The Bramble-Pasciak^{+} preconditioner for saddle point problems

Martin Stoll & Andy Wathen

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The linear system

The Problem

We want to solve $\mathcal{A}x = b$ where

$$\underbrace{\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}}_{\mathcal{A}} \tag{1}$$

with $A \in \mathbb{R}^{n,n}$ symmetric and positive definite and $C \in \mathbb{R}^{m,m}$ symmetric negative semi-definite. $B \in \mathbb{R}^{m,n}$ is assumed to have full rank.

Saddle point problems

Saddle point problems arise in a variety of applications such as

- Mixed finite element methods for Fluid and Solid mechanics
- Interior point methods in optimisation

See Benzi, Golub, Liesen (2005), Elman, Silvester, Wathen (2005), Brezzi, Fortin (1991), Nocedal, Wright (1999)

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Some Background – Basic relations

We introduce the bilinear form induced by $\mathcal{H}$

$$\langle x, y \rangle_{\mathcal{H}} := x^T \mathcal{H} y$$

which is an inner product iff $\mathcal{H}$ is positive definite. A matrix $\mathcal{A} \in \mathbb{R}^{n \times n}$ is self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ if and only if

$$\langle \mathcal{A}x, y \rangle_{\mathcal{H}} = \langle x, \mathcal{A}y \rangle_{\mathcal{H}} \qquad \text{for all } x, y$$

which can be reformulated to

$$\mathcal{A}^T \mathcal{H} = \mathcal{H} \mathcal{A}.$$

Some Background – Solvers

- CG needs symmetry in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ **plus** positive definiteness in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$
- MINRES needs the symmetry $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ but **no** definiteness in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$

Spectral properties of $\mathcal{A}$ can be enhanced by preconditioning, ie. considering

$$\widehat{\mathcal{A}} = \mathcal{P}^{-1}\mathcal{A}$$

instead of $\mathcal{A}$.

Original matrix $\mathcal{A}$ is symmetric in $\langle \cdot, \cdot \rangle_I \Rightarrow$ MINRES can be used.

What about the symmetry of $\widehat{\mathcal{A}}$?

The Bramble-Pasciak CG

We consider saddle point problem

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}$$

with a block-triangular preconditioner

$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ B & -I \end{bmatrix}$$

which results in

$$\widehat{\mathcal{A}} = \mathcal{P}^{-1}\mathcal{A} = \begin{bmatrix} A_0^{-1}A & A_0^{-1}B^T \\ BA_0^{-1}A - B & BA_0^{-1}B^T + C \end{bmatrix}.$$

The Bramble-Pasciak CG

The preconditioned matrix

$$\widehat{\mathcal{A}} = \mathcal{P}^{-1}\mathcal{A} = \begin{bmatrix} A_0^{-1}A & A_0^{-1}B^T \\ BA_0^{-1}A - B & BA_0^{-1}B^T + C \end{bmatrix}$$

is self-adjoint in the bilinear form defined by

$$\mathcal{H} = \begin{bmatrix} A - A_0 & 0 \\ 0 & I \end{bmatrix}.$$

Under certain conditions for A_0 $\mathcal{H}$ defines an inner product and $\widehat{\mathcal{A}}$ is also positive definite in this inner product, e.g. $A_0 = .5A$.

The condition for A_0 usually involves the solution of an eigenvalue problem which can be expensive.

The Bramble-Pasciak^{+} CG

We always want an inner product for symmetric and positive definite A_0

$$\mathcal{H}^{+} = \begin{bmatrix} A + A_0 & \\ & I \end{bmatrix}.$$

Therefore, new preconditioner $\mathcal{P}^{+}$

$$\mathcal{P}^{+} = \begin{bmatrix} A_0 & 0 \\ -B & I \end{bmatrix}$$

is required. The preconditioned matrix

$$\widehat{\mathcal{A}} = \left(\mathcal{P}^{+}\right)^{-1}\mathcal{A} = \begin{bmatrix} A_0^{-1}A & A_0^{-1}B^T \\ BA_0^{-1}A + B & BA_0^{-1}B^T - C \end{bmatrix}$$

is self-adjoint in this inner product.

Definiteness in $\mathcal{H}^+$

If we split

$$\widehat{\mathcal{A}}^T\mathcal{H}^+ = \begin{bmatrix} AA_0^{-1}A + A & AA_0^{-1}B^T + B^T \\ BA_0^{-1}A + B & BA_0^{-1}B^T - C \end{bmatrix}$$

as

$$\begin{bmatrix} I & \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} AA_0^{-1}A + A & \\ & -BA_0^{-1}B^T - C \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ & I \end{bmatrix}$$

we see that since this is a congruence transformation the matrix is always indefinite. This means:

- No reliable CG can be applied
- In practice CG quite often works fine
- Augmented methods might be used.

An $\mathcal{H}^+$-inner product implementation of MINRES

Use that $\widehat{\mathcal{A}}$ symmetric in $\mathcal{H}$-inner product and therefore implement a version of **Lanczos process** with $\mathcal{H}$-inner product which gives

$$\widehat{\mathcal{A}} V_k = V_k T_k + \beta_k v_{k+1} e_k^T$$

with

$$T_k = \begin{bmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{k-1} \\ & & \beta_{k-1} & \alpha_k \end{bmatrix} \text{ and } V_k = [v_1, v_2, \ldots, v_k]$$

as well as $V_k^T \mathcal{H}^+ V_k = I$.

An $\mathcal{H}^+$-inner product implementation of MINRES

The following condition holds for the residual

$$\begin{aligned}
\|r_k\|_{\mathcal{H}^+} &= \|b - Ax_k\|_{\mathcal{H}} &&= \|b - Ax_0 - AV_k y_k\|_{\mathcal{H}^+} \\
&= \|r_0 - V_{k+1} T_{k+1} y_k\|_{\mathcal{H}^+} &&= \left\| V_{k+1}(V_{k+1}^T \mathcal{H}^+ r_0 - T_{k+1} y_k) \right\|_{\mathcal{H}^+} \\
&= \left\| V_{k+1}^T \mathcal{H}^+ r_0 - T_{k+1} y_k \right\|_{\mathcal{H}^+} &&= \| \|r_0\| e_1 - T_{k+1} y_k \|_{\mathcal{H}^+} .
\end{aligned}$$

Minimizing $\| \|r_0\| e_1 - T_{k+1} y_k \|_{\mathcal{H}^+}$ can be done by the standard updated-QR factorization technique. Implementation details can be found in Greenbaum (1997).

The simplified Lanczos method

The non-symmetric Lanczos process generates two sequences of vectors where the following condition holds

$$v_j = \phi_j(\widehat{\mathcal{A}})v_1 \text{ and } w_j = \gamma_j \phi_j(\widehat{\mathcal{A}}^T)w_1$$

where ϕ is a polynomial of degree $j-1$ the so-called *Lanczos polynomial*. Setting $w_1 = \mathcal{H}v_1$ and using the self-adjointness of $\widehat{\mathcal{A}}$ in $\mathcal{H}^+$, ie. $\widehat{\mathcal{A}}^T\mathcal{H}^+ = \mathcal{H}^+\widehat{\mathcal{A}}$, gives

$$w_j = \gamma_j\phi_j(\widehat{\mathcal{A}}^T)w_1 = \gamma_j\phi_j(\widehat{\mathcal{A}}^T)\mathcal{H}^+v_1 = \gamma_j\mathcal{H}^+\phi_j(\widehat{\mathcal{A}})v_1 = \gamma_j\mathcal{H}^+v_j.$$

Therefore the non-symmetric Lanczos process can be simplified, ie. multiplications with $\widehat{\mathcal{A}}^T$ can be exchanged for multiplication by $\mathcal{H}^+$.

The ideal transpose-free QMR method (ITFQMR)

Based on the QMR method Freund (1994) a transpose-free QMR method with an implementation derived from the BICG procedure. Here, we use matrix formulation of the non-symmetric Lanczos process

$$\widehat{\mathcal{A}} V_k = V_{k+1} H_k$$

and

$$r_k = V_{k+1}(\|r_0\| \, e_1 - H_k y_k).$$

Ignoring the term V_{k+1} gives QMR method. Using simplification of the Lanczos process gives ITFQMR .

Eigenvalue analysis for $A_0 = A$

To get some insight into the convergence behaviour we the eigenvalues of

$$\widehat{\mathcal{A}} = \left(\mathcal{P}^{+}\right)^{-1} \mathcal{A} = \begin{bmatrix} I & A^{-1}B^T \\ 2B & BA^{-1}B^T \end{bmatrix}.$$

For the eigenpair $(\lambda, \begin{bmatrix} x \\ y \end{bmatrix})$ of $\widehat{\mathcal{A}}$ we know that

$$\begin{bmatrix} I & A^{-1}B^T \\ 2B & BA^{-1}B^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + A^{-1}B^T y \\ 2Bx + BA^{-1}B^T y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

For $\lambda = 1$ we get

$$Ax + B^T y = Ax$$

which gives $B^T y = 0$ and $y = 0$ iff $Bx = 0$.

Since $\dim(\ker(B)) = n - m$ multiplicity of $\lambda = 1$ is $n - m$.

Eigenvalue analysis for $A_0 = A$

For $\lambda \neq 1$, we get that $x = \frac{1}{\lambda-1} A^{-1} B^T y$ which gives

$$BA^{-1}B^T y = \frac{\lambda(\lambda-1)}{\lambda+1} y.$$

For an eigenvalue σ of $BA^{-1}B^T$ we get

$$\sigma = \frac{\lambda(\lambda-1)}{\lambda+1}.$$

Eigenvalues of $\widehat{\mathcal{A}}$ become

$$\lambda_{1,2} = \frac{1+\sigma}{2} \pm \sqrt{\frac{(1+\sigma)^2}{4} + \sigma}.$$

Since $\sigma > 0$ we have m negative eigenvalues.

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Numerical Experiments – Stokes problem

We are going to solve saddle point systems coming from the finite element method for the Stokes problem

$$\begin{aligned} -\nabla^2 u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \end{aligned}$$

The linear system governing the finite element method for the Stokes problem is a saddle point problem

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}$$

where $C \neq 0$ for stabilized systems. In our examples $C = 0$.

All examples come from the IFISS package.

Block diagonal preconditioning

Silvester and Wathen (1993,1994) use preconditioner

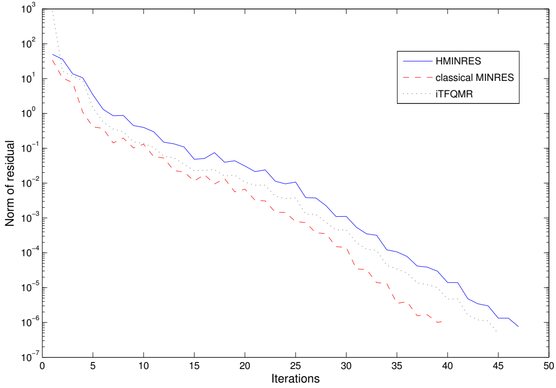
$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ 0 & S_0 \end{bmatrix}$$

which is symmetric positive definite.

$\Longrightarrow$ Preconditioned MINRES can be applied.

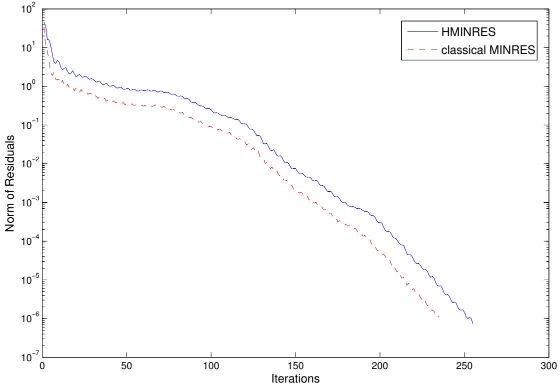
Example 1 – Stokes problem Channel domain

Results for $\mathcal{H}$-MINRES and classical Preconditioned MINRES with problem dimension 9539. Preconditioner $A_0 = A$ and S_0 being the Gramian (Mass matrix).



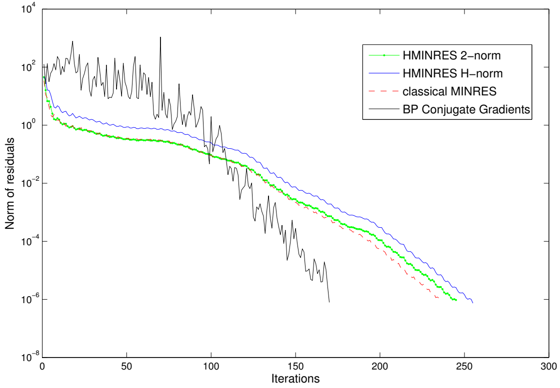
Example 2 – Stokes problem Channel domain

Results for $\mathcal{H}$-MINRES and classical Preconditioned MINRES with problem dimension 9539. Preconditioner A_0 is Incomplete Cholesky of A and S_0 being the Gramian (Mass matrix).



Example 3 – Stokes problem Channel domain

Again $\mathcal{H}$-MINRES and classical Preconditioned MINRES for problem dimension 9539. Preconditioner A_0 is Incomplete Cholesky of A and S_0 being the Gramian (Mass matrix). Additionally, $\mathcal{H}$-MINRES residual in the 2-norm and the classical Bramble-Pasciak CG.



Conclusions

- We presented a alternative approach
- Method could be used with augmented techniques to become competitive
- Presented algorithm could be used for combination preconditioning

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Thank you for your attention!

Difficult questions can be discussed in 20 minutes in the pub!