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Ehrenfeucht-Fraïssé Games

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1 Back and Forth

Proposition 1. Given two countable unbounded dense linear orderings $(A, <)$ and $(B, <)$, there is an order preserving bijection $f : A \to B$.

Proof. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be enumerations of the elements of A and B. We define new enumerations $a'_1, a'_2, \ldots$ and $b'_1, b'_2, \ldots$ such that for any pair of indices i and j, $a'_i < a'_j$ if and only if $b'_i < b'_j$. Having done this we define the function f by $f(a'_i) = b'_i$ for each $i = 1, 2, \ldots$.

We define the a'_i and b'_i by strong induction via a *back and forth* construction. Suppose we have defined $a'_1, \ldots, a'_n$ and $b'_1, \ldots, b'_n$. If n is even then we define a'_{n+1} to be the first element of the original enumeration $a_1, a_2, \ldots$ that does not appear among $a'_1, \ldots, a'_n$. We then define b'_{n+1} such that $a'_i < a'_{n+1}$ if and only if $b'_i < b'_{n+1}$ for $1 \leq i \leq n$. We can do this because $(B, <)$ is a dense linear order. On the other hand, if n is odd then we define b'_{n+1} to be the first element of the original enumeration $b_1, b_2, \ldots$ that does not appear among $b'_1, \ldots, b'_n$. We then define a'_{n+1} such that $a'_i < a'_{n+1}$ if and only if $b'_i < b'_{n+1}$ for $1 \leq i \leq n$. We can do this because $(A, <)$ is a dense linear order. Proceeding in this way, we obtain new enumerations $a'_1, a'_2, \ldots$ and $b'_1, b'_2, \ldots$ with the desired properties. □

Proposition 2. Any two countable models $\mathcal{G} = (V, E)$ and $\mathcal{G}' = (V', E')$ of the theory $\boldsymbol{T}_{\mathrm{RG}}$ of the random graph are isomorphic as graphs, i.e., there exists a bijection $f : V \to V'$ such that $(u, v) \in E$ iff $(f(u), f(v)) \in E'$ for all $u, v \in V$.

Proof. We can use a similar back-and-forth construction as for the proof of Proposition 1. □

2 Ehrenfeucht-Fraïssé Games

In this section we define a notion of "back-and-forth" equivalence between structures that is weaker than isomorphism, but is nevertheless sufficient for showing that two structures satisfy the same formulas. We use this to give alternative proofs of the completeness of the theory of unbounded dense linear orders and the theory of the random graph.

Let σ be a *relational* signature, that is, σ has no functional symbols—only relation and constant symbols. Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures with respective universes A and B. For all $k, m \in \mathbb{N}$ and tuples $\boldsymbol{a} \in A^m$ and $\boldsymbol{b} \in B^m$, we define the equivalence relation $(\mathcal{A}, \boldsymbol{a}) \sim_k (\mathcal{B}, \boldsymbol{b})$ by induction on k as follows:

1. $(\mathcal{A}, \boldsymbol{a}) \sim_0 (\mathcal{B}, \boldsymbol{b})$ iff for any atomic σ-formula φ with free variables among $x_1, \ldots, x_m$ we have $\mathcal{A} \models \varphi(\boldsymbol{a})$ iff $\mathcal{B} \models \varphi(\boldsymbol{b})$.
2. $(\mathcal{A}, \boldsymbol{a}) \sim_{k+1} (\mathcal{B}, \boldsymbol{b})$ iff (i) for all $a' \in A$ there exists $b' \in B$ such that $(\mathcal{A}, \boldsymbol{a}a') \sim_k (\mathcal{B}, \boldsymbol{b}b')$ and (ii) for all $b' \in B$ there exists $a' \in A$ such that $(\mathcal{A}, \boldsymbol{a}a') \sim_k (\mathcal{B}, \boldsymbol{b}b')$.

If $\boldsymbol{a} \in A^m$ is the empty tuple (that is, $m = 0$) then we sometimes denote the tuple $(\mathcal{A}, \boldsymbol{a})$ simply by $\mathcal{A}$. Thus we may write $\mathcal{A} \sim_k \mathcal{B}$, etc.

Intuitively, $(\mathcal{A}, \boldsymbol{a}) \sim_k (\mathcal{B}, \boldsymbol{b})$ means that Duplicator can maintain a partial isomorphism for k further rounds of play. Next we characterise the above-defined family of equivalence relations $\{\sim_k\}_{k\in\mathbb{N}}$ in terms of games.

Definition 3. Fix $k, m \in \mathbb{N}$. Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures and let $\boldsymbol{a} \in A^m$ and $\boldsymbol{b} \in B^m$ respectively. The k-round *Ehrenfeucht-Fraïssé game* $G_k((\mathcal{A}, \boldsymbol{a}), (\mathcal{B}, \boldsymbol{b}))$ is defined as follows. There are two players: Spoiler and Duplicator. In each round $i = 1, \ldots, k$, Spoiler picks either an element $a \in A$ or $b \in B$ and then Duplicator responds with an element of the other structure. After k rounds, let $\boldsymbol{a'} \in A^k$ and $\boldsymbol{b'} \in B^k$ be the tuples of elements generated by the play of the game. Then Duplicator wins the play if for all atomic formulas $\varphi(x_1, \ldots, x_{m+k})$ we have $\mathcal{A} \models \varphi(\boldsymbol{aa'})$ iff $\mathcal{B} \models \varphi(\boldsymbol{bb'})$.

A *strategy* of Duplicator is a function that inputs a configuration of the game $((\mathcal{A}, \boldsymbol{a}), (\mathcal{B}, \boldsymbol{b}))$ and a move of Spoiler (either $a \in A$ or $b \in B$) and outputs a response of Duplicator. Such a strategy is said to be *winning* if Duplicator wins every play in which they follow the strategy. The following is immediate from the definitions.

Proposition 4. For all σ-structures $\mathcal{A}$ and $\mathcal{B}$ and all tuples $\boldsymbol{a} \in A^m$ and $\boldsymbol{b} \in B^m$, $(\mathcal{A}, \boldsymbol{a}) \sim_k (\mathcal{B}, \boldsymbol{b})$ iff Duplicator has a winning strategy in $G_k((\mathcal{A}, \boldsymbol{a}), (\mathcal{B}, \boldsymbol{b}))$.

Example 5. Let σ be the empty signature. Given two σ-structures $\mathcal{A}$ and $\mathcal{B}$ (i.e., sets), Duplicator wins the game $G_k(\mathcal{A}, \mathcal{B})$ if and only if either $|A| = |B|$ or $|A|, |B| \geq k$. There are several cases to consider: we consider two by way of example. First, if $|A| < |B|$ and $|A| < k$, then Spoiler can pick a sequence of distinct elements in $\mathcal{B}$, which Duplicator cannot match. Second, if $|A|, |B| \geq k$ or $|A| = |B|$, then Duplicator's winning strategy is to ensure that the game state $(\boldsymbol{a}, \boldsymbol{b})$ after each round satisfies the condition $a_i = a_j$ if and only if $b_i = b_j$ for all i, j.

Example 6. Let σ be the signature with a binary relation symbol $<$. Given two finite linear orders $\mathcal{A}$ and $\mathcal{B}$ such that $|A|, |B| \geq 2^k - 1$, Duplicator wins $G_k(\mathcal{A}, \mathcal{B})$. The Winning strategy of Duplicator is to ensure that after round $\ell = 0, 1, \ldots, k$ the tuples $\boldsymbol{a} = (a_{-1}, a_0, \ldots, a_\ell)$ and $\boldsymbol{b} = (b_{-1}, b_0, b_1, \ldots, b_\ell)$ satisfy the following condition: (i) $a_i < a_j$ iff $b_i < b_j$ for all $i, j \in \{1, \ldots, \ell\}$; (ii) $d(a_i, a_j) < 2^{k-\ell} - 1$ implies $d(a_i, a_j) = d(b_i, b_j)$; (iii) $d(a_i, a_j) \geq 2^{k-\ell} - 1$ implies $d(b_i, b_j) \geq 2^{k-\ell} - 1$. Here $a_{-1} := \min(A), a_0 := \max(A)$ and $b_{-1} := \min(B), b_0 := \max(B)$ are not moves played in the game and only serve to express the invariant. Also d is the distance relation that arises by viewing a linear order as a graph in which two elements are neighbours iff they are adjacent in the order. By assumption, Conditions (i)–(iii) hold for $\ell = 0$. We leave it as an exercise to show how Duplicator can inductively maintain this invariant, no matter how Spoiler moves.

Fix $k, m \in \mathbb{N}$. Write $\mathrm{FO}^\sigma_{k,m}$ for the set of σ-formulas in free variables $x_1, \ldots, x_m$ that have *quantifier-depth* at most k. (i.e., maximum nesting depth of quantifiers).

Proposition 7. Fix $k, m \in \mathbb{N}$. There are finitely many formulas in $\mathrm{FO}^\sigma_{k,m}$ up to logical equivalence.

Proof. The proof is by induction on the quantifier depth k. For $k = 0$ every formula in $\mathrm{FO}^\sigma_{k,m}$ is a boolean combination of atomic formulas in the free variables $x_1, \ldots, x_m$, of which there are finitely many. For the induction step, we note that every formula in $\mathrm{FO}^\sigma_{k+1,m}$ is a boolean combination of formulas of the form $\exists x_{m+1}\, \varphi$, where $\varphi \in \mathrm{FO}^\sigma_{k,m+1}$. But by the induction hypothesis, there are only finitely many choices of φ up to logical equivalence. Since logical equivalence is a congruence with respect to the logical connectives and quantifiers ($\varphi \equiv \varphi'$ implies $\exists x_{m+1}\, \varphi \equiv \exists x_{m+1}\, \varphi'$, etc.), the argument is complete. ☐

Write $(\mathcal{A}, \boldsymbol{a}) \equiv_k (\mathcal{B}, \boldsymbol{b})$ to denote that $\mathcal{A} \models \varphi(\boldsymbol{a})$ iff $\mathcal{B} \models \varphi(\boldsymbol{b})$ for all formulas $\varphi \in \mathrm{FO}^{\sigma}_{k,m}$. The following is the main result of this lecture.

Theorem 8 (Ehrenfeucht-Fraïssé)**.** Fix $k, m \in \mathbb{N}$. Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures, $\boldsymbol{a} \in A^m$, and $\boldsymbol{b} \in B^m$. Then $(\mathcal{A}, \boldsymbol{a}) \equiv_k (\mathcal{B}, \boldsymbol{b})$ iff $(\mathcal{A}, \boldsymbol{a}) \sim_k (\mathcal{B}, \boldsymbol{b})$.

Proof. The proof is by induction on k. The case $k = 0$ is trivial since $\sim_0$ and $\equiv_0$ have exactly the same definition.

The inductive case is as follows. Suppose that $(\mathcal{A}, \boldsymbol{a}) \sim_{k+1} (\mathcal{B}, \boldsymbol{b})$. We show that $(\mathcal{A}, \boldsymbol{a}) \equiv_{k+1} (\mathcal{B}, \boldsymbol{b})$. To this end, take a formula $\exists x_{m+1}\, \varphi \in \mathrm{FO}^{\sigma}_{k+1,m}$ and suppose that $\mathcal{A} \models (\exists x_{m+1}\, \varphi)(\boldsymbol{a})$. Then there exists $a' \in A$ such that $\mathcal{A} \models \varphi(\boldsymbol{a}a')$. Since $(\mathcal{A}, \boldsymbol{a}) \sim_{k+1} (\mathcal{B}, \boldsymbol{b})$ there exists b' such that $(\mathcal{A}, \boldsymbol{a}a') \sim_k (\mathcal{B}, \boldsymbol{b}b')$. By the induction hypothesis we have $(\mathcal{A}, \boldsymbol{a}a') \equiv_k (\mathcal{B}, \boldsymbol{b}b')$, which entails $\mathcal{B} \models \varphi(\boldsymbol{b}b')$ and hence $\mathcal{B} \models (\exists x_{m+1}\, \varphi)(\boldsymbol{b})$. Since every formula in $\mathrm{FO}^{\sigma}_{k+1,m}$ is a Boolean combination of formulas $\exists x_{m+1}\varphi$ for $\varphi \in \mathrm{FO}^{\sigma}_{k,m+1}$, we conclude that $(\mathcal{A}, \boldsymbol{a}) \equiv_{k+1} (\mathcal{B}, \boldsymbol{b})$.

Conversely, suppose that $(\mathcal{A}, \boldsymbol{a}) \equiv_{k+1} (\mathcal{B}, \boldsymbol{b})$. We show that $(\mathcal{A}, \boldsymbol{a}) \sim_{k+1} (\mathcal{B}, \boldsymbol{b})$. To this end, pick $a' \in A$. We must find $b' \in B$ such that $(\mathcal{A}, \boldsymbol{a}a') \sim_k (\mathcal{B}, \boldsymbol{b}b')$. Let $\psi_1, \ldots, \psi_{\ell_1}$ be an enumeration of all formulas in $\mathrm{FO}^{\sigma}_{k,m+1}$ up to logical equivalence that are satisfied by $(\mathcal{A}, \boldsymbol{a}a')$. Let also $\psi'_1, \ldots, \psi'_{\ell_2}$ be an enumeration of all formulas in $\mathrm{FO}^{\sigma}_{k,m+1}$ up to logical equivalence that are not satisfied by $(\mathcal{A}, \boldsymbol{a}a')$. Let $\varphi := \bigwedge_{i=1}^{\ell_1} \psi_i \wedge \bigwedge_{i=1}^{\ell_2} \neg\psi'_i$. Then $\mathcal{A} \models (\exists x_{m+1}\, \varphi)(\boldsymbol{a})$. Since $\exists x_{m+1}\, \varphi \in \mathrm{FO}_{k+1,m}$, we have $\mathcal{B} \models (\exists x_{m+1}\, \varphi)(\boldsymbol{b})$. We conclude that there exists $b' \in B$ with $\mathcal{B} \models \varphi(\boldsymbol{b}b')$. By definition of φ we thus have $(\mathcal{A}, \boldsymbol{a}a') \equiv_k (\mathcal{B}, \boldsymbol{b}b')$. Now by the induction hypothesis we have $(\mathcal{A}, \boldsymbol{a}a') \sim_k (\mathcal{B}, \boldsymbol{b}b')$. By symmetry we also show that for all $b' \in B$ there exists $a' \in A$ with $(\mathcal{A}, \boldsymbol{a}a') \sim_k (\mathcal{B}, \boldsymbol{b}b')$. We conclude that $(\mathcal{A}, \boldsymbol{a}) \sim_{k+1} (\mathcal{B}, \boldsymbol{b})$. ☐

3 Expressiveness of First-Order Logic

The Ehrenfeucht-Fraïssé Theorem can be used to show that certain properties cannot be expressed in first-order logic. For example, following Example 6 we can show that the property that a finite linear order has even cardinality cannot be expressed in first-order logic over the signature with a binary relation symbol. Indeed, the example shows that for every k there are structures $\mathcal{A} \equiv_k \mathcal{B}$ (i.e., that cannot be distinguished by sentences of quantifier depth at most k) such that $\mathcal{A}$ has an odd number of elements and $\mathcal{B}$ has an even number of elements.

Using similar ideas to the proof of Proposition 1, it can be shown that for any two unbounded dense linear orders $\mathcal{A}$ and $\mathcal{B}$, and all k, we have $\mathcal{A} \sim_k \mathcal{B}$. It follows from the Ehrenfeucht-Fraïssé Theorem that the theory of unbounded dense linear orders is complete. We can likewise prove the completeness of the theory of the random graph. Previously we have proven completeness in both cases using quantifier elimination.

For further applications of Ehrenfeucht-Fraïssé games, it will be useful to develop some ideas concerning locality in first-order logic. We will express these ideas in terms of graphs, although they can be easily generalised to the context of general relational structures.

3.1 Neighbourhoods in Graphs

Let $\mathcal{G} = (V, E)$ be a graph. Define the distance $d(u, u')$ of two vertices $u, u' \in V$ to be the length of shortest path connecting them. If no such path exists then we say that the distance is infinity. The *d-neighbourhood* of $u \in V$ is the subgraph $N^{\mathcal{G}}(u, d)$ of $\mathcal{G}$ induced by the set of vertices at distance at most d to u. More generally, the d-neighbourhood of a tuple of vertices $\boldsymbol{u} = (u_1, \ldots, u_s)$ of $\mathcal{G}$ is the subgraph $N^{\mathcal{G}}(\boldsymbol{u}, d)$ induced the set of vertices at distance at most d from some element of $\boldsymbol{u}$. Given two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, tuples of vertices $\boldsymbol{u} = (u_1, \ldots, u_s)$ from $\mathcal{G}_1$ and $\boldsymbol{v} = (v_1, \ldots, v_s)$ from $\mathcal{G}_2$, we write

$$N^{\mathcal{G}_1}(\boldsymbol{u}, d) \cong N^{\mathcal{G}_2}(\boldsymbol{v}, d)$$

to mean that there is a graph isomorphism Φ from $N^{\mathcal{G}_1}(\boldsymbol{u}, d)$ to $N^{\mathcal{G}_2}(\boldsymbol{v}, d)$ such that $\Phi(u_i) = v_i$ for $i \in \{1, \ldots, s\}$.

The following theorem concerns the expressiveness of first-order logic over the usual signature for graphs, with a single binary relation symbol E. The idea is that two graphs that are "locally the same" are indistinguishable in first-order logic.

Theorem 9. Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two graphs. Given $k \in \mathbb{N}$, $\mathcal{G}_1 \equiv_k \mathcal{G}_2$ if the following condition holds for all $0 \leq d \leq \frac{3^k - 1}{2}$ and tuples of vertices and $\boldsymbol{u} = (u_1, \ldots, u_{k-1})$ in V_1 and $\boldsymbol{v} = (v_1 \ldots, v_{k-1})$ in V_2 such that $N^{\mathcal{G}_1}(\boldsymbol{u}, d) \cong N^{\mathcal{G}_2}(\boldsymbol{v}, d)$;

1. for all $u \in V_1$ such that $d(u, \boldsymbol{u}) > d$, there exists $v \in V_2$ such that $d(v, \boldsymbol{v}) > d$ and $N^{\mathcal{G}_1}(u, d) \cong N^{\mathcal{G}_2}(v, d)$;
2. for all $v \in V_2$ such that $d(v, \boldsymbol{v}) > d$, there exists $u \in V_1$ such that $d(u, \boldsymbol{u}) > d$ and $N^{\mathcal{G}_1}(u, d) \cong N^{\mathcal{G}_2}(v, d)$;

Proof. We show that Duplicator wins the k-round EF game on the pair of structures $\mathcal{G}_1$ and $\mathcal{G}_2$. Define $d_s := \frac{3^{k-s} - 1}{2}$ for $s = 1, \ldots, k$. Duplicator maintains the invariant that after $s \in \{1, \ldots, k\}$ rounds, if the generated tuples are $\boldsymbol{u} = (u_1, \ldots, u_s)$ in $\mathcal{G}_1$ and $\boldsymbol{v} = (v_1, \ldots, v_s)$ in $\mathcal{G}_2$, then $N^{\mathcal{G}_1}(\boldsymbol{u}, d_s) \cong N^{\mathcal{G}_2}(\boldsymbol{v}, d_s)$. Note that $N^{\mathcal{G}_1}(\boldsymbol{u}, 0) \cong N^{\mathcal{G}_2}(\boldsymbol{v}, 0)$ implies that $(\mathcal{G}_1, \boldsymbol{u}) \equiv_0 (\mathcal{G}_2, \boldsymbol{v})$. Since $d_k = 0$, Duplicator wins any play in which the invariant is maintained.

After s rounds, the invariant gives an isomorphism Φ from $N^{\mathcal{G}_1}(\boldsymbol{u}, d_s)$ to $N^{\mathcal{G}_2}(\boldsymbol{v}, d_s)$ such that $\Phi(u_i) = v_i$ for $i \in \{1, \ldots, s\}$. By symmetry, we assume without loss of generality that Spoiler plays in $\mathcal{G}_1$. Roughly speaking, Duplicator's strategy to maintain the invariant is to use the isomorphism Φ to match Spoiler's move if Spoiler plays sufficiently "close" to $\boldsymbol{u}$, and otherwise to use Condition 2 to find a response if Spoiler plays "far away" from $\boldsymbol{u}$.

Let us now describe Duplicator's strategy in more detail. If Spoiler plays $u_{s+1} \in V_1$ at distance at most $2d_{s+1} + 1$ from $\boldsymbol{u}$, then Duplicator responds with $v_{s+1} := \Phi(u_{s+1}) \in V_2$. In this case $N^{\mathcal{G}_1}(\boldsymbol{u}u_{s+1}, d_{s+1})$ is contained in $N^{\mathcal{G}_1}(\boldsymbol{u}, d_s)$ and $N^{\mathcal{G}_2}(\boldsymbol{v}v_{s+1}, d_{s+1})$ is contained in $N^{\mathcal{G}_2}(\boldsymbol{v}, d_s)$. Hence Φ restricts to an isomorphism from $N^{\mathcal{G}_1}(\boldsymbol{u}u_{s+1}, d_{s+1})$ to $N^{\mathcal{G}_2}(\boldsymbol{v}v_{s+1}, d_{s+1})$ and the invariant is maintained.

On the other hand, if Spoiler plays u_{s+1} at distance $> 2d_{s+1} + 1$ from $\boldsymbol{u}$ then Duplicator uses Condition 2 in the criterion in the statement of the Theorem (padding the list $v_1, \ldots, v_s$ with repetitions if necessary) with $d := 2d_{s+1} + 1$ to choose a response $v_{s+1} \in V_2$ that is also at distance $> 2d_{s+1} + 1$ from $\boldsymbol{v}$ such that

$$N^{\mathcal{G}_1}(u_{s+1}, d) \cong N^{\mathcal{G}_2}(v_{s+1}, d).$$

By restricting the above isomorphism to radius $d_{s+1} \leq d$ we obtain $N^{\mathcal{G}_1}(u_{s+1}, d_{s+1}) \cong N^{\mathcal{G}_2}(v_{s+1}, d_{s+1})$. Moreover, since $d(u_{s+1}, \boldsymbol{u}) > 2d_{s+1}+1$ and $d(v_{s+1}, \boldsymbol{v}) > 2d_{s+1}+1$, the neighbourhoods $N^{\mathcal{G}_1}(\boldsymbol{u}, d_{s+1})$ and $N^{\mathcal{G}_1}(u_{s+1}, d_{s+1})$ are disjoint and non-adjacent, and likewise for $N^{\mathcal{G}_2}(\boldsymbol{v}, d_{s+1})$ and $N^{\mathcal{G}_2}(v_{s+1}, d_{s+1})$.

Figure 1: All sufficiently small neighbourhoods in a cycle look like a path, so first-order logic cannot distinguish one long cycle from two disjoint long cycles.

We can therefore "glue" the restriction of Φ on $N^{\mathcal{G}_1}(\boldsymbol{u}, d_{s+1})$ with the fresh isomorphism on $N^{\mathcal{G}_1}(u_{s+1}, d_{s+1})$ to obtain an isomorphism

$$N^{\mathcal{G}_1}(\boldsymbol{u}u_{s+1}, d_{s+1}) \cong N^{\mathcal{G}_2}(\boldsymbol{v}v_{s+1}, d_{s+1}).$$

This shows that the invariant is also maintained in the second case. □

Corollary 10. The property of connectedness is not first-order definable over finite graphs (in the usual signature for graphs).

Proof. Suppose, for contradiction, that there is a first-order sentence φ_{conn} that defines connectedness on finite graphs. Let k be the quantifier depth of φ_{conn}.

Let $r := \frac{3^k-1}{2}$ as in Theorem 9 and choose an integer L such that $L > k(2r+1)$. Consider the graphs

$$\mathcal{G}_1 := C_{2L} \qquad \text{and} \qquad \mathcal{G}_2 := C_L \sqcup C_L.$$

The graph $\mathcal{G}_1$ is connected, while $\mathcal{G}_2$ is disconnected. Since $L > 2r+1$, the r-neighbourhood of any vertex in either graph is (as an induced subgraph) a simple path of length $2r$; in particular, for all vertices u of $\mathcal{G}_1$ and v of $\mathcal{G}_2$ we have $N^{\mathcal{G}_1}(u, r) \cong N^{\mathcal{G}_2}(v, r)$.

We claim that the criterion of Theorem 9 holds for $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $0 \le d \le r$. For (1), given $u_1, \ldots, u_{k-1} \in V_1$ and $v \in V_2$, each ball of radius d around some u_i contains at most $2d+1$ vertices, so at most $(k-1)(2d+1)$ vertices are within distance $\le r$ of the tuple $(u_1, \ldots, u_{k-1})$. Since $|V_1| = 2L > (k-1)(2d+1)$, there exists $u \in V_1$ with $d(u, u_i) > d$ for all i and, automatically, $N^{\mathcal{G}_1}(u, d) \cong N^{\mathcal{G}_2}(v, d)$. The argument for (2) is symmetric: working inside $\mathcal{G}_2$, and using the fact that each connected component of $\mathcal{G}_2$ has size $L > (k-1)(2r+1)$, there is always some vertex outside the union of the radius-d balls around $v_1, \ldots, v_{k-1}$.

By Theorem 9 we therefore have $\mathcal{G}_1 \equiv_k \mathcal{G}_2$. But then φ_{conn} cannot distinguish $\mathcal{G}_1$ from $\mathcal{G}_2$, contradicting that φ_{conn} defines connectedness. □