Elimination of Complex RIAs without Automata

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Abstract. We present an algorithm that eliminates complex role inclusion axioms (RIAs) from a $\mathcal{SROIQ}$ ontology preserving all logical consequences not involving non-simple roles. Unlike other existing methods, our algorithm does not explicitly construct finite automata recognizing the languages generated by the RIAs. Instead, it is formulated as a recursive expansion of universal restrictions, similar to well-known encodings of transitivity axioms.

1 Introduction

Complex role inclusion axioms (RIAs) $R_1 \circ \ldots \circ R_n \sqsubseteq R$ are an important feature by which the web ontology language OWL 2 [2], based on the description logic (DL) $\mathcal{SROIQ}$ [5], extends the earlier standard OWL DL. Unrestricted use of complex RIAs causes undecidability of the basic reasoning tasks already in case of fairly inexpressive DLs and modal logics such as $\mathcal{ALC}$ [1]; decidability is recovered by requiring that RIAs, when viewed as context-free grammar rules $R \to R_1 \ldots R_n$, generate regular languages [3, 4, 7]. However, checking if a given set of RIAs has this property is already difficult; this problem is related to checking regularity of *pure* context-free grammars [10] (which do not distinguish terminal and non-terminal symbols) whose decidability appears to have been long open [7]. To avoid this difficulty, a stronger syntactic *regularity* condition that requires RIAs to be acyclic (apart from a few selected cases such as transitivity axioms) is imposed in $\mathcal{SROIQ}$. This condition is easy to check and allows for an effective construction of the underlying finite automata [5].

The standard automata construction for $\mathcal{SROIQ}$ [5] is an inductive procedure that performs non-trivial manipulations such as taking the mirrored copy and the disjoint union of previously constructed automata. To implement the procedure directly, developers of OWL 2 reasoners have to either look for suitable third-party libraries that support such operations, or resort to writing their own automata library. In this paper, we present an algorithm that eliminates complex RIAs from a $\mathcal{SROIQ}$ ontology without explicitly using finite automata. Instead, our algorithm is formulated as a simple recursive expansion of universal restrictions, similar to well-known encodings of transitivity axioms, using acyclicity of RIAs directly to ensure termination. For this reason, we believe that our method might be easier to implement in practice. Furthermore, we illustrate that, by introducing new rules for handling universal restrictions, the tableau algorithm for $\mathcal{SROIQ}$ [5] can be modified to apply our elimination on the fly. In that case, no pre-construction due to complex RIAs is needed at all.

Our recursive expansion of universal restrictions is inspired by and directly simulates the recursion in the standard automata construction. Therefore, for many purposes, the result of our elimination algorithm can be regarded as equivalent to the standard automata-based encoding [6].

2 Preliminaries

2.1 The DL $\mathcal{SROIQ}$

For a gentle introduction to DLs we refer the readers to the DL primer [8]. In this section we merely recall the definition (syntax only) of $\mathcal{SROIQ}$ [5], together with the notions of a regular RBox and of polarity of concept occurrence. We follow the approach of Shearer [11] in assuming that the set of simple roles is given in the signature and calling a RIA $w \sqsubseteq R$ complex iff R is non-simple (even if w is of length 1).

A signature $\Sigma = \langle \Sigma_S, \Sigma_R, \Sigma_C, \Sigma_I \rangle$ consists of mutually disjoint sets of *atomic roles* Σ_R, *atomic concepts* Σ_C, and *individuals* Σ_I, together with a distinguished subset $\Sigma_S \subseteq \Sigma_R$ of *simple atomic roles*. The set of *roles* (over Σ) is $\mathbf{R} := \Sigma_R \cup \{R^- \mid R \in \Sigma_R\}$; the set of *simple roles* is $\mathbf{S} := \Sigma_S \cup \{S^- \mid S \in \Sigma_S\}$. A *role chain* is an expression of the form $R_1 \cdot \ldots \cdot R_n$ with $n \geq 1$ and each $R_i \in \mathbf{R}$. The function $\mathsf{inv}(\cdot)$ is defined on roles by $\mathsf{inv}(R) := R^-$ and $\mathsf{inv}(R^-) := R$ where $R \in \Sigma_R$, and extended to role chains by $\mathsf{inv}(R_1 \cdot \ldots \cdot R_n) := \mathsf{inv}(R_n) \cdot \ldots \cdot \mathsf{inv}(R_1)$.

The set $\mathbf{C}$ of $\mathcal{SROIQ}$ *concepts* (over Σ) is defined recursively as follows:

$$\mathbf{C} := \Sigma_C \,|\, \{\Sigma_I\} \,|\, (\mathbf{C} \sqcap \mathbf{C}) \,|\, (\mathbf{C} \sqcup \mathbf{C}) \,|\, \neg\mathbf{C} \,|\, \exists\mathbf{R}.\mathbf{C} \,|\, \forall\mathbf{R}.\mathbf{C} \,|\, \geqslant n\,\mathbf{S}.\mathbf{C} \,|\, \leqslant n\,\mathbf{S}.\mathbf{C} \,|\, \exists\mathbf{S}.\mathit{Self}.$$

A *role inclusion axiom* (RIA) is either a *simple RIA* of the form $S_1 \sqsubseteq S_2$ where $S_1, S_2 \in \mathbf{S}$, or a *complex RIA* of the form $w \sqsubseteq R$ where w is a role chain and $R \in \mathbf{R} \setminus \mathbf{S}$. A *role assertion* is an axiom of the form $\mathsf{Ref}(R)$ (reflexivity), $\mathsf{Irr}(S)$ (irreflexivity), $\mathsf{Uni}(R)$ (universality), or $\mathsf{Dis}(S_1, S_2)$ (role disjointness), where $R \in \mathbf{R}$ and $S_{(i)} \in \mathbf{S}$. Transitivity and symmetry must be expressed as $R \cdot R \sqsubseteq R$ and $\mathsf{inv}(R) \sqsubseteq R$ respectively. An *RBox* is a finite set of RIAs and role assertions.

A *regular order* $\prec$ is an irreflexive transitive binary relation on the set of roles $\mathbf{R}$ satisfying $R_1 \prec R_2$ iff $\mathsf{inv}(R_1) \prec \mathsf{inv}(R_2)$. An RBox $\mathcal{R}$ is $\prec$*-regular* if each RIA in $\mathcal{R}$ is of one of the following forms:

(R1) $R_1 \cdot \ldots \cdot R_n \sqsubseteq R$ with $R_i \prec R$ for all $1 \leq i \leq n$;
(R2) $R \cdot R_1 \cdot \ldots \cdot R_n \sqsubseteq R$ with $R_i \prec R$ for all $1 \leq i \leq n$;
(R3) $R_1 \cdot \ldots \cdot R_n \cdot R \sqsubseteq R$ with $R_i \prec R$ for all $1 \leq i \leq n$;
(R4) $R \cdot R \sqsubseteq R$;
(R5) $\mathsf{inv}(R) \sqsubseteq R$.

An RBox $\mathcal{R}$ is *regular* if it is $\prec$-regular for some regular order $\prec$. For a regular RBox $\mathcal{R}$, let $\prec_{\mathcal{R}}$ be the intersection of all regular orders $\prec$ such that $\mathcal{R}$ is $\prec$-regular; the *depth* of $\mathcal{R}$ is the maximal n for which there exists a sequence $R_1 \prec_{\mathcal{R}} \ldots \prec_{\mathcal{R}} R_n$. It is easy to show that if $\mathcal{R}$ is regular, then it is $\prec_{\mathcal{R}}$-regular.

A *TBox* is a finite set of *general concepts inclusions* (GCIs) of the form $C \sqsubseteq D$ where $C, D \in \mathbf{C}$. To keep the presentation simple, we do not allow ABox assertions; these can be expressed as GCIs using nominals. A $\mathcal{SROIQ}$ ontology (over Σ) is a pair $\mathcal{O} = \langle \mathcal{R}, \mathcal{T} \rangle$ where $\mathcal{R}$ is a regular RBox and $\mathcal{T}$ a TBox.

Polarities of occurrences of $\mathcal{SROIQ}$ concepts in concepts and GCIs are defined inductively as follows: C occurs positively in C. If C occurs positively (resp. negatively) in C', then C occurs positively (resp. negatively) in $C' \sqcap D$, $D \sqcap C'$, $C' \sqcup D$, $D \sqcup C'$, $\exists R.C'$, $\forall R.C'$, $\geqslant n\, R.C'$, and $D \sqsubseteq C'$, and C occurs negatively (resp. positively) in $\neg C'$, $\leqslant n\, R.C'$, and $C' \sqsubseteq D$.

2.2 Conservative Encodings

We use the framework of conservative extensions [9] to prove correctness of our encoding of RIAs. Let $\mathcal{O}$ be an ontology over a signature $\Sigma = \langle \Sigma_S, \Sigma_R, \Sigma_C, \Sigma_I \rangle$, and let $\mathcal{Q}$ be an ontology over a (not necessarily strictly) larger signature. Then $\mathcal{Q}$ is a *conservative encoding* of $\mathcal{O}$ (also $\mathcal{Q}$ is conservative over $\mathcal{O}$) if

(i) for every model $\mathcal{I}$ of $\mathcal{O}$ there exists a model $\mathcal{J}$ of $\mathcal{Q}$ such that $\mathcal{I}$ and $\mathcal{J}$ have the same domain and coincide on the interpretation of Σ_R, Σ_C, and Σ_I, and
(ii) for every model $\mathcal{J}$ of $\mathcal{Q}$ there exists a model $\mathcal{I}$ of $\mathcal{O}$ such that $\mathcal{I}$ and $\mathcal{J}$ have the same domain and coincide on the interpretation of Σ_R, Σ_C, and Σ_I.

Since this definition is sensitive to Σ, to avoid ambiguity, we will assume that each ontology carries its signature with it, so that each ontology is over only one signature. The signature may, however, contain symbols not occurring in the ontology. Note that, unlike the standard notion of a conservative extension, the above notion of a conservative encoding does not require that $\mathcal{Q}$ contains all axioms from $\mathcal{O}$.

We define a *simple-conservative encoding* analogously except that the models $\mathcal{I}$ and $\mathcal{J}$ are only required to coincide on Σ_S, Σ_C, and Σ_I. As observed by Lutz et al. [9], this model-theoretic notion of conservativity implies that the two ontologies entail the same consequences over Σ_S, Σ_C, and Σ_I. We will prove that our encoding of complex RIAs produces a simple-conservative encoding of the input ontology. Thus, in particular, the two ontologies have the same classification (ignoring the extra atomic concepts introduced in the encoding). Furthermore, by introducing new concept definitions $A \equiv C$ and $B \equiv D$ in the original ontology, one can check subsumptions even between concepts C and D which contain non-simple roles.

2.3 Languages Generated by RIAs

Each RIA $w \sqsubseteq R$ can be expressed equivalently as $\mathsf{inv}(w) \sqsubseteq \mathsf{inv}(R)$; to avoid having to keep this in mind, let $\mathcal{R}^c := \mathcal{R} \cup \{\mathsf{inv}(w) \sqsubseteq \mathsf{inv}(R) \mid w \sqsubseteq R \in \mathcal{R}\}$ be the *completion* of the RBox $\mathcal{R}$. Note that $\mathcal{R}$ and $\mathcal{R}^c$ are equivalent and $\mathcal{R}$ is $\prec$-regular iff $\mathcal{R}^c$ is $\prec$-regular.

The languages $L_\mathcal{R}(R)$ are defined inductively by (i) $R \in L_\mathcal{R}(R)$ for each role R, and (ii) if $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^c$ and $w_i \in L_\mathcal{R}(R_i)$ for all $1 \leq i \leq n$, then $w_1 \cdot \ldots \cdot w_n \in L_\mathcal{R}(R)$. Intuitively, $L_\mathcal{R}(R)$ is the language generated from the role R by the grammar rules $\{R \rightarrow w \mid w \sqsubseteq R \in \mathcal{R}^c\}$. Horrocks et al. [5] showed that if $\mathcal{R}$ is regular, then each $L_\mathcal{R}(R)$ is a regular language, the finite automata recognizing $L_\mathcal{R}(R)$ can be effectively constructed by induction over $\prec_\mathcal{R}$, and the size of these automata is at most exponential in the depth of $\mathcal{R}$.

An interpretation function $\cdot^\mathcal{I}$ is extended to the languages $L_\mathcal{R}(R)$ as follows:

$$L_\mathcal{R}(R)^\mathcal{I} = \{\langle x, y\rangle \mid \text{there exists } w \in L_\mathcal{R}(R) \text{ such that } \langle x, y\rangle \in w^\mathcal{I}\}. \qquad (1)$$

One can easily prove by induction on the definition of $L_\mathcal{R}(R)$ that $w \in L_\mathcal{R}(R)$ implies $\mathcal{R} \models w \sqsubseteq R$. The following proposition is a direct consequence of this fact.

Proposition 1. *If $\mathcal{I} \models \mathcal{R}$, then $L_\mathcal{R}(R)^\mathcal{I} = R^\mathcal{I}$.*

3 Motivation

In this section we motivate and present (first approximation of) our RIA-elimination algorithm. For simplicity, we do not yet consider symmetry axioms (R5) in this section.

Many $\mathcal{SROIQ}$ constructors are restricted to simple roles and therefore do not interact with complex RIAs. The key step in eliminating complex RIAs is to capture the propagation of universal restrictions over non-simple roles. Consider the following property:

$$\text{if } x \in (\forall R.C)^{\mathcal{I}} \text{ and } \langle x, y\rangle \in L_{\mathcal{R}}(R)^{\mathcal{I}}, \text{ then } y \in C^{\mathcal{I}}. \quad (2)$$

Every model $\mathcal{I}$ of the RBox $\mathcal{R}$ satisfies (2) simply because $L_{\mathcal{R}}(R)^{\mathcal{I}} = R^{\mathcal{I}}$ by Proposition 1, in which case (2) coincides with semantics of universal restrictions. The main idea behind all methods for dealing with complex RIAs (e.g., [4–6]) is to axiomatise (using a finite number of GCIs) the property (2) for all $\forall R.C$ occurring in the ontology, and use this axiomatisation to simulate the presence of complex RIAs from $\mathcal{R}$.

In the simplest case, when all RIAs in $\mathcal{R}$ are of the form (R1), this can be achieved by a simple recursive expansion of all universal restrictions occurring in the ontology. To expand $\forall R.C$, for each RIA $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$ introduce the axiom

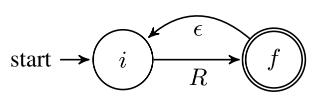
$$\forall R.C \sqsubseteq \forall R_1.\forall R_2 \ldots \forall R_n.C, \quad (3)$$

and recursively expand all the nested universal restrictions on the right-hand side of (3). If all RIAs are of the form (R1), then $R_i \prec_{\mathcal{R}} R$ for each role R_i in (3), so the depth of the recursion is bounded by the depth of $\mathcal{R}$ and the expansion terminates.

In case there are other forms of RIAs in $\mathcal{R}$, e.g., transitivity axioms, the recursion would never terminate. On the other hand, transitivity axioms can be eliminated using several well-known encodings. For example, to capture (2) for the concept $\forall R.C$ with respect to the transitivity axiom $R \cdot R \sqsubseteq R$, introduce two new atomic concepts I and F not in the signature of the the ontology and assert

$$\forall R.C \sqsubseteq I, \quad F \sqsubseteq C, \quad I \sqsubseteq \forall R.F, \quad F \sqsubseteq I. \quad (4)$$

This encoding is inspired by the fact that the RIA $R \cdot R \sqsubseteq R$ generates the regular language R^+ which is recognised by the following finite automaton:

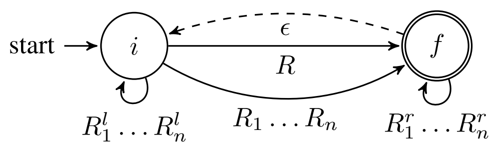


The encoding simulates the run of this automaton from all instances of $\forall R.C$ in a model of (4). Concepts I and F respectively correspond to the initial state i and the final state f, the first axiom initialises the automaton at all $x \in (\forall R.C)^{\mathcal{I}}$, the second axiom ensures that $y \in C^{\mathcal{I}}$ for all y in the final state, and the last two axioms encode the transitions of the automaton. This method generalises easily to all cases when the language $L_{\mathcal{R}}(R)$ is given by a finite automaton [6].

We propose a RIA-elimination algorithm that does not assume that the automaton is already fully constructed. Our method recursively expands all universal restrictions similarly to (3), but uses a two-state automaton at each step to handle the cyclic forms of RIAs similarly to (4). More specifically, to expand the universal restriction $\forall R.C$, introduce two new atomic concepts I and F, assert

(E0) $\forall R.C \sqsubseteq I$, $F \sqsubseteq C$, and $I \sqsubseteq \forall R.F$,
(E1) $I \sqsubseteq \forall R_1.\ldots.\forall R_n.F$ for each RIA $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$ of form (R1);
(E2) $F \sqsubseteq \forall R_1^r.\ldots.\forall R_n^r.F$ for each RIA $R \cdot R_1^r \cdot \ldots \cdot R_n^r \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$ of form (R2),
(E3) $I \sqsubseteq \forall R_1^l.\ldots.\forall R_n^l.I$ for each RIA $R_1^l \cdot \ldots \cdot R_n^l \cdot R \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$ of form (R3),
(E4) $F \sqsubseteq I$ if $R \cdot R \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$,

and recursively expand all the universal restrictions introduced in (E1)–(E3), but not the $\forall R.F$ introduced in (E0). Regularity of $\mathcal{R}$ ensures that the depth of the recursion is bounded by the depth of $\mathcal{R}$. This encoding is inspired by the following automaton (the ϵ-edge is present iff $R \cdot R \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$) used in the construction by Horrocks et al. [5] that recognises the language generated from R by the RIAs referred to in (E1)–(E4) :



Example 1. We will now demonstrate the RIA-elimination algorithm on an example. Let $\Sigma_S = \{P, R\}$, $\Sigma_R = \{P, R, S, T\}$, $\Sigma_C = \{A, B, C, D\}$, $\Sigma_I = \emptyset$, and let $\mathcal{O} = \langle \mathcal{R}, \mathcal{T} \rangle$ be the following ontology over the signature $\Sigma = \langle \Sigma_S, \Sigma_R, \Sigma_C, \Sigma_I \rangle$:

$$\mathcal{R} = \{T \cdot S \sqsubseteq T, \quad T \cdot T \sqsubseteq T, \quad R \cdot S \sqsubseteq S, \quad P \sqsubseteq R\}, \tag{5}$$

$$\mathcal{T} = \{A \sqsubseteq D \sqcup \forall T.\neg C, \quad B \sqsubseteq \exists T.\exists P.\exists S.C\}. \tag{6}$$

Note that $\mathcal{R}$ is regular and $P \prec_{\mathcal{R}} R \prec_{\mathcal{R}} S \prec_{\mathcal{R}} T$. The RIA $P \sqsubseteq R$ is simple; the remaining RIAs in $\mathcal{R}$ are complex. To expand the universal restriction $\forall T.\neg C$ occurring in $\mathcal{T}$, introduce new atomic concepts I_1 and F_1, and assert the following axioms:

$$\forall T.\neg C \sqsubseteq I_1, \quad F_1 \sqsubseteq \neg C, \quad I_1 \sqsubseteq \forall T.F_1 \qquad \text{by (E0)} \tag{7}$$

$$F_1 \sqsubseteq \forall S.F_1 \qquad \text{by (E2) for } T \cdot S \sqsubseteq T \tag{8}$$

$$F_1 \sqsubseteq I_1 \qquad \text{by (E4) for } T \cdot T \sqsubseteq T \tag{9}$$

Then, to recursively expand the new universal restriction $\forall S.F_1$ occurring in (8), introduce new atomic concepts I_2 and F_2, and assert the following axioms:

$$\forall S.F_1 \sqsubseteq I_2, \quad F_2 \sqsubseteq F_1, \quad I_2 \sqsubseteq \forall S.F_2 \qquad \text{by (E0)} \tag{10}$$

$$I_2 \sqsubseteq \forall R.I_2 \qquad \text{by (E3) for } R \cdot S \sqsubseteq S \tag{11}$$

Finally, since we intend to keep all simple RIAs in the RBox and the role R is simple, the new universal restriction $\forall R.I_2$ introduced in (11) does not need to be further expanded. Let $\mathcal{U}$ be the TBox consisting of the new axioms (7)–(11). The results of the next section will establish that the ontology $\mathcal{Q} = \langle \{P \sqsubseteq R\}, \mathcal{T} \cup \mathcal{U} \rangle$ is simple-conservative over $\mathcal{O}$, so, in particular, the two ontologies entail the same consequences over Σ_S, Σ_C, and Σ_I. For example, both $\mathcal{O}$ and $\mathcal{Q}$ entail $P \sqsubseteq R$ and $A \sqcap B \sqsubseteq D$. Note that this cannot be strengthened to all consequences over Σ since, for example, $\mathcal{O}$ entails $T \cdot P \cdot S \sqsubseteq T$ and $B \sqsubseteq \exists T.C$, but $\mathcal{Q}$ does not entail either of these axioms.

4 The RIA-Elimination Algorithm

In this section we formally present our RIA-elimination algorithm and prove that it produces a simple-conservative encoding of the input ontology. Some of the proofs are rather technical, in those cases we present only brief sketches here. More detailed proofs can be found in the appendix.

There are several important points about the algorithm that were omitted in the previous section in favour of simplicity; this is amended in this section. Firstly, if R is a symmetric role, i.e., $\mathsf{inv}(R) \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$, then the concept $\forall R.C$ is additionally expanded in the same way as $\forall\mathsf{inv}(R).C$ would be. Secondly, since we do not assume that the ontology is in negation normal form, we have to treat negative occurrences of existential restrictions similarly to positive occurrences of universal restrictions. Finally, the expansion rule (E0) is inefficient because it introduces the axiom $\forall R.C \sqsubseteq I$ in which $\forall R.C$ occurs negatively. Negative occurrences of universal restrictions are not Horn, i.e., they lead to non-determinism in reasoning. To avoid this problem, instead of asserting $\forall R.C \sqsubseteq I$, the algorithm *replaces* all positive occurrences of $\forall R.C$ in the original ontology by I. This way we obtain a Horn-preserving encoding.

To keep track of the progress of the algorithm, we label those concepts $\forall R.C$ and $\exists R.C$ that still need to be expanded with $\mathcal{R}$ as defined below.

Definition 1 ($\mathcal{R}$-labelled concepts). *Given an RBox $\mathcal{R}$, we introduce new concept constructors $\forall_{\mathcal{R}} R.C$ and $\exists_{\mathcal{R}} R.C$ called* $\mathcal{R}$-labelled universals *and* $\mathcal{R}$-labelled existentials *respectively. Their semantics is defined as follows:*

$$(\forall_{\mathcal{R}} R.C)^{\mathcal{I}} = \{x \mid \forall y : \langle x, y\rangle \in L_{\mathcal{R}}(R)^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}\},$$
$$(\exists_{\mathcal{R}} R.C)^{\mathcal{I}} = \{x \mid \exists y : \langle x, y\rangle \in L_{\mathcal{R}}(R)^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}.$$

$\mathcal{R}$-labelled concepts *are $\mathcal{SROIQ}$ concepts that* may *additionally contain $\mathcal{R}$-labelled universals and existentials. To distinguish them from normal $\mathcal{SROIQ}$ concepts, we sometimes call the latter* unlabelled*. Similarly, we speak of $\mathcal{R}$-labelled (resp. unlabelled) ontologies.*

Note that the semantics of $\mathcal{R}$-labelled concepts is irrelevant for the execution of the algorithm. It does, however, greatly simplify the proofs, since with this semantics we can prove that each intermediate expansion step of the algorithm already produces a simple-conservative encoding of the input ontology.

Given an input ontology $\mathcal{O} = \langle\mathcal{R}, \mathcal{T}\rangle$, the initial step of the algorithm is to remove all complex RIAs from $\mathcal{O}$ (keeping all simple RIAs and all role assertions) and label all positive occurrences of universal restrictions and all negative occurrences of existential restrictions in $\mathcal{O}$ with $\mathcal{R}$ to indicate that they need to be expanded. This is defined more formally in the following two definitions.

Definition 2 (labelling). *Let $\mathcal{R}$ be an RBox. For x an unlabelled concept or a TBox, let $\sigma_{\mathcal{R}}(x)$ be the result of labelling each positive occurrence of each universal restriction and each negative occurrence of each existential restriction in x with $\mathcal{R}$. Dually, let $\bar{\sigma}_{\mathcal{R}}(x)$ be the result of labelling each negative occurrence of each universal restriction and each positive occurrence of each existential restriction in x with $\mathcal{R}$.*

Definition 3 (initialisation). *Let $\mathcal{O} = \langle \mathcal{R}, \mathcal{T} \rangle$ be an unlabelled $\mathcal{SROIQ}$ ontology over a signature Σ. Let $\mathcal{R}^s \coloneqq \mathcal{R} \setminus \{w \sqsubseteq R \in \mathcal{R} \mid w \sqsubseteq R$ is a complex RIA$\}$. The* initialisation *of $\mathcal{O}$ is the $\mathcal{R}$-labelled ontology $\langle \mathcal{R}^s, \sigma_{\mathcal{R}}(\mathcal{T}) \rangle$ over the same signature Σ.*

The next theorem proves that initialisation produces a simple-conservative encoding of $\mathcal{O}$. This captures the intuition that positive universal restrictions and negative existential restrictions are the only $\mathcal{SROIQ}$ features that interact with complex RIAs.

Theorem 1. *The initialisation of $\mathcal{O}$ is simple-conservative over $\mathcal{O}$.*

Proof (sketch). We must show that (i) for each model $\mathcal{I}$ of $\mathcal{O}$ there is a model $\mathcal{J}$ of $\mathcal{Q} = \langle \mathcal{R}^s, \sigma_{\mathcal{R}}(\mathcal{T}) \rangle$ that agrees with $\mathcal{I}$ on Σ_S, Σ_C, and Σ_I, and (ii) vice versa.

For (i), we show that each model $\mathcal{I}$ of $\mathcal{O}$ is already a model of $\mathcal{Q}$. Trivially, $\mathcal{I} \models \mathcal{R}$ implies $\mathcal{I} \models \mathcal{R}^s$ since $\mathcal{R}^s \subseteq \mathcal{R}$. By Proposition 1, we have $L_{\mathcal{R}}(R)^{\mathcal{I}} = R^{\mathcal{I}}$, so $(\forall R.C)^{\mathcal{I}} = (\forall_{\mathcal{R}} R.C)^{\mathcal{I}}$ and $(\exists R.C)^{\mathcal{I}} = (\exists_{\mathcal{R}} R.C)^{\mathcal{I}}$ for all concepts $\forall R.C$ and $\exists R.C$. This means that $\mathcal{R}$-labelling does not affect the interpretation of concepts in $\mathcal{I}$, so $\mathcal{I} \models \mathcal{T}$ implies $\mathcal{I} \models \sigma_{\mathcal{R}}(\mathcal{T})$. Therefore $\mathcal{I} \models \mathcal{Q}$.

For (ii), each model $\mathcal{J}$ of $\mathcal{Q}$ can be transformed to a model $\mathcal{I}$ of $\mathcal{O}$ by extending the interpretation of roles to $R^{\mathcal{I}} = L_{\mathcal{R}}(R)^{\mathcal{J}}$. From $\mathcal{J} \models \mathcal{R}^s$ one proves that $\mathcal{I}$ and $\mathcal{J}$ agree on simple roles. Checking that $\mathcal{I} \models \mathcal{R}$ is routine. The key step to prove $\mathcal{I} \models \mathcal{T}$ is to show by structural induction for each unlabelled concept D that $\sigma_{\mathcal{R}}(D)^{\mathcal{J}} \subseteq D^{\mathcal{I}} \subseteq \bar{\sigma}_{\mathcal{R}}(D)^{\mathcal{J}}$; since $\sigma_{\mathcal{R}}(\mathcal{T}) = \{\bar{\sigma}_{\mathcal{R}}(C) \sqsubseteq \sigma_{\mathcal{R}}(D) \mid C \sqsubseteq D \in \mathcal{T}\}$, we can then infer $\mathcal{I} \models \mathcal{T}$ from $\mathcal{J} \models \sigma_{\mathcal{R}}(\mathcal{T})$. Hence $\mathcal{I} \models \mathcal{O}$. □

After initialisation, the algorithm repeatedly expands all $\mathcal{R}$-labelled universals and existentials until it arrives at an unlabelled ontology. Before we define these expansions, in the next definition we first introduce an auxiliary function $\mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F)$ that encodes the axiom $I \sqsubseteq \forall_{\mathcal{R}} R.F$ only using universals $\forall_{\mathcal{R}} S.D$ with $S \prec_{\mathcal{R}} R$. The function implements the transition function of the two-state automaton from the previous section, and additionally deals with symmetric roles R by expanding $I \sqsubseteq \forall_{\mathcal{R}} R.F$ in the same way as $I \sqsubseteq \forall_{\mathcal{R}} \mathsf{inv}(R).F$. The correctness of this encoding is expressed in the following Proposition 2. Its proof can be found in the appendix.

Definition 4 (expand). *Let $\mathcal{R}$ be a regular RBox, R a role, and I and F atomic concepts. We define $\mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} R.F)$ to be the set consisting of the following GCIs:*

1. *$I \sqsubseteq \forall R.F$,*
2. *$I \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_n.F$ for each $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^c$,*
3. *$F \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_n.F$ for each $R \cdot R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^c$,*
4. *$I \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_n.I$ for each $R_1 \cdot \ldots \cdot R_n \cdot R \sqsubseteq R \in \mathcal{R}^c$,*
5. *$F \sqsubseteq I$ if $R \cdot R \sqsubseteq R \in \mathcal{R}^c$,*

where each R_i is distinct from R. Finally, we define $\mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F)$ to be

$$\begin{cases} \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} R.F) \cup \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} \mathsf{inv}(R).F) & \text{if } \mathsf{inv}(R) \sqsubseteq R \in \mathcal{R}^c, \\ \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} R.F) & \text{otherwise}. \end{cases}$$

Proposition 2. *If $\mathcal{I} \models \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F)$, then $\mathcal{I} \models I \sqsubseteq \forall_{\mathcal{R}} R.F$.*

We are now ready to define the expansions of $\mathcal{R}$-labelled concepts. Similarly to structural transformation, $\forall_\mathcal{R}$-expansion uses atomic concepts I and F as new names for the concepts $\forall_\mathcal{R} R.C$ and C respectively, replaces all positive occurrences of $\forall_\mathcal{R} R.C$ by I, adds $F \sqsubseteq C$, and, instead of asserting $I \sqsubseteq \forall_\mathcal{R} R.F$, it uses $\mathsf{expand}(I \sqsubseteq \forall_\mathcal{R} R.F)$ to encode the same property. $\exists_\mathcal{R}$-expansion works similarly but it additionally uses the equivalence of $\exists R.I \sqsubseteq F$ with $I \sqsubseteq \forall \mathsf{inv}(R).F$; it uses atomic concepts I and F as new names for the concepts C and $\exists_\mathcal{R} R.C$ respectively, adds $C \sqsubseteq I$, replaces all negative occurrences of $\exists_\mathcal{R} R.C$ by F, and, instead of asserting $\exists_\mathcal{R} R.I \sqsubseteq F$, it uses $\mathsf{expand}(I \sqsubseteq \forall_\mathcal{R} \mathsf{inv}(R).F)$ to encode the same property. More formal definitions follow.

Definition 5 (substitution). *For concepts C_{old} and C_{new}, and x a concept or a TBox, let $\rho^+[C_{new} \,/\, C_{old}](x)$ resp. $\rho^-[C_{new} \,/\, C_{old}](x)$ be the result of simultaneously replacing each positive resp. negative occurrence of C_{old} in x by C_{new}.*

Definition 6 ($\forall_\mathcal{R}$- and $\exists_\mathcal{R}$-expansions). *Let $\mathcal{R}$ be a regular RBox and let $\mathcal{O} = \langle \mathcal{R}', \mathcal{T} \rangle$ be an $\mathcal{R}$-labelled ontology over a signature $\Sigma = \langle \Sigma_S, \Sigma_R, \Sigma_C, \Sigma_I \rangle$.*

$\forall_\mathcal{R}$-expansion:
*Let $\forall_\mathcal{R} R.C$ be an $\mathcal{R}$-labelled universal occurring in $\mathcal{O}$. Let I and F be two different atomic concepts not in Σ_C. The $\forall_\mathcal{R} R.C$-*expansion *of $\mathcal{O}$ is the ontology*

$$\langle \mathcal{R}',\ \rho^+[I \,/\, \forall_\mathcal{R} R.C](\mathcal{T}) \cup \{F \sqsubseteq C\} \cup \mathsf{expand}(I \sqsubseteq \forall_\mathcal{R} R.F)\ \rangle.$$

$\exists_\mathcal{R}$-expansion:
*Let $\exists_\mathcal{R} R.C$ be an $\mathcal{R}$-labelled existential occurring in $\mathcal{O}$. Let I and F be two different atomic concepts not in Σ_C. The $\exists_\mathcal{R} R.C$-*expansion *of $\mathcal{O}$ is the ontology*

$$\langle \mathcal{R}',\ \rho^-[F \,/\, \exists_\mathcal{R} R.C](\mathcal{T}) \cup \{C \sqsubseteq I\} \cup \mathsf{expand}(I \sqsubseteq \forall_\mathcal{R} \mathsf{inv}(R).F)\ \rangle.$$

In both cases, the resulting ontology is over the signature $\langle \Sigma_S, \Sigma_R, \Sigma_C \cup \{I, F\}, \Sigma_I \rangle$.

Note that the initialisation of an ontology contains only positive occurrences of $\mathcal{R}$-labelled universals and only negative occurrences of $\mathcal{R}$-labelled existentials. Furthermore, both expansions introduce only positive occurrences of $\mathcal{R}$-labelled universals. Therefore, when used in the context of the RIA-elimination algorithm, the above expansions will actually eliminate *all* occurrences of $\forall_\mathcal{R} R.C$ resp. $\exists_\mathcal{R} R.C$ from the ontology. The reason for making the substitutions polarity-sensitive was to make the following theorem hold for arbitrary ontologies.

Theorem 2. *Let $\mathcal{R}$ be a regular RBox, let $\mathcal{O}$ be an $\mathcal{R}$-labelled ontology, and let $\mathcal{Q}$ be a $\forall_\mathcal{R} R.C$- or $\exists_\mathcal{R} R.C$-expansion of $\mathcal{O}$. Then $\mathcal{Q}$ is conservative over $\mathcal{O}$.*

Proof (sketch). Let $\mathcal{O} = \langle \mathcal{R}', \mathcal{T}_1 \rangle$ be an $\mathcal{R}$-labelled ontology over Σ and let $\mathcal{Q} = \langle \mathcal{R}', \mathcal{T}_2 \rangle$ be a $\forall_\mathcal{R} R.C$-expansion of $\mathcal{O}$. We need to show that (i) for each model $\mathcal{I}$ of $\mathcal{O}$ there exists a model $\mathcal{J}$ of $\mathcal{Q}$ that agrees with $\mathcal{I}$ on Σ, and (ii) vice versa.

For (i), each model $\mathcal{I}$ of $\mathcal{O}$ can be extended to a model of $\mathcal{Q}$ by interpreting the new concepts $I^\mathcal{I} := (\forall_\mathcal{R} R.C)^\mathcal{I}$ and $F^\mathcal{I} := (\exists_\mathcal{R} \mathsf{inv}(R).\forall_\mathcal{R} R.C)^\mathcal{I}$. The substitution merely replaces some occurrences of $\forall_\mathcal{R} R.C$ by I, and $(\forall_\mathcal{R} R.C)^\mathcal{I} = I^\mathcal{I}$, so $\mathcal{I} \models \mathcal{T}_1$ implies $\mathcal{I} \models \rho^+[I \,/\, \forall_\mathcal{R} R.C](\mathcal{T}_1)$. To conclude that $\mathcal{I}$ is a model of $\mathcal{Q}$, it remains to check that

$\mathcal{I}$ satisfies each axiom in $\{F \sqsubseteq C\} \cup \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F)$. This is done using the definition of $I^{\mathcal{I}}$ and $F^{\mathcal{I}}$; for example, $\mathcal{I} \models F \sqsubseteq C$ holds because $\exists_{\mathcal{R}} \mathsf{inv}(R).\forall_{\mathcal{R}} R.C \sqsubseteq C$ is a tautology. The remaining axioms can be checked similarly.

For (ii), we show that each model $\mathcal{J}$ of $\mathcal{Q}$ is already a model of $\mathcal{O}$. To prove $\mathcal{J} \models \mathcal{T}_1$, the key step is to prove by structural induction for all concepts D over Σ that

$$\rho^{+}[I \,/\, \forall_{\mathcal{R}} R.C](D)^{\mathcal{J}} \subseteq D^{\mathcal{J}} \subseteq \rho^{-}[I \,/\, \forall_{\mathcal{R}} R.C](D)^{\mathcal{J}}; \qquad (12)$$

then $\mathcal{J} \models \rho^{+}[I \,/\, \forall_{\mathcal{R}} R.C](\mathcal{T}_1) \subseteq \mathcal{T}_2$ implies $\mathcal{J} \models \mathcal{T}_1$. The only non-trivial case in the induction is $D = \forall_{\mathcal{R}} R.C$, where (12) reduces to $I^{\mathcal{J}} \subseteq (\forall_{\mathcal{R}} R.C)^{\mathcal{J}}$; to show this, we apply Proposition 2 to $\mathcal{J} \models (\{F \sqsubseteq C\} \cup \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F)) \subseteq \mathcal{T}_2$ to infer $\mathcal{J} \models I \sqsubseteq \forall_{\mathcal{R}} R.C$, which is equivalent to the required $I^{\mathcal{J}} \subseteq (\forall_{\mathcal{R}} R.C)^{\mathcal{J}}$.

The proof of the case when $\mathcal{Q}$ is an $\exists_{\mathcal{R}} R.C$-expansion of $\mathcal{O}$, is similar: Each model $\mathcal{I}$ of $\mathcal{O}$ can be extended to a model of $\mathcal{Q}$ by interpreting $I^{\mathcal{I}} := (\forall_{\mathcal{R}} \mathsf{inv}(R).\exists_{\mathcal{R}} R.C)^{\mathcal{I}}$ and $F^{\mathcal{I}} := (\exists_{\mathcal{R}} R.C)^{\mathcal{I}}$. The substitution merely replaces some occurrences of $\exists_{\mathcal{R}} R.C$ by F, and $(\exists_{\mathcal{R}} R.C)^{\mathcal{I}} = F^{\mathcal{I}}$, so $\mathcal{I} \models \mathcal{T}_1$ implies $\mathcal{I} \models \rho^{-}[F \,/\, \exists_{\mathcal{R}} R.C](\mathcal{T}_1)$. To conclude that $\mathcal{I}$ is a model of $\mathcal{Q}$, one can check that the definition of $I^{\mathcal{I}}$ and $F^{\mathcal{I}}$ satisfies each axiom in $\{C \sqsubseteq I\} \cup \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} \mathsf{inv}(R).F)$.

To show that each model $\mathcal{J}$ of $\mathcal{Q}$ is a model of $\mathcal{O}$, prove by structural induction for all concepts D over Σ that $\rho^{-}[F \,/\, \exists_{\mathcal{R}} R.C](D)^{\mathcal{J}} \subseteq D^{\mathcal{J}} \subseteq \rho^{+}[F \,/\, \exists_{\mathcal{R}} R.C](D)^{\mathcal{J}}$. This, in the only non-trivial case $D = \exists_{\mathcal{R}} R.C$, reduces to $(\exists_{\mathcal{R}} R.C)^{\mathcal{J}} \subseteq F^{\mathcal{J}}$; to show this, apply Proposition 2 to $\mathcal{J} \models (\{C \sqsubseteq I\} \cup \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} \mathsf{inv}(R).F)) \subseteq \mathcal{T}_2$ to infer $\mathcal{J} \models C \sqsubseteq \forall_{\mathcal{R}} \mathsf{inv}(R).F$, which is equivalent to the required $(\exists_{\mathcal{R}} R.C)^{\mathcal{J}} \subseteq F^{\mathcal{J}}$. □

The following theorem is our main result. It ensures that the RIA-elimination algorithm produces a simple-conservative encoding of the input ontology, and that the number of expansions is at most exponential in the depth of $\mathcal{R}$. Therefore, since each expansion is linear in the size of $\mathcal{R}$, the algorithm can be implemented to run in time exponential in the depth of $\mathcal{R}$, which is optimal since complex RIAs are known to incur an exponential increase in the complexity of reasoning [6].

Theorem 3. *Let $\mathcal{O} = \langle \mathcal{R}, \mathcal{T} \rangle$ be an unlabelled $\mathcal{SROIQ}$ ontology, let $\langle \mathcal{R}^{s}, \mathcal{T}_0 \rangle$ be the initialisation of $\mathcal{O}$, and let $(\mathcal{T}_i)_{i=1}^{n}$ be any sequence of TBoxes such that $\langle \mathcal{R}^{s}, \mathcal{T}_{i+1} \rangle$ is obtained from $\langle \mathcal{R}^{s}, \mathcal{T}_i \rangle$ by a $\forall_{\mathcal{R}}$- or $\exists_{\mathcal{R}}$-expansion. Then $\langle \mathcal{R}^{s}, \mathcal{T}_n \rangle$ is simple-conservative over $\mathcal{O}$. Moreover, n is bounded by $\|\mathcal{T}\| \cdot (2 \cdot \|\mathcal{R}\|)^{d}$ where d is the depth of $\mathcal{R}$.*

Proof. The proof uses the observation that if $\mathcal{O}_1$ is simple-conservative over $\mathcal{O}$, and $\mathcal{O}_2$ is conservative over $\mathcal{O}_1$, then $\mathcal{O}_2$ is also simple-conservative over $\mathcal{O}$, which follows directly from the respective definitions. With this, it is easy to prove by induction on i that each $\langle \mathcal{R}^{s}, \mathcal{T}_i \rangle$ is simple-conservative over $\mathcal{O}$: the base case is established in Theorem 1, and the induction step follows from Theorem 2 and the above observation. Therefore, in particular, $\langle \mathcal{R}^{s}, \mathcal{T}_n \rangle$ is simple-conservative over $\mathcal{O}$.

To obtain the bound on n, let $r = 2 \cdot \|\mathcal{R}\|$. As remarked earlier, each $\forall_{\mathcal{R}} R.C$- resp. $\exists_{\mathcal{R}} R.C$-expansion eliminates all occurrences of $\forall_{\mathcal{R}} R.C$ resp. $\exists_{\mathcal{R}} R.C$ from the ontology, and introduces at most r new concepts $\forall_{\mathcal{R}} S.D$ (the factor 2 is due to symmetric roles) all satisfying $S \prec_{\mathcal{R}} R$. Therefore, each $\forall R.C$ and $\exists R.C$ occurring in $\mathcal{O}$ (of which there are at most $\|\mathcal{T}\|$) can altogether generate at most $1 + r + r^2 + \ldots + r^{d-1} < r^{d}$ $\mathcal{R}$-labelled universals and existentials, which yields the required bound of $\|\mathcal{T}\| \cdot r^{d}$. □

I-intro:	if	$\forall R.C \in \mathcal{L}(x)$,
	then	$\mathcal{L}(x)$ += $I[\forall R.C]$, and $\mathcal{L}(x)$ += $I[\forall \mathsf{inv}(R).C]$ if $\mathsf{inv}(R) \sqsubseteq R \in \mathcal{R}^{c}$.
F-intro:	if	$I[\forall R.C] \in \mathcal{L}(x)$ and ($R \in \mathcal{L}(x,y)$ or $\mathsf{inv}(R) \in \mathcal{L}(y,x)$),
	then	$\mathcal{L}(y)$ += $F[\forall R.C]$.
F-elim:	if	$F[\forall R.C] \in \mathcal{L}(y)$,
	then	$\mathcal{L}(y)$ += C.
I-exp:	if	$I[\forall R.C] \in \mathcal{L}(x)$,
	then	$\mathcal{L}(x)$ += $\forall R_1 \ldots \forall R_n.F[\forall R.C]$ for each $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^{c}$, and $\mathcal{L}(x)$ += $\forall R_1 \ldots \forall R_n.I[\forall R.C]$ for each $R \cdot R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^{c}$.
F-exp:	if	$F[\forall R.C] \in \mathcal{L}(y)$,
	then	$\mathcal{L}(y)$ += $\forall R_1 \ldots \forall R_n.F[\forall R.C]$ for each $R_1 \cdot \ldots \cdot R_n \cdot R \sqsubseteq R \in \mathcal{R}^{c}$, and $\mathcal{L}(y)$ += $I[\forall R.C]$ if $R \cdot R \sqsubseteq R \in \mathcal{R}^{c}$.

Fig. 1. Tableau rules for expansion of complex RIAs

Finally, as already observed in Example 1, the algorithm can be optimised by replacing all concepts $\forall_{\mathcal{R}} S.C$ and $\exists_{\mathcal{R}} S.C$ with a simple role S directly by $\forall S.C$ and $\exists S.C$ respectively, omitting their expansion. Interestingly, this makes the algorithm work without further modifications even in the presence of arbitrary cyclic simple RIAs; it is enough that complex RIAs are acyclic to ensure termination.

5 Elimination of Complex RIAs in the Tableau Algorithm

In this section we briefly sketch how the tableau algorithm for $\mathcal{SROIQ}$ [5] can be modified to perform our encoding of complex RIAs on the fly. We assume that the readers are already familiar with the tableau algorithm. We use the standard notation $\mathcal{L}(x)$ and $\mathcal{L}(x,y)$ for labels of nodes and edges in the completion graph. We assume that with each concept $\forall R.C$ we can uniquely associate new concepts $I[\forall R.C]$ and $F[\forall R.C]$; these will be used in the expansion of $\forall R.C$. Since the tableau algorithm operates with concepts in negation normal form, it can never encounter a negative occurrence of an existential restriction, therefore expansion is only applicable to universal restrictions.

To obtain the modified tableau algorithm, replace all rules relating to universal restrictions (rules $\forall_1, \forall_2, \forall_3$ in [5]) by the rules in Fig. 1, where $\mathcal{L}(x)$ += C is a shorthand for $\mathcal{L}(x) \coloneqq \mathcal{L}(x) \cup \{C\}$ and each R_i is implicitly distinct from R. These rules can be readily compared to the expansion rules (E0)–(E4) from Section 3: rules I-intro, F-intro, and F-elim implement the axioms $\forall R.C \sqsubseteq I$, $I \sqsubseteq \forall R.F$, and $F \sqsubseteq C$ of (E0), rule I-exp implements the expansions (E1) and (E3), and rule F-exp implements the expansions (E2) and (E4). The rules can be extended with blocking conditions as usual.

Note that the first three rules together subsume the standard $\forall$-rule, but additionally introduce $I[\forall R.C]$ in $\mathcal{L}(x)$ and $F[\forall R.C]$ in $\mathcal{L}(y)$, even in case there are no complex RIAs in the ontology at all. To eliminate this overhead, similarly to the optimisation above, one can restrict rule I-intro to non-simple roles R, and apply the standard $\forall$-rule to universal restrictions with simple roles.

6 Conclusions

We presented an algorithm that encodes complex RIAs in $\mathcal{SROIQ}$ without constructing finite automata. The algorithm can also be applied in weaker DLs: apart from GCIs involving atomic concepts and concepts already occurring in the ontology, the algorithm introduces only GCIs of the form $I \sqsubseteq \forall R.F$ where I and F are atomic concepts, and R a possibly inverse role. Inverse roles are not strictly required either: if desired, each $I \sqsubseteq \forall R^-.F$ with an inverse role R^- can be replaced by the equivalent $\exists R.I \sqsubseteq F$.

Our algorithm shares many theoretical properties of the traditional approaches based on automata, e.g., it is Horn-preserving and runs in time exponential in the depth of the RBox. On the other hand, a notable difference between the two approaches is that, in the automata construction, one can apply standard techniques for minimising the number of automata states and thus potentially reduce the number of new concepts introduced in the encoding. While it might be difficult to provide similarly robust optimisation for our algorithm, several simple optimisations, such as the one presented here that restricts expansion of universal restrictions to non-simple roles, might already help in many realistic cases. Experimental evaluation of the algorithm is left for future work.

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A Two Lemmas about the Languages $L_{\mathcal{R}}(R)$

We first prove two simple lemmas related to the languages $L_{\mathcal{R}}(R)$, which we then use in the proofs in the later part of the appendix.

Lemma 1. *If* $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^c$ *and* $\langle x_{i-1}, x_i\rangle \in L_{\mathcal{R}}(R_i)^{\mathcal{I}}$ *for all* $1 \leq i \leq n$*, then* $\langle x_0, x_n\rangle \in L_{\mathcal{R}}(R)^{\mathcal{I}}$.

Proof. Suppose $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$ and $\langle x_{i-1}, x_i\rangle \in L_{\mathcal{R}}(R_i)^{\mathcal{I}}$ for all $1 \leq i \leq n$. Then, for each i, by the definition of $L_{\mathcal{R}}(R_i)^{\mathcal{I}}$, there exists $w_i \in L_{\mathcal{R}}(R_i)$ such that $\langle x_{i-1}, x_i\rangle \in w^{\mathcal{I}}$. Let $w = w_1 \cdot \ldots \cdot w_n$. Then, by semantics of role chains, we have $\langle x_0, x_n\rangle \in w_1^{\mathcal{I}} \circ \ldots \circ w_n^{\mathcal{I}} = (w_1 \cdot \ldots \cdot w_n)^{\mathcal{I}} = w^{\mathcal{I}}$. By the definition of $L_{\mathcal{R}}(R)$, $R_1 \cdot \ldots \cdot R_n \in \mathcal{R}^{\mathrm{c}}$ and $w_i \in L_{\mathcal{R}}(R_i)$ together imply $w \in L_{\mathcal{R}}(R)$. Therefore, since $\langle x_0, x_n\rangle \in w^{\mathcal{I}}$ and $w \in L_{\mathcal{R}}(R)$, we have $\langle x_0, x_n\rangle \in L_{\mathcal{R}}(R)^{\mathcal{I}}$. □

Lemma 2. *If* $\langle x, y\rangle \in L_{\mathcal{R}}(R)^{\mathcal{I}}$*, then* $\langle y, x\rangle \in L_{\mathcal{R}}(\mathsf{inv}(R))^{\mathcal{I}}$.

Proof. The key step is to show that $w \in L_{\mathcal{R}}(R)$ implies $\mathsf{inv}(w) \in L_{\mathcal{R}}(\mathsf{inv}(R))$. This is proved by induction on the definition of $w \in L_{\mathcal{R}}(R)$ as follows. For the base case $R \in L_{\mathcal{R}}(R)$ we trivially have $\mathsf{inv}(R) \in L_{\mathcal{R}}(\mathsf{inv}(R))$. For the induction step we consider $w = w_1 \cdot \ldots \cdot w_n$, $R_1 \cdot \ldots \cdot R_n \in R \sqsubseteq \mathcal{R}^{\mathrm{c}}$, and $w_i \in L_{\mathcal{R}}(R_i)$ for all $1 \leq i \leq n$. By the induction hypothesis we have $\mathsf{inv}(w_i) \in L_{\mathcal{R}}(\mathsf{inv}(R_i))$ for each i. Furthermore, $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$ implies $\mathsf{inv}(R_n) \cdot \ldots \cdot \mathsf{inv}(R_1) \sqsubseteq \mathsf{inv}(R) \in \mathcal{R}^{\mathrm{c}}$, so $\mathsf{inv}(w) = \mathsf{inv}(w_n) \cdot \ldots \cdot \mathsf{inv}(w_1) \in L_{\mathcal{R}}(\mathsf{inv}(R))$ follows from the definition of $L_{\mathcal{R}}(\mathsf{inv}(R))$. This shows that $w \in L_{\mathcal{R}}(R)$ implies $\mathsf{inv}(w) \in L_{\mathcal{R}}(\mathsf{inv}(R))$.

To conclude the lemma, suppose $\langle x, y\rangle \in L_{\mathcal{R}}(R)^{\mathcal{I}}$. Then, by the definition of $L_{\mathcal{R}}(R)^{\mathcal{I}}$, there exists $w \in L_{\mathcal{R}}(R)$ such that $\langle x, y\rangle \in w^{\mathcal{I}}$. Then $\langle y, x\rangle \in \mathsf{inv}(w)^{\mathcal{I}}$ by semantics of inverse roles, and, since $\mathsf{inv}(w) \in L_{\mathcal{R}}(\mathsf{inv}(R))$, the conclusion $\langle y, x\rangle \in L_{\mathcal{R}}(\mathsf{inv}(R))^{\mathcal{I}}$ follows from the definition of $L_{\mathcal{R}}(\mathsf{inv}(R))^{\mathcal{I}}$. □

B Proof of Theorem 1

Let $\mathcal{O}$ be an unlabelled $\mathcal{SROIQ}$ ontology over a signature $\Sigma = \langle \Sigma_S, \Sigma_R, \Sigma_C, \Sigma_I\rangle$, and let $\mathcal{Q} = \langle \mathcal{R}^{\mathrm{s}}, \sigma_{\mathcal{R}}(\mathcal{T})\rangle$ be the initialisation of $\mathcal{O}$. We will show that $\mathcal{Q}$ is simple-conservative over $\mathcal{O}$. This requires showing that (i) for each model $\mathcal{I}$ of $\mathcal{O}$ there is a model $\mathcal{J}$ of $\mathcal{Q}$ that agrees with $\mathcal{I}$ on Σ_S, Σ_C, and Σ_I, and (ii) vice versa.

For (i), let $\mathcal{I}$ be an arbitrary model of $\mathcal{O}$. We will show that $\mathcal{I}$ is already a model of $\mathcal{Q}$. Trivially, $\mathcal{I} \models \mathcal{R}$ implies $\mathcal{I} \models \mathcal{R}^{\mathrm{s}}$ since $\mathcal{R}^{\mathrm{s}} \subseteq \mathcal{R}$. By Proposition 1, we have $L_{\mathcal{R}}(R)^{\mathcal{I}} = R^{\mathcal{I}}$, so $(\forall R.C)^{\mathcal{I}} = (\forall_{\mathcal{R}} R.C)^{\mathcal{I}}$ and $(\exists R.C)^{\mathcal{I}} = (\exists_{\mathcal{R}} R.C)^{\mathcal{I}}$ for all concepts $\forall R.C$ and $\exists R.C$. This means that $\mathcal{R}$-labelling does not affect the interpretation of concepts in $\mathcal{I}$, so $\mathcal{I} \models \mathcal{T}$ implies $\mathcal{I} \models \sigma_{\mathcal{R}}(\mathcal{T})$. Therefore $\mathcal{I} \models \mathcal{Q}$.

For (ii), let $\mathcal{J}$ be an arbitrary model of $\mathcal{Q}$. We will show that $\mathcal{J}$ can be transformed to a model $\mathcal{I}$ of $\mathcal{O}$ by extending the interpretation of atomic roles to

$$R^{\mathcal{I}} = L_{\mathcal{R}}(R)^{\mathcal{J}}; \tag{13}$$

the interpretation of atomic concepts and individuals in $\mathcal{I}$ coincides with $\mathcal{J}$. From Lemma 2 it follows that (13) also holds for inverse roles R. Since $\mathcal{R}^{\mathsf{s}}$ contains all simple RIAs from $\mathcal{R}$, we have $L_{\mathcal{R}^{\mathsf{s}}}(S) = L_{\mathcal{R}}(S)$ for each simple role S. Then $S^{\mathcal{I}} = L_{\mathcal{R}}(S)^{\mathcal{J}} = L_{\mathcal{R}^{\mathsf{s}}}(S)^{\mathcal{J}} = S^{\mathcal{J}}$, where the last equality is by Proposition 1. Since $R \in L_{\mathcal{R}}(R)$ implies $R^{\mathcal{J}} \subseteq L_{\mathcal{R}}(R)^{\mathcal{J}}$ for any role R, it follows that $R^{\mathcal{J}} \subseteq R^{\mathcal{I}}$ by (13). We repeat that for each simple role S and each role R we have

$$S^{\mathcal{J}} = S^{\mathcal{I}} \quad \text{and} \quad R^{\mathcal{J}} \subseteq R^{\mathcal{I}}. \tag{14}$$

In the remainder of this section we prove that $\mathcal{I} \models \mathcal{O}$. First, we consider all RIAs in $\mathcal{R}$. Let $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}$. To show that $\mathcal{I} \models R_1 \cdot \ldots \cdot R_n \sqsubseteq R$, we consider arbitrary $x_0, \ldots, x_n$ such that $\langle x_{i-1}, x_i \rangle \in R_i^{\mathcal{I}}$ for each $1 \leq i \leq n$, and show that $\langle x_0, x_n \rangle \in R^{\mathcal{I}}$. By (13), $\langle x_{i-1}, x_i \rangle \in L_{\mathcal{R}}(R_i)^{\mathcal{J}}$ for each i. Then by Lemma 1, we have $\langle x_0, x_n \rangle \in L_{\mathcal{R}}(R)^{\mathcal{J}}$, whence $\langle x_0, x_n \rangle \in R^{\mathcal{I}}$ follows by (13). This shows that $\mathcal{I} \models R_1 \cdot \ldots \cdot R_n \sqsubseteq R$.

Second, we consider all role assertions in $\mathcal{R}$. Since $S^{\mathcal{I}} = S^{\mathcal{J}}$ for all simple roles by (14), we have $\mathcal{J} \models \mathsf{Irr}(S)$ iff $\mathcal{I} \models \mathsf{Irr}(S)$, and $\mathcal{J} \models \mathsf{Dis}(S_1, S_2)$ iff $\mathcal{I} \models \mathsf{Dis}(S_1, S_2)$. Note that a superset of a reflexive relation is also reflexive and a superset of a total relation is also total. Thus, since $R^{\mathcal{J}} \subseteq R^{\mathcal{I}}$ for all roles R by (14), we have $\mathcal{J} \models \mathsf{Ref}(R)$ implies $\mathcal{I} \models \mathsf{Ref}(R)$, and $\mathcal{J} \models \mathsf{Uni}(R)$ implies $\mathcal{I} \models \mathsf{Uni}(R)$. Therefore, since $\mathcal{J} \models \mathcal{R}^{\mathsf{s}}$ and $\mathcal{R}^{\mathsf{s}}$ contains all role assertions from $\mathcal{R}$, it follows that $\mathcal{I}$ satisfies all role assertions in $\mathcal{R}$. Thus $\mathcal{I} \models \mathcal{R}$.

To show $\mathcal{I} \models \mathcal{T}$ we need the following property for each unlabelled concept D:

$$\sigma_{\mathcal{R}}(D)^{\mathcal{J}} \subseteq D^{\mathcal{I}} \subseteq \bar{\sigma}_{\mathcal{R}}(D)^{\mathcal{J}}. \tag{15}$$

The proof of (15) is by structural induction on D. The base cases $D = A$ and $D = \{a\}$ are trivial and, for most concept constructors, the induction step simply lifts the induction hypothesis (15) from subexpressions of D to D itself. The only interesting cases are $D = \exists R.C$ and $D = \forall R.C$ which are the concepts that are being labelled by $\mathcal{R}$; we consider these cases next.

<u>Case $D = \exists R.C$:</u>

$\sigma_{\mathcal{R}}$ labels negative occurrences of existential restrictions, $\bar{\sigma}_{\mathcal{R}}$ labels positive occurrences of existential restrictions, the occurrence of $\exists R.C$ in itself is positive, and existential restrictions preserve polarities of their components, so

$$\sigma_{\mathcal{R}}(\exists R.C) = \exists R.\sigma_{\mathcal{R}}(C) \quad \text{and} \quad \bar{\sigma}_{\mathcal{R}}(\exists R.C) = \exists_{\mathcal{R}} R.\bar{\sigma}_{\mathcal{R}}(C). \tag{16}$$

Since $R^{\mathcal{J}} \subseteq R^{\mathcal{I}}$ by (14) and $\sigma_{\mathcal{R}}(C)^{\mathcal{J}} \subseteq C^{\mathcal{I}}$ by the induction hypothesis (15), it follows that $(\exists R.\sigma_{\mathcal{R}}(C))^{\mathcal{J}} \subseteq (\exists R.C)^{\mathcal{I}}$, which, together with (16), proves the first inclusion in (15). Since $R^{\mathcal{I}} = L_{\mathcal{R}}(R)^{\mathcal{J}}$ by (13) and $C^{\mathcal{I}} \subseteq \bar{\sigma}_{\mathcal{R}}(C)^{\mathcal{J}}$ by the induction hypothesis (15), it follows that $(\exists R.C)^{\mathcal{I}} \subseteq (\exists_{\mathcal{R}} R.\bar{\sigma}_{\mathcal{R}}(C))^{\mathcal{J}}$, which, together with (16), proves the second inclusion of (15).

<u>Case $D = \forall R.C$:</u>

$\sigma_{\mathcal{R}}$ labels positive occurrences of universal restrictions, $\bar{\sigma}_{\mathcal{R}}$ labels negative occurrences of universal restrictions, the occurrence of $\forall R.C$ in itself is positive, and universal restrictions preserve polarities of their components, so

$$\sigma_{\mathcal{R}}(\forall R.C) = \forall_{\mathcal{R}} R.\sigma_{\mathcal{R}}(C) \quad \text{and} \quad \bar{\sigma}_{\mathcal{R}}(\forall R.C) = \forall R.\bar{\sigma}_{\mathcal{R}}(C). \tag{17}$$

Since $L_{\mathcal{R}}(R)^{\mathcal{J}} = R^{\mathcal{I}}$ by (13) and $\sigma_{\mathcal{R}}(C)^{\mathcal{J}} \subseteq C^{\mathcal{I}}$ by the induction hypothesis (15), it follows that $(\forall_{\mathcal{R}} R.\sigma_{\mathcal{R}}(C))^{\mathcal{J}} \subseteq (\forall R.C)^{\mathcal{I}}$, which, together with (17), proves the first inclusion in (15). Since $R^{\mathcal{I}} \supseteq R^{\mathcal{J}}$ by (14) and $C^{\mathcal{I}} \subseteq \bar{\sigma}_{\mathcal{R}}(C)^{\mathcal{J}}$ by the induction hypothesis (15), it follows that $(\forall R.C)^{\mathcal{I}} \subseteq (\forall R.\bar{\sigma}_{\mathcal{R}}(C))^{\mathcal{J}}$, which, together with (17), proves the second inclusion of (15).

Finally, having proved (15), we are ready to show that $\mathcal{I} \models \mathcal{T}$. Consider an arbitrary GCI $C \sqsubseteq D$ in $\mathcal{T}$. Then, $(\bar{\sigma}_{\mathcal{R}}(C) \sqsubseteq \sigma_{\mathcal{R}}(D)) \in \sigma_{\mathcal{R}}(\mathcal{T})$, so, since $\mathcal{J} \models \sigma_{\mathcal{R}}(\mathcal{T})$, we have $\bar{\sigma}_{\mathcal{R}}(C)^{\mathcal{J}} \subseteq \sigma_{\mathcal{R}}(D)^{\mathcal{J}}$. Moreover, by (15), we have $C^{\mathcal{I}} \subseteq \bar{\sigma}_{\mathcal{R}}(C)^{\mathcal{J}}$ and $\sigma_{\mathcal{R}}(D)^{\mathcal{J}} \subseteq D^{\mathcal{I}}$. The last three inclusions imply $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, so $\mathcal{I} \models C \sqsubseteq D$.

This concludes the proof of $\mathcal{I} \models \mathcal{O}$, so $\mathcal{Q}$ is simple-conservative over $\mathcal{O}$.

C Proof of Proposition 2

Let $\mathcal{R}$ be a regular RBox. We assume that $\mathcal{I} \models \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F)$ and show that $\mathcal{I} \models I \sqsubseteq \forall_{\mathcal{R}} R.F$ follows.

Let $[\![R]\!] := \{R, \mathsf{inv}(R)\}$ if $\mathsf{inv}(R) \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$, and $[\![R]\!] := \{R\}$ otherwise. Then $\mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F) = \bigcup_{S \in [\![R]\!]} \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} S.F)$, so the assumption implies

$$\mathcal{I} \models \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} S.F) \text{ for each } S \in [\![R]\!]. \tag{18}$$

We will prove for all $S \in [\![R]\!]$, all role chains w, and all $x, y \in \Delta^{\mathcal{I}}$ that

$$\text{if } x \in I^{\mathcal{I}},\ \langle x, y\rangle \in w^{\mathcal{I}}, \text{ and } w \in L_{\mathcal{R}}(S), \text{ then } y \in F^{\mathcal{I}}. \tag{19}$$

Since the required $\mathcal{I} \models I \sqsubseteq \forall_{\mathcal{R}} R.F$ is equivalent to (19) for $S = R$, this will conclude the proof of Proposition 2.

The proof of (19) is by induction on the definition of $w \in L_{\mathcal{R}}(S)$. In the base case $w = S$ property (19) is equivalent to $\mathcal{I} \models I \sqsubseteq \forall S.F$, which follows from (18) since $I \sqsubseteq \forall S.F \in \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} S.F)$.

In the induction step we have $w = w_1 \cdot \ldots \cdot w_n$, $R_1 \cdot \ldots \cdot R_n \sqsubseteq S \in \mathcal{R}^{\mathrm{c}}$, and $w_i \in L_{\mathcal{R}}(R_i)$ for all $1 \leq i \leq n$. To prove (19), we assume $x^{\mathcal{I}}$ and $\langle x, y\rangle \in w^{\mathcal{I}}$, and we show that $y \in F^{\mathcal{I}}$ follows. Now $\langle x, y\rangle \in w^{\mathcal{I}}$ and $w = w_1 \cdot \ldots \cdot w_n$ imply that there exist $x_0, \ldots, x_n$ such that $x_0 = x$, $x_n = y$, and for all $1 \leq i \leq n$

$$\langle x_{i-1}, x_i\rangle \in w_i^{\mathcal{I}}, \quad \text{and recall that each } w_i \in L_{\mathcal{R}}(R_i). \tag{20}$$

Since $\mathcal{R}$ is regular, the RIA $R_1 \cdot \ldots \cdot R_n \sqsubseteq S$ is of one of the following forms.

(R1) $R_i \neq S$ for all $1 \leq i \leq n$:
In this case $I \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_n.F \in \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} S.F)$, so, by (18), we have $\mathcal{I} \models I \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_n.F$. Then $x_0 = x \in I^{\mathcal{I}}$ and (20) imply the required $y = x_n \in F^{\mathcal{I}}$.

(R2) $R_1 = S$ and $R_i \neq R$ for $1 < i \leq n$:
In this case $F \sqsubseteq \forall_{\mathcal{R}} R_2 \ldots \forall_{\mathcal{R}} R_n.F \in \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} S.F)$, so, by (18), we have $\mathcal{I} \models F \sqsubseteq \forall_{\mathcal{R}} R_2 \ldots \forall_{\mathcal{R}} R_n.F$. Since $x_0 \in I^{\mathcal{I}}$, $\langle x_0, x_1\rangle \in w_1^{\mathcal{I}}$, and $w_1 \in L_{\mathcal{R}}(R_1) = L_{\mathcal{R}}(S)$, we can apply the induction hypothesis (19) to S, w_1, and $\langle x_0, x_1\rangle$ to obtain $x_1 \in F^{\mathcal{I}}$. Then (20) and $\mathcal{I} \models F \sqsubseteq \forall_{\mathcal{R}} R_2 \ldots \forall_{\mathcal{R}} R_n.F$ imply the required $y = x_n \in F^{\mathcal{I}}$.

(R3) $R_n = S$ and $R_i \neq R$ for $1 \leq i < n$:
In this case $I \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_{n-1}.I \in \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} S.F)$, so, by (18), we have $\mathcal{I} \models I \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_{n-1}.I$. Then $x_0 = x \in I^{\mathcal{I}}$ and (20) imply $x_{n-1} \in I^{\mathcal{I}}$. Since $x_{n-1} \in I^{\mathcal{I}}$, $\langle x_{n-1}, x_n \rangle \in w_n$, and $w_n \in L_{\mathcal{R}}(R_n) = L_{\mathcal{R}}(S)$, we can apply the induction hypothesis (19) to S, w_n, and $\langle x_{n-1}, x_n \rangle$ to get the required $y = x_n \in F^{\mathcal{I}}$.

(R4) $n = 2$ and $R_1 = R_2 = S$:
In this case $F \sqsubseteq I \in \mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} S.F)$, so, by (18), we have $\mathcal{I} \models F \sqsubseteq I$. Since $x_0 = x \in I^{\mathcal{I}}$, $\langle x_0, x_1 \rangle \in w_1$, and $w_1 \in L_{\mathcal{R}}(R_1) = L_{\mathcal{R}}(S)$, we can apply the induction hypothesis (19) to S, w_1, and $\langle x_0, x_1 \rangle$ to obtain $x_1 \in F^{\mathcal{I}}$. Then, since $\mathcal{I} \models I \sqsubseteq F$, it follows that $x_1 \in I^{\mathcal{I}}$. Now, since $x_1 \in I^{\mathcal{I}}$, $\langle x_1, x_2 \rangle \in w_2$, and $w_2 \in L_{\mathcal{R}}(R_2) = L_{\mathcal{R}}(S)$, we can again apply the the induction hypothesis (19) to S, w_2, and $\langle x_1, x_2 \rangle$ to obtain the required $y = x_2 \in F^{\mathcal{I}}$.

(R5) $n = 1$ and $R_1 = \mathsf{inv}(S)$:
This means that $\mathsf{inv}(S) \sqsubseteq S \in \mathcal{R}^{\mathrm{c}}$ which, since S is either R or $\mathsf{inv}(R)$, implies $\mathsf{inv}(R) \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$. Then $[\![R]\!] = \{R, \mathsf{inv}(R)\}$ and therefore $\mathsf{inv}(S) \in [\![R]\!]$. Furthermore, since $x = I^{\mathcal{I}}$, $\langle x, y \rangle = \langle x_0, x_1 \rangle \in w_1$, and $w_1 \in L_{\mathcal{R}}(R_1) = L_{\mathcal{R}}(\mathsf{inv}(S))$, we can apply the induction hypothesis (19) to $\mathsf{inv}(S)$, w_1, and $\langle x, y \rangle$, to obtain the required $y \in F^{\mathcal{I}}$.

This concludes the inductive proof of (19), and thus proves the proposition.

D Proof of Theorem 2

Let $\mathcal{R}$ be a regular RBox, let $\mathcal{O} = \langle \mathcal{R}', \mathcal{T}_1 \rangle$ be an $\mathcal{R}$-labelled ontology over Σ, and let $\mathcal{Q} = \langle \mathcal{R}', \mathcal{T}_2 \rangle$ be the $\forall_{\mathcal{R}} R.C$-expansion of $\mathcal{O}$. We will show that $\mathcal{Q}$ is conservative over $\mathcal{O}$. This requires showing that (i) for each model $\mathcal{I}$ of $\mathcal{O}$ there exists a model $\mathcal{J}$ of $\mathcal{Q}$ that agrees with $\mathcal{I}$ on Σ, and (ii) vice versa.

We write ρ^+ and ρ^- to abbreviate $\rho^+[I \,/\, \forall_{\mathcal{R}} R.C]$ and $\rho^-[I \,/\, \forall_{\mathcal{R}} R.C]$ respectively. Recall that, with this notation, we have

$$\mathcal{T}_2 = \rho^+(\mathcal{T}_1) \cup \{F \sqsubseteq C\} \cup \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F). \tag{21}$$

For (i), let $\mathcal{I}$ be an arbitrary model of $\mathcal{O}$. We will show that $\mathcal{I}$ can be extended to a model $\mathcal{J}$ of $\mathcal{Q}$ by interpreting the new atomic concepts I and F as

$$I^{\mathcal{J}} = (\forall_{\mathcal{R}} R.C)^{\mathcal{I}} \quad \text{and} \quad F^{\mathcal{J}} = (\exists_{\mathcal{R}} \mathsf{inv}(R).\forall_{\mathcal{R}} R.C)^{\mathcal{I}}; \tag{22}$$

the interpretation of all symbols from Σ in $\mathcal{J}$ coincides with $\mathcal{I}$. Since $\mathcal{I}$ and $\mathcal{J}$ agree on Σ, we have $D^{\mathcal{I}} = D^{\mathcal{J}}$ for all concepts D over Σ. Then, by (22), we have

$$I^{\mathcal{J}} = (\forall_{\mathcal{R}} R.C)^{\mathcal{J}} \quad \text{and} \quad F^{\mathcal{J}} = (\exists_{\mathcal{R}} \mathsf{inv}(R).\forall_{\mathcal{R}} R.C)^{\mathcal{J}} = (\exists_{\mathcal{R}} \mathsf{inv}(R).I)^{\mathcal{J}}. \tag{23}$$

We will now show that $\mathcal{J} \models \mathcal{T}_2$. Since the substitutions ρ^+ and ρ^- merely replace some occurrences of $\forall_{\mathcal{R}} R.C$ by I and these two concepts have the same interpretation in $\mathcal{J}$, we have $D^{\mathcal{I}} = D^{\mathcal{J}} = \rho^+(D)^{\mathcal{J}} = \rho^-(D)^{\mathcal{J}}$ for all concepts D over Σ. Therefore $\mathcal{I} \models \mathcal{T}_1$ implies $\mathcal{J} \models \rho^+(\mathcal{T}_1)$. Moreover, the axiom $\exists_{\mathcal{R}} \mathsf{inv}(R).\forall_{\mathcal{R}} R.C \sqsubseteq C$ is a tautology, so, since $F^{\mathcal{J}} = (\exists_{\mathcal{R}} \mathsf{inv}(R).\forall_{\mathcal{R}} R.C)^{\mathcal{J}}$ by (23), we have $\mathcal{J} \models F \sqsubseteq C$. To finish the proof of $\mathcal{J} \models \mathcal{T}_2$, by (21), it remains to show that $\mathcal{J} \models \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F)$. First, we consider each GCI in $\mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} R.F)$ in turn.

$I \sqsubseteq \forall R.F$:
The axiom $I \sqsubseteq \forall R.\exists_{\mathcal{R}} \mathsf{inv}(R).I$ is a tautology, so it holds in $\mathcal{J}$. This, since $F^{\mathcal{J}} = (\exists_{\mathcal{R}} \mathsf{inv}(R).I)^{\mathcal{J}}$ by (23), is equivalent to the required $\mathcal{J} \models I \sqsubseteq \forall R.F$.

$I \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_n.F$ with $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$:
To show that $\mathcal{J}$ satisfies the above GCI, we consider arbitrary $x_0, \ldots, x_n \in \Delta^{\mathcal{J}}$ such that $x_0 \in I^{\mathcal{J}}$ and $\langle x_{i-1}, x_i \rangle \in L_{\mathcal{R}}(R_i)^{\mathcal{J}}$ for all $1 \leq i \leq n$, and prove that $x_n \in F^{\mathcal{J}}$. By Lemma 1, $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$ and $\langle x_{i-1}, x_i \rangle \in L_{\mathcal{R}}(R_i)^{\mathcal{J}}$ together imply $\langle x_0, x_n \rangle \in L_{\mathcal{R}}(R)^{\mathcal{J}}$. Then, by Lemma 2, we have $\langle x_n, x_0 \rangle \in L_{\mathcal{R}}(\mathsf{inv}(R))^{\mathcal{J}}$. Then, since $x_0 \in I^{\mathcal{J}}$, we have the required $x_n \in (\exists_{\mathcal{R}} \mathsf{inv}(R).I)^{\mathcal{J}} = F^{\mathcal{J}}$ by (23).

$F \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_n.F$ with $R \cdot R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$:
To show that $\mathcal{J}$ satisfies the above GCI, we consider arbitrary $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$ such that $x_0 \in F^{\mathcal{J}}$ and $\langle x_{i-1}, x_i \rangle \in L_{\mathcal{R}}(R_i)$ for all $1 \leq i \leq n$, and prove that $x_n \in F^{\mathcal{J}}$. Since $x_0 \in F^{\mathcal{J}} = (\exists_{\mathcal{R}} \mathsf{inv}(R).I)^{\mathcal{J}}$ by (23), there exists some y such that $\langle x_0, y \rangle \in L_{\mathcal{R}}(\mathsf{inv}(R))^{\mathcal{J}}$ and $y \in I^{\mathcal{J}}$. Then, by Lemma 2, $\langle y, x_0 \rangle \in L_{\mathcal{R}}(R)$. By Lemma 1, $R \cdot R_1 \cdot \ldots \cdot R_n \in \mathcal{R}^{\mathrm{c}}$, $\langle y, x_0 \rangle \in L_{\mathcal{R}}(R)^{\mathcal{J}}$, and $\langle x_{i-1}, x_i \rangle \in L_{\mathcal{R}}(R_i)^{\mathcal{J}}$ together imply $\langle y, x_n \rangle \in L_{\mathcal{R}}(R)^{\mathcal{J}}$, so, by Lemma 2, $\langle x_n, y \rangle \in L_{\mathcal{R}}(\mathsf{inv}(R))^{\mathcal{J}}$. Then, since $y \in I^{\mathcal{J}}$, we have the required $x_n \in (\exists_{\mathcal{R}} \mathsf{inv}(R).I)^{\mathcal{J}} = F^{\mathcal{J}}$ by (23).

$I \sqsubseteq \forall_{\mathcal{R}} R_1 \ldots \forall_{\mathcal{R}} R_n.I$ with $R_1 \cdot \ldots \cdot R_n \cdot R \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$:
To show that $\mathcal{J}$ satisfies the above GCI, we consider arbitrary $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$ such that $x_0 \in I^{\mathcal{J}}$ and $\langle x_{i-1}, x_i \rangle \in L_{\mathcal{R}}(R_i)^{\mathcal{J}}$ for all $1 \leq i \leq n$, and show that $x_n \in I^{\mathcal{J}}$. By (23), this is equivalent to $x_n \in (\forall_{\mathcal{R}} R.C)^{\mathcal{J}}$; to show this, we consider arbitrary y such that $\langle x_n, y \rangle \in L_{\mathcal{R}}(R)^{\mathcal{J}}$ and show that $y \in C^{\mathcal{J}}$. By Lemma 1, $R_1 \cdot \ldots \cdot R_n \cdot R \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$, $\langle x_{i-1}, x_i \rangle \in L_{\mathcal{R}}(R_i)$, and $\langle x_n, y \rangle \in L_{\mathcal{R}}(R)$ together imply $\langle x_0, y \rangle \in L_{\mathcal{R}}(R)$. Then, since $x_0 \in I^{\mathcal{J}} = (\forall_{\mathcal{R}} R.C)^{\mathcal{J}}$ by (23), the required $y \in C^{\mathcal{J}}$ follows.

$F \sqsubseteq I$ with $R \cdot R \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$:
To show that $\mathcal{J}$ satisfies the above GCI, we consider arbitrary $x \in F^{\mathcal{J}}$ and show that $x \in I^{\mathcal{J}}$. By (23), this is equivalent to $x \in (\forall_{\mathcal{R}} R.C)^{\mathcal{J}}$; to show this, we consider arbitrary y such that $\langle x, y \rangle \in L_{\mathcal{R}}(R)^{\mathcal{J}}$ and show that $y \in C^{\mathcal{J}}$. Since $x \in F^{\mathcal{J}} = (\exists_{\mathcal{R}} \mathsf{inv}(R).I)^{\mathcal{J}}$ by (23), there exists z such that $\langle x, z \rangle \in L_{\mathcal{R}}(\mathsf{inv}(R))^{\mathcal{J}}$ and $z \in I^{\mathcal{J}}$. Then, by Lemma 2, we have $\langle z, x \rangle \in L_{\mathcal{R}}(R)^{\mathcal{J}}$. By Lemma 1, $R \cdot R \sqsubseteq R \in \mathcal{R}$, $\langle z, x \rangle \in L_{\mathcal{R}}(R)$, and $\langle x, y \rangle \in L_{\mathcal{R}}(R)^{\mathcal{J}}$ together imply $\langle z, y \rangle \in L_{\mathcal{R}}(R)$. Then, since $z \in I^{\mathcal{J}} = (\forall_{\mathcal{R}} R.C)^{\mathcal{J}}$ by (23), the required $y \in C^{\mathcal{J}}$ follows.

Second, if $\mathsf{inv}(R) \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$, then we must also consider each GCI in $\mathsf{expand}'(I \sqsubseteq \forall_{\mathcal{R}} \mathsf{inv}(R).F)$. This is analogous to the cases above with $\mathsf{inv}(R)$ in place of R since, if $\mathsf{inv}(R) \sqsubseteq R \in \mathcal{R}^{\mathrm{c}}$, then we have $I^{\mathcal{J}} = (\forall_{\mathcal{R}} \mathsf{inv}(R).C)^{\mathcal{J}}$ and $F^{\mathcal{J}} = (\exists_{\mathcal{R}} R.I)^{\mathcal{J}}$, which we use to substitute the property (23) in the above argument.

This shows that $\mathcal{J} \models \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F)$ and thus proves $\mathcal{J} \models \mathcal{T}_2$. Clearly, $\mathcal{I} \models \mathcal{R}'$ implies $\mathcal{J} \models \mathcal{R}'$. Therefore $\mathcal{J} \models \mathcal{Q}$.

For (ii), let $\mathcal{J}$ be an arbitrary model of $\mathcal{Q}$. We will show that $\mathcal{J}$ is already a model of $\mathcal{O}$. The key step is to prove for all concepts D over the signature Σ that

$$\rho^{+}(D)^{\mathcal{J}} \subseteq D^{\mathcal{J}} \subseteq \rho^{-}(D)^{\mathcal{J}}. \tag{24}$$

The proof of (24) is by structural induction on D. The base cases $D = A$ and $D = \{a\}$ are trivial and, for most concept constructors, the induction step simply lifts the induction hypothesis (24) from subexpressions of D to D itself. The only interesting case in the induction is $D = \forall_{\mathcal{R}} R.C$ which is the concepts that is being replaced by I; this case we consider next. Since the occurrence of $\forall_{\mathcal{R}} R.C$ in itself is positive, we have $\rho^+(\forall_{\mathcal{R}} R.C) = I$ and $\rho^-(\forall_{\mathcal{R}} R.C) = \forall_{\mathcal{R}} R.C$. Therefore, to show (24), we need $I^{\mathcal{J}} \subseteq (\forall_{\mathcal{R}} R.C)^{\mathcal{J}}$. Since $\mathcal{J} \models \mathcal{T}_2$, we have, by (21), $\mathcal{J} \models F \sqsubseteq C$ and $\mathcal{J} \models \mathsf{expand}(I \sqsubseteq \forall_{\mathcal{R}} R.F)$. The latter, by Proposition 2, implies $\mathcal{J} \models I \sqsubseteq \forall_{\mathcal{R}} R.F$, which together with the former $\mathcal{J} \models F \sqsubseteq C$ implies $\mathcal{J} \models I \sqsubseteq \forall_{\mathcal{R}} R.C$. This is equivalent to the required $I^{\mathcal{J}} \subseteq (\forall_{\mathcal{R}} R.C)^{\mathcal{J}}$, which proves (24) for the case $D = \forall_{\mathcal{R}} R.C$.

Finally, with (24), we can show that $\mathcal{J} \models \mathcal{T}_1$. Let $D_1 \sqsubseteq D_2$ be an arbitrary GCI in $\mathcal{T}_1$. Then $(\rho^-(D_1) \sqsubseteq \rho^+(D_2)) \in \rho^+(\mathcal{T}_1) \subseteq \mathcal{T}_2$, so, since $\mathcal{J} \models \mathcal{T}_2$, we have $\rho^-(D_1)^{\mathcal{J}} \subseteq \rho^+(D_2)^{\mathcal{J}}$. Further, by (24), $D_1^{\mathcal{J}} \subseteq \rho^-(D_1)^{\mathcal{J}}$ and $\rho^+(D_2)^{\mathcal{J}} \subseteq D_2^{\mathcal{J}}$. The last three inclusions imply $D_1^{\mathcal{J}} \subseteq D_2^{\mathcal{J}}$, so $\mathcal{J} \models D_1 \sqsubseteq D_2$. This shows that $\mathcal{J} \models \mathcal{O}$.

This shows that $\mathcal{Q}$ is conservative over $\mathcal{O}$, and thus proves Theorem 2 for $\forall_{\mathcal{R}}$-expansions. The proof for $\exists_{\mathcal{R}}$-expansions is similar, with the important differences as outlined in the sketch in the main body of the paper.