Logic and Proof Hilary 2026

The Compactness Theorem

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1 The Compactness Theorem

In this lecture we prove a fundamental result about propositional logic called the *Compactness Theorem*. This will play an important role in the second half of the course when we study predicate logic. This is due to our use of *Herbrand's Theorem* to reduce reasoning about formulas of predicate logic to reasoning about infinite sets of formulas of propositional logic.

Before stating and proving the Compactness Theorem we need to introduce one new piece of terminology. A *partial assignment* is a function $\mathsf{v} : D \to \{0, 1\}$, where D is either the infinite set $\{p_1, p_2, \ldots\}$ of all propositional variables or a finite initial segment $\{p_1, \ldots, p_n\}$ thereof. The set D is called the *domain* of v and is denoted $\mathrm{dom}(\mathsf{v})$. Given partial assignments v and v', we say that v' *extends* v if $\mathrm{dom}(\mathsf{v}) \subseteq \mathrm{dom}(\mathsf{v}')$ and if $\mathsf{v}[\![p_i]\!] = \mathsf{v}'[\![p_i]\!]$ for all $p_i \in \mathrm{dom}(\mathsf{v})$. Sometimes we refer to partial assignments simply as assignments.

Recall that a set of formulas $\boldsymbol{S}$ is satisfiable if there is an assignment that satisfies every formula in $\boldsymbol{S}$. For example, the set of formulas

$$\boldsymbol{S} = \{p_1 \vee p_2,\ \neg p_1 \vee \neg p_2,\ p_2 \vee p_3,\ \neg p_2 \vee \neg p_3, \ldots\}$$

is satisfied by the assignment v such that $\mathsf{v}[\![p_i]\!] = 1$ if i is odd and $\mathsf{v}[\![p_i]\!] = 0$ if i is even.

Theorem 1 (Compactness Theorem)**.** A set of formulas $\boldsymbol{S}$ is satisfiable if and only if every finite subset of $\boldsymbol{S}$ is satisfiable.

Proof. One direction of the theorem is obvious: if $\boldsymbol{S}$ is satisfiable then every finite subset is certainly satisfiable. The non-trivial direction is the converse.

Let $\boldsymbol{S}$ be a set of formulas such that every finite subset of $\boldsymbol{S}$ is satisfiable. Say that a partial assignment v is *good* if it satisfies any formula $F \in \boldsymbol{S}$ that only mentions propositional variables in the domain of v. We first observe that for each $n \in \mathbb{N}$ there is a partial assignment v with $\mathrm{dom}(\mathsf{v}) = \{p_1, p_2, \ldots, p_n\}$ that is good. To see this, consider the subset $\boldsymbol{S}' \subseteq \boldsymbol{S}$ consisting of all formulas that mention only propositional variables $p_1, p_2, \ldots, p_n$. Now $\boldsymbol{S}'$ may be an infinite set, but it only contains finitely many formulas *up to logical equivalence* since there are only finitely many formulas on propositional variables $p_1, p_2, \ldots, p_n$ up to logical equivalence (2^{2^n} formulas to be precise). Since all finite subsets of $\boldsymbol{S}$ are satisfiable we conclude that $\boldsymbol{S}'$ is satisfiable by some partial assignment v with $\mathrm{dom}(\mathsf{v}) = \{p_1, p_2, \ldots, p_n\}$. By construction such an assignment is good.

The central idea of the proof is to construct a sequence of good partial assignments $\mathsf{v}_0, \mathsf{v}_1, \mathsf{v}_2, \ldots$ such that $\mathrm{dom}(\mathsf{v}_n) = \{p_1, \ldots, p_n\}$ and v_{n+1} extends v_n for each n. We construct the v_n in sequence, starting with v_0, and maintaining the following induction hypothesis: $(*)$ there are infinitely many good partial assignments that extend v_n.

For the base step we define v_0 to be the assignment with empty domain. Since there is a good assignment with domain $\{p_1, \ldots, p_n\}$ for every n, there are infinitely many good assignments that extend v_0; thus v_0 satisfies $(*)$.

For the induction step, suppose that we have constructed assignments $\mathsf{v}_0, \ldots, \mathsf{v}_n$ such that v_n satisfies $(*)$. Consider the two assignments v, v' that extend v_n with $\mathrm{dom}(\mathsf{v}) = \mathrm{dom}(\mathcal{B}') = \{p_1, p_2, \ldots, p_{n+1}\}$

(say $\mathsf{v}[\![p_{n+1}]\!] = 0$ and $\mathsf{v}'[\![p_{n+1}]\!] = 1$.) Since any proper extension of v_n is an extension of either v or v', it follows that one (or both) of v and v' has infinitely many good extensions. Define v_{n+1} to be v if v has infinitely many good extensions; otherwise define v_{n+1} to be v'. Then v_{n+1} satisfies $(*)$ by construction.

There is a unique (total) assignment v that extends all the v_n—it is defined by $\mathsf{v}[\![p_n]\!] := \mathsf{v}_n[\![p_n]\!]$ for each $n \in \mathbb{N}$. We claim that v satisfies all formulas in $\boldsymbol{S}$. Indeed if $F \in \boldsymbol{S}$ mentions propositional variables $\{p_1, \ldots, p_n\}$ then v_n satisfies F. It follows that v also satisfies F, since v extends v_n. Thus v satisfies all formulas in $\boldsymbol{S}$ and the proof is concluded. □

The importance of the Compactness Theorem may be more apparent from the contrapositive formulation: *if a set of formulas $\boldsymbol{S}$ is unsatisfiable then some finite subset of $\boldsymbol{S}$ is already unsatisfiable.* This suggests a procedure by which we can show that an infinite set of formulas $\boldsymbol{S}$ is unsatisfiable. Suppose that $\boldsymbol{S}$ can be enumerated by some algorithm as

$$\boldsymbol{S} = \{F_1, F_2, F_3, \ldots\}$$

Then for each $n \in \mathbb{N}$ we test whether the finite set $\{F_1, \ldots, F_n\}$ is unsatisfiable (using, say, truth tables or some other method). The Compactness Theorem guarantees that if $\boldsymbol{S}$ is not satisfiable we will detect that fact after a finite amount of time. On the other hand if $\boldsymbol{S}$ is satisfiable then the above procedure will not terminate.

2 Application: Graph Colouring

Let's consider an application of the compactness theorem to prove a purely combinatorial result. Recall that a graph $G = (V, E)$ is *k-colourable* if there is a function $c : V \to \{1, \ldots, k\}$ mapping the set of vertices to a set of k colours such that adjacent vertices do not have the same colour, i.e., $(u, v) \in E$ implies $c(u) \neq c(v)$. Let us say that $H = (V_1, E_1)$ is a *subgraph* of G if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Theorem 2. Let $G = (V, E)$ be a graph with set of vertices $V = \{v_i : i \in \mathbb{N}\}$. Suppose that every finite subgraph of G is k-colourable. Then G is k-colourable.

Proof. Recall how we reduced k-colouring to propositional satisfiability. Introduce propositional variables $P_{v,i}$, for each $v \in V$ and $1 \leq i \leq k$, interpreted as "vertex v has colour i". We consider the following propositions:

- $F_v := \bigvee_{i=1}^{k} P_{v,i}$ (vertex v has some colour)
- $G_v := \bigwedge_{i=1}^{k} \bigwedge_{j=i+1}^{k} \neg P_{v,i} \vee \neg P_{v,j}$ (vertex v has at most one colour)
- $H_{u,v} := \bigwedge_{i=1}^{k} \neg P_{u,i} \vee \neg P_{v,i}$ (vertices u and v don't have the same colour)

Now define $\boldsymbol{S} = \{F_v, G_v : v \in V\} \cup \{H_{u,v} : (u, v) \in E\}$. We claim that $\boldsymbol{S}$ is satisfiable if and only if the graph G has a k-colouring. Indeed, given such a colouring c, define an assignment v by $\mathsf{v}[\![P_{v,i}]\!] = 1$ if and only if $c(v) = i$. Then it is clear that v satisfies $\boldsymbol{S}$. Conversely, given an assignment v satisfying $\boldsymbol{S}$ we can define a k-colouring c by $c(v) = i$ iff $\mathsf{v}[\![P_{v,i}]\!] = 1$.

By assumption, every every finite subgraph of G has a k-colouring. It follows that every finite subset of $\boldsymbol{S}$ is satisfiable. By the Compactness Theorem it must be that $\boldsymbol{S}$ is satisfiable, and thus G itself is k-colourable. □