Consequence-Based Reasoning beyond Horn Ontologies *

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Abstract

Consequence-based ontology reasoning procedures have so far been known only for Horn ontology languages. A difficulty in extending such procedures is that non-Horn axioms seem to require reasoning by case, which causes non-determinism in tableau-based procedures. In this paper we present a consequence-based procedure for $\mathcal{ALCH}$ that overcomes this difficulty by using rules similar to ordered resolution to deal with disjunctive axioms in a deterministic way; it retains all the favourable attributes of existing consequence-based procedures, such as goal-directed "one pass" classification, optimal worst-case complexity, and "pay-as-you-go" behaviour. Our preliminary empirical evaluation suggests that the procedure scales well to non-Horn ontologies.

1 Introduction and Motivation

Description logics (DLs) [Baader *et al.*, 2007] are a family of logic-based formal languages, which provide theoretical underpinning for modern ontology languages such as OWL 2 [Cuenca Grau *et al.*, 2008] and serve as the basis for the development of ontology reasoning procedures and tools. One of the key DL reasoning tasks is *ontology classification*: computing all subsumption relations between atomic concepts implied by an ontology. Most modern ontology reasoners, such as FaCT++,[1] HermiT,[2] Pellet,[3] and RacerPro,[4] are based on optimized tableau-based procedures, or variations thereof, which compute classification by trying to build counter-models for candidate subsumption relations. Recently, another type of reasoning procedure has been introduced. Instead of building counter-models for subsumption relations, such procedures derive logical consequences of axioms in the ontology using inference rules. These rules are designed to produce all implied subsumption relations, while guaranteeing that only a bounded number of axioms is derived. Because the rules produce logical consequences of axioms, such procedures are sometimes referred to as *consequence-based* procedures.

Consequence-based procedures were first introduced for the family of polynomial DLs $\mathcal{EL}^{++}$ [Baader *et al.*, 2005], but later were extended to Horn-$\mathcal{SHIQ}$ [Kazakov, 2009] and even Horn-$\mathcal{SROIQ}$ [Ortiz *et al.*, 2010]. Although these DLs are no longer polynomial, the extended procedures remain computationally optimal and exhibit "pay-as-you-go" behaviour, e.g., remain polynomial for $\mathcal{EL}$ ontologies.

One limitation of consequence-based procedures is that, up until now, they only supported Horn DLs, and, in particular, could not handle disjunctions. In tableau procedures, disjunctive axioms such as $A \sqsubseteq B \sqcup C$ result in non-deterministic inferences: in order to satisfy A one tries to satisfy either B or C. A direct reformulation of this idea as a non-deterministic rule producing consequences would not work: if $A \sqsubseteq B \sqcup C$ holds then it is not true that either $A \sqsubseteq B$ or $A \sqsubseteq C$ holds. In this paper we demonstrate how disjunctions can be handled in a deterministic way using inference rules reminiscent of ordered resolution (see, e.g., [Bachmair and Ganzinger, 2001]). To focus on the problem, we will consider a relatively simple DL $\mathcal{ALCH}$ featuring disjunction and negation. We formulate a consequence-based classification procedure for $\mathcal{ALCH}$, prove its soundness and completeness, describe optimizations, and present first experimental results which suggest that the procedure scales well to non-Horn ontologies without adversely affecting performance on Horn ontologies.

$\mathcal{ALCH}$ is interesting not only from a theoretical point of view. Although many existing ontologies are Horn, in particular the largest ones such as SNOMED CT[5] and GALEN,[6] this is often for historical reasons, and advances in reasoning systems for expressive DLs have led many ontology developers to consider the use of new language features. One example of this phenomenon, which is directly relevant to this paper, is the latest initiative to remodel the anatomical part of SNOMED CT. Presently, the anatomical model in SNOMED CT uses the so-called SEP-triplet encoding (see, e.g., [Suntisrivaraporn *et al.*, 2007]), which encodes "part-of" relations as "is-a" relations. For example, "finger" is modelled using a triple of concepts: S-finger representing the *structure* of fin-

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[1] owl.man.ac.uk/factplusplus/
[2] hermit-reasoner.com/
[3] clarkparsia.com/pellet/
[4] www.racer-systems.com/
[5] www.ihtsdo.org/snomed-ct/
[6] www.opengalen.org/

	Syntax	Semantics
Roles:		
atomic role	R	$R^{\mathcal{I}}$
Concepts:		
atomic concept	A	$A^{\mathcal{I}}$
top	$\top$	$\Delta^{\mathcal{I}}$
bottom	$\bot$	$\emptyset$
negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
existential restriction	$\exists R.C$	$\{x \mid R^{\mathcal{I}}(x) \cap C^{\mathcal{I}} \neq \emptyset\}$
universal restriction	$\forall R.C$	$\{x \mid R^{\mathcal{I}}(x) \subseteq C^{\mathcal{I}}\}$
Axioms:		
concept inclusion	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
role inclusion	$R \sqsubseteq S$	$R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$

Table 1: The syntax and semantics of $\mathcal{ALCH}$

ger, which subsumes E-finger representing the *entire* finger and P-finger representing the *parts* of finger. The fact that finger is a part of hand is expressed as S-finger $\sqsubseteq$ P-hand.

Unfortunately, interactions between the SEP triplet encoding and other axioms result in undesirable artefacts [Suntisrivaraporn *et al.*, 2007]. For example, if an axiom says that the index finger is a finger (E-index-finger $\sqsubseteq$ E-finger), this does not imply the same for its parts or structures. To address this and other related problems, a new version of the SNOMED CT anatomical model is being developed using axioms that fully define the S- and P- concepts using disjunctions and the transitive part-of relation, for example:

$$\text{S-finger} \equiv \text{E-finger} \sqcup \text{P-finger}$$
$$\text{P-finger} \equiv \exists \text{part-of}.\text{E-finger}.$$

We have been granted access to a preliminary version of the ontology featuring this encoding, and were able to classify it in under 2 minutes using our new procedure. In comparison, the fastest tableau-based reasoners required over 35 minutes to classify this ontology.

2 $\mathcal{ALCH}$ and Horn-$\mathcal{ALCH}$

The vocabulary of $\mathcal{ALCH}$ consists of countably infinite sets N_R of *(atomic) roles* and N_C of *atomic concepts*. Complex *concepts* and *axioms* are defined recursively using the constructors in Table 1. We use the letters R, S for roles, C, D for concepts and A, B for atomic concepts. An *ontology* is a finite set of axioms. Given an ontology $\mathcal{O}$, we write $\sqsubseteq_{\mathcal{O}}$ for the smallest reflexive transitive binary relation over roles such that $R \sqsubseteq_{\mathcal{O}} S$ holds for all $R \sqsubseteq S \in \mathcal{O}$.

$\mathcal{ALCH}$ has Tarski-style set-theoretic semantics. An *interpretation* $\mathcal{I}$ consists of a non-empty set $\Delta^{\mathcal{I}}$ called the domain of $\mathcal{I}$ and an interpretation function $\cdot^{\mathcal{I}}$ that assigns to each R a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and to each A a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$. The interpretation function is extended to complex concepts as shown in Table 1, where $R^{\mathcal{I}}(x) = \{y \mid (x, y) \in R^{\mathcal{I}}\}$.

An interpretation $\mathcal{I}$ *satisfies* an axiom α (written $\mathcal{I} \models \alpha$) if the corresponding condition in Table 1 holds. If an interpretation $\mathcal{I}$ satisfies all axioms in an ontology $\mathcal{O}$, then $\mathcal{I}$ is a *model* of $\mathcal{O}$ (written $\mathcal{I} \models \mathcal{O}$). An axiom α is a *consequence* of an ontology $\mathcal{O}$ (written $\mathcal{O} \models \alpha$) if every model of $\mathcal{O}$ satisfies α. A concept C is *subsumed* by D w.r.t. $\mathcal{O}$ if $\mathcal{O} \models C \sqsubseteq D$, and C is *unsatisfiable* w.r.t. $\mathcal{O}$ if $\mathcal{O} \models C \sqsubseteq \bot$. *Classification* is the task of computing all subsumptions $A \sqsubseteq B$ between atomic concepts such that $\mathcal{O} \models A \sqsubseteq B$.

The *polarities* of (syntactic) occurrences of $\mathcal{ALCH}$ concepts in concepts and axioms are defined recursively as follows: C occurs positively in C. If C occurs positively (resp. negatively) in C', then C occurs positively (resp. negatively) in $C' \sqcap D$, $D \sqcap C'$, $C' \sqcup D$, $D \sqcup C'$, $\exists R.C'$, $\forall R.C'$ and $D \sqsubseteq C'$, and C occurs negatively (resp. positively) in $\neg C'$ and $C' \sqsubseteq D$. Horn-$\mathcal{ALCH}$ is the fragment of $\mathcal{ALCH}$ in which axioms with positive occurrences of $C \sqcup D$ or negative occurrences of $\neg C$ or $\forall R.C$ are disallowed.

The notation $\bigsqcap_{i=1}^{n} C_i$ and $\bigsqcup_{i=1}^{n} C_i$, omitting the range when irrelevant, is used for finite n-ary conjunctions and disjunctions with the usual semantics. We do not distinguish between conjunctions and disjunctions with different order or multiplicity of elements and use set-theoretic operators $\in$, $\subseteq$, $\cap$ on them as if they were sets. The empty conjunction is identified with $\top$ and the empty disjunction with $\bot$.

3 Consequence-Based Procedure for Horn-$\mathcal{ALCH}$

Horn-$\mathcal{ALCH}$ can be seen as a fragment of the DL Horn-$\mathcal{SHIQ}$, which additionally allows for inverse roles, transitive roles, and (qualified) functionality restrictions. A consequence-based procedure for Horn-$\mathcal{SHIQ}$ ontologies was recently presented by Kazakov [2009]. In this section we outline a restriction of this procedure to Horn-$\mathcal{ALCH}$ and discuss how it can be extended to handle disjunctions.

To classify a Horn-$\mathcal{ALCH}$ ontology, the procedure first applies normalization rules to obtain an ontology $\mathcal{O}$ containing only axioms of the form $\bigsqcap A_i \sqsubseteq C$, $A \sqsubseteq \exists R.B$, $\exists R.A \sqsubseteq B$,[7] $A \sqsubseteq \forall R.B$ or $R \sqsubseteq S$, where C can be either atomic or $\bot$. The procedure then applies the rules in Table 2 to derive axioms of the form $H \sqsubseteq C$ or $H \sqsubseteq \exists R.K$, where H, K are conjunctions of atomic concepts and C is either atomic or $\bot$.

Note that the rule $\mathbf{R}^{\mathbf{n}}_{\sqcap}$ applies to n premises; when $n = 0$, it has no premises and uses the side condition $\top \sqsubseteq C \in \mathcal{O}$ to derive $H \sqsubseteq C$ for every H. The procedure is sound and complete for classification in the sense that $\mathcal{O} \models A \sqsubseteq B$ if and only if either $A \sqsubseteq B$ or $A \sqsubseteq \bot$ is derived. The number of (non-equivalent) axioms of the form $H \sqsubseteq C$ and $H \sqsubseteq \exists R.K$ is exponential in the number of atomic concepts and the procedure terminates in time at most exponential in the size of the input ontology.

In order to extend the procedure from Horn-$\mathcal{ALCH}$ to $\mathcal{ALCH}$, we need to deal with axioms involving disjunctions. For this purpose, we generalize the form of derivable axioms from $H \sqsubseteq C$ and $H \sqsubseteq \exists R.K$ to

$$H \sqsubseteq M \qquad \text{and} \qquad H \sqsubseteq N \sqcup \exists R.K, \tag{1}$$

where M, N are disjunctions of atomic concepts. Most rules in Table 2 are easily generalized to operate on axioms of this

[7] written as $A \sqsubseteq \forall R^{-}.B$ in the original presentation

$$\mathsf{R_A}\ \frac{}{H \sqsubseteq A} : A \in H$$

$$\mathsf{R}^{\mathsf{n}}_{\sqcap}\ \frac{\{H \sqsubseteq A_i\}_{i=1}^{n}}{H \sqsubseteq C} : \textstyle\bigsqcap_{i=1}^{n} A_i \sqsubseteq C \in \mathcal{O}$$

$$\mathsf{R}^{+}_{\exists}\ \frac{H \sqsubseteq A}{H \sqsubseteq \exists R.B} : A \sqsubseteq \exists R.B \in \mathcal{O}$$

$$\mathsf{R}^{-}_{\exists}\ \frac{H \sqsubseteq \exists R.K \quad K \sqsubseteq A}{H \sqsubseteq B} : \begin{array}{l}\exists S.A \sqsubseteq B \in \mathcal{O}\\ R \sqsubseteq_{\mathcal{O}} S\end{array}$$

$$\mathsf{R}^{\perp}_{\exists}\ \frac{H \sqsubseteq \exists R.K \quad K \sqsubseteq \perp}{H \sqsubseteq \perp}$$

$$\mathsf{R}_{\forall}\ \frac{H \sqsubseteq \exists R.K \quad H \sqsubseteq A}{H \sqsubseteq \exists R.(K \sqcap B)} : \begin{array}{l}A \sqsubseteq \forall S.B \in \mathcal{O}\\ R \sqsubseteq_{\mathcal{O}} S\end{array}$$

Table 2: The inference rules for Horn-$\mathcal{ALCH}$

form. For example, $\mathsf{R}^{\mathsf{n}}_{\sqcap}$ with $n = 2$ can be written as

$$\frac{H \sqsubseteq N_1 \sqcup A_1 \quad H \sqsubseteq N_2 \sqcup A_2}{H \sqsubseteq N_1 \sqcup N_2 \sqcup M} : A_1 \sqcap A_2 \sqsubseteq M \in \mathcal{O},$$

where $N \sqcup A$ stands for a disjunction containing A (not necessarily at the last position).

The main difficulty lies in generalizing the rule $\mathsf{R}^{-}_{\exists}$. Consider the premises $H \sqsubseteq \exists R.K$ and $K \sqsubseteq N \sqcup A$ and the side-condition $\exists R.A \sqsubseteq B \in \mathcal{O}$. If N is empty, $\mathsf{R}^{-}_{\exists}$ should produce $H \sqsubseteq B$ as before. If, however, N is not empty, this inference is unsound. We can take N into account by deriving a weaker conclusion $H \sqsubseteq B \sqcup \exists R.(K \sqcap N)$, which is now correct. Unfortunately, this strategy introduces both conjunctions and disjunctions under existential restrictions, and it is difficult to obtain optimal (exponential) complexity bounds for the procedure. Our solution is to derive an axiom $H \sqsubseteq B \sqcup \exists R.(K \sqcap \neg A)$ instead, which is equivalent to the previous axiom given $K \sqsubseteq N \sqcup A$. To capture the new form of axioms, we generalize H, K in (1) to stand for conjunctions of *literals*—atomic or negated atomic concepts. The number of such axioms remains exponential in the size of $\mathcal{O}$.

4 Consequence-Based Procedure for $\mathcal{ALCH}$

In this section we present the consequence-based procedure for $\mathcal{ALCH}$ ontologies based on the ideas from the previous section. The procedure consists of a *normalization stage*, during which structural transformation is used to simplify the form of axioms in the ontology, and a *saturation stage*, which derives new axioms using inference rules.

4.1 Normalization

Given an $\mathcal{ALCH}$ ontology $\mathcal{O}$, introduce a new atomic concept $[C]$ for each concept C occurring in $\mathcal{O}$. *Structural transformation* of C, denoted by $st(C)$, is defined as follows:

$$\begin{array}{ll} st(A) := A & st(C \sqcap D) := [C] \sqcap [D] \\ st(\top) := \top & st(C \sqcup D) := [C] \sqcup [D] \\ st(\perp) := \perp & st(\exists R.C) := \exists R.[C] \\ st(\neg C) := \neg [C] & st(\forall R.C) := \forall R.[C]. \end{array}$$

The *structural transformation* of $\mathcal{O}$ is a new ontology $st(\mathcal{O})$ containing the same role inclusion axioms as $\mathcal{O}$ and

- $st(C) \sqsubseteq [C]$ for every C occurring negatively in $\mathcal{O}$,
- $[D] \sqsubseteq st(D)$ for every D occurring positively in $\mathcal{O}$,
- $[C] \sqsubseteq [D]$ for each axiom $C \sqsubseteq D$ in $\mathcal{O}$.

Notice that the size of $st(\mathcal{O})$ is linear in the size of $\mathcal{O}$ and $st(\mathcal{O})$ can be computed from $\mathcal{O}$ in polynomial time.

We prove correctness of structural transformation by showing that $st(\mathcal{O})$ is conservative over $\mathcal{O}$. An ontology $\mathcal{Q}$ is *conservative over* $\mathcal{O}$ if every model of $\mathcal{Q}$ is also a model of $\mathcal{O}$, and every model of $\mathcal{O}$ can be turned into a model of $\mathcal{Q}$ by interpreting the atomic concepts that do not occur in $\mathcal{O}$. It is easy to see that in such case $\mathcal{O} \models A \sqsubseteq B$ iff $\mathcal{Q} \models A \sqsubseteq B$ for all A and B occurring in $\mathcal{O}$. To classify $\mathcal{O}$ it is thus sufficient to classify $\mathcal{Q}$ and take the subsumptions between the original concepts.

Lemma 1. *Let $\mathcal{I}$ be a model of $st(\mathcal{O})$. Then:*

(a) if C occurs negatively in $\mathcal{O}$, then $C^{\mathcal{I}} \subseteq [C]^{\mathcal{I}}$;

(b) if D occurs positively in $\mathcal{O}$, then $[D]^{\mathcal{I}} \subseteq D^{\mathcal{I}}$;

(c) if $C \sqsubseteq D \in \mathcal{O}$ then $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

Proof. Points (a) and (b) are proved by structural induction on C and D. To prove (c), let $C \sqsubseteq D \in \mathcal{O}$. Then C occurs negatively in $\mathcal{O}$, D occurs positively in $\mathcal{O}$ and $[C] \sqsubseteq [D] \in st(\mathcal{O})$. By (a), (b), and since $\mathcal{I} \models st(\mathcal{O})$, we have $C^{\mathcal{I}} \subseteq [C]^{\mathcal{I}}$, $[D]^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and $[C]^{\mathcal{I}} \subseteq [D]^{\mathcal{I}}$. Thus, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. □

Lemma 2. *$st(\mathcal{O})$ is conservative over $\mathcal{O}$.*

Proof. By Lemma 1 (c), every model $\mathcal{I}$ of $st(\mathcal{O})$ is a model of $\mathcal{O}$. Conversely, every model $\mathcal{I}$ of $\mathcal{O}$ can be turned into a model of $st(\mathcal{O})$ by interpreting $[C]^{\mathcal{I}} := C^{\mathcal{I}}$. □

We say that an $\mathcal{ALCH}$ ontology is *normalized* if it only contains axioms of the form $\bigsqcap A_i \sqsubseteq \bigsqcup B_j$, $A \sqsubseteq \exists R.B$, $\exists R.A \sqsubseteq B$, $A \sqsubseteq \forall R.B$ or $R \sqsubseteq S$.

Proposition 3. *For every $\mathcal{ALCH}$ ontology $\mathcal{O}$ there exists a normalized ontology $\mathcal{Q}$ conservative over $\mathcal{O}$. Moreover, the size of $\mathcal{Q}$ is linear in the size of $\mathcal{O}$ and $\mathcal{Q}$ can be computed from $\mathcal{O}$ in polynomial time.*

Proof. Given an $\mathcal{ALCH}$ ontology $\mathcal{O}$, replace every negative occurrence of a concept of the form $\forall R.C$ in $\mathcal{O}$ by the equivalent concept $\neg\exists R.\neg C$. From now on, we assume that $\mathcal{O}$ contains no negative occurrences of universal restrictions. Let $\mathcal{Q} := st(\mathcal{O})$. Its concept inclusions are of the form $st(C) \sqsubseteq [C]$, $[D] \sqsubseteq st(D)$, or $[C] \sqsubseteq [D]$, where C is not of the form $\forall R.D$. Axioms $\perp \sqsubseteq [\perp]$ and $[\top] \sqsubseteq \top$ are always

$$\mathsf{R}^+_{\mathsf{A}}\ \frac{}{H \sqsubseteq A} : A \in H \qquad \mathsf{R}^-_{\mathsf{A}}\ \frac{H \sqsubseteq N \sqcup A}{H \sqsubseteq N} : \neg A \in H$$

$$\mathsf{R}^{\mathsf{n}}_{\sqcap}\ \frac{\{H \sqsubseteq N_i \sqcup A_i\}_{i=1}^n}{H \sqsubseteq \bigsqcup_{i=1}^n N_i \sqcup M} : \textstyle\bigsqcap_{i=1}^n A_i \sqsubseteq M \in \mathcal{O}$$

$$\mathsf{R}^+_{\exists}\ \frac{H \sqsubseteq N \sqcup A}{H \sqsubseteq N \sqcup \exists R.B} : A \sqsubseteq \exists R.B \in \mathcal{O}$$

$$\mathsf{R}^-_{\exists}\ \frac{H \sqsubseteq M \sqcup \exists R.K \quad K \sqsubseteq N \sqcup A}{H \sqsubseteq M \sqcup B \sqcup \exists R.(K \sqcap \neg A)} : \begin{matrix}\exists S.A \sqsubseteq B \in \mathcal{O}\\ R \sqsubseteq_{\mathcal{O}} S\end{matrix}$$

$$\mathsf{R}^\perp_{\exists}\ \frac{H \sqsubseteq M \sqcup \exists R.K \quad K \sqsubseteq \perp}{H \sqsubseteq M}$$

$$\mathsf{R}_{\forall}\ \frac{H \sqsubseteq M \sqcup \exists R.K \quad H \sqsubseteq N \sqcup A}{H \sqsubseteq M \sqcup N \sqcup \exists R.(K \sqcap B)} : \begin{matrix}A \sqsubseteq \forall S.B \in \mathcal{O}\\ R \sqsubseteq_{\mathcal{O}} S\end{matrix}$$

Table 3: The inference rules for $\mathcal{ALCH}$

satisfied and can be removed from $\mathcal{Q}$. Of the remaining axioms, the following are not normalized but can be rewritten into normalized axioms as indicated:

$$[C \sqcap D] \sqsubseteq [C] \sqcap [D] \rightsquigarrow [C \sqcap D] \sqsubseteq [C] \text{ and } [C \sqcap D] \sqsubseteq [D]$$
$$[C] \sqcup [D] \sqsubseteq [C \sqcup D] \rightsquigarrow [C] \sqsubseteq [C \sqcup D] \text{ and } [D] \sqsubseteq [C \sqcup D]$$
$$[\neg C] \sqsubseteq \neg[C] \rightsquigarrow [\neg C] \sqcap [C] \sqsubseteq \perp$$
$$\neg[C] \sqsubseteq [\neg C] \rightsquigarrow \top \sqsubseteq [\neg C] \sqcup [C].$$

□

4.2 Saturation

The inference rules given in Table 3 are applied to a normalized ontology $\mathcal{O}$ and derive axioms of the form (1), where H, K are conjunctions of literals and M, N disjunctions of atomic concepts. We write $\mathcal{O} \vdash \alpha$ if the axiom α is derivable using these rules with side conditions from $\mathcal{O}$. It is easy to see that the inference system is sound: if $\mathcal{O} \vdash \alpha$ then $\mathcal{O} \models \alpha$. Although the converse is in general not true, the inference system is *refutationally complete* in the following sense:

Theorem 4. *Let $\mathcal{O}$ be a normalized $\mathcal{ALCH}$ ontology and H a conjunction of literals. Then $\mathcal{O} \vdash H \sqsubseteq \perp$ if $\mathcal{O} \models H \sqsubseteq \perp$.*

4.3 Proof of Refutational Completeness

The proof of Theorem 4 is by canonical model construction, similar to the case of $\mathcal{EL}^{++}$ and Horn-$\mathcal{SHIQ}$ [Baader *et al.*, 2005; Kazakov, 2009]. We will demonstrate that $\mathcal{O} \nvdash H \sqsubseteq \perp$ implies $\mathcal{O} \not\models H \sqsubseteq \perp$. W.l.o.g., $\mathcal{O} \nvdash H \sqsubseteq \perp$ for some H, for otherwise this implication is trivial. We will construct a *canonical model* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{O}$ such that:

$$\text{for every } H \text{ with } \mathcal{O} \nvdash H \sqsubseteq \perp \text{ there exists } x_H \in H^{\mathcal{I}}. \quad (2)$$

It will then follow that if $\mathcal{O} \nvdash H \sqsubseteq \perp$, then $H^{\mathcal{I}} \neq \emptyset$ by (2), so $\mathcal{I} \not\models H \sqsubseteq \perp$, and so $\mathcal{O} \not\models H \sqsubseteq \perp$ provided that $\mathcal{I} \models \mathcal{O}$.

We introduce a distinct individual x_H for each H such that $\mathcal{O} \nvdash H \sqsubseteq \perp$ and define the domain of $\mathcal{I}$ by

$$\Delta^{\mathcal{I}} := \{x_H \mid \mathcal{O} \nvdash H \sqsubseteq \perp\}. \quad (3)$$

It is not empty since by assumption $\mathcal{O} \nvdash H \sqsubseteq \perp$ for some H.

To define the interpretation of atomic concepts, for every $x_H \in \Delta^{\mathcal{I}}$, we construct a set $\mathcal{I}_H$ of atomic concepts such that $A \in \mathcal{I}_H$ iff $x_H \in A^{\mathcal{I}}$. Intuitively, $\mathcal{I}_H$ is defined to satisfy all derivable axioms of the form $H \sqsubseteq M$. Let us fix some total ordering of atomic concepts $B_1 \prec B_2 \prec \ldots$. We write $M \prec B_n$ if $M \subseteq \{B_1, \ldots, B_{n-1}\}$. We define $\mathcal{I}_H$ as the limit $\mathcal{I}_H := \bigcup_{i \geq 0} \mathcal{I}_H^{(i)}$, where $\mathcal{I}_H^{(0)} := \emptyset$ and for $i \geq 1$,

$$\mathcal{I}_H^{(i)} := \begin{cases} \mathcal{I}_H^{(i-1)} \cup \{B_i\} \text{ if there exists } M \prec B_i \text{ such that} \\ \qquad \mathcal{O} \vdash H \sqsubseteq M \sqcup B_i \text{ and } M \cap \mathcal{I}_H^{(i-1)} = \emptyset, \\ \mathcal{I}_H^{(i-1)} \text{ otherwise.} \end{cases}$$

From the definition of $\mathcal{I}_H$ it is easy to see that:

$$\text{if } \mathcal{O} \nvdash H \sqsubseteq \perp \text{ and } \mathcal{O} \vdash H \sqsubseteq M, \text{ then } M \cap \mathcal{I}_H \neq \emptyset. \quad (4)$$

The interpretation of atomic concepts is now defined by

$$A^{\mathcal{I}} := \{x_H \mid A \in \mathcal{I}_H\}. \quad (5)$$

We interpret roles to satisfy all role inclusion axioms and all derivable axioms of the form $H \sqsubseteq M \sqcup \exists R.K$. For every role R and every conjunction H such that $\mathcal{O} \nvdash H \sqsubseteq \perp$, define

$$\mathcal{I}_H^R := \{K \mid \exists M \colon \mathcal{O} \vdash H \sqsubseteq M \sqcup \exists R.K,\ M \cap \mathcal{I}_H = \emptyset\}. \quad (6)$$

We say that K is *maximal* in $\mathcal{I}_H^R$ if $K \in \mathcal{I}_H^R$ and there is no $K' \in \mathcal{I}_H^R$ with $K \subsetneq K'$. From (4) and (6) using $\mathsf{R}^\perp_\exists$ it follows that $\mathcal{O} \nvdash K \sqsubseteq \perp$ for every $K \in \mathcal{I}_H^R$, so x_K is a well-defined element in $\Delta^{\mathcal{I}}$. The interpretation of roles is now defined by

$$S^{\mathcal{I}} := \bigcup_{R \sqsubseteq_{\mathcal{O}} S} \{(x_H, x_K) \mid K \text{ is maximal in } \mathcal{I}_H^R\}. \quad (7)$$

Since $\mathcal{I}_H^R$ is finite (every $K \in \mathcal{I}_H^R$ contains only atomic concepts that occur in $\mathcal{O}$), it follows that:

$$\text{every } K \in \mathcal{I}_H^R \text{ is a subset of some maximal } K' \in \mathcal{I}_H^R. \quad (8)$$

Lemma 5. *(a) For every $x_H \in A^{\mathcal{I}}$ there exists $M \prec A$ such that $\mathcal{O} \vdash H \sqsubseteq M \sqcup A$ and $M \cap \mathcal{I}_H = \emptyset$; (b) for every $(x_H, x_K) \in S^{\mathcal{I}}$ there exist $R \sqsubseteq_{\mathcal{O}} S$ and M such that K is maximal in $\mathcal{I}_H^R$, $\mathcal{O} \vdash H \sqsubseteq M \sqcup \exists R.K$ and $M \cap \mathcal{I}_H = \emptyset$.*

Now we are in a position to establish (2):

Lemma 6. *Let $x_H \in \Delta^{\mathcal{I}}$. Then $x_H \in H^{\mathcal{I}}$.*

Proof. We will show that $x_H \in C^{\mathcal{I}}$ for all conjuncts $C \in H$.
- $C = A$. Then $\mathcal{O} \vdash H \sqsubseteq A$ by $\mathsf{R}^+_{\mathsf{A}}$, $A \in \mathcal{I}_H$ by (4), so $x_H \in A^{\mathcal{I}}$ by (5).
- $C = \neg A$. If $x_H \in A^{\mathcal{I}}$, then by Lemma 5 (a) there exists $M \prec A$ such that $\mathcal{O} \vdash H \sqsubseteq M \sqcup A$ and $M \cap \mathcal{I}_H = \emptyset$. By $\mathsf{R}^-_{\mathsf{A}}$ we obtain $\mathcal{O} \vdash H \sqsubseteq M$, so $M \cap \mathcal{I}_H \neq \emptyset$ by (4). Contradiction. So $x_H \notin A^{\mathcal{I}}$ and therefore $x_H \in (\neg A)^{\mathcal{I}}$. □

We conclude the proof of Theorem 4 by showing that:

Theorem 7. *$\mathcal{I}$ is a model of $\mathcal{O}$.*

Proof. We show that $\mathcal{I} \models \alpha$ for all $\alpha \in \mathcal{O}$. Since $\mathcal{O}$ is normalized, α is of one of the following forms:
- $\alpha = \bigsqcap_{i=1}^n A_i \sqsubseteq M, n \geq 0$. Take any $x_H \in \Delta^{\mathcal{I}}$ with $x_H \in A_i$ for all i $(1 \leq i \leq n)$. To prove $\mathcal{I} \models \alpha$, we show

that $x_H \in M^{\mathcal{I}}$. By Lemma 5 (a), for every i there exists $N_i \prec A_i$ such that $\mathcal{O} \vdash H \sqsubseteq N_i \sqcup A_i$ and $N_i \cap \mathcal{I}_H = \emptyset$. By $\mathsf{R}^{\mathsf{n}}_{\sqcap}$ with side condition α, we obtain $\mathcal{O} \vdash H \sqsubseteq \bigsqcup_{i=1}^{n} N_i \sqcup M$. By (4), $(\bigsqcup_{i=1}^{n} N_i \sqcup M) \cap \mathcal{I}_H \neq \emptyset$. Since $N_i \cap \mathcal{I}_H = \emptyset$ for all i $(1 \leq i \leq n)$, we have $M \cap \mathcal{I}_H \neq \emptyset$. So $x_H \in M^{\mathcal{I}}$ by (5).

- $\alpha = A \sqsubseteq \exists R.B$. Take any $x_H \in A^{\mathcal{I}}$. To prove $\mathcal{I} \models \alpha$, we show that $x_H \in (\exists R.B)^{\mathcal{I}}$. By Lemma 5 (a), there exists $N \prec A$ such that $\mathcal{O} \vdash H \sqsubseteq N \sqcup A$ and $N \cap \mathcal{I}_H = \emptyset$. By $\mathsf{R}^{+}_{\exists}$ with side condition α, we obtain $\mathcal{O} \vdash H \sqsubseteq N \sqcup \exists R.B$. Since $N \cap \mathcal{I}_H = \emptyset$, $B \in \mathcal{I}_H^R$. By (8) there exists a maximal $K \in \mathcal{I}_H^R$ containing B. Then $(x_H, x_K) \in R^{\mathcal{I}}$ by (7). By Lemma 6, $x_K \in K^{\mathcal{I}}$, so $x_K \in B^{\mathcal{I}}$ as $B \in K$. Therefore $x_H \in (\exists R.B)^{\mathcal{I}}$ by the semantics of existential restrictions.

- $\alpha = \exists S.A \sqsubseteq B$. Take any $x_H \in (\exists S.A)^{\mathcal{I}}$. To prove $\mathcal{I} \models \alpha$, we show that $x_H \in B^{\mathcal{I}}$. By the semantics of existential restrictions, there exists $x_K \in \Delta^{\mathcal{I}}$ with $(x_H, x_K) \in S^{\mathcal{I}}$ and $x_K \in A^{\mathcal{I}}$. By Lemma 5 (b), there exists $R \sqsubseteq_{\mathcal{O}} S$ and M such that K is maximal in $\mathcal{I}_H^R$, $\mathcal{O} \vdash H \sqsubseteq M \sqcup \exists R.K$ and $M \cap \mathcal{I}_H = \emptyset$. By Lemma 5 (a), there exists $N \prec A$ such that $\mathcal{O} \vdash K \sqsubseteq N \sqcup A$. By $\mathsf{R}^{-}_{\exists}$ with side condition α, we obtain $\mathcal{O} \vdash H \sqsubseteq M \sqcup B \sqcup \exists R.(K \sqcap \neg A)$. By maximality of K, either $\neg A \in K$ or $(K \sqcap \neg A) \notin \mathcal{I}_H^R$. Assume the former. By Lemma 6, $x_K \in K^{\mathcal{I}} \subseteq (\neg A)^{\mathcal{I}}$, contradicting $x_K \in A^{\mathcal{I}}$. Therefore $(K \sqcap \neg A) \notin \mathcal{I}_H^R$, so $(M \sqcup B) \cap \mathcal{I}_H \neq \emptyset$. Since $M \cap \mathcal{I}_H = \emptyset$, $B \in \mathcal{I}_H$. Then $x_H \in B^{\mathcal{I}}$ by (5).

- $\alpha = A \sqsubseteq \forall S.B$. Take any $x_H \in A^{\mathcal{I}}$. To prove $\mathcal{I} \models \alpha$, we show that $x_H \in (\forall S.B)^{\mathcal{I}}$, that is, $x_K \in B^{\mathcal{I}}$ for every x_K such that $(x_H, x_K) \in S^{\mathcal{I}}$. Consider any such x_K. By Lemma 5 (b), there exist R and M such that $R \sqsubseteq_{\mathcal{O}} S$, K is maximal in $\mathcal{I}_H^R$, $\mathcal{O} \vdash H \sqsubseteq M \sqcup \exists R.K$ and $M \cap \mathcal{I}_H = \emptyset$, and by Lemma 5 (a) applied to $x_H \in A^{\mathcal{I}}$, there exists $N \prec A$ such that $\mathcal{O} \vdash H \sqsubseteq N \sqcup A$ and $N \cap \mathcal{I}_H = \emptyset$. By $\mathsf{R}_{\forall}$ with side condition α we obtain $\mathcal{O} \vdash H \sqsubseteq M \sqcup N \sqcup \exists R.(K \sqcap B)$. Then $(K \sqcap B) \in \mathcal{I}_H^R$ as $M \cap \mathcal{I}_H = N \cap \mathcal{I}_H = \emptyset$. By maximality of K, $B \in K$. By Lemma 6, $x_K \in K^{\mathcal{I}}$, so $x_K \in B^{\mathcal{I}}$.

- $\alpha = R \sqsubseteq S$. $\mathcal{I} \models \alpha$ follows immediately from (7). □

Remark 8. Note that the completeness proof applies the rules $\mathsf{R}^{-}_{\mathsf{A}}$, $\mathsf{R}^{\mathsf{n}}_{\sqcap}$, $\mathsf{R}^{+}_{\exists}$, $\mathsf{R}^{-}_{\exists}$, $\mathsf{R}_{\forall}$ only when $N \prec A$. Just like for ordered resolution [Bachmair and Ganzinger, 2001], this means that the rules in Table 3 remain refutationally complete even under *ordering restrictions* $N \prec A$ for some total ordering $\prec$ on atomic concepts. In fact, as seen from the proof, different orderings $\prec$ can be used for different left-hand sides H.

Remark 9. Recall that (8) requires $\mathcal{O}$ to contain only finitely many atomic concepts. For Theorem 4, however, this assumption is not needed. Indeed, by compactness of first-order logic, if $\mathcal{O} \models H \sqsubseteq \bot$, then there exists a finite subset $\mathcal{Q} \subseteq \mathcal{O}$ such that $\mathcal{Q} \models H \sqsubseteq \bot$, and by applying Theorem 4 for $\mathcal{Q}$, we have $\mathcal{Q} \vdash H \sqsubseteq \bot$, which implies $\mathcal{O} \vdash H \sqsubseteq \bot$.

5 Implementation and Optimizations

The rules in Table 3 can be used for classification through the equivalence of $A \sqsubseteq B$ with $A \sqcap \neg B \sqsubseteq \bot$. Since the number of derivable axioms is exponential in the number of atomic concepts, the procedure can be implemented to run in time exponential in the size of the input ontology. This is theoretically optimal because checking a single subsumption between a pair of atomic concepts w.r.t. an $\mathcal{ALCH}$ ontology is already ExpTime-complete (see, e.g., [Baader *et al.*, 2007]). However, a straightforward implementation of the procedure is impractical as, e.g., rule $\mathsf{R}^{+}_{\mathsf{A}}$ alone generates all possible conjunctions H, and further optimizations are needed if we are to use the procedure in a practical reasoning system.

5.1 Goal-Directed Introduction of Contexts

We speak of the conjunctions on the left-hand sides of axioms $H \sqsubseteq M$ and $H \sqsubseteq M \sqcup \exists R.K$ as *contexts* and of the disjunctions on their right-hand sides as *clauses*. Similarly to Horn-$\mathcal{SHIQ}$ [Kazakov, 2009], new contexts can be introduced in a goal-directed way. During saturation we maintain a list of *active* contexts and apply rule $\mathsf{R}^{+}_{\mathsf{A}}$ (and $\mathsf{R}^{\mathsf{n}}_{\mathsf{A}}$ with $n = 0$) only to those. The list is initialized by contexts that are relevant to the reasoning task, e.g., for classification, contexts H of the form $H = A \sqcap \neg B$, where A and B are atomic concepts occurring in the input ontology. A new context K becomes active when some clause of the form $M \sqcup \exists R.K$ is derived, as K can then be required in the second premise of rules $\mathsf{R}^{-}_{\exists}$ and $\mathsf{R}^{\bot}_{\exists}$. Note that, since it introduces quadratically many contexts $A \sqcap \neg B$, this approach is still impractical for classification. In the next section we will show how we address this issue.

5.2 Context Representations

The inference rules in Table 3 are monotone w.r.t. context, i.e., if $H \sqsubseteq M$ is derived, then $H' \sqsubseteq M$ will also be derived for every H' such that $H \subseteq H'$. To avoid recomputation, the rules in Table 3 can be implemented using a shared representation where clauses derived for contexts are implicitly present for their super-contexts. Specifically, we operate with *context representations* H^+ which represent all contexts H' such that $H \subseteq H'$, i.e., we write $H^+ \sqsubseteq M$ and $H^+ \sqsubseteq M \sqcup \exists R.K$ to represent every axiom $H' \sqsubseteq M$ and respectively $H' \sqsubseteq M \sqcup \exists R.K$ with $H \subseteq H'$. The inference rules can easily be reformulated to operate directly on context representations. For example, rule $\mathsf{R}^{\mathsf{n}}_{\sqcap}$ with $n = 2$ becomes

$$\frac{H_1^+ \sqsubseteq N_1 \sqcup A \quad H_2^+ \sqsubseteq N_2 \sqcup B}{(H_1 \sqcap H_2)^+ \sqsubseteq N_1 \sqcup N_2 \sqcup M} : A \sqcap B \sqsubseteq M \in \mathcal{O}, \quad (9)$$

i.e., if a clause $N_1 \sqcup A$ is derivable for all contexts H' with $H_1 \subseteq H'$ and $N_2 \sqcup B$ is derivable for all H' with $H_2 \subseteq H'$, then $N_1 \sqcup N_2 \sqcup M$ is derivable for all H' satisfying both of these properties. Other rules are reformulated similarly, e.g.,

$$\mathsf{R}^{+}_{\mathsf{A}} \ \frac{}{A^+ \sqsubseteq A} \quad \text{and} \quad \mathsf{R}^{-}_{\mathsf{A}} \ \frac{H^+ \sqsubseteq N \sqcup A}{(H \sqcap \neg A)^+ \sqsubseteq N}. \quad (10)$$

All reformulated rules are shown in Table 4.

Note that $\mathsf{R}^{+}_{\mathsf{A}}$ now generates only representations of the form A^+; a representation of the form $(A \sqcap \neg B)^+$ is introduced by $\mathsf{R}^{-}_{\mathsf{A}}$ only when some $A^+ \sqsubseteq N \sqcup B$ with $N \prec B$ is derived by the rules, which, in practice, happens only for a small proportion of pairs (A, B). This avoids unnecessary satisfiability checks and classifies the ontology in "one pass".

$$\mathsf{R}_{\mathsf{A}}^{+}\ \frac{}{A^+ \sqsubseteq A} \qquad \mathsf{R}_{\mathsf{A}}^{-}\ \frac{H^+ \sqsubseteq N \sqcup A}{(H \sqcap \neg A)^+ \sqsubseteq N}$$

$$\mathsf{R}_{\sqcap}^{\mathsf{n}}\ \frac{\{H_i^+ \sqsubseteq N_i \sqcup A_i\}_{i=1}^n}{(\bigsqcap_{i=1}^n H_i)^+ \sqsubseteq \bigsqcup_{i=1}^n N_i \sqcup M} : \bigsqcap_{i=1}^n A_i \sqsubseteq M \in \mathcal{O}$$

$$\mathsf{R}_{\exists}^{+}\ \frac{H^+ \sqsubseteq N \sqcup A}{H^+ \sqsubseteq N \sqcup \exists R.B} : A \sqsubseteq \exists R.B \in \mathcal{O}$$

$$\mathsf{R}_{\exists}^{-}\ \frac{H^+ \sqsubseteq M \sqcup \exists R.K' \quad K^+ \sqsubseteq N \sqcup A}{H^+ \sqsubseteq M \sqcup B \sqcup \exists R.(K' \sqcap \neg A)} : \begin{array}{l} K \subseteq K' \\ \exists S.A \sqsubseteq B \in \mathcal{O} \\ R \sqsubseteq_{\mathcal{O}} S \end{array}$$

$$\mathsf{R}_{\exists}^{\perp}\ \frac{H^+ \sqsubseteq M \sqcup \exists R.K' \quad K^+ \sqsubseteq \perp}{H^+ \sqsubseteq M} : K \subseteq K'$$

$$\mathsf{R}_{\forall}\ \frac{H_1^+ \sqsubseteq M \sqcup \exists R.K \quad H_2^+ \sqsubseteq N \sqcup A}{(H_1 \sqcap H_2)^+ \sqsubseteq M \sqcup N \sqcup \exists R.(K \sqcap B)} : \begin{array}{l} A \sqsubseteq \forall S.B \in \mathcal{O} \\ R \sqsubseteq_{\mathcal{O}} S \end{array}$$

Table 4: The inference rules with context representations

5.3 Context Partitioning

One drawback of context representations is that, unlike $\mathsf{R}_{\sqcap}^{\mathsf{n}}$ in Table 3, (9) causes an interaction between different H_1 and H_2, and, in particular, can introduce new $H_1 \sqcap H_2$; this makes it difficult to implement the rule in a goal-directed way, i.e., to avoid generating conclusions of (9) in which $(H_1 \sqcap H_2)^+$ will never represent any active context. While an efficient implementation of such rules is yet to be found, we have studied an alternative strategy which avoids many unnecessary inferences. Intuitively, we divide the set of contexts into several partitions and make sure that inferences between clauses derived for different partitions are never made. We achieve this by relaxing H^+ to represent only those H' with $H \subseteq H'$ that are in the same partition as H.

Formally, fix a set of partitions $P \neq \emptyset$ and a function $p(\cdot)$ that assigns to every context H its partition $p(H) \in P$ such that if $p(H_1) = p(H_2)$, then $p(H_1 \sqcap H_2) = p(H_1) = p(H_2)$. Given this function, let H^+ represent every context H' such that $H \subseteq H'$ and $p(H) = p(H')$. Then the inference (9) can be restricted by an additional side condition $p(H_1) = p(H_2)$, for otherwise there is no context H' represented both by H_1^+ and H_2^+. The two strategies for rule applications (direct and using context representations) correspond to two extreme choices of the partition function. If we choose a singleton partition for every context, e.g., assign $p(H) := H$, then H^+ represents just H, and we arrive at our original rules in Table 3. On the other hand, if we assign the same partition to every context, then the restriction $p(H_1) = p(H_2)$ has no effect, and we arrive at rules such as (9) and (10).

In our implementation, we chose the partitioning function $p_{\prec}(H) := \max_{\prec}\{A \mid A \in H\}$, where $\prec$ is some total ordering of atomic concepts and $\max(\emptyset) = \top$. This way, in particular, we can still represent all contexts $A \sqcap \neg B$ by A^+ and there is no interaction between A^+ and B^+ for different A, B. In fact, it can be shown that this strategy simulates the original $\mathcal{EL}$ classification procedure when applied to $\mathcal{EL}$ ontologies, which gives us nice "pay-as-you-go" behaviour. The inference rules that correspond to our partitioning function $p_{\prec}$ are listed Table 5.

$$\mathsf{R}_{\mathsf{A}}^{+}\ \frac{}{A^+ \sqsubseteq A} \quad \text{and} \quad \frac{}{(A \sqcap B)^+ \sqsubseteq B} : B \prec A$$

$$\mathsf{R}_{\mathsf{A}}^{-}\ \frac{H^+ \sqsubseteq N \sqcup A}{(H \sqcap \neg A)^+ \sqsubseteq N}$$

$$\mathsf{R}_{\sqcap}^{\mathsf{0}}\ \frac{}{\top^+ \sqsubseteq M \quad A^+ \sqsubseteq M} : \top \sqsubseteq M \in \mathcal{O}$$

$$\mathsf{R}_{\sqcap}^{\mathsf{n}}\ \frac{\{H_i^+ \sqsubseteq N_i \sqcup A_i\}_{i=1}^n}{(\bigsqcap_{i=1}^n H_i)^+ \sqsubseteq \bigsqcup_{i=1}^n N_i \sqcup M} : \begin{array}{l} p_{\prec}(H_i) = p_{\prec}(H_1) \\ \bigsqcap_{i=1}^n A_i \sqsubseteq M \in \mathcal{O} \\ n \geq 1 \end{array}$$

$$\mathsf{R}_{\exists}^{+}\ \frac{H^+ \sqsubseteq N \sqcup A}{H^+ \sqsubseteq N \sqcup \exists R.B} : A \sqsubseteq \exists R.B \in \mathcal{O}$$

$$\mathsf{R}_{\exists}^{-}\ \frac{H^+ \sqsubseteq M \sqcup \exists R.K' \quad K^+ \sqsubseteq N \sqcup A}{H^+ \sqsubseteq M \sqcup B \sqcup \exists R.(K' \sqcap \neg A)} : \begin{array}{l} K \subseteq K' \\ p_{\prec}(K) = p_{\prec}(K') \\ \exists S.A \sqsubseteq B \in \mathcal{O} \\ R \sqsubseteq_{\mathcal{O}} S \end{array}$$

$$\mathsf{R}_{\exists}^{\perp}\ \frac{H^+ \sqsubseteq M \sqcup \exists R.K' \quad K^+ \sqsubseteq \perp}{H^+ \sqsubseteq M} : \begin{array}{l} K \subseteq K' \\ p_{\prec}(K) = p_{\prec}(K') \end{array}$$

$$\mathsf{R}_{\forall}\ \frac{H_1^+ \sqsubseteq M \sqcup \exists R.K \quad H_2^+ \sqsubseteq N \sqcup A}{(H_1 \sqcap H_2)^+ \sqsubseteq M \sqcup N \sqcup \exists R.(K \sqcap B)} : \begin{array}{l} p_{\prec}(H_1) = p_{\prec}(H_2) \\ A \sqsubseteq \forall S.B \in \mathcal{O} \\ R \sqsubseteq_{\mathcal{O}} S \end{array}$$

Table 5: The inference rules with context partitioning using the partitioning function $p_{\prec}$

5.4 Avoiding Clauses for the Empty Context

Although context representations can largely prevent duplication of clauses, this is not the case for the clauses derivable for the empty context $\top$. Since such clauses are derivable for every context, they are recomputed in every partition (see, e.g., $\mathsf{R}_{\sqcap}^{\mathsf{0}}$ in Table 5). Axioms of the form $\top \sqsubseteq M \in \mathcal{O}$ can arise from negative occurrences of negations and these, in turn, arise from negative occurrences of universal restrictions in the input ontology. Recall from the proof of Proposition 3 that every negative occurrence of $\forall R.C$ in the input ontology is first rewritten to $\neg\exists R.\neg C$ from which $\top \sqsubseteq [\neg\exists R.\neg C] \sqcup [\exists R.\neg C]$ is then produced by structural transformation. We use well-known absorption techniques [Hudek and Weddell, 2006] to avoid such axioms whenever possible, e.g., rewrite $A \sqcap \forall R.B \sqsubseteq C$ into $A \sqsubseteq C \sqcup \exists R.\neg B$.

It is also possible to avoid duplicating the clauses derivable for $\top$ in other partitions by deriving them explicitly only for $\top$ and assuming that they are implicitly present for every other context. However, in practice, we haven't observed a significant difference in the performance of the procedure after implementing this optimization.

5.5 Subsumption Deletion

Observe that $H \sqsubseteq M$ implies $H \sqsubseteq N$ when $M \subsetneq N$. In this case, it is often said that the clause N *is subsumed by* M. We can safely delete axioms having clauses subsumed by other clauses for the same context since such axioms are never used in the proof of Theorem 4. Similarly, $H \sqsubseteq N \sqcup \exists R.K$ can be deleted when either $H \sqsubseteq M$ has been derived with $M \subseteq N$, or $H \sqsubseteq M \sqcap \exists R.K'$ has been derived with $M \subseteq N$, $K \subseteq K'$ provided that at least one of these inclusions is strict.

6 Evaluation

Previous experimental evidence [Suntisrivaraporn, 2009; Kazakov, 2009] suggests that, where applicable, specialized consequence-based procedures often outperform the more general-purpose tableau-based procedures. The main goal of our evaluation was to test whether this can also be said about non-Horn ontologies, and whether the implementation overhead of supporting disjunctions would impair the performance of the procedure on Horn ontologies.

We have implemented the procedure and the optimizations described in this paper in a prototype reasoner ConDOR.[8] The reasoner uses a well-known preprocessing step to eliminate transitive roles (see, e.g., [Kazakov, 2009]) and thus supports the DL $\mathcal{SH}$ ($\mathcal{ALCH}$ + transitivity axioms). We compared the performance of ConDOR with the tableau-based reasoners FaCT++ 1.5.0, HermiT 1.3.2 and Pellet 2.2.2, and the consequence-based Horn-$\mathcal{SHIQ}$ reasoner CB r.12. All experiments were run on a PC with a 2.5GHz CPU and 4GB RAM running 64bit Fedora 13. We set a time-out of 1 hour and Java heap space to 4GB. We ran ConDOR and CB through their command-line interface and measured the total run-time including input and output. We accessed the remaining reasoners through the OWL API 3.1.0 and only measured the time spent inside the classification method.

Many existing ontologies were either created by translations from less expressive knowledge-representation formalisms, which do not support disjunctions, or designed directly in OWL and contain many other constructors. Consequently, there are very few ontologies that contain disjunctions but are still in $\mathcal{SH}$; in fact we found only one large $\mathcal{SH}$ ontology with a significant number of disjunctions, namely the new SNOMED CT anatomical model mentioned in Section 1, which we call here SCT-SEP.[8] The ontology contains 54,973 concepts, of which 18,323 are defined using disjunctions, and 9 roles, including one transitive role. To obtain additional test data, we took four non-Horn ontologies that have been widely used in evaluations of DL reasoners (OBI,[9] FMA-Constitutional,[10] NCI-2[10] and Wine[10]) and reduced them to $\mathcal{SH}$ by replacing unsupported complex roles and concepts with fresh atomic ones, and removing unsupported axioms. In order to evaluate the performance of our reasoner on Horn ontologies, our test suite also included the official SNOMED CT ontology (SCT) and the $\mathcal{EL}$ version of GALEN (GLN-EL).[8]

The results of our experiments are shown in Table 6. Its first part contains the ontologies fully supported by ConDOR, and we consider it to be the main indicator of the performance of our procedure. On the two Horn ontologies ConDOR shows the same improvement in performance over tableau-based reasoners as CB. Moreover, ConDOR retains the improvement even on SCT-SEP, reducing the classification time from over 35 minutes (for FaCT++) to under 2 minutes.

	ConDOR	FaCT++	HermiT	Pellet	CB
SCT	40.4	650.1	-	-	51.8
SCT-SEP	88.9	2324.1	-	-	n/a
GLN-EL	4.9	-	-	-	4.6
OBI	0.6	153.8	2.5	11.8	n/a
FMA-C	11.7	-	-	-	n/a
NCI-2	-	7.6	89.7	22.4	n/a
Wine	0.2	0.1	0.7	1.7	n/a

Table 6: Classification times in seconds; "-" indicates that the reasoner failed the test due to time-out or memory exhaustion

The ontologies in the second part of Table 6 are those that were first reduced to $\mathcal{SH}$. Note that ConDOR is the only reasoner that could classify the reduced version of FMA-Constitutional. Interestingly, its original version can be classified by both FaCT++ and HermiT in about 10-15 minutes. This is because the original version contains many unsatisfiable concepts, all of which became satisfiable after the reduction. On the other hand, NCI-2 turned out to be difficult for ConDOR. This ontology contains large groups of equivalent concepts, and ConDOR explicitly derives all (quadratically-many) subsumptions between them. Further optimizations, e.g., merging of equivalent concepts, could resolve this issue.

7 Discussion and Related Work

We have demonstrated that it is possible to develop practical consequence-based procedures even for a DL that supports disjunctions. Currently we are exploring ways of extending our procedure to more expressive DLs. By combining our result with the techniques proposed for Horn-$\mathcal{SHIQ}$ [Kazakov, 2009] and Horn-$\mathcal{SROIQ}$ [Ortiz *et al.*, 2010], it should be relatively straightforward to support additional Horn features such as inverse and functional roles. Extensions by non-Horn features, e.g., number restrictions, are under investigation.

7.1 Relation to First-Order Resolution

Consequence-based procedures are closely related to procedures based on resolution, a general theorem-proving method for first-order logic (see, e.g., [Bachmair and Ganzinger, 2001]). Similarly to our procedure, resolution works by deriving new clauses that are consequences of the original axioms, is refutationally complete, and allows for many optimizations, such as ordering restrictions and subsumption deletion. Resolution has been used as a decision procedure for many fragments of first-order logic, modal logics and DLs (see, e.g., [de Nivelle *et al.*, 2000]).

Resolution-based procedures for DLs translate DL axioms into first-order clauses and apply specific resolution strategies which ensure that only a bounded number of clauses are derived, and thus guarantee termination and, in many cases, even optimal worst-case complexity. In particular, an

[8] the reasoner and the ontologies SCT-SEP and GLN-EL are available at `condor-reasoner.googlecode.com/`

[9] `obi-ontology.org/`

[10] for further information about these ontologies we refer the reader to the paper by Motik *et al.* [2009]

optimal resolution-based procedure has been formulated for the expressive DL $\mathcal{SHIQ}$ and implemented in the reasoner KAON2 [Hustadt *et al.*, 2008].

Although theoretically optimal, resolution-based procedures do not seem to be able to compete with modern tableau and consequence-based reasoners in practice. For example, KAON2 was not able to classify any medical ontology in a recent evaluation [Suntisrivaraporn, 2009]. The reason seems to be that, despite optimizations, resolution still performs too many inferences. For example, consider the following pair of commonly occurring DL definitions:

$$A_1 \equiv B_1 \sqcap \exists R.C_1, \tag{11}$$

$$A_2 \equiv B_2 \sqcap \exists R.C_2. \tag{12}$$

Axioms (11) and (12) are unrelated except for having a common role R and they do not interact in tableau and consequence-based procedures. However, they result in an application of the resolution rule. The inclusion $A_1 \sqsubseteq B_1 \sqcap \exists R.C_1$ that is a part of (11) is translated (amongst others) to the clause

$$\neg A_1(x) \vee \underline{R(x, f(x))}, \tag{13}$$

and the inclusion $B_2 \sqcap \exists R.C_2 \sqsubseteq A_2$ that is a part of (12) is translated to the clause

$$\neg B_2(x) \vee \neg \underline{R(x, y)} \vee \neg C_2(y) \vee A_2(x). \tag{14}$$

(13) and (14) will usually be resolved by a resolution theorem prover on the (maximal) underlined literals to produce

$$\neg A_1(x) \vee \neg B_2(x) \vee \neg C_2(f(x)) \vee A_2(x).$$

It is common for an ontology to contain many definitions (11) and (12) but only a few roles, which leads to a quadratic number of such inferences. In fact, for SNOMED CT we estimate this number to be in the order of hundreds of millions, and this interaction is one of the main factors that prevent KAON2 from classifying this ontology.

7.2 Relation to the Automata Approach

Both consequence-based and resolution-based procedures can be regarded as implementations of the automata approach for DLs. Traditionally, methods based on a reduction to the emptiness problem for finite automata on infinite trees, e.g., looping tree automata [Vardi and Wolper, 1994], have been used in the DL community for proving worst-case complexity bounds [Lutz and Sattler, 2000; Tobies, 2001]. Intuitively, given an ontology $\mathcal{O}$, one constructs a tree automaton $\mathcal{A}$ whose states correspond to the logical types that are consistent with $\mathcal{O}$, and whose runs correspond to tree-shaped models of $\mathcal{O}$. Consistency of $\mathcal{O}$ then reduces to non-emptiness of $\mathcal{A}$, which is usually computed by a bottom-up propagation of inconsistent states.

The above approach is not suitable for a direct implementation because it involves an explicit construction of an automaton whose size is exponential in the size of the input ontology, but one can overcome this difficulty by constructing an efficient representation of the automaton "on-the-fly" while performing the emptiness test [Gerth *et al.*, 1995; Pan *et al.*, 2006]. Many saturation-based procedures can be seen in the light of the automata approach as implementing a similar kind of propagation of inconsistencies (see, e.g., [Baader and Tobies, 2001]).

Our procedure can also be viewed this way. An automaton is represented by normalized axioms, rules $\mathbf{R}_\mathbf{A}^+$ and $\mathbf{R}_\mathbf{A}^-$ detect obviously inconsistent types, and the remaining rules propagate inconsistencies. For example, if both $H \sqcap \neg N_1 \sqcap \neg A_1$ and $H \sqcap \neg N_2 \sqcap \neg A_2$ are known to be inconsistent, then, given $A_1 \sqcap A_2 \sqsubseteq M \in \mathcal{O}$, rule $\mathbf{R}_\sqcap^\mathbf{n}$ deduces that $H \sqcap \neg N_1 \sqcap \neg N_2 \sqcap \neg M$ is also inconsistent.

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