Logic and Proof Hilary 2026

Cutting Planes

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In this lecture we introduce the cutting-plane proof system and show that it can polynomially simulate resolution. We show also that the pigeonhole principle admits polynomial-length cutting-plane proofs—so cutting-plane proofs may be exponentially shorter than resolution proofs. Finally, we prove a version of interpolation for cutting planes, which is the key to obtaining lower bounds on the length of cutting-plane proofs.

1 Cutting-Plane Proof System

There is a straightforward reduction of the SAT problem for CNF formulas to Integer Programming—the problem of solving a system of linear inequalities over integers. For a propositional variable x_i the literals x_i and $\neg x_i$ are respectively translated as x_i and $1-x_i$, where now x_i is considered as an integer variable subject to constraints $0 \leq x_i \leq 1$. A clause, comprising a disjunction of literals, is translated to an inequality, expressing that the sum of the literals is at least 1. For example, we translate the clause $\neg x_1 \vee x_2 \vee x_3$ as $(1-x_1)+x_2+x_3 \geq 1$. Notice that, considering 0 as false and 1 as true, the set of satisfying assignments is preserved by this translation. In particular, an unsatisfiable CNF translates to an unsatisfiable system of linear inequalities.

We now introduce the cutting-plane proof system, which can be used to prove unsatisfiability of a system of linear inequalities over integers. This consists of the following rules, where $x=(x_1,\ldots,x_n)$ is a vector of integer variables:

Sum rule: Let $f(x)=\sum_{i=1}^n a_i x_i$ and $g(x)=\sum_{i=1}^n b_i x_i$ be linear forms with integer coefficients and let $c,d,\lambda,\mu \in \mathbb{Z}$ with $\lambda,\mu \geq 0$. Then we have,

$$\frac{f(x) \geq c \qquad g(x) \geq d}{\lambda f(x) + \mu g(x) \geq \lambda c + \mu d}$$

Division rule: Given a linear form $f(x)=\sum_{i=1}^n a_i x_i$ and $c,\lambda \in \mathbb{Z}$ such that $\lambda > 0$ divides every a_i, we have:

$$\frac{f(x) \geq c}{\frac{1}{\lambda} f(x) \geq \lceil \frac{c}{\lambda} \rceil}$$

The rules are sound: for all integer valuations of x, if the premises of a rule hold then so does its conclusion. Soundness for the division rule relies on the fact that $\frac{1}{\lambda}f(x)$ is an integer and so rounds up to itself. Figure 1 gives an example of the above two rules, together with a geometric illustration.

A *cutting-plane proof* from a system of linear inequalities $F=\{f_1(x)\geq b_1,\ldots,f_m(x)\geq b_m\}$, is a sequence of inequalities, $g_1(x)\geq c_1,\ldots,g_\ell(x)\geq c_\ell$ in which each element either belongs to F, or follows from previous elements by the sum and division rules. A proof that ends in a contraction $0 \geq c$, with c strictly positive, is called a *refutation*.

The number ℓ of elements of the proof is called its *length*. Each element of the proof is assigned a *rank*, where an element has rank 0 if it lies in F and, if it follows from an application of the sum or division rules, it has rank one more than the maximum of the rank of its antecedents.

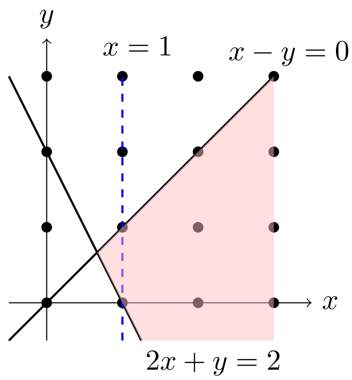

Figure 1: The shaded region is defined by the inequalities $2x + y \geq 2$ and $x - y \geq 0$. The sum rule yields $3x \geq 2$, and then the division rule gives $x \geq 1$. This inequality holds for all points to the right of the dashed blue *cutting plane* $x = 1$. Notice that all integer points in the shaded region satisfy the derived constraint, but part of the pink region is cut out.

Example 1. The one-step resolution proof

$$\frac{x_1 \vee x_2 \vee x_3 \qquad \neg x_2 \vee x_3}{x_1 \vee x_3}$$

translates to the cutting-plane proof

$$\frac{\dfrac{\dfrac{}{x_1 + x_2 + x_3 \geq 1} \qquad \dfrac{x_1 \geq 0 \qquad -x_2 + x_3 \geq 0}{x_1 - x_2 + x_3 \geq 0}}{2x_1 + 2x_3 \geq 1}}{x_1 + x_3 \geq 1}$$

Example 2 (The "triangle trick")**.** Let I_1, I_2, I_3 be three disjoint sets of indices. Then from the inequalities

$$\sum_{i \in I_1 \cup I_2} x_i \leq 1, \qquad \sum_{i \in I_1 \cup I_3} x_i \leq 1, \qquad \sum_{i \in I_2 \cup I_3} x_i \leq 1$$

we can derive the inequality

$$\sum_{i \in I_1 \cup I_2 \cup I_3} x_i \leq 1$$

in three steps by adding the three previous inequalities, dividing by two and rounding down.

2 Proof Complexity Upper Bounds

The following proposition shows, following Example 1, that cutting planes can polynomially simulate resolution.

Proposition 3. If a CNF formula F on n variables has a resolution refutation of length ℓ, then it has a cutting-plane refutation of length $O(\ell n)$.

Proof. Evidently, one step of resolution can be simulated by at most $2n$ steps of the following two propositional logic proof rules, where C is a clause and L is a literal:

Weakening:

$$\frac{C}{C \cup \{L\}}$$

Restricted resolution:

$$\frac{C \cup \{L\} \qquad C \cup \{\overline{L}\}}{C}$$

Next we show that both the above rules can be simulated by at most two cutting-plane steps. We first consider restricted resolution. Let c be the number of variables occurring negatively in C. Then $C \cup \{x_i\}$ translates to an inequality of the form $f + x_i \geq 1 - c$, while $C \cup \{\neg x_i\}$ translates to $f - x_i \geq -c$ for some linear form f. Summing the previous two inequalites gives $2f \geq 1 - 2c$. Dividing by 2 and rounding up then gives $f \geq 1 - c$, corresponding to formula derived by the resolution step.

Thanks to the presence of the axioms $x_i \geq 0$ and $1 - x_i \geq 0$ weakening can be simulated by the sum rule. □

The following proposition refers to the propositional formulation of the pigeonhole principle, as defined in Lecture 3. Translating to the language of linear inequalities, the formula PHP_n ("*there is an injective assignment of* $n+1$ *pigeons to* n *holes*") is as follows (where $x_{i,j}$ stands for "*pigeon* i *is in hole* j"):

1. $\sum_{j=1}^{n} x_{i,j} \geq 1$ for $1 \leq i \leq n+1$ (every pigeon is in some hole);
2. $x_{i,j} + x_{k,j} \leq 1$ for $1 \leq i < k \leq n+1, 1 \leq j \leq n$ (each hole has at most one pigeon);
3. $x_{i,j} \geq 0$, $x_{i,j} \leq 1$ for $1 \leq i \leq n+1, 1 \leq j \leq n$ (variables are 0-1 valued).

Proposition 4. For all n the formula PHP_n has a cutting-plane refutation of length $O(n^3)$.

Proof. For each $j \in \{1, \ldots, n\}$ we derive $x_{1,j} + \cdots + x_{m,j} \leq 1$ by induction on $m = 1, \ldots, n+1$. For the base case we note that $x_{1,j} \leq 1$ occurs as an axiom in Item 3. For the induction step, suppose we have established $x_{1j,} + \cdots + x_{m,j} \leq 1$. Using the sum rule we multiple the last inequality by $m-1$ and add the inequalities $x_{i,j} + x_{m+1,j} \leq 1$ for $i = 1, \ldots, m$ from Item 2, to obtain

$$m x_{1,j} + \cdots + m x_{m+1,j} \leq 2m - 1\,.$$

Applying the division rule gives

$$x_{1,j} + \cdots + x_{m+1,j} \leq \lfloor (2m-1)/m \rfloor = 1\,. \tag{1}$$

Next we sum the inequalities (1) for $j = 1, \ldots, n$ to get $\sum_{i=1}^{n+1} \sum_{j=1}^{n} x_{i,j} \leq n$. On the other hand, summing the inequalities in Item 1 for $i = 1, \ldots, n+1$ yields $\sum_{i=1}^{n+1} \sum_{j=1}^{n} x_{i,j} \geq n+1$. A final application of the sum rule gives $-1 \geq 0$, a contradiction. □

3 Interpolation Algorithm

Let x, y, z be disjoint vectors of propositional variables. Recall that given an unsatisfiable CNF formula $F(x,z) \wedge G(y,z)$, an interpolant is a boolean function $I(z)$ such that for any truth assignment α to z:

- If $I(\alpha) = 0$ then $F(x, \alpha)$ is unsatisfiable;
- If $I(\alpha) = 1$ then $G(y, \alpha)$ is unsatisfiable.

The next result shows that, as with resolution, we can efficiently compute interpolants from cutting-plane proofs.

Theorem 5. There is a polynomial-time procedure that inputs a cutting-plane refutation of a CNF formula $F(x,z) \wedge G(y,z)$ together with a valuation α of the common variables z, and outputs the value $I(\alpha)$ of the interpolant.

Proof. We describe a procedure that inputs a cutting-plane refutation of $F \wedge G$ of the form $\{f_i(x) + g_i(y) + h_i(z) \geq c_i\}_{i=1}^{\ell}$ and outputs a proof $\{f_i(x) \geq d_i\}_{i=1}^{\ell}$ from $F(x,\alpha)$ and a proof $\{g_i(y) \geq e_i\}_{i=1}^{\ell}$ from $G(y,\alpha)$ such that at least one of two proofs ends in a contradiction. If the first proof ends in a contradiction we output $I(\alpha)$ equals 0 and otherwise we output that $I(\alpha)$ equals 1.

The idea is to split each inequality $f_i(x) + g_i(y) + h_i(z) \geq c_i$ into an "x-part" $f_i(x) \geq d_i$ and a "y-part" $g_i(y) \geq e_i$ while maintaining the invariant

$$d_i + e_i \geq c_i - h_i(\alpha)\,, \tag{2}$$

which ensures that

$$\{f_i(x) \geq d_i,\, g_i(y) \geq e_i\} \models f_i(x) + g_i(y) + h_i(\alpha) \geq c_i\,. \tag{3}$$

The construction, which is solely determined by the choice of constants d_i and e_i, is by induction on i:

1. Suppose that $f_i(x) + h_i(z) \geq c_i$ is a hypothesis from F. Then the x-part is $f_i(x) \geq c_i - h_i(\alpha)$ and the y-part is $0 \geq 0$; i.e., $d_i := c_i - h_i(\alpha)$ and $e_i = 0$. Clearly (2) is respected.

2. In case $g_i(y) + h_i(z) \geq d_i$ is a hypothesis from G, we split it into x-part $0 \geq 0$ and y-part $g_i(y) \geq d_i - h_i(\alpha)$, i.e., $d_i := 0$ and $e_i = c_i - h_i(\alpha)$. Again, (2) is immediate.

3. Suppose that $f_i(x)+g_i(y)+h_i(z) \geq c_i$ follows from $f_j(x)+g_j(y)+h_j(z) \geq c_j$ and $f_k(x)+g_k(y)+h_k(z) \geq c_k$, for some $j,k < i$, by an application of the sum rule with coefficients $\lambda, \mu \geq 0$. Then we obtain the x-part of $f_i(x) + g_i(y) + h_i(z)$ by applying the sum rule to the respective x-parts of $f_j(x) + g_j(y) + h_j(z) \geq c_j$ and $f_k(x) + g_k(y) + h_k(z) \geq c_k$. We treat the y-parts similarly. Thus we have $d_i := \lambda d_j + \mu d_k$ and $e_i = \lambda d_j + \mu d_k$. Again, it is straightforward to verify that the invariant (2) is preserved.

4. Suppose $f_i(x) + g_i(y) + h_i(z) \geq c_i$ follows from $f_j(x) + g_j(y) + h_j(z) \geq c_j$ by dividing by $\lambda > 0$ and rounding up (that is, an application of the division rule). We obtain the respective x- and y-parts of $f_i(x)+g_i(y)+h_i(z) \geq c_i$ by applying the the same operation to the x-part $f_j(x) \geq d_j$ and y-part $g_j(y) \geq d_j$ of $f_j(x) + g_j(y) + h_j(z) \geq c_j$. Thus we have

$$\begin{aligned}
d_i + e_i &= \lceil d_j/\lambda \rceil + \lceil e_j/\lambda \rceil && \text{(definition of } d_i \text{ and } e_i\text{)}\\
&\geq \lceil (d_j + e_j)/\lambda \rceil && \text{(since } \lceil x \rceil + \lceil y \rceil \geq \lceil x + y \rceil \text{ for all } x, y\text{)}\\
&\geq \lceil (c_j - h_j(\alpha))/\lambda \rceil && \text{(induction hypothesis)}\\
&= \lceil c_j/\lambda \rceil - h_i(\alpha) && \text{(since } \lambda h_i(\alpha) = h_j(\alpha)\text{)}\\
&= c_i - h_i(\lambda) && \text{(definition of } c_i\text{)}
\end{aligned}$$

Since the input proof ends in a contradiction, (3) implies that at least one of the x-part of the y-part also ends in a contradiction. □

We conclude by remarking that the above result can be used as part of a proof of an exponential lower bound on the length of a cutting-plane refutation of the CNF formula expressing that no $(k-1)$-colourable graph has a k-clique. The proof has a similar flavour to the corresponding result for resolution, described in Lecture 3. We do not give details here.