Open Problems in Topology
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Chapter 14

The Point-Countable Base Problem

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Introduction

It is the purpose of this article to show how the problem arose, to place it in the context of the most natural of structuring mechanisms, to indicate some powerful metrisation theorems which are available on the addition of a single condition, and to review some recent work which gives some partial answers and suggests lines of enquiry which should be further pursued. Despite the attention shown to the problem over the last five years, we would not wish to claim that the results presented here go more than a small distance towards a solution. We believe that some new ideas are required and would encourage our colleagues to provide them.

Since the review paper Collins [19∞] was given at the Baku Topology Symposium in 1987, new insights have encouraged us to vary the presentation and to include hitherto unpublished material.

All spaces will have the T_1 separation axiom and $\mathbb{N}$ will denote the set of natural numbers.

1. Origins

The structuring mechanism which spawned the problem arose in the search for a simple, yet natural, condition which would produce a countable basis in a separable space. The model was, naturally enough, a standard elementary proof for a metric space with countable dense subset A.

If, in this context, x belongs to open U and a is an element of $A \cap S_\epsilon(x)$, where the open ball $S_{3\epsilon}(x) \subseteq U$, then $x \in S_r(y) \subseteq U$ for any $y \in S_\epsilon(x)$ and any rational r such that $\epsilon < r < 2\epsilon$. The picture is given in Figure 1.

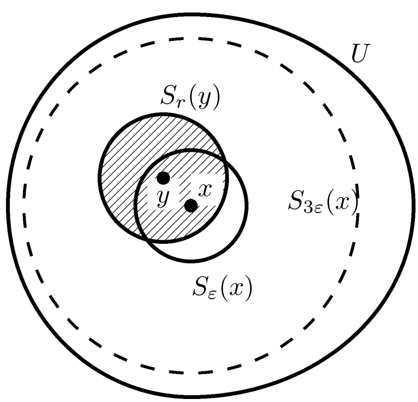

Figure 1: The proof that a separable metric space is second countable

Then $\{S_r(a) : a \in A \wedge r \in \mathbb{Q}^+\}$, where $\mathbb{Q}^+$ is the set of positive rational numbers, is a countable basis.

An essential feature of the above proof, which must be borne clearly in mind in constructing generalisations, is the need, not only for $S_r(y)$ to be small enough to be within U, but also large enough to 'pick up' x. We shall return to this point later.

The first generalisation to be investigated (COLLINS and ROSCOE [1984]), which allows an immediate parody of the above proof, runs as follows. For each x in a space X, let

$$\mathbf{W}(x) = \{W(n, x) : n \in \mathbb{N}\}$$

be a countable family of subsets of X, each containing x. $\mathcal{W} = \{\mathbf{W}(x) : x \in X\}$ is said *to satisfy* (A)[1] if it satisfies

(A) if $x \in U$ and U is open, then there exist a positive integer $s = s(x, U)$ and an open set $V = V(x, U)$ containing x such that $x \in W(s, y) \subseteq U$ whenever $y \in V$.

The picture is the same as Figure 1, once one sets $V = S_\epsilon(x)$ and $W(s, y) = S_r(y)$. Second countability follows from separability when each $W(n, x)$ is open, or indeed is a neighbourhood of x ($\mathcal{W}$ satisfies '*open* (A)', or '*neighbourhood* (A)'). In fact, one can go further if $\mathcal{W}$ satisfies '*neighbourhood decreasing* (A)', that is, if $W(n + 1, x) \subseteq W(n, x)$ holds for each x and n in addition to the $W(n, x)$ being neighbourhoods of x.

1. THEOREM (COLLINS and ROSCOE [1984]). *In order that X be metrisable it is necessary and sufficient that X has $\mathcal{W}$ satisfying neighbourhood decreasing (A).* □

In COLLINS and ROSCOE [1984], it is shown that *eventually decreasing neighbourhood* (A) will not suffice for metrisability. Theorem 1 is proved in one page, relying on no other results, and a number of classical metrisation theorems, such as those of Nagata-Smirnov and of Moore-Arkhangel′skiĭ-Stone, are quickly deduced.

We should like to stress how natural condition (A) is by restating Theorem 1 to provide a set-theoretic model for metric spaces.

2. THEOREM (COLLINS and ROSCOE [1984]). *Suppose that for each x in a set X there is a decreasing sequence $\mathbf{W}(x) = \{W(n, x) : n \in \mathbb{N}\}$ of subsets of X, each containing x, such that*

(1) *given x and y, $x \neq y$, there exists a positive integer m with $y \notin W(m, x)$,*
(2) *given x in X and a positive integer n, there exist positive integers $r = r(n, x)$ and $s = s(n, x)$ such that $y \in W(r, x)$ implies that $x \in W(s, y) \subseteq W(n, x)$.*

[1] The names (A), (F) and (G) used for conditions in this paper are taken from COLLINS and ROSCOE [1984].

Then there is a metric for X such that, for each x in X, $\mathbf{W}(x)$ is a basis for the neighbourhood system of x in the metric topology. □

Condition (2) is just (A) re-stated in terms of the $W(n,x)$'s and obviously strengthens the usual neighbourhood axioms for a topological space. Condition (1) ensures appropriate separation.

In the proof of second countability of a separable metric space given in the last section, the same r sufficed for each y in $S_\epsilon(x)$. This is reflected in condition (A) where $s = s(x,U)$ does not depend on $y \in V$. It is natural to ask what happens if s also depends on y. We say (Collins and Roscoe [1984]) that $\mathcal{W}$ satisfies (G) if it satisfies

(G) if $x \in U$ and U is open, then there exists an open $V = V(x,U)$ containing x such that $x \in W(s,y) \subseteq U$ for some $s = s(x,y,U) \in \mathbb{N}$ whenever $y \in V$.

The picture is much the same as before (Figure 2).

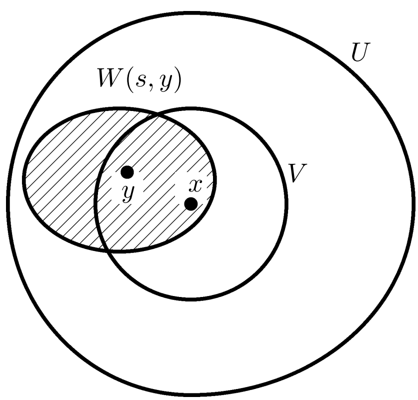

Figure 2: Condition (G)

Again, with analogous definitions, if $\mathcal{W}$ satisfies open (G), then separability implies second countability (Lemma 3 of Collins and Roscoe [1984]). How finely conditions (A) and (G) are balanced on the point of what is possible in metrisation theory may be judged from the following results.

3. Theorem (Collins and Roscoe [1984]). *There is a space X (the 'bow-tie' space of* L. F. McAuley *[1955]) which has $\mathcal{W}$ satisfying* neighbourhood *decreasing (G) but which is not metrisable.* □

4. Theorem (Collins, Reed, Roscoe and Rudin [1985]). *In order that X be metrisable it is necessary and sufficient that X has $\mathcal{W}$ satisfying* open *decreasing (G).* □

It should be noted that Theorems 1 and 4 are *not* inter-dependent.

The value of considering generalisations of open decreasing (G) is exemplified by the next result.

5. THEOREM (BALOGH [1985], COLLINS, REED, ROSCOE and RUDIN [1985]). *A space is stratifiable(and hence a Nagata space if first countable) if and only if it has $\mathcal{W}$ satisfying decreasing (G) and has countable pseudocharacter.* □

Comparison of Theorems 4 and 5 prompts the following open question.

? **377. Problem 1.** (COLLINS and ROSCOE [1984]) *Which spaces are characterised as having $\mathcal{W}$ satisfying neighbourhood decreasing (G)?*

It is known (see COLLINS, REED, ROSCOE and RUDIN [1985]) that there are stratifiable (indeed, Nagata) spaces which do not have neighbourhood decreasing (G).

It is another generalisation of open decreasing (G) which provides the title of this article and the next section.

2. The point-countable base problem

Whilst investigating the structuring mechanism described in the last section, the authors made a number of conjectures, many of which have now been answered by theorems or counterexamples. Of those that remain, the point-countable base problem is the most intriguing, both because of the number of partial solutions that have been discovered and because of the effort that has been expended on it.

A basis for a space X is *point-countable* if every element of X is contained in at most a countable number of elements of the basis. It may quickly be deduced that, if X has such a basis $\mathcal{B}$, then X has $\mathcal{W}$ satisfying open (G) (by defining $\mathbf{W}(x) = \{B \in \mathcal{B} : x \in B\}$). The converse remains an open question.

? **378. Problem 2.** (The Point-Countable Base Problem (COLLINS, REED, ROSCOE and RUDIN [1985])) *If X has $\mathcal{W}$ satisfying open (G), need X have a point-countable basis?*

Note that it is not possible usefully to reduce 'open non-decreasing $\mathbf{W}(x)$' to 'open decreasing $\mathbf{V}(x)$' (which one might hope to do, so as to apply results of the last section) by the formula

$$V(n,x) = \bigcap_{i=1}^{n} W(i,x)$$

since, even if $\mathcal{W}$ satisfies (G), $\mathcal{V}$ may not, as the $V(s,x)$ may not 'pick up' x. (See our comment in Section 2.)

That a space be meta-Lindelöf (a condition clearly implied by the existence of a point-countable basis) does not require *open* (G), as the following result shows. This result is not only useful, but exemplifies a common style of proof found when (G) and related conditions are used.

6. LEMMA (MOODY, REED, ROSCOE and COLLINS [19∞]). *If the space X has $\mathcal{W}$ satisfying (G), then X is hereditarily meta-Lindelöf (i.e., each open cover has a point-countable open refinement).*

PROOF. Suppose $\mathcal{G} = \{U_\alpha : \alpha \in \gamma\}$ is an open cover of X, enumerated using some ordinal γ. For each $\alpha \in \gamma$, define the set

$$S_\alpha = \bigcup \{V(x, U_\alpha) : x \in U_\alpha \setminus \bigcup_{\beta \in \alpha} U_\beta\}$$

where $V(x, U)$ is given by (G). By construction, $\mathcal{S} = \{S_\alpha : \alpha \in \gamma\}$ is an open cover of X. We claim that $\mathcal{S}$ is point-countable. If y belongs to S_α, then there is $x_\alpha \in U_\alpha \setminus \bigcup_{\kappa \in \alpha} U_\kappa$ such that $y \in V(x_\alpha, U_\alpha)$, and hence there is $W_\alpha \in \mathbf{W}(y)$ such that

$$x_\alpha \in W_\alpha \subseteq U_\alpha.$$

There can only be countably many such α, for otherwise there would be two ordinals α, β (with $\alpha \in \beta$, say) such that $W_\alpha = W_\beta$. But then $x_\beta \in W_\beta \subseteq U_\alpha$, giving a contradiction. So, $\mathcal{S}$ is point-countable as claimed.

The fact that X is *hereditarily* meta-Lindelöf follows simply from the observation that any subspace Y trivially has $\mathcal{W}'$ satisfying (G). □

A large number of partial answers have been found to the point-countable base problem. Many of them turn out to be consequences of a simple Lemma (which was actually discovered after many of its consequences). To state it, we need the concept of a pointed open cover. A *pointed open cover* for a space X with topology $\mathcal{T}$ is a subset $\mathcal{P}$ of $X \times \mathcal{T}$ such that $\{U : \exists x \in X \ (x, U) \in \mathcal{P}\}$ is a cover for X. $\mathcal{P}$ is said to be *point-countable* if $\{(x, U) \in \mathcal{P} : y \in U\}$ is countable for all y, and *dense* if

$$y \in \overline{\{x : (x, U) \in \mathcal{P} \wedge y \in U\}}$$

for all y. Note that we have not insisted that $(x, U) \in \mathcal{P}$ implies $x \in U$.[2]

7. LEMMA (MOODY, REED, ROSCOE and COLLINS [19∞]). *If the space X has $\mathcal{W}$ satisfying open* (G), *then X has a point-countable base if and only if X has a dense, point-countable, pointed open cover.*

[2]Indeed, it is often more natural to generate these pointed open covers in such a way that $x \notin U$ for some (x, U). However it is easy to show that if a space has $\mathcal{W}$ satisfying open (G) and a dense, point-countable pointed open cover $\mathcal{P}$, then there is another one, $\mathcal{P}'$, where all (x, U) have $x \in U$.

PROOF. First suppose that X has a point-countable base $\mathcal{B}$. For each non-empty element U of $\mathcal{B}$, pick an $x_U \in U$, and define $\mathcal{P} = \{(x_U, U) : U \in \mathcal{B} \setminus \{\emptyset\}\}$. It can be easily verified that $\mathcal{P}$ is a dense, point-countable, pointed open cover. Conversely, define

$$\mathcal{B} = \{U \cap W : \exists x\ (x, U) \in \mathcal{P} \wedge W \in \mathbf{W}(x)\}.$$

Clearly, $\mathcal{B}$ is a point-countable collection of open sets. To see that $\mathcal{B}$ is a base, consider any $x \in X$ and any open set O containing x. Since $\mathcal{P}$ is dense, there must exist a $(y, U) \in \mathcal{P}$ such that $x \in U$ and $y \in V(x, O)$. Pick a $W \in \mathbf{W}(y)$ for which $x \in W \subseteq O$. Then $x \in U \cap W \subseteq O$ and $U \cap W \in \mathcal{B}$, so that $\mathcal{B}$ is a base as required. □

Notice how this proof follows the one that a separable space with $\mathcal{W}$ satisfying open (G) is second-countable. In fact, possessing a dense, point-countable pointed open cover is a natural generalisation of separability: there are countably many points available to have each point as a limit, only now, *which* points are available varies from place to place. (Each point is available in an open set.) Every separable space X with countable dense subset D has such a pointed open cover $\{(x, X) : x \in D\}$.

Given these two lemmas, it is possible to establish a number of results rather easily. We now give sketch proofs of the three such theorems, in each case showing how the dense, point-countable, pointed open covers are constructed.

8. THEOREM (MOODY, REED, ROSCOE and COLLINS [19∞]). *If the space X has $\mathcal{W}$ satisfying open (G) and has density $\leq \aleph_1$, then it has a point-countable basis.*

PROOF. If X is separable then we already know it is second countable, so we may assume it has a dense subset $D = \{x_\alpha : \alpha \in \omega_1\}$. It is then easy to show that the pointed open cover

$$\mathcal{P} = \{(x_\alpha, X \setminus \overline{\{x_\beta : \beta \in \alpha\}}) : \alpha \in \omega_1\}\,.$$

is point-countable and dense. □

Considerable work has been done to show that, under the assumption that large cardinals exist, if certain topological properties are true of all subsets with cardinal $\leq \aleph_1$ of a given space, then they are true of the space (see F. D. Tall's questions on reflection, this volume). In this vein, it has been conjectured that, if every $\leq \aleph_1$ subset of a first countable regular space X has a point-countable base, then X has a point-countable base. If this could be proved, then, of course, Theorem 8 would answer the point-countable base problem in the affirmative for regular spaces, on the assumption of large cardinals.

9. THEOREM (MOODY, REED, ROSCOE and COLLINS [19∞]). *If X is semi-stratifiable and has $\mathcal{W}$ satisfying open (G), then it has a point-countable basis.*

PROOF. We shall use the characterisation of semi-stratifiable given by G. D. CREEDE in [1970], which states that a space X with topology $\mathcal{T}$ is semi-stratifiable if and only if there exists a function g from $\mathbb{N} \times X$ to $\mathcal{T}$ such that

(i) $\{x\} = \bigcap\{g(n,x) : n \in \mathbb{N}\}$, and

(ii) if $y \in X$ and (x_n) is a sequence of points in X such that $y \in g(n, x_n)$ for all n, then (x_n) converges to y.

Let g be such a function. By Lemma 6 we may let $\mathcal{U}_n$ be a point-countable open refinement of $\{g(n,x) : x \in X\}$. For each $U \in \mathcal{U}_n$, choose x_U such that $U \subseteq g(n, x_U)$. Define $\mathcal{P} = \big\{(x_U, U) : U \in \mathcal{U}_n \wedge n \in \mathbb{N}\big\}$. By construction, $\mathcal{P}$ is a point-countable, pointed open cover. It also follows easily from (ii) above that $\mathcal{P}$ is dense. □

10. THEOREM (MOODY, REED, ROSCOE and COLLINS [19∞]). *If X has $\mathcal{W}$ satisfying open (G) and is the locally countable sum of spaces which have point-countable bases (i.e., $X = \bigcup\{X_\lambda : \lambda \in \Lambda\}$, where each subspace X_λ has a point-countable base and where there is a neighbourhood $N(x)$ of each x which meets only countably many X_λ), then X has a point-countable base.*

PROOF. In fact, we will show, without using the openness of $\mathcal{W}$, that if each X_λ has a dense, point-countable, pointed open cover $\mathcal{P}_\lambda$ then so does X. If U is a set open in one of the X_λ, let U^X denote some set chosen to be open in X and such that $U^X \cap X_\lambda = U$. The dense, point-countable, pointed open cover of X is then given by

$$\{(x, \bigcup\{V(y, N(y) \cap U^X) : y \in U\}) : \lambda \in \Lambda,\ (x, U) \in P_\lambda\}\,.$$

□

G. Gruenhage has solved the point-countable base problem for GO-spaces.

11. THEOREM (GRUENHAGE [19∞]). *Every GO-space with $\mathcal{W}$ satisfying open (G) has a point-countable base.* □

And P. J. NYIKOS [1986] and one of us (AWR) have established the following result (which does not use *open* (G)).

12. THEOREM. *If X is a first countable, non-archimedean space which has $\mathcal{W}$ satisfying* (G)*, then X has a point-countable base.* □

We have demonstrated that the answer to the point-countable base problem is 'yes' in a number of cases where there is extra structure for is available. Our next result demonstrates that any counter-example must be particularly unpleasant. It is a consequence of Lemma 6 and Theorem 10.

13. THEOREM (MOODY, REED, ROSCOE and COLLINS [19∞]). *If the space X is a counterexample (i.e., has $\mathcal{W}$ satisfying open (G) but no point-countable base) then there is a non-empty subspace X' of X, every non-empty open subset of which is also a counterexample.* □

Thus, if there is a counter-example, then it has a subspace which is a counterexample and none of whose open sets (viewed as subspaces)

(1) have density $\leq \aleph_1$,
(2) are semi-stratifiable
(3) are GO-spaces
(4) are non-archimedean
(5) or are locally countable sums of such spaces,

which excludes many common ways of constructing counter-examples.

It is worth remarking that the property of having $\mathcal{W}$ satisfying open (G) shares a number of other properties with that of having a point-countable basis. For example, both are countably productive and both are hereditary. In [19∞] GRUENHAGE showed (i) that a submetacompact β-space with $\mathcal{W}$ satisfying open (G) is developable (and hence has a point-countable base; this result actually generalises Theorem 7 above), and (ii) that a countably compact space with $\mathcal{W}$ satisfying open (G) is metric.

The next result gives a little more insight into the problem.

14. THEOREM (MOODY, REED, ROSCOE and COLLINS [19∞]). *If X is a space with $\mathcal{W}$ satisfying (G), then X has a point-countable pointed open cover $\mathcal{P}$ such that*

(i) $(x, U) \in \mathcal{P} \Rightarrow x \in U$, *and*
(ii) $\{x : (x, U) \in \mathcal{P}\}$ *is dense in X.*

Thus, any space with $\mathcal{W}$ satisfying open (G) has a dense subspace with a point-countable base (the set in (ii)). □

The same techniques used in the proof of Theorem 8 demonstrate that the point-countable base $\mathcal{B}$ for the dense subset D can be lifted to a point-countable set $\mathcal{B}'$ of subsets of X which are a basis for its topology at all points in D. However, there is no obvious way of making them into a basis for the whole of X.

The condition (G) may be strengthened to (G′) as follows:

(G′) if $x \in U$ and U is open, then there exists an open $V = V(x, U) \subseteq U$ containing x such that $x \in W(s, y) \subseteq V$ for some $s = s(x, y, U) \in \mathbb{N}$ whenever $y \in V$.

The picture here has changed in that now $W(s, y) \subseteq V$ rather than U. It is easy to see that the $\mathcal{W}$ constructed earlier, for spaces with point-countable bases, satisfies (G′). In fact, it is possible to prove the following result.

15. Theorem (Moody, Reed, Roscoe and Collins [19∞]). *X has $\mathcal{W}$ satisfying (G′) if, and only if, it has a dense, point-countable, pointed open cover.* □

Thus, if it also has $\mathcal{W}'$ (not necessarily equal to $\mathcal{W}$) satisfying open (G), then it has a point-countable base. Unfortunately, the techniques used in the proof of Theorem 15 do not seem to generalise to the weaker condition (G). However, that result does give rise to the following problem (an affirmative answer to which would solve the point-countable base problem):

Problem 3. *Does every space with $\mathcal{W}$ satisfying (G) have a dense, point-countable, pointed open cover?* **379. ?**

Since almost all of our positive results about the point-countable base problem are consequences of Lemma 7 after constructing a dense, point-countable, pointed open cover, there is reason to believe that this problem may be best attacked via Problem 3. Direct analogues of all of Theorems 8–13 hold for Problem 3. (We have shown that Gruenhage's proof of Theorem 11 can be adapted to show that any GO-space with $\mathcal{W}$ satisfying (G) has a dense, point-countable, pointed open cover.) The single caveat is in the case of Theorem 8, where the proof relies on first countability (implied by open (G) but not (G)). However any first countable space with density $\leq \aleph_1$ has a dense, point-countable, pointed open cover, as does *any* space with *cardinality* $\leq \aleph_1$.

We have already remarked that the property of having a point-countable, pointed open cover is rather like separability. And, like separability, it is not in general hereditary: a counterexample can be constructed by assuming $\mathfrak{c} = \aleph_2$ and using the Sierpinski construction of a topology on the real line where a neighbourhood of a point x consists of all points within $\epsilon > 0$ which are not less than x in an ω_2 well-order. This space does not have such a pointed open cover, but by adding the rational points of the plane in a suitable way the space becomes separable. However, if Problem 3 were to have a positive answer, then any space with $\mathcal{W}$ satisfying (G) would have this property hereditarily. A simple modification to the proof of Theorem 1 of Collins, Reed, Roscoe and Rudin [1985] shows that the property is hereditary if the space has $\mathcal{W}$ satisfying (G) (i.e., if X has such a $\mathcal{W}$ and a dense, point-countable pointed open cover, then so does every subspace). This is a small piece of positive evidence towards the conjecture.

3. Postscript: a general structuring mechanism

We have already seen that conditions (A) and (G) give a powerful structuring mechanism for topological spaces when we impose various conditions on the $\mathbf{W}(x)$. This mechanism can be further extended when we relax the condition that each $\mathbf{W}(x)$ is countable. If $\mathbf{W}(x)$ is, for each x in a space X, a set of subsets of X containing x, we say $\mathcal{W} = \{\mathbf{W}(x) : x \in X\}$ *satisfies* (F) when it satisfies

(F) if $x \in U$ and U is open, then there exists an open $V = V(x, U)$ containing x such that $x \in W \subseteq U$ for some $W \in \mathbf{W}(y)$ whenever $y \in V$.

The picture here is the same as in Figure 2. Every topological space clearly has $\mathcal{W}$ satisfying open (F), and metrisability is given when X has $\mathcal{W}$ satisfying open decreasing (G), of which open (F) is a generalisation. Therefore, it should not surprise the reader that restrictions on $\mathcal{W}$ satisfying (F) relate naturally to certain well-known generalised metric spaces. $\mathcal{W}$ satisfies *chain* (F) if each $\mathbf{W}(x)$ is a chain with respect to inclusion.

16. THEOREM (MOODY, REED, ROSCOE and COLLINS [19∞]). *If X has $\mathcal{W}$ satisfying chain (F) and each $\mathbf{W}(x) = \mathbf{W}_1(x) \cup \mathbf{W}_2(x)$, where $\mathbf{W}_1(x)$ consists of neighbourhoods of x and $\mathbf{W}_2(x)$ is well-ordered by $\supseteq$, then X is paracompact.* □

Not all paracompact spaces have such $\mathcal{W}$.

? **380. Problem 4.** *Characterise the spaces which have $\mathcal{W}$ satisfying chain (F), where the $\mathbf{W}(x)$ are*
(i) *all neighbourhoods,*
(ii) *well-ordered by $\supseteq$, or*
(iii) *as in the statement of Theorem 14.*

17. THEOREM (COLLINS and ROSCOE [1984]). *If X has $\mathcal{W}$ satisfying chain (F), then X is monotonically normal* (in the sense of R. W. HEATH, D. J. LUTZER and P. L. ZENOR [1973]]). □

It is possible to characterise the spaces that have chain (F): we define a space X to be *acyclically monotonically normal* if there is, for each x and open U such that $x \in U$, an open set $V(x, U)$ such that
(i) $x \in U_1 \subseteq U_2 \Rightarrow V(x, U_1) \subseteq V(x, U_2)$
(ii) $x \neq y \Rightarrow V(x, X \setminus \{y\}) \cap V(y, X \setminus \{x\}) = \emptyset$
(iii) If $n \geq 2$, $x_0, \ldots x_{n-1}$ are all distinct and $x_n = x_0$, then

$$\bigcap_{r=0}^{n-1} V(x_r, X \setminus \{x_{r+1}\}) = \emptyset\,.$$

Conditions (i) and (ii) are just the usual conditions for monotone normality, and (iii) is an extension of (ii) (notice that (ii) is just condition (iii) when $n = 2$). The effect of (iii) is to ban certain types of cycles, hence the name.

18. THEOREM. *A space X is acyclically monotonically normal if and only if it has $\mathcal{W}$ satisfying chain (F).*

However, we do not know if there are any monotonically normal spaces which do not have chain (F). This leads to our final problem.

Problem 5. *Is every monotonically normal space acyclically monotonically normal?* **381. ?**

(The definition of acyclic monotone normality, Theorem 15 and Problem 5 all first appeared in Roscoe [1984] and were further discussed in Moody, Reed, Roscoe and Collins [19∞].)

GO spaces and stratifiable spaces, the two best known classes of monotonically normal spaces, are both acyclically monotonically normal (Moody, Reed, Roscoe and Collins [19∞]), as are elastic spaces (Moody [1989]). It is known that no counter-example can be scattered. In his thesis [1989], P. J. Moody did a considerable amount of work on this problem and proved that acyclic monotone normality has many of the same properties enjoyed by monotone normality. He showed that there is a close relationship between this problem and the problem of E. K. van Douwen [1975] of whether every monotonically normal space is K_0, since he observed that every acyclically monotonically normal space is K_0. He also showed that a counter-example exists to van Douwen's problem, and hence to ours, if there is what he terms a λ-Gower space (see J. van Mill [1982]) which is monotonically normal, for any infinite cardinal λ. However, it is not known if such a space exists.

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