \approx w@^{u}_{\leqslant 1R.D}\} \subseteq \mathcal{A}_{n-1}$ such that $s = u.\langle R,D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, t is a blockable individual and neither v nor t is indirectly blocked. We apply IH (1) to $v \approx w@^{u}_{\leqslant 1R.D}$ to obtain $[v]^{\mathcal{I}} = [u.\langle R,D\rangle]^{\mathcal{I}}$. We apply IH (2) to $\mathsf{ar}(S,v,t)$ to conclude $[v]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$. Consequently $[s]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$.
   (b) $\{\mathsf{ar}(S,s,v), v \approx w@^{u}_{\leqslant 1R.D}\} \subseteq \mathcal{A}_{n-1}$ such that $t = u.\langle R,D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, s is a blockable individual and neither v nor s is indiredctly blocked. Analogous to the previous case.

Remark: both S and $\mathsf{Inv}(S)$ are necessarily unsafe roles in case 2.

IH (3): $s \approx t \in \mathcal{A}_n$ only if

1. (Hyp-rule) $C(x) \wedge \mathsf{ar}(R,x,y) \wedge D(y) \wedge \mathsf{ar}(R,x,z) \wedge D(z) \to z \approx y \in \mathcal{C}$ and $\{C(u), \mathsf{ar}(R,u,s), D(s), \mathsf{ar}(R,u,t), D(t)\} \subseteq \mathcal{A}_{n-1}$. We apply IH (2) to $\mathsf{ar}(\mathsf{Inv}(R),s,u)$ and $\mathsf{ar}(R,u,t)$ and obtain $[s]^{\mathcal{I}} \subseteq (\exists \mathsf{Inv}(R).[u])^{\mathcal{I}}$ and $[u]^{\mathcal{I}} \subseteq (\exists R.[t])^{\mathcal{I}}$. Applying IH (7) to $C(u)$, $D(s)$ and $D(t)$ we obtain $[u]^{\mathcal{I}} \subseteq C^{\mathcal{I}}$, $[s]^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $[t]^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. We can conclude that $[s]^{\mathcal{I}} \neq \emptyset$ by IH (A.1). Consequently $[s]^{\mathcal{I}} \subseteq (D \sqcap \exists \mathsf{Inv}(R).(C \sqcap \exists R.(D \sqcap [t])))^{\mathcal{I}}$. Thus $[s]^{\mathcal{I}} \subseteq [t]^{\mathcal{I}}$ since $C(x) \wedge \mathsf{ar}(R,x,y) \wedge D(y) \wedge \mathsf{ar}(R,x,z) \wedge D(z) \to z \approx y \in \mathcal{C}$. An analogous argument can be made to show that $[t]^{\mathcal{I}} \subseteq [s]^{\mathcal{I}}$. Hence $[s]^{\mathcal{I}} = [t]^{\mathcal{I}}$.
2. ($\approx$-rule) Two possible cases arise:
   (a) $\{s \approx u, u \approx t\} \subseteq \mathcal{A}_{n-1}$ such that $t \in N_{\mathsf{I}}$, u is a descendant of t, or $t \in N_{\mathsf{O}}$ and $u \notin N_{\mathsf{I}}$. We apply IH (3) to both $s \approx u$ and $u \approx t$ to obtain $[s]^{\mathcal{I}} = [u]^{\mathcal{I}}$ and $[u]^{\mathcal{I}} = [t]^{\mathcal{I}}$. Hence $[s]^{\mathcal{I}} = [t]^{\mathcal{I}}$.
   (b) $\{s \approx u, u \approx t\} \subseteq \mathcal{A}_{n-1}$ such that $s \in N_{\mathsf{I}}$, u is a descendant of s, or $s \in N_{\mathsf{O}}$ and $u \notin N_{\mathsf{I}}$. Analogous to the previous case.
3. (NI-rule) Two possible cases arise:

(a) $\{v \approx t, v \approx w@^u_{\leqslant 1R.D}\} \subseteq \mathcal{A}_{n-1}$ such that $s = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, t is a blockable individual and neither v nor t is indirectly blocked. We apply (1) to $v \approx w@^u_{\leqslant 1R.D}$ to conclude $[v]^{\mathcal{I}} = [u.\langle R, D\rangle]^{\mathcal{I}}$. We apply (3) to $v \approx t$ to obtain $[v]^{\mathcal{I}} = [t]^{\mathcal{I}}$. Consequently, $[s]^{\mathcal{I}} = [t]^{\mathcal{I}}$.

(b) $\{s \approx v, v \approx w@^u_{\leqslant 1R.D}\} \subseteq \mathcal{A}_{n-1}$ such that $t = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, s is a blockable individual and neither v nor s is indirectly blocked. Analogous to the previous one.

Remark: IH (2) can be applied in case 1 since $R \prec^*_{\mathcal{O}} R$. Also, note that $\mathsf{ar}(R, u, s) = \mathsf{ar}(\mathsf{Inv}(R), s, u)$.

IH (4) : $s \rightsquigarrow^S_{\mathcal{A}_n} t$ only if

1. (Hyp-rule) $\mathsf{ar}(R, x, y) \rightarrow \mathsf{ar}(S, x, y) \in \mathcal{C}$, $\mathsf{ar}(R, s, t) \in \mathcal{A}_{n-1}$ and $s \rightsquigarrow^R_{\mathcal{A}_{n-1}} t$. We apply IH (4) to $s \rightsquigarrow^R_{\mathcal{A}_{n-1}} t$ to obtain $[s]^{\mathcal{I}} \subseteq (\exists R.[t])^{\mathcal{I}}$. This implies $[s]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$ since $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$.
2. ($\exists$-rule) Two possible cases arise:
   (a) $\exists S.D(s) \in \mathcal{A}_{n-1}$. We can apply IH (7) to $\exists S.D(s)$ to infer $[s]^{\mathcal{I}} \subseteq \exists S.D^{\mathcal{I}}$. Note that $t = s.\langle S, D\rangle$ or $t = \alpha_{SD}$ and consequently $[t] = \exists \mathsf{Inv}(S).[s] \sqcap D$ or $[t] = \exists \mathsf{Inv}(S).\top \sqcap D$. Either way $[s]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$.
   (b) $\exists \mathsf{Inv}(S).D(t) \in \mathcal{A}_{n-1}$. Then $s = t.\mathsf{Inv}(S)D$, $[s] = \exists S.[t] \sqcap D$ and hence $[s]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$.
3. ($\approx$-rule) Two possible cases arise:
   (a) $t \approx u \in \mathcal{A}_{n-1}$ and $s \rightsquigarrow^S_{\mathcal{A}_{n-1}} u$ such that $t \in N_{\mathsf{I}}$, u is a descendant of t, or $t \in N_{\mathsf{O}}$ and $u \notin N_{\mathsf{I}}$. We apply IH (4) to $s \rightsquigarrow^S_{\mathcal{A}_{n-1}} u$ to obtain $[s]^{\mathcal{I}} \subseteq (\exists S.[u])^{\mathcal{I}}$ and IH (3) to $u \approx t$ to obtain $[u]^{\mathcal{I}} = [t]^{\mathcal{I}}$. Hence $[s]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$.
   (b) $s \approx u \in \mathcal{A}_{n-1}$ and $u \rightsquigarrow^S_{\mathcal{A}_{n-1}} t$ such that $s \in N_{\mathsf{I}}$, u is a descendant of s, or $s \in N_{\mathsf{O}}$ and $u \notin N_{\mathsf{I}}$. Analogous to the previous case.
4. (NI-rule) Two possible cases arise:
   (a) $v \rightsquigarrow^S_{\mathcal{A}_{n-1}} t$ and $v \approx w@^u_{\leqslant 1R.D} \in \mathcal{A}_{n-1}$ such that $s = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, t is a blockable individual and neither v nor t is indirectly blocked. We apply (1) to $v \approx w@^u_{\leqslant 1R.D}$ to conclude $[v]^{\mathcal{I}} = [u.\langle R, D\rangle]^{\mathcal{I}}$. We apply (3) to $v \rightsquigarrow^S_{\mathcal{A}_{n-1}} t$ to obtain $[v]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$. Consequently, $[s]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$.
   (b) $s \rightsquigarrow^S_{\mathcal{A}_{n-1}} v$ and $v \approx w@^u_{\leqslant 1R.D} \in \mathcal{A}_{n-1}$ such that $t = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, s is a blockable individual and neither v nor s is indirectly blocked. Analogous to the previous case.

IH (5) $\mathsf{ar}(S,s,t) \in \mathcal{A}_n$ and $s \not\rightsquigarrow^{S}_{\mathcal{A}_n} t$ only if

1. (Hyp-rule) $\mathsf{ar}(R,x,y) \to \mathsf{ar}(S,x,y) \in \mathcal{C}$ and $\mathsf{ar}(R,s,t) \in \mathcal{A}_{n-1}$. Note that necessarily $s \not\rightsquigarrow^{R}_{\mathcal{A}_{n-1}} t$ as otherwise $s \rightsquigarrow^{S}_{\mathcal{A}_n} t$. We apply IH (5) to $\mathsf{ar}(R,s,t)$ to obtain $V \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(R)$ which implies $V \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(S)$.
2. ($\exists$-rule) $\exists \mathsf{Inv}(S).D(t) \in \mathcal{A}_{n-1}$. Then $s = \alpha_{\mathsf{Inv}(S)D}$ and $V \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(S)$ for a safe role $V = \mathsf{Inv}(S)$.
3. ($\approx$-rule) Two possible cases arise:
   (a) $\{s \approx u, \mathsf{ar}(S,u,t)\} \subseteq \mathcal{A}_{n-1}$ such that $s \in N_{\mathsf{I}}$ or $s \in N_{\mathsf{O}}$ and $u \notin N_{\mathsf{I}}$. Note that necessarily $u \not\rightsquigarrow^{S}_{\mathcal{A}_{n-1}} t$ as otherwise $s \rightsquigarrow^{S}_{\mathcal{A}_n} t$. We apply IH (5) to $\mathsf{ar}(S,u,t)$ to obtain $V \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(S)$ for some safe role V.
   (b) $\{\mathsf{ar}(S,s,u), u \approx t\} \subseteq \mathcal{A}_{n-1}$ and $s \rightsquigarrow_{\mathcal{A}} u$ such that $t \in N_{\mathsf{I}}$, or $t \in N_{\mathsf{O}}$ and $u \notin N_{\mathsf{I}}$. Analogous to the previous case.
4. (NI-rule) Analogous to case 2.a.

IH (6): $\mathsf{ar}(R,s,t) \in \mathcal{A}_n$ only if

1. (Hyp-rule) $\mathsf{ar}(S,x,y) \to \mathsf{ar}(R,x,y) \in \mathcal{C}$ and $\mathsf{ar}(S,s,t) \in \mathcal{A}_{n-1}$. We apply IH (6) to $\mathsf{ar}(S,s,t) \in \mathcal{A}_{n-1}$ to obtain $[s]^{\mathcal{I}} \subseteq \exists S.\top^{\mathcal{I}}$. We conclude $[s]^{\mathcal{I}} \subseteq \exists R.\top^{\mathcal{I}}$ since $S^{\mathcal{I}} \subseteq R^{\mathcal{I}}$.
2. ($\exists$-rule) Two possible cases arise:
   (a) $\exists R.D(s) \in \mathcal{A}_{n-1}$. We apply IH (7) to $\exists R.D(s)$ to obtain $[s]^{\mathcal{I}} \subseteq \exists R.D^{\mathcal{I}}$. Hence, $[s]^{\mathcal{I}} \subseteq \exists R.\top$.
   (b) $\exists \mathsf{Inv}(R).D(t) \in \mathcal{A}_{n-1}$. Then $s = t.\mathsf{Inv}(R)D$ or $s = \alpha_{\mathsf{Inv}(R)D}$ and consequently $[s] = \exists R.[t] \sqcap D$ or $[s] = \exists R.\top \sqcap D$. Either way $[s]^{\mathcal{I}} \subseteq \exists R.\top$.
3. ($\approx$-rule) Two possible cases arise:
   (a) $\{\mathsf{ar}(R,s,u), u \approx t\} \subseteq \mathcal{A}_{n-1}$ such that t is named or u is a descendant of t. We apply IH (6) to $\mathsf{ar}(R,s,u) \in \mathcal{A}_{n-1}$ to obtain $[s]^{\mathcal{I}} \subseteq (\exists R.[u])^{\mathcal{I}}$. Hence $s^{\mathcal{I}} \subseteq (\exists R.\top)^{\mathcal{I}}$
   (b) $\{s \approx u, \mathsf{ar}(R,u,t)\} \in \mathcal{A}_{n-1}$ such that s is named or u is a descendant of s. We apply IH (6) to $\mathsf{ar}(R,u,t) \in \mathcal{A}_{n-1}$ to obtain $[u]^{\mathcal{I}} \subseteq (\exists R.\top)^{\mathcal{I}}$ and IH (3) to $s \approx u$ to obtain $[s]^{\mathcal{I}} = [u]^{\mathcal{I}}$. Hence $s^{\mathcal{I}} \subseteq (\exists R.\top)^{\mathcal{I}}$.
4. (NI-rule) Two possible cases arise:
   (a) $\mathsf{ar}(R,v,t)$, $v \approx w@^{u}_{\leqslant 1R.D} \in \mathcal{A}_{n-1}$ such that $s = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, t is a blockable individual and neither v nor t is indirectly blocked. We apply (1) to $v \approx w@^{u}_{\leqslant 1R.D}$ to conclude $[v]^{\mathcal{I}} = [u.\langle R, D\rangle]^{\mathcal{I}}$. We apply (6) to $\mathsf{ar}(R,v,t)$ to obtain $[v]^{\mathcal{I}} \subseteq (\exists S.\top)^{\mathcal{I}}$. Consequently, $[s]^{\mathcal{I}} \subseteq (\exists S.[t])^{\mathcal{I}}$.
   (b) $\{\mathsf{ar}(R,s,v), v \approx w@^{u}_{\leqslant 1R.D}\} \subseteq \mathcal{A}_{n-1}$ such that $t = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, s is a blockable individual and neither v nor s is indirectly blocked. We apply (6) to $\mathsf{ar}(R,s,v)$ to obtain $[s]^{\mathcal{I}} \subseteq (\exists S.\top)^{\mathcal{I}}$.

IH (7) $D(s) \in \mathcal{A}_n$ only if

1. (Hyp-rule) Five possible cases arise:
   (a) $\bigwedge C_i(x) \to D(x) \in \mathcal{C}$ and $C_i(s) \in \mathcal{A}_{n-1}$. We apply (7) to all $C_i(s) \in \mathcal{A}_{n-1}$ to obtain $[s]^{\mathcal{I}} \subseteq C_i^{\mathcal{I}}$. Hence $[s]^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
   (b) $R(x, y) \wedge C(y) \to D(x) \in \mathcal{C}$ and $\{C(s), \mathsf{ar}(R, s, t)\} \subseteq \mathcal{A}_{n-1}$. We apply (7) to $C(s) \in \mathcal{A}_{n-1}$ to obtain $[s]^{\mathcal{I}} \subseteq C$. Two cases arise:
      i. $s \rightsquigarrow^R_{\mathcal{A}_{n-1}} t$. We apply (4) to $\mathsf{ar}(R, s, t)$ to obtain $[s]^{\mathcal{I}} \subseteq (\exists R.[t]^{\mathcal{I}})$. Hence, $[s]^{\mathcal{I}} \subseteq D$
      ii. $s \not\rightsquigarrow^R_{\mathcal{A}_{n-1}} t$. We apply (5 to $\mathsf{ar}(R, s, t)$ to obtain $V \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(R)$ for some safe role V. Note that $\mathsf{ar}(R, x, y) \wedge C(y) \to D(x) \in \mathcal{C}$ implies $\exists R.C \sqsubseteq D \in \mathcal{O}$. Since $V \sqsubseteq^*_{\mathcal{R}} \mathsf{Inv}(R)$ for V a safe role we can conclude that $C = \top$ by contradiction. We apply IH (6) to $\mathsf{ar}(R, s, t)$ to obtain $[s]^{\mathcal{I}} \subseteq (\exists R.[t])^{\mathcal{I}}$ and conclude $[s]^{\mathcal{I}} \subseteq D$.
   (c) $C(x) \wedge R(x, y) \to D(y) \in \mathcal{C}$ and $\{C(t), R(t, s)\} \subseteq \mathcal{A}_{n-1}$. Analogous to the previous case. Note that $C \sqsubseteq \forall R.D \equiv \exists R^-.C \sqsubseteq D$.
   (d) $C(x) \to \geqslant nR.D(x) \in \mathcal{C}$ and $C(s) \in \mathcal{A}_n$. Analogous to case 1.a.
   (e) $\to D(x)$ and s occurs in $\mathcal{A}_n$. Trivial.
2. ($\geqslant$-rule) $\geqslant nR.D(t) \in \mathcal{A}_{n-1}$. Then $s = t.iRD$ or $s = \alpha_{iRD}$. Either way $[s]^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
3. ($\approx$-rule) $\{D(u), u \approx s\} \subseteq \mathcal{A}_{n-1}$ such that such that $u \in N_{\mathsf{I}}$, s is a descendant of u, or $u \in N_{\mathsf{O}}$ and $s \notin N_{\mathsf{I}}$. We apply IH (3) to $u \approx s$ to obtain $[u]^{\mathcal{I}} = [s]^{\mathcal{I}}$ and IH (7) to $D(u)$ to conclude $u^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Hence $[s]^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
4. (NI-rule) $\{D(v), v \approx w@^u_{\leqslant 1R.D}\} \subseteq \mathcal{A}_{n-1}$ such that $s = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, t is a blockable individual and neither v nor t is indirectly blocked. We apply (1) to $v \approx w@^u_{\leqslant 1R.D}$ to conclude $[v]^{\mathcal{I}} = [u.\langle R, D\rangle]^{\mathcal{I}}$. We also apply IH (7) to $D(v)$ to conclude $v^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Hence $[s]^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

IH (8) $s \not\approx t \in \mathcal{A}_n$ only if:

1. ($\geqslant$-rule) $\geqslant nR.D(u) \in \mathcal{A}_n$ such that $n > 2$. Then $s = u.(i, R, D)$ and $t = u.(j, R, D)$, or $s = \alpha_{iRD}$ and $t = \alpha_{jRD}$. Either way $[s]^{\mathcal{I}} \neq [t]^{\mathcal{I}}$.
2. ($\approx$-rule) Two possible cases arise:
   (a) $\{s \not\approx v, v \approx t\} \subseteq \mathcal{A}_{n-1}$ such that $t \in N_{\mathsf{I}}$, v is a descendant of t, or $t \in N_{\mathsf{O}}$ and $v \notin N_{\mathsf{I}}$. We apply IH (8) to $s \not\approx v$ to obtain $[s]^{\mathcal{I}} \neq [v]^{\mathcal{I}}$ and IH (3) to $v \approx t$ and obtain $[v]^{\mathcal{I}} = [t]^{\mathcal{I}}$. Consequently, $[s]^{\mathcal{I}} \neq tv]^{\mathcal{I}}$
   (b) $\{s \approx v, v \not\approx t\} \subseteq \mathcal{A}_{n-1}$ such that $s \in N_{\mathsf{I}}$, u is a descendant of s, or $s \in N_{\mathsf{O}}$ and $u \notin N_{\mathsf{I}}$. Analogous to the previous case.
3. (NI-rule) Analogous to the previous case, just make use of IH 1 instead of 3.

IH (9) claim:s occurs syntactically in $\mathcal{A}_n$ implies $[s]^{\mathcal{I}} \neq \emptyset$. We only need to verify this when s appears for the first time in $\mathcal{A}_n$, as for all other cases we can just use this IH to verify the claim. Individual s appears for the first time in $\mathcal{A}_n$ only if:

1. ($\geqslant$-rule): $\geqslant nR.D(t) \in \mathcal{A}_{n-1}$. Then $s = \alpha_{iRD}$ or $s = t.(i, R, D)$. We apply IH 9 over t to conclude non-emptiness of $[t]^{\mathcal{I}}$. Hence, $[s]^{\mathcal{I}}$ must be non-empty as well.
2. (NI-rule): $v \approx w@^{u}_{\leqslant 1R.D}\} \subseteq \mathcal{A}_{n-1}$ such that $s = u.\langle R, D\rangle$, u is a root individual, v is a blockable individual that is not a successor of u, t is a blockable individual and neither v nor t is indirectly blocked. We apply IH 9 over v to conclude non-emptiness of $[v]^{\mathcal{I}}$. By IH 1 we have that $[v]^{\mathcal{I}} = [s]^{\mathcal{I}}$. Hence, $[s]^{\mathcal{I}}$ must be non-empty as well.

IH (A.1) claim: $\bot \notin \mathcal{A}_n$. $\bot \notin \mathcal{A}_n$ only if $D(x) \rightarrow \in \mathcal{C}$ and $D(s) \in \mathcal{A}_{n-1}$ or $s \not\approx s$ for some s. Note that by IH (9) we have that $[s]^{\mathcal{I}}$ must be non-empty. Hence, $s \not\approx s \notin \mathcal{A}_{n-1}$ by contradiction, as we have that by IH (8) this would imply that $s^{\mathcal{I}} \neq s^{\mathcal{I}}$. Again, by contradiction $D(s) \notin \mathcal{A}_{n-1}$. This would imply that $[s]^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ by IH (8) from which we would conclude unsatisfiability of K. Since we know that K is satisfiable (statement (3)) the is must not be the case. □□

It is straightforward to see the previously modified algorithm produces the same kind of ABoxes for K than the regular algorithm would for $\Psi(\mathcal{K})$ where auxiliary individuals α_{iRD} are substituted by the newly introduced nominals $\{c_{iRD}\}$ (after some extra rule applications). Hence, the previous argument can be used to show equisatisfiability of K and $\Psi(\mathcal{K})$. Since classification is reduced to equisatisfiability, we can always do this reduction to compute the class hierarchy of an ontology. Reusing the previous argument we also proof the following theorem:

Lemma 6. *$K \models C(a)$ if and only if $\Psi(\mathcal{K}) \models C(a)$.*

Proof. The previously (lengthy) argument can be rewritten reformulating IH 8 into

$$D(s) \in \mathcal{A}_n \text{ implies } \mathcal{K} \models [s] \subseteq D$$

Hence, for all $s \in N_{\mathsf{I}}$, $C(s) \in \mathcal{A}_n$ implies $K \models C(s)$. For the completeness part of the argument we can always proof the contrapositive. Namely, that if $D(t) \notin \mathcal{A}_n$ then there exist a model for which $t^{\mathcal{I}} \notin C^{\mathcal{I}}$. This model can be constructed from $\mathcal{A}_n$ as shown in the proof of Lemma 6 in [15]. □

Hence, when all roles are safe and the ontology can be translated into OWL RL, we can perform instance retrieval in "one pass" transforming the DL clauses into datalog rules and materializing all consequences. Equality is transformed in the usual way, adding the following set of predicates to the program:

$$\rightarrow x \approx x$$
$$x \approx y \rightarrow y \approx x$$
$$x \approx y \wedge y \approx z \rightarrow x \approx z$$
$$P(x_1, \ldots, x_i, \ldots x_n) \wedge x_i \approx y \rightarrow P(x_1, \ldots, y, \ldots x_n)$$

for every predicate P and every variable x_i.