Expressive Completeness for Metric Temporal Logic

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***Abstract*—Metric Temporal Logic (MTL) is a generalisation of Linear Temporal Logic in which the Until and Since modalities are annotated with intervals that express metric constraints. A seminal result of Hirshfeld and Rabinovich shows that over the reals, first-order logic with binary order relation $<$ and unary function $+1$ is strictly more expressive than MTL with integer constants. Indeed they prove that no temporal logic whose modalities are definable by formulas of bounded quantifier depth can be expressively complete for $FO(<, +1)$. In this paper we show the surprising result that if we allow unary functions $+q$, $q \in \mathbb{Q}$, in first-order logic and correspondingly allow rational constants in MTL, then the two logics have the same expressive power. This gives the first generalisation of Kamp's theorem on the expressive completeness of LTL for $FO(<)$ to the quantitative setting. The proof of this result involves a generalisation of Gabbay's notion of separation.**

I. Introduction

One of the best-known and most widely studied logics in specification and verification is *Linear Temporal Logic (LTL)*: temporal logic with the modalities *Until* and *Since*. For discrete-time systems one considers interpretations of *LTL* over the integers $(\mathbb{Z}, <)$, and for continuous-time systems one considers interpretations over the reals $(\mathbb{R}, <)$. A celebrated result of Kamp [1] is that, over both $(\mathbb{Z}, <)$ and $(\mathbb{R}, <)$, *LTL* has the same expressiveness as the *Monadic Logic of Order ($FO(<)$)*: first-order logic with binary order relation $<$ and uninterpreted monadic predicates. Thus we can benefit from the appealing variable-free syntax and elementary decision procedures of *LTL*, while retaining the expressiveness and canonicity of first-order logic.

Over the reals $FO(<)$ cannot express metric properties, such as, "every request is followed by a response within one time unit". This motivates the introduction of *Monadic Logic of Order and Metric ($FO(<, +\mathbb{Q})$)*, which augments $FO(<)$ with a family of unary function symbols $+q$, $q \in \mathbb{Q}$. Correspondingly, there have been a variety of proposals of quantitative temporal logics, with modalities definable in $FO(<, +\mathbb{Q})$ (see, e.g., [2], [3], [4], [5], [6], [7]). Sometimes attention is restricted to $FO(<, +1)$—the fragment of $FO(<, +\mathbb{Q})$ with only the $+1$ function—and to temporal logics definable in this fragment. Typically these temporal logics can be seen as quantitative extensions of *LTL*. However, until now there has been no fully satisfactory counterpart to Kamp's theorem in the quantitative setting.

The best-known quantitative temporal logic is *Metric Temporal Logic (MTL)*, introduced over 20 years ago in [8]. *MTL* arises by annotating the temporal modalities of *LTL* with intervals with rational endpoints, representing metric constraints. Since the *MTL* operators are definable in $FO(<, +\mathbb{Q})$, it is immediate that one can translate *MTL* into $FO(<, +\mathbb{Q})$. The main result of this paper shows the converse, that *MTL* is expressively complete for $FO(<, +\mathbb{Q})$.

The generality of allowing rational constants is crucial for expressive completeness: our translation from $FO(<, +\mathbb{Q})$ to *MTL* does not preserve the granularity of timing constraints. Indeed, it is known that *MTL* with integer constants is not expressively complete for $FO(<, +1)$. More generally, a seminal result of Hirshfeld and Rabinovich [9, Theorem] asserts that no temporal logic whose modalities are definable by a (possibly infinite) set of formulas of $FO(<, +1)$ of bounded quantifier depth can be expressively complete for $FO(<, +1)$. Since the modalities of *MTL* are definable by formulas of quantifier depth two, necessarily an *MTL* formula equivalent to a given $FO(<, +1)$ formula may require rational constants and itself only be definable in $FO(<, +\mathbb{Q})$.

Two of the key ideas underlying our proof of expressive completeness are *boundedness* and *separation*. Given $N \in \mathbb{N}$, an $FO(<, +\mathbb{Q})$ formula $\varphi(x)$ is N-bounded if all quantifiers are relativised to the interval $(x - N, x + N)$. Exploiting a normal form for $FO(<)$, due to Gabbay, Pnueli, Shelah and Stavi [10], we show how to translate bounded $FO(<, +\mathbb{Q})$ formulas into *MTL*. Extending this translation to arbitrary $FO(<, +\mathbb{Q})$ formulas requires an appropriate metric analog of Gabbay's notion of *separation* [11].

Gabbay [11] shows that every *LTL* formula can be equivalently rewritten as a Boolean combination of formulas, each of which depends only on the past, present or future. This seemingly innocuous separation property has several far-reaching consequences (see the survey of Hodkinson and Reynolds [12]). In particular, the fact *LTL* has the property is a key lemma in an inductive translation from $FO(<)$ to *LTL*. We prove an analogous result for *MTL*: every *MTL* formula can be equivalently rewritten as a Boolean combination of formulas, each of which is either bounded (i.e., refers to the *near present*) or refers to the *distant future* or *distant past*. Crucially, while the distant past and distant future are disjoint, they are both allowed to overlap with near present, unlike in Gabbay's result. We exploit our result in like manner to Gabbay to give an inductive translation of $FO(<, +\mathbb{Q})$ to *MTL*. Here it is vital that we already have a translation of bounded $FO(<, +\mathbb{Q})$ formulas to *MTL*.

Related Work

A more elaborate quantitative extension of *LTL* is *Timed Propositional Temporal Logic (TPTL)*, which expresses timing constraints using variables and freeze quantification [4]. From the respective definitions of the logics the following inclusions in expressiveness are straightforward:

$$MTL \subseteq TPTL \subseteq FO(<, +\mathbb{Q})\,. \tag{1}$$

Bouyer, Chevalier and Markey [13] showed that the inclusion between *MTL* and *TPTL* is strict if only future temporal connectives are considered, confirming a conjecture of [4]. However they left open the case in which both past and future connectives are allowed. Our main result shows that in this case the chain of inclusions (1) collapses, resolving this open question.

In fact, *TPTL* has already been shown to be expressively complete for $FO(<, +\mathbb{Q})$ in [14]. Notwithstanding this result, we regard the result in the present paper as the first fully satisfactory analog of Kamp's Theorem for $FO(<, +\mathbb{Q})$. This is because *TPTL* is a hybrid between first-order logic and temporal logic, featuring variables and quantification in addition to temporal modalities [15].

The logics considered in this paper are all undecidable. Adding $+1$ to $FO(<)$ or $\Diamond_{=1}\varphi$ (φ will be true in exactly one time unit) to *LTL* already leads to an undecidable satisfiability problem over the reals. Intuitively, the source of undecidability is the ability to express *punctual* metric constraints, such as "every request is followed by a response in exactly one time unit". The expressiveness of decidable quantitative temporal logics has also been investigated in [16], [9]. The main results present a hierarchy of decidable temporal logics with *counting modalities*, and characterise their expressiveness in terms of fragments of $FO(<, +1)$. The most basic such logic arises by adding the modality $\Diamond_{<1}\varphi$ to *LTL*, expressing that φ will be true within one time unit in the future, and is equivalent in expressiveness to the logic MITL [17].

Yet another approach to expressive completeness is taken in our previous work [18]. This paper considers the fragment of $FO(<, +\mathbb{Q})$ with only the $+1$ function. Likewise it restricts to *MTL* formulas in which intervals have integer endpoints. Recall that in this setting expressive completeness fails over unbounded domains such as $(\mathbb{R}, <)$ and $(\mathbb{R}_{\geq 0}, <)$. However [18] shows that expressive completeness holds over each bounded time domain $([0, N), <)$. While some of the ideas from [18] are used in the present paper, our results differ substantially. While [18] relies on a bounded time domain, the present paper considers syntactically bounded formulas on an unbounded domain. Even the fact that *MTL* is expressively complete for syntactically bounded $FO(<, +1)$ formulas crucially uses the fact that we allow fractional constants.

II. Definitions and Main Results

A. First-order logic

Formulas of *Monadic Logic of Order and Metric* ($FO(<, +\mathbb{Q})$) are first-order formulas over a signature with a binary relation symbol $<$, an infinite collection of unary predicate symbols $P_1, P_2, \ldots$, and an infinite family of unary function symbols $+q$, $q \in \mathbb{Q}$. Formally, the terms of $FO(<, +\mathbb{Q})$ are generated by the grammar $t ::= x \mid t + q$, where x is a variable and $q \in \mathbb{Q}$. Formulas of $FO(<, +\mathbb{Q})$ are given by the following syntax:

$$\varphi ::= \mathbf{true} \mid P_i(t) \mid t < t \mid \varphi \wedge \varphi \mid \neg\varphi \mid \exists x\, \varphi\,,$$

where x denotes a variable and t a term.

We consider interpretations of $FO(<, +\mathbb{Q})$ over the real line[1], $\mathbb{R}$, with the natural interpretations of $<$ and $+q$. It follows that a structure for $FO(<, +\mathbb{Q})$ is determined by an interpretation of the monadic predicates.

Of particular importance is $FO(<, +1)$, the fragment of $FO(<, +\mathbb{Q})$ that omits all the $+q$ functions except $+1$. For simplicity, when considering formulas of $FO(<, +1)$ we will often use standard arithmetical notation as a shorthand, for example,

$$x - y > 2 \quad \equiv \quad (y + 1) + 1 < x\,.$$

B. Metric Temporal Logic

Given a set $\boldsymbol{P}$ of atomic propositions, the formulas of *Metric Temporal Logic (MTL)* are built from $\boldsymbol{P}$ using Boolean connectives and time-constrained versions of the *Until* and *Since* operators $\mathbf{U}$ and $\mathbf{S}$ as follows:

$$\varphi ::= \mathbf{true} \mid P \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi\, \mathbf{U}_I\, \varphi \mid \varphi\, \mathbf{S}_I\, \varphi\,,$$

where $P \in \boldsymbol{P}$ and $I \subseteq (0, \infty)$ is an interval with endpoints in $\mathbb{Q}_{\geq 0} \cup \{\infty\}$.

Intuitively, the meaning of $\varphi_1\, \mathbf{U}_I\, \varphi_2$ is that φ_2 will hold at some time in the interval I, and until then φ_1 holds. More precisely, the semantics of *MTL* are defined as follows. A *signal* is a function $f : \mathbb{R} \to 2^{\boldsymbol{P}}$. Given a signal f and $r \in \mathbb{R}$, we define the satisfaction relation $f, r \models \varphi$ by induction over φ as follows:

- $f, r \models p$ iff $p \in f(r)$,
- $f, r \models \neg\varphi$ iff $f, r \not\models \varphi$,
- $f, r \models \varphi_1 \wedge \varphi_2$ iff $f, r \models \varphi_1$ and $f, r \models \varphi_2$,
- $f, r \models \varphi_1\, \mathbf{U}_I\, \varphi_2$ iff there exists $t > r$ such that $t - r \in I$, $f, t \models \varphi_2$ and $f, u \models \varphi_1$ for all u, $r < u < t$,
- $f, r \models \varphi_1\, \mathbf{S}_I\, \varphi_2$ iff there exists $t < r$ such that $r - t \in I$, $f, t \models \varphi_2$ and $f, u \models \varphi_1$ for all u, $t < u < r$.

LTL can be seen as a restriction of *MTL* with only the interval $I = (0, \infty)$. Indeed, if $I = (0, \infty)$ then we omit the annotation I in the corresponding temporal operator since the constraint is vacuous. We also use arithmetic expressions to denote intervals. For example, we write $\mathbf{U}_{<3}$ for $\mathbf{U}_{(0,3)}$ and $\mathbf{U}_{=1}$ for $\mathbf{U}_{\{1\}}$. We say the $\mathbf{U}_I$ and $\mathbf{S}_I$ operators are *bounded* if I is bounded, otherwise we say that the operators are *unbounded*.

We introduce the derived connectives $\Diamond_I\varphi := \mathbf{true}\ \mathbf{U}_I\ \varphi$ (φ will be true at some point in interval I) and $\diamondminus_I\varphi :=$

[1] Our results carry over to subintervals of $\mathbb{R}$, such as the non-negative reals $\mathbb{R}_{\geq 0}$.

$\mathbf{true}\ \mathbf{S}_I\ \varphi$ (φ was true at some point in interval I in the past). We also have the dual connectives $\Box_I\varphi := \neg\Diamond_I\neg\varphi$ (φ will hold at all times in interval I in the future) and $\boxminus_I := \neg\diamondminus_I\neg\varphi$ (φ was true at all times in interval I in the past).

C. Expressive Equivalence

Given a set $\boldsymbol{P} = \{P_1, \dots, P_m\}$ of monadic predicates, a signal $f : \mathbb{R} \to 2^{\boldsymbol{P}}$ defines an interpretation of each P_i, where $P_i(r)$ if and only if $P_i \in f(r)$. As observed earlier, this is sufficient to define the model-theoretic semantics of $FO(<, +\mathbb{Q})$, enabling us to relate the semantics of $FO(<, +\mathbb{Q})$ and *MTL*.

Let $\varphi(x)$ be an $FO(<, +\mathbb{Q})$ formula with one free variable and ψ an *MTL* formula. We say φ and ψ are *equivalent* if for all signals f and $r \in \mathbb{R}$:

$$f \models \varphi[r] \iff f, r \models \psi.$$

Example 1. *Consider the following formula, which says that P will be true at two points within the next time unit:*

$$\varphi(x) := \exists y\, \exists z\, ((x < y < z < x + 1) \wedge P(y) \wedge P(z)) .$$

It was shown in [9] that φ cannot be expressed in MTL using only integer constants[2]*. To see this, consider the signal f in which the predicate P is true exactly at the points $\frac{2n}{3}$, $n \in \mathbb{N}$. It can be shown by induction that for every MTL formula φ with integer constants there exists $t_0 > 0$ and a predicate θ that is either* $\mathbf{true}$, $\mathbf{false}$, P, $\neg P$, *or* $\Diamond_{=1}P$, *such that for all $t > t_0$, $f, r \models \varphi$ iff $f, r \models \theta$. On the other hand, for $2n \equiv 1 \pmod 3$, φ is continuously true on the interval $(\frac{2n-1}{3}, \frac{2n}{3})$ and false on the boundary of the interval.*

As observed in [13], we can, however, express $\varphi(x)$ in MTL by using fractional constants. The idea is to consider three cases according to whether P is true twice in the interval $(x, x + \frac{1}{2}]$, twice in the interval $[x + \frac{1}{2}, x + 1)$, or once each in $(x, x + \frac{1}{2})$ and $(x + \frac{1}{2}, x + 1)$. We are thus led to define the MTL formula

$$\begin{aligned}\varphi^\dagger := {} & \Diamond_{(0,\frac{1}{2})}(P \wedge \Diamond_{(0,\frac{1}{2})}P) \vee \\ & \Diamond_{=1}(\diamondminus_{(0,\frac{1}{2})}(P \wedge \diamondminus_{(0,\frac{1}{2})}P)) \vee \\ & (\Diamond_{(0,\frac{1}{2})}P \wedge \Diamond_{(\frac{1}{2},1)}P) ,\end{aligned}$$

which is equivalent to φ.

The following is straightforward.

Proposition 2. *For every MTL formula φ there is an equivalent $FO(<, +\mathbb{Q})$ formula $\varphi^*(x)$.*

Our main result is the converse:

Theorem 3. *For every $FO(<, +\mathbb{Q})$ formula $\varphi(x)$ there is an equivalent MTL formula $\varphi^\dagger$.*

As we now explain, by a simple scaling argument it suffices to prove Theorem 3 in the special case for which φ is an $FO(<, +1)$ formula. Let f be a signal and $r \in \mathbb{Q}_{>0}$. We define the signal $r \cdot f$ by $r \cdot f(s) := f(\frac{s}{r})$. Given either an $FO(<, +\mathbb{Q})$ formula $\varphi(x)$ or an *MTL* formula φ, we say that the formula φ^r is a *scale* of φ by $r \in \mathbb{Q}_{>0}$, if for all signals f and all $s \in \mathbb{R}$,

$$f, s \models \varphi \quad \Longleftrightarrow \quad r \cdot f, rs \models \varphi^r .$$

It is straightforward that $FO(<, +\mathbb{Q})$ and *MTL* are both closed under scaling: in each case the required formula φ^r is obtained by multiplying all constants occurring in φ by r.

Now we show how to deduce expressive completeness of *MTL* for $FO(<, +\mathbb{Q})$ from the fact that *MTL* is at least as expressive as the fragment $FO(<, +1)$. Given an $FO(<, +\mathbb{Q})$ formula $\varphi(x)$, pick r such that φ^r is an $FO(<, +1)$ formula and translate φ^r to an equivalent *MTL* formula ψ. Then rescaling ψ by $1/r$, we obtain an *MTL* formula $\psi^{1/r}$ that is equivalent to the original formula φ.

We will see later that the translation from $FO(<, +1)$ to *MTL* already involves temporal operators whose constraining intervals have fractional endpoints, as suggested by Example 1.

III. Syntactic Separation of MTL

In [19], Gabbay *et al.* showed that *LTL* formulas over Dedekind-complete domains are equivalent to Boolean combinations of formulas that depend exclusively on one of the past, present, or future. We state this result as it applies to continuous domains (the formulation in the discrete setting is slightly more straightforward). To state the result we recall the *right-limit* modality K^+ and *left-limit* modality K^-, respectively defined as:

$$K^+\varphi := \neg(\neg\varphi\ \mathbf{U}\ \mathbf{true}) \qquad K^-\varphi := \neg(\neg\varphi\ \mathbf{S}\ \mathbf{true}) .$$

The formula $K^+\varphi$ states that φ is true arbitrarily close in the future and $K^-\varphi$ asserts that φ is true arbitrarily close in the past.

Theorem 4 ([19]). *Over Dedekind-complete domains, every LTL formula is equivalent to a Boolean combination of:*

- *atomic formulas,*
- *formulas of the form $\varphi_1\ \mathbf{U}\ \varphi_2$ such that φ_1 and φ_2 use only $\mathbf{U}$ and K^-,*
- *formulas of the form $\varphi_1\ \mathbf{S}\ \varphi_2$ such that φ_1 and φ_2 use only $\mathbf{S}$ and K^+.*

Note that the three classes of formulas in Theorem 4 respectively refer to the present, future and past. In this section we derive an analogous result for *MTL*. We show that every *MTL* formula can be written as a Boolean combination of *bounded*, *distant future* and *distant past formulas*. Just as Gabbay *et al.* used syntactic forms for future and past representations, our plan is to use natural forms for bounded, distant future and distant past formulas. Crucially, the distant future and distant past are allowed to overlap with the bounded present, unlike in the result of Gabbay *et al*.

Given an *MTL* formula φ, we define the *future-reach* $fr(\varphi)$ and *past-reach* $pr(\varphi)$ inductively as follows:

- $fr(p) = pr(p) = 0$ for all propositions p,

[2] In fact [9] did not consider so-called *punctual* operators, i.e., singleton constraining intervals. But their argument goes through *mutatis mutandis*.

- $\mathit{fr}(\mathbf{true}) = \mathit{pr}(\mathbf{true}) = 0$,
- $\mathit{fr}(\neg\varphi) = \mathit{fr}(\varphi)$, $\mathit{pr}(\neg\varphi) = \mathit{pr}(\varphi)$,
- $\mathit{fr}(\varphi \wedge \psi) = \max\{\mathit{fr}(\varphi), \mathit{fr}(\psi)\}$,
- $\mathit{pr}(\varphi \wedge \psi) = \max\{\mathit{pr}(\varphi), \mathit{pr}(\psi)\}$,
- If $n = \inf(I)$ and $m = \sup(I)$:
  - $\mathit{fr}(\varphi \, \mathbf{U}_I \, \psi) = m + \max\{\mathit{fr}(\varphi), \mathit{fr}(\psi)\}$,
  - $\mathit{pr}(\varphi \, \mathbf{S}_I \, \psi) = m + \max\{\mathit{pr}(\varphi), \mathit{pr}(\psi)\}$,
  - $\mathit{fr}(\varphi \, \mathbf{S}_I \, \psi) = \max\{\mathit{fr}(\varphi), \mathit{fr}(\psi) - n\}$,
  - $\mathit{pr}(\varphi \, \mathbf{U}_I \, \psi) = \max\{\mathit{pr}(\varphi), \mathit{pr}(\psi) - n\}$.

Intuitively the future-reach indicates how much of the future is required to determine the truth of an *MTL* formula, and likewise for the past-reach. Note that if φ contains an unbounded $\mathbf{U}$ operator then $\mathit{fr}(\varphi) = \infty$ and likewise if φ contains an unbounded $\mathbf{S}$ operator, $\mathit{pr}(\varphi) = \infty$.

We say an *MTL* formula is *syntactically separated* if it is a Boolean combination of the following

- $\Diamond_{=N}\varphi$ where $\mathit{pr}(\varphi) < N - 1$,
- $\diamondminus_{=N}\varphi$ where $\mathit{fr}(\varphi) < N - 1$,
- φ, where all intervals occurring in temporal operators are bounded.

We call formulas of the third kind above *bounded*. Note that formulas with no occurrences of $\mathbf{U}_I$ and $\mathbf{S}_I$ are included in the definition of bounded formulas.

Example 5. *Consider the formula $\varphi = \Diamond\boxminus(p \to \diamondminus_{=1}p)$. Then $\mathit{fr}(\varphi) = \mathit{pr}(\varphi) = \infty$. We define an equivalent separated formula as follows. First, write $\psi = p \to \diamondminus_{=1}p$. Then φ is equivalent to*

$$\begin{aligned}&\diamondminus_{=1}(\psi \wedge \boxminus\psi) \wedge \boxminus_{(0,1)}\psi \wedge \psi \\ &\wedge \big((\psi \, \mathbf{U}_{\leq 2} \, \psi) \vee (\Box_{\leq 2}\psi \wedge \Diamond_{=2}(\psi \, \mathbf{U} \, \psi))\big)\,.\end{aligned}$$

Theorem 6. *Every MTL formula is equivalent to one which is syntactically separated.*

To prove Theorem 6 our strategy is as follows:

Step 1. Remove all unbounded $\mathbf{U}$ and $\mathbf{S}$ operators from within the scope of bounded operators.

Step 2. Treating bounded formulas as atoms, apply Theorem 4 to remove unbounded $\mathbf{U}$ operators from the scope of unbounded $\mathbf{S}$ operators and vice versa.

Step 3. Divide the top-level unbounded operators into formulas bounded by N and formulas at least N away for sufficiently large N to separate these formulas. This step may also place unbounded operators within the scope of bounded operators, but still maintains the separation of unbounded $\mathbf{U}$ and unbounded $\mathbf{S}$ operators. Using Step 1, and observing that this does not introduce any new unbounded operators, we can move these unbounded operators to the top level and recursively apply the division to completely separate the formula.

Step 0. Translation to Normal Form: We first introduce a normal form for *MTL* formulas. In defining this we regard $\mathbf{U}_I$, $\mathbf{S}_I$, $\Box_I$, $\Diamond_I$, $\boxminus_I$, and $\diamondminus_I$ as primitive operators. Then an *MTL* formula is said to be in *normal form* if the following all hold:

(i) The formula is written using the Boolean operators and the temporal connectives $\mathbf{U}_{(0,\gamma)}$, $\mathbf{S}_{(0,\gamma)}$, $\Box_{(0,\gamma)}$, $\boxminus_{(0,\gamma)}$, where $\gamma \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$, and $\Diamond_{=q}$ and $\diamondminus_{=q}$, where $q \in \mathbb{Q}_{\geq 0}$;

(ii) In any subformula $\varphi_1 \, \mathbf{U}_I \, \varphi_2$ or $\varphi_1 \, \mathbf{S}_I \, \varphi_2$, the outermost connective of φ_1 is not conjunction and the outermost connective of φ_2 is not disjunction;

(iii) No temporal operator occurs in the scope of $\Diamond_{=q}$ or $\diamondminus_{=q}$;

(iv) Negation is only applied to propositional variables and bounded temporal operators.

We can transform an *MTL* formula into an equivalent normal form as follows. To satisfy (i) we eliminate connectives $\mathbf{U}_I$ and $\mathbf{S}_I$ in which the interval I does not have left endpoint 0 using the equivalences

$$\begin{aligned}\varphi \, \mathbf{U}_{(p,q)} \, \psi \quad &\longleftrightarrow \quad \Box_{(0,p)}\varphi \wedge \Diamond_{=p}\big(\varphi \wedge (\varphi \, \mathbf{U}_{(0,q-p)} \, \psi)\big) \\ \varphi \, \mathbf{S}_{(p,q)} \, \psi \quad &\longleftrightarrow \quad \boxminus_{(0,p)}\varphi \wedge \diamondminus_{=p}\big(\varphi \wedge (\varphi \, \mathbf{S}_{(0,q-p)} \, \psi)\big)\end{aligned}$$

and corresponding equivalences for left-closed and right-closed intervals.

To satisfy (ii) we use the equivalences

$$\begin{aligned}\varphi \, \mathbf{U}_I \, (\psi \vee \theta) \quad &\longleftrightarrow \quad (\varphi \, \mathbf{U}_I \, \psi) \vee (\varphi \, \mathbf{U}_I \, \theta) \\ (\varphi \wedge \psi) \, \mathbf{U}_I \, \theta \quad &\longleftrightarrow \quad (\varphi \, \mathbf{U}_I \, \theta) \wedge (\psi \, \mathbf{U}_I \, \theta)\end{aligned}$$

and their corresponding versions for $\mathbf{S}_I$,

$$\begin{aligned}\varphi \, \mathbf{S}_I \, (\psi \vee \theta) \quad &\longleftrightarrow \quad (\varphi \, \mathbf{S}_I \, \psi) \vee (\varphi \, \mathbf{S}_I \, \theta) \\ (\varphi \wedge \psi) \, \mathbf{S}_I \, \theta \quad &\longleftrightarrow \quad (\varphi \, \mathbf{S}_I \, \theta) \wedge (\psi \, \mathbf{S}_I \, \theta)\,.\end{aligned}$$

To satisfy (iii) we use the equivalences

$$\begin{aligned}\Diamond_{=q}(\varphi \wedge \psi) \quad &\longleftrightarrow \quad \Diamond_{=q}\varphi \wedge \Diamond_{=q}\psi \\ \Diamond_{=q}(\neg\varphi) \quad &\longleftrightarrow \quad \neg\Diamond_{=q}\varphi \\ \Diamond_{=q}(\varphi \, \mathbf{U}_I \, \psi) \quad &\longleftrightarrow \quad \Diamond_{=q}\varphi \, \mathbf{U}_I \, \Diamond_{=q}\psi \\ \Diamond_{=q}(\varphi \, \mathbf{S}_I \, \psi) \quad &\longleftrightarrow \quad \Diamond_{=q}\varphi \, \mathbf{S}_I \, \Diamond_{=q}\psi\end{aligned}$$

and the corresponding equivalences for $\diamondminus_{=q}$ to distribute $\Diamond_{=q}$ and $\diamondminus_{=q}$ across all other operators.

To satisfy (iv) we observe that the K^+ and K^- operators can be defined as bounded formulas, *viz.*

$$K^+(\varphi) \leftrightarrow \neg(\neg\varphi \, \mathbf{U}_{<1} \, \mathbf{true}) \qquad K^-(\varphi) \leftrightarrow \neg(\neg\varphi \, \mathbf{S}_{<1} \, \mathbf{true})\,.$$

Then we use the equivalences

$$\begin{aligned}\neg(\varphi \, \mathbf{U} \, \psi) \quad &\longleftrightarrow \quad \Box\neg\psi \vee K^+(\neg\varphi) \vee \\ &\qquad\quad (\neg\psi \, \mathbf{U} \, (\neg\psi \wedge (\neg\varphi \vee K^+(\neg\varphi)))) \\ \neg\Box\varphi \quad &\longleftrightarrow \quad \mathbf{true} \, \mathbf{U} \, \neg\varphi\end{aligned}$$

and their corresponding past versions to rewrite any subformula in which negation is applied to an unbounded temporal operator.

Step 1. Extracting unbounded Until and Since

Our goal in this subsection is the following lemma.

Lemma 7. *Every MTL formula φ is equivalent to one in which no unbounded temporal operator occurs within the scope of a bounded temporal operator.*

The proof of this lemma relies on Proposition 8, whose proof is straightforward.

Proposition 8. *For all $q \in \mathbb{Q}_{\geq 0}$, the following equivalences and their temporal duals hold over all signals.*

(i)
$$\begin{gathered} \theta \, \mathbf{U}_{<q} \, \big((\varphi \, \mathbf{U} \, \psi) \wedge \chi\big) \\ \leftrightarrow \\ \theta \, \mathbf{U}_{<q} \, \big((\varphi \, \mathbf{U}_{<q} \, \psi) \wedge \chi\big) \quad \vee \\ \Big(\big(\theta \, \mathbf{U}_{<q} \, (\Box_{<q}\varphi \wedge \chi)\big) \;\wedge\; \Diamond_{=q}(\varphi \, \mathbf{U} \, \psi)\Big) \end{gathered}$$

(ii)
$$\begin{gathered} \theta \, \mathbf{U}_{<q} \, (\Box\varphi \wedge \chi) \\ \leftrightarrow \\ \big(\theta \, \mathbf{U}_{<q} \, (\Box_{<q}\varphi \wedge \chi)\big) \wedge \Diamond_{=q}\Box\varphi \end{gathered}$$

(iii)
$$\begin{gathered} \theta \, \mathbf{U}_{<q} \, \big((\varphi \, \mathbf{S} \, \psi) \wedge \chi\big) \\ \leftrightarrow \\ \theta \, \mathbf{U}_{<q} \, \big((\varphi \, \mathbf{S}_{<q} \, \psi) \wedge \chi\big) \quad \vee \\ \Big(\big(\theta \, \mathbf{U}_{<q} \, (\boxminus_{<q}\varphi \wedge \chi)\big) \;\wedge\; \varphi \, \mathbf{S} \, \psi\Big) \end{gathered}$$

(iv)
$$\begin{gathered} \theta \, \mathbf{U}_{<q} \, (\boxminus\varphi \wedge \chi) \\ \leftrightarrow \\ \big(\theta \, \mathbf{U}_{<q} \, (\boxminus_{<q}\varphi \wedge \chi)\big) \wedge \boxminus\varphi \end{gathered}$$

(v)
$$\begin{gathered} \big((\varphi \, \mathbf{U} \, \psi) \vee \chi\big) \, \mathbf{U}_{<q} \, \theta \\ \leftrightarrow \\ \big((\varphi \, \mathbf{U}_{<q} \, \psi) \vee \chi\big) \, \mathbf{U}_{<q} \, \theta \quad \vee \\ \Big[\big((\varphi \, \mathbf{U}_{<q} \, \psi) \vee \chi\big) \, \mathbf{U}_{<q} \, (\Box_{<q}\varphi) \\ \wedge \Diamond_{<q}\theta \;\wedge\; \Diamond_{=q}(\varphi \, \mathbf{U} \, \psi)\Big] \end{gathered}$$

(vi)
$$\begin{gathered} \big((\Box\varphi) \vee \chi\big) \, \mathbf{U}_{<q} \, \theta \\ \leftrightarrow \\ \chi \, \mathbf{U}_{<q} \, \theta \quad \vee \\ \big(\chi \, \mathbf{U}_{<q} \, (\Box_{<q}\varphi) \;\wedge\; \Diamond_{<q}\theta \;\wedge\; \Diamond_{=q}(\Box\varphi)\big) \end{gathered}$$

(vii)
$$\begin{gathered} \big((\varphi \, \mathbf{S} \, \psi) \vee \chi\big) \, \mathbf{U}_{<q} \, \theta \\ \leftrightarrow \\ \big((\varphi \, \mathbf{S}_{<q} \, \psi) \vee \chi\big) \, \mathbf{U}_{<q} \, \theta \quad \vee \\ \Big[\big(\boxminus_{<q}\varphi \vee (\varphi \, \mathbf{S}_{<q} \, \psi) \vee \chi\big) \, \mathbf{U}_{<q} \, \theta \;\wedge\; (\varphi \, \mathbf{S} \, \psi)\Big] \end{gathered}$$

(viii)
$$\begin{gathered} \big(\boxminus\varphi \vee \chi\big) \, \mathbf{U}_{<q} \, \theta \\ \leftrightarrow \\ \chi \, \mathbf{U}_{<q} \, \theta \vee \\ \Big[\big((\boxminus_{<q}\varphi \vee \chi) \, \mathbf{U}_{<q} \, \theta\big) \;\wedge\; \boxminus\varphi\Big]. \end{gathered}$$

Proof of Lemma 7: Define the *unbounding depth* $ud(\varphi)$ of an *MTL* formula φ to be the modal depth of φ, counting only unbounded temporal operators. Thus we have

$$ud(\varphi_1 \, \mathbf{U}_I \, \varphi_2) = \begin{cases} \max(ud(\varphi_1), ud(\varphi_2)) & I \text{ bounded} \\ \max(ud(\varphi_1), ud(\varphi_2)) + 1 & \text{otherwise} \end{cases}$$

with similar clauses for the other temporal operators.

Now suppose that φ is an *MTL* formula in normal form in which some unbounded temporal operator occurs within the scope of a bounded temporal operator. Then some subformula of φ (or its temporal dual) matches the top side of one of the equivalences in Proposition 8. Pick such a subformula ψ with maximum unbounding depth $ud(\psi)$ and replace it with the bottom side ψ' of the corresponding equivalence. Notice that all subformulas of ψ' whose outermost connective is a bounded temporal operator other than $\Diamond_{=q}$ and $\diamondminus_{=q}$ have unbounding depth strictly less than $ud(\psi)$. Finally rewrite ψ' to normal form, in particular pushing the newly introduced $\Diamond_{=q}$ and $\diamondminus_{=q}$ operators inward. Notice that this last step does not increase the maximum unbounding depth.

This rewriting process must eventually terminate, yielding a formula in which no unbounded operator remains within the scope of a bounded operator.

Step 2. Extracting Since from Until and vice-versa

Now suppose we have an *MTL* formula in which no unbounded temporal operator occurs within the scope of a bounded operator. If we replace each bounded subformula θ with a new proposition P_θ, the resulting formula is now an *LTL* formula equivalent to our original formula for suitable interpretations of the P_θ. From Theorem 4 we know that this formula is equivalent to a Boolean combination of:

- atomic formulas,
- formulas of the form $\varphi_2 \mathbf{U} \varphi_1$ such that φ_1 and φ_2 use only $\mathbf{U}$ and K^-,
- formulas of the form $\varphi_2 \mathbf{S} \varphi_1$ such that φ_1 and φ_2 use only $\mathbf{S}$ and K^+.

Recalling from Step 0 that we can express the operators K^+ and K^- using bounded operators, and also replacing each proposition P_θ with its associated bounded formula θ, we obtain:

Lemma 9. *Every MTL formula is equivalent to a Boolean combination of:*

- *bounded formulas,*
- *formulas that use arbitrary $\mathbf{U}_I$ but only bounded $\mathbf{S}_I$,*
- *formulas that use arbitrary $\mathbf{S}_I$ but only bounded $\mathbf{U}_I$*

Step 3. Completing the separation

Now suppose we have an *MTL* formula θ that does not contain unbounded $\mathbf{S}$. We prove by induction on the number of unbounded $\mathbf{U}$ operators that θ is equivalent to a syntactically separated formula. Clearly if θ contains no unbounded $\mathbf{U}$ operators then it is bounded and therefore syntactically separated. Otherwise, by applying Lemma 7 and observing that it does not introduce unbounded $\mathbf{U}$ operators, we may assume that $\theta = \varphi \mathbf{U} \psi$ where φ and ψ have strictly fewer unbounded $\mathbf{U}$ operators than θ. As θ does not contain unbounded $\mathbf{S}$ operators, $pr(\theta)$ is finite, so choose $N > pr(\theta) + 1$. Next we apply the following equivalence

$$\varphi \mathbf{U} \psi \quad \longleftrightarrow \quad \varphi \mathbf{U}_{<N} \psi \\ \vee\big(\Box_{<N}\varphi \wedge \Diamond_{=N}(\psi \vee (\varphi \wedge \varphi \mathbf{U} \psi))\big) .$$

Now $pr(\psi \vee (\varphi \wedge \varphi \mathbf{U} \psi)) = pr(\theta) < N-1$, and the subformulas $\varphi \mathbf{U}_{<N} \psi$ and $\Box_{<N}\varphi$ have strictly fewer unbounded $\mathbf{U}$ operators than θ. So by the induction hypothesis the formula on the right hand side of the above equivalence is equivalent to one that is syntactically separated, completing the inductive step. Similarly $\mathbf{S}$ formulas that do not contain unbounded $\mathbf{U}$ operators are equivalent to syntactically separated formulas. Applying these observations to Lemma 9 gives our main result, which we repeat here for completeness.

Theorem 6. *Every MTL formula is equivalent to a Boolean combination of:*

- *$\Diamond_{=N}\varphi$ where $pr(\varphi) < N-1$,*
- *$\overleftarrow{\Diamond}_{=N}\varphi$ where $fr(\varphi) < N-1$, and*
- *φ where all intervals occurring in the temporal operators are bounded.*

IV. Expressive completeness on bounded formulas

In this section we show expressive completeness of *MTL* for a fragment of $FO(<,+1)$ consisting of *bounded formulas*, i.e., formulas $\varphi(x)$ that refer only to a bounded interval around x.

Given terms t_2 and t_2, define $\mathrm{Bet}(t_1,t_2)$ to consist of $FO(<,+1)$ formulas in which

(i) each subformula $\exists z\, \psi$ has the form $\exists z\,((t_1 \leq z < t_2) \wedge \chi)$, i.e., each quantifier is relativized to the half-open interval between t_1 (inclusive) and t_2 (exclusive);

(ii) in each atomic subformula $P(t)$ the term t is a bound occurrence of a variable.

Clauses (i) and (ii) ensure that a formula in $\mathrm{Bet}(t_1,t_2)$ only refers to the values of monadic predicates on points in the half-open interval $[t_1,t_2)$. We say that a formula $\varphi(x)$ in $\mathrm{Bet}(x-N,x+N)$ is *N-bounded* and that $\varphi(x)$ in $\mathrm{Bet}(x,x+1)$ is a *unit formula*.

Observe that in a unit formula the only essential use of the $+1$ function is in specifying the range of the quantified variables. More precisely, we have the following proposition, where $\psi[t/y]$ denotes the formula obtained by substituting term t for all free occurrences of variable y in ψ:

Proposition 10. *For any unit formula $\varphi(x)$ there is an $FO(<)$ formula $\psi \in \mathrm{Bet}(x,y)$ such that φ is equivalent to $\psi[(x+1)/y]$.*

Proof. We show that all uses of the $+1$ function in φ other than to specify the range of quantified variables can be eliminated.

Let u,v be bound variables and $k_1,k_2 \in \mathbb{N}$. Since u,v range over an open interval of length 1 an inequality of the form $u+k_1 < v+k_2$ can be replaced by (i) $u < v$, if $k_1 = k_2$; (ii) $\mathbf{true}$, if $k_1 < k_2$; and (iii) $\mathbf{false}$ otherwise. Likewise an equality of the form $u+k_1 = v+k_2$ can be replaced by $u = v$ if $k_1 = k_2$, and $\mathbf{false}$ otherwise. ☐

The main result of this section is:

Theorem 11. *For every N-bounded formula $\varphi(x)$ there exists an equivalent MTL formula $\varphi^\dagger$.*

In [18] it was shown that *MTL* is expressively complete for $FO(<,+1)$ on bounded domains of the form $[0,N)$. Theorem 11 is subtly different from that result, which used the definability of the point 0 in a crucial way. In particular, unlike [18], in the present setting we require *MTL* operators whose constraining intervals have fractional endpoints to achieve expressive completeness.

The proof of Theorem 11 has the following structure:

Step 1. By introducing extra predicates, we rewrite each N-bounded formula as a Boolean combination of unit formulas and atoms.

Step 2. Using a normal form of Gabbay, Pnueli, Shelah, and Stavi [10] (see also Hodkinson [20]) we give a translation of unit formulas to *MTL*. This step reveals a connection between the granularity of *MTL* and the quantifier depth of the unit formulas.

Step 3. We complete the translation by removing the new predicate symbols introduced in Step 1.

Step 1. Translation to unit formulas and atoms

We translate an N-bounded formula $\varphi(x)$ into a formula $\overline{\varphi}(x)$ that is a Boolean combination of unit formulas and atoms.

Let $\varphi(x)$ mention monadic predicates $P_1, \ldots, P_m$. For each predicate P_i we introduce an indexed family of new predicates P_i^j, where $-N \leq j < N$. Intuitively, $P_i^j(y)$ stands for $P_i(y+j)$. Formally, given a signal f that interprets the P_i we define a signal $\overline{f}$ that interprets the P_i^j by

$$P_i^j \in \overline{f}(r) \Longleftrightarrow P_i \in f(r+j)$$

for all $r \in \mathbb{R}$.

Next we define a formula $\overline{\varphi}$ such that $f, r \models \varphi$ if and only if $\overline{f}, r \models \overline{\varphi}$. To obtain $\overline{\varphi}$ we recursively replace every instance of a subformula

$$\exists y\,((x - N \leq y < x + N) \wedge \psi)$$

in φ by the formula

$$\exists y\,\big((x \leq y < x{+}1) \wedge (\psi[(y{-}N)/y] \vee \ldots \vee \psi[(y{+}(N{-}1))/y])\big).$$

Having carried out these substitutions, we use simple arithmetic to rewrite every term in φ as $z+k$, where z is a variable and $k \in \mathbb{Z}$ is an integer constant. Every use of monadic predicates in φ now has the form $P_i(z+k)$, for $-N \leq k < N$. Replace every such predicate by $P_i^k(z)$.

After the above operations the resulting formula is a Boolean combination of unit formulas and atomic formulas.

Step 2. Translating unit formulas to MTL

In the next stage of the proof we show how to translate unit formulas into equivalent *MTL* formulas. Critical to this step is the following definition and lemma from [10]. Lemma 12 is the main technical lemma in the expressive completeness proof of *LTL* for FO($<$) in [10].

A *decomposition formula* $\delta(x, y)$ is any formula of the form

$$\begin{aligned} & x < y \wedge \exists z_0 \ldots \exists z_n\,(x = z_0 < \cdots < z_n = y) \\ & \quad \wedge \bigwedge \{\varphi_i(z_i) : 0 \leq i < n\} \\ & \quad \wedge \bigwedge \{\forall u\,((z_{i-1} < u < z_i) \rightarrow \psi_i(u)) : 0 < i \leq n\} \end{aligned}$$

where φ_i and ψ_i are *LTL* formulas regarded as unary predicates.

Lemma 12 ([10]). *Over any domain with a complete linear order, every FO($<$) formula $\psi(x, y)$ in* $\mathrm{Bet}(x, y)$ *is equivalent to a Boolean combination of decomposition formulas $\delta(x, y)$.*

Recall from Proposition 10 that for any unit formula $\theta(x)$ there exists an *FO*($<$) formula $\psi \in \mathrm{Bet}(x, y)$ such that $\psi[(x{+}1)/y]$ is equivalent to $\theta(x)$. Thus, in light of Lemma 12, to translate unit formulas to *MTL* it suffices to consider unit formulas of the form $\delta[(x + 1)/y]$ where $\delta(x, y)$ is a a decomposition formula.

Proposition 13. *Let $\delta(x, y)$ be a decomposition formula and consider the unit formula $\theta(x) = \delta[(x + 1)/y]$. Then there is an MTL formula equivalent to $\theta(x)$.*

Proof. We proceed by induction on the number n of existential quantifiers in $\delta(x, y)$.

Base case: Let $\delta(x, y) = \varphi(x) \wedge \forall u\,(x < u < y \rightarrow \psi(u))$, where φ and ψ are *LTL* formulas. Clearly the *MTL* formula $\varphi \wedge \Box_{(0,1)}\psi$ is equivalent to $\delta[(x + 1)/y]$.

Inductive case: Let $\delta(x, y)$ have the form

$$\begin{aligned} & x < y \wedge \exists z_0 \ldots \exists z_n\,(x = z_0 < \cdots < z_n = y) \\ & \quad \wedge \bigwedge \{\varphi_i(z_i) : 0 \leq i < n\} \\ & \quad \wedge \bigwedge \{\forall u\,((z_{i-1} < u < z_i) \rightarrow \psi_i(u)) : 0 < i \leq n\}\,. \end{aligned}$$

Consider the unit formula $\theta(x) := \delta[(x{+}1)/y]$. The idea is to define *MTL* formulas α_k, β_k, $0 \leq k < 2n$, whose disjunction is equivalent to θ. The definition of these formulas is based on a case analysis of the values of the existentially quantified variables $z_1, \ldots, z_{n-1}$ in δ, similar to the idea of Example 1. To this end, consider the following $2n$ half-open subintervals of $[x, x{+}1)$: $[x, x{+}\frac{1}{2n}), [x{+}\frac{1}{2n}, x{+}\frac{2}{2n}), \ldots, [x{+}\frac{2n-1}{2n}, x{+}1)$. We identify three mutually exclusive cases according to the distribution of the z_i among these intervals:

1) $\{z_1, \ldots, z_{n-1}\} \subseteq [x + \frac{k}{2n}, x + \frac{k+1}{2n})$ for some $k < n$;
2) $\{z_1, \ldots, z_{n-1}\} \subseteq [x + \frac{k}{2n}, x + \frac{k+1}{2n})$ for some k, $n \leq k < 2n$;
3) There exists k, $1 \leq k < 2n$, and l, $1 \leq l < n - 1$, such that $z_l < x + \frac{k}{2n} \leq z_{l+1}$ (i.e., $z_1, \ldots, z_{n-1}$ are not all contained in a single interval).

a) Case 1.: Assume that $k < n$ and consider the following *MTL* formula:

$$\begin{aligned} \alpha_k \;:=\; & \varphi_0 \wedge \psi_1\, \mathbf{U}_{[\frac{k}{2n}, \frac{k+1}{2n})} \\ & \quad (\varphi_1 \wedge (\psi_2\, \mathbf{U}_{(0, \frac{1}{2n})} \\ & \quad\quad (\varphi_2 \wedge (\psi_3\, \mathbf{U}_{(0, \frac{1}{2n})} \\ & \quad\quad\quad \ddots \\ & \quad\quad\quad (\varphi_{n-2} \wedge (\psi_{n-1}\, \mathbf{U}_{(0, \frac{1}{2n})} \\ & \quad\quad\quad\quad (\varphi_{n-1} \wedge \Box_{(0, \frac{1}{2n})}\psi_n)) \cdots) \\ \wedge\; & \Box_{(\frac{k+1}{2n}, 1)}\psi_n\,. \end{aligned}$$

By construction, if α_k holds at a point x then the formulas $\varphi_0, \psi_1, \varphi_1, \ldots, \varphi_{n-1}, \psi_n$ hold in sequence along the interval $[x, x + 1)$. In particular, ψ_n holds on the interval starting at the time that the subformula $\Box_{(0, \frac{1}{2n})}\psi_n$ begins to hold and extending to time $x + 1$ (thanks to the "overlapping" subformula $\Box_{(\frac{k+1}{2n}, 1)}\psi_n$). Thus α_k implies θ. Conversely, if θ holds with the existentially quantified variables $z_1, \ldots, z_{n-1}$ all lying in the interval $(x + \frac{k}{2n}, x + \frac{k+1}{2n})$, then clearly α_k also holds. $\Box$

b) Case 2.: Suppose that $n \leq k < 2n$ and consider the following *MTL* formula:

$$
\begin{aligned}
\alpha_k \;:=\; & \Diamond_{=1}\big[\psi_n\, \mathbf{S}_{(\frac{2n-k-1}{2n},\frac{2n-k}{2n})} \\
& \quad (\varphi_{n-1} \wedge (\psi_{n-1}\, \mathbf{S}_{(0,\frac{1}{2n})} \\
& \quad\quad (\varphi_{n-2} \wedge (\psi_{n-2}\, \mathbf{S}_{(0,\frac{1}{2n})} \\
& \quad\quad\quad \ddots \\
& \quad\quad\quad\quad (\varphi_2 \wedge (\psi_2\, \mathbf{S}_{(0,\frac{1}{2n})} \\
& \quad\quad\quad\quad\quad (\varphi_1 \wedge \boxminus_{(0,\frac{1}{2n})}\psi_1)) \cdots)\big] \\
& \wedge \quad \Box_{(0,\frac{k}{2n})}\psi_1 \quad \wedge \quad \varphi_0\,.
\end{aligned}
$$

The definition of α_k is according to similar principles as in Case 1. If it holds at a point x then the sequence of past operators ensures that the formulas $\psi_n, \varphi_{n-1}, \psi_{n-1}, \ldots, \varphi_1, \psi_1, \varphi_0$ hold in sequence, backward from $x+1$ to x. Thus α_k implies θ. Conversely, if θ holds with the existentially quantified variables $z_1, \ldots, z_{n-1}$ all lying in the interval $[x+\frac{k}{2n}, x+\frac{k+1}{2n})$, $n \leq k < 2n$, then clearly α_k also holds.

c) Case 3.: Suppose that $z_l < x + \frac{k}{2n} \leq z_{l+1}$ for some k, $1 \leq k < 2n$, and l, $1 \leq l < n-1$.

The idea is, for each choice of l, to decompose θ into a property σ_l holding on the interval $[x, x+\frac{k}{2n})$ and a property τ_l holding on the interval $[x+\frac{k}{2n}, x+1)$. We then apply the induction hypothesis to transform σ_l and τ_l to equivalent *MTL* formulas. To this end, define

$$
\begin{aligned}
\sigma_l(x) \;:=\; & \exists z_0 \ldots \exists z_{l+1}(x = z_0 < \cdots < z_{l+1} = x + \tfrac{k}{2n}) \\
& \wedge \bigwedge\{\varphi_i(z_i) : 0 \leq i \leq l\} \\
& \wedge \bigwedge\{\forall u((z_{i-1} < u < z_i) \rightarrow \psi_i(u)) : 1 \leq i \leq l+1\}
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_l(x) \;:=\; & \exists z_l \ldots \exists z_n(x = z_l < \cdots < z_n = x + \tfrac{2n-k}{2n}) \\
& \wedge \bigwedge\{\varphi_i(z_i) : l+1 \leq i < n\} \\
& \wedge \bigwedge\{\forall u((z_{i-1} < u < z_i) \rightarrow \psi_i(u)) : l < i \leq n\}\,.
\end{aligned}
$$

We can turn σ_l into an equivalent *MTL* formula σ_l^* by the following sequence of transformations: scale by $\frac{2n}{k}$ to obtain a unit formula, apply the induction hypothesis to transform the unit formula to an equivalent *MTL* formula, finally scale the resulting *MTL* formula by $\frac{k}{2n}$. We likewise transform τ_l into an equivalent *MTL* formula τ_l^*.

We now define

$$
\beta_k := \bigvee_{1 \leq l < n-1} \Big(\sigma_l^* \wedge \Diamond_{=\frac{k}{2n}}\big((\psi_{l+1} \wedge \tau_l^*) \vee (\varphi_{l+1} \wedge \tau_{l+1}^*)\big)\Big)\,.
$$

From the definition of σ_l it is clear that β_k matches θ on $[x, x+\frac{k}{2n})$. For the remaining interval $[x+\frac{k}{2n}, x+1)$ we distinguish between two cases: if $x+\frac{k}{2n} < z_{l+1}$, then $\Diamond_{=\frac{k}{2n}}(\psi_{l+1} \wedge \tau_l^*)$ agrees with θ; and if $x+\frac{k}{2n} = z_{l+1}$ then $\Diamond_{=\frac{k}{2n}}(\varphi_{l+1} \wedge \tau_{l+1}^*)$ agrees with θ. Thus β_k implies θ. Conversely if θ holds with the existentially variables $z_1, \ldots, z_{n-1}$ satisfying the conditions of Case 3 then one of the disjuncts, and hence β_k, must hold.

Step 3. Completing the translation

After Step 2 we have an *MTL* formula equivalent to the formula $\overline{\varphi}(x)$ obtained in Step 1. It remains only to eliminate the extra predicates introduced in Step 1. To this end, for each predicate P and $j \geq 0$, replace P^j by $\Diamond_{=j}P$, and for $j < 0$ replace P^j by $\diamondminus_{=-j}P$. Finally we obtain an *MTL* formula $\varphi^\dagger$ equivalent to the original N-bounded formula $\varphi(x)$.

Theorem 11. *For every N-bounded $FO(<,+1)$ formula $\varphi(x)$ there exists an equivalent MTL formula $\varphi^\dagger$.*

V. Expressive completeness of MTL

Our next step towards proving the expressive completeness of *MTL* is to show that it is able to express all of $FO(<,+1)$.

Lemma 14. *For every $FO(<,+1)$ formula $\varphi(x)$ there is an equivalent MTL formula $\varphi^\dagger$.*

Proof. The proof is by induction on the quantifier depth n of φ.

Base case, $n = 0$: All atoms are of the form $P_i(x)$, $x = x$, $x < x$, $x+1 = x$. We replace these by P_i, $\mathbf{true}$, $\mathbf{false}$, $\mathbf{false}$ respectively and obtain an *MTL* formula which is clearly equivalent to φ.

Inductive case: Without loss of generality we may assume $\varphi = \exists y.\psi(x,y)$, where $\psi(x,y)$ has quantifier depth $n-1$. We would like to remove x from ψ. To this end we take a disjunction over all possible choices for $\gamma : \{P_1(x), \ldots P_m(x)\} \rightarrow \{\mathbf{true}, \mathbf{false}\}$, and use γ to determine the value of $P_i(x)$ in each disjunct via the formula $\theta_\gamma := \bigwedge_{i=1}^m (P_i(x) \leftrightarrow \gamma(P_i))$. Thus we can equivalently write φ in the form

$$
\bigvee_\gamma \big(\theta_\gamma(x) \wedge \exists y.\psi_\gamma(x,y)\big)\,, \tag{2}
$$

where the propositions $P_i(x)$ do not appear in the ψ_γ.

Now in each ψ_γ, x appears only in atoms of the form $x = z$, $x < z$, $x > z$, $x+1 = z$, $x = z+1$ for some variable z. We next introduce new monadic propositions $P_=$, $P_<$, $P_>$, P_+ and P_-, and replace each of the atoms containing x in ψ_γ with the corresponding proposition. That is, $x = z$ becomes $P_=(z)$, $x < z$ becomes $P_<(z)$ and so on. This yields a formula $\psi'_\gamma(y)$ in which x does not occur, such that $\psi'_\gamma(y)$ has the same truth value as $\psi_\gamma(x,y)$ if the interpretations of the new propositions are consistent with x. Thus for each value of x, (2) has the same truth value as

$$
\bigvee_\gamma (\theta_\gamma(x) \wedge \exists y.\psi'_\gamma(y))\,. \tag{3}
$$

for suitable intepretations of the new propositions.

By the induction hypothesis, for each γ there is an *MTL* formula $\theta_\gamma^\dagger$ equivalent to $\theta_\gamma(x)$, and an *MTL* formula $\psi_\gamma^\dagger$ equivalent to $\psi'_\gamma(y)$. Then our original formula φ has the same truth value at each point x as

$$
\varphi' := \bigvee_\gamma \big(\theta_\gamma^\dagger \wedge (\diamondminus\psi_\gamma^\dagger \vee \psi_\gamma^\dagger \vee \Diamond\psi_\gamma^\dagger)\big)
$$

for suitable interpretations of $\{P_=, P_<, P_>, P_+, P_-\}$.

By Theorem 6, φ' is equivalent to a Boolean combination of formulas

(I) $\Diamond_{=N}\theta$ where $pr(\theta) < N-1$,

(II) $\overleftarrow{\Diamond}_{=N}\theta$ where $fr(\theta) < N-1$, and

(III) θ where all intervals occurring in the temporal operators are bounded.

Now in formulas of type (I) above, we know the intended value of each of the propositional variables $P_=, P_<, P_>, P_+, P_-$: they are all $\mathbf{false}$ except $P_<$, which is $\mathbf{true}$. So we can replace these propositional atoms by $\mathbf{true}$ and $\mathbf{false}$ as appropriate and obtain an equivalent *MTL* formula which does not mention the new variables. Likewise we know the value of each of propositional variables in formulas of type (II): all are $\mathbf{false}$ except $P_>$, which is $\mathbf{true}$; so we can again obtain an equivalent *MTL* formula which does not mention the new variables. It remains to deal with each of the bounded formulas, θ. From Proposition 2, there exists a formula $\theta^*(x)$ in $FO(<, +\mathbb{Q})$, with predicates from $\{P_=, P_<, P_>, P_+, P_-\}$, which is equivalent to θ. It is not difficult to see that as θ is bounded, there is an N such that θ^* is N-bounded. We now unsubstitute each of the introduced propositional variables. That is, replace in $\theta^*(x)$ all occurrences of $P_=(z)$ with $z = x$, all occurrences of $P_<(z)$ with $x < z$ etc. The result is an equivalent formula $\theta^+ \in FO(<, +\mathbb{Q})$, which is still N-bounded as we have not removed any constraints on the variables of θ^*. From Theorem 11, it follows that there exists an *MTL* formula δ that is equivalent to θ^+, i.e., equivalent to θ. □

Finally, recall from Section II-C how a translation from $FO(<, +1)$ to *MTL* can be lifted to a translation $FO(<, +\mathbb{Q})$ to *MTL* via a simple scaling argument. Thus Lemma 14 entails our main result:

Theorem 3. *For every $FO(<, +\mathbb{Q})$ formula $\varphi(x)$ there is an equivalent MTL formula $\varphi^\dagger$.*

VI. Conclusion

In general, the theory of real-time verification lacks the stability and canonicity of the classical theory, and has tended to suffer from a proliferation of competing and mismatching formalisms. Thus it was a pleasant surprise to discover that *MTL* is expressively complete for first-order logic, particularly in view of the extensive literature on the former and the fact that the latter is a natural yardstick against which to measure expressiveness.

We are currently investigating the full extent of this result, including a version for *MTL* with integer constants, equipped with counting modalities.

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