(A shorter version has been published in: 238–264, Papers on General Topology and Applications — Eleventh Summer Conference at the University of Southern Maine, Annals of the New York Academy of Sciences, Volume 806, 1996.)

On transition systems and non-well-founded sets

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Abstract

(Labelled) transition systems are relatively common in theoretical computer science, chiefly as vehicles for operational semantics. The first part of this paper constructs a hierarchy of canonical transition systems and associated maps, aiming to give a strongly extensional theory of transition systems, where any two points with equivalent behaviours are identified. The cornerstone of the development is a notion of convergence in arbitrary transition systems, generalising the idea of finite (n-step) approximations to a given point. In particular, our canonical transition systems are also uniform spaces.

The resulting hierarchy has very rich combinatorial (and topological) structure, and a lot of the first part of the paper is devoted to its study. We also discuss fixed points in this framework.

This kind of study of transition systems is very closely connected to non-well-founded set theory. In the second part of the paper, we show how to obtain a model of set theory with Aczel's Anti-Foundation Axiom (AFA) from canonical transition systems constructed earlier. We study further the structure of the model thus obtained, and also give a few more abstract results, concerning consistency and independence in the presence of AFA.

1991 Mathematics subject classification: 68Q90 (also 04A99, 54E15).

Key words: transition systems, bisimulation, fixed points, non-well-founded set theory, uniform spaces.

1 Introduction

(Labelled) transition systems are relatively common in theoretical computer science, chiefly as vehicles for operational semantics. The first part of this paper, which grew out of some work done by the second author in [Rosc 82, 215–230] and more recently in [Rosc 88a, Rosc 88b], constructs a hierarchy of canonical transition systems and associated maps, aiming to give a strongly extensional theory of transition systems, where any two points with equivalent behaviours are identified. The cornerstone of the development is a notion of convergence in arbitrary transition

[1]The authors gratefully acknowledge that the work reported in this paper was supported by a grant from Hajrija & Boris Vukobrat and Copechim France S.A. (to R.S. Lazić) and ONR grant N00014-87-G-0242 (to A.W. Roscoe).

systems, generalising the idea of finite (n-step) approximations to a given point. In particular, our canonical transition systems are also uniform spaces.

The resulting hierarchy has very rich combinatorial (and topological) structure, and a lot of the first part of the paper is devoted to its study. We also discuss fixed points in this framework.

Aczel (among others) observed that this kind of study of transition systems is very closely connected to non-well-founded set theory. Based on Milner's work on operational semantics of Synchronous CCS, in [Acz 88] he gave a quotient construction of a model of set theory with the Axiom of Foundation replaced by an Anti-Foundation Axiom (AFA).

In the second part of this paper, we show how to obtain a model of set theory with AFA from canonical transition systems constructed earlier. This gives the model a rich structure, which we then study further, building on the work in the first part. We also give a few more abstract results, concerning consistency and independence in the presence of AFA.

Non-well-founded set theory has been worked on long before [Acz 88]. In particular, AFA was probably first introduced as X_1 by Forti and Honsell in [FH 83], which investigates a number of axioms derived from a Free Construction Principle. In the subsequent papers (especially in [FH 89] and [FH 92]), they study structures which correspond to our canonical transition systems at regular cardinal heights (and for a singleton alphabet). They regard them primarily as quotients of a universe satisfying AFA.

Working in set theory with AFA, Aczel and Mendler obtain a Terminal Coalgebra Theorem (see Chapter 7 of [Acz 88], and [AczM 89]), which can be used to obtain spaces for semantics of process algebras based on the idea that $[\![P]\!] = \{\langle\delta, [\![P']\!]\rangle \mid P \xrightarrow{\delta} P'\}$ (see Chapter 8 of [Acz 88], and e.g. [RT 93]). By generalising the structures from [FH 89] and [FH 92], Forti, Honsell and Lenisa arrive at hyperuniverses, which are models of a strong Comprehension Scheme with topological structure, and can be similarly used for denotational semantics — see [FHL 94].

2 Transition systems, morphisms and bisimulations

We work within ZFC^-, i.e. Zermelo–Fraenkel set theory with the Axiom of Choice, but without the Axiom of Foundation. We drop the Axiom of Foundation, because we will sometimes want to adopt an axiom which contradicts it as one of our basic axioms. We follow the usual use of proper classes, i.e. classes which are not sets.[2]

We fix a set $\Sigma^+ = \Sigma \cup \{\tau\}$ of events, where $\tau \notin \Sigma$.[3] This will be an implicit parameter of almost everything we do from now on.

Definition 1 A *transition system* is a class $\mathcal{S}$ together with a family of binary relations $\xrightarrow{\delta}$ on $\mathcal{S}$ indexed by Σ^+, such that $a_{\mathcal{S},\delta} = \{b \in \mathcal{S} \mid a \xrightarrow{\delta} b\}$ is a set for all $a \in \mathcal{S}$, $\delta \in \Sigma^+$. A transition system is *small* iff its underlying class is a set.

If $\mathcal{S}$ is a transition system and $\mathcal{S}' \subseteq \mathcal{S}$, then $\mathcal{S}'$ is a *subsystem* of $\mathcal{S}$ iff $a \in \mathcal{S}' \wedge b \in \mathcal{S} \wedge a \xrightarrow{\delta} b \Rightarrow b \in \mathcal{S}'$, i.e. iff $a_{\mathcal{S}',\delta} = a_{\mathcal{S},\delta}$ for all $a \in \mathcal{S}'$, $\delta \in \Sigma^+$.

An *accessible pointed system (aps)* is a transition system $\mathcal{S}$ with a designated *point* $a \in \mathcal{S}$ such that, given any $b \in \mathcal{S}$, there is a finite sequence of transitions $a \xrightarrow{\delta_0} \ldots \xrightarrow{\delta_{n-1}} b$. Given a transition system $\mathcal{S}$ and any $a \in \mathcal{S}$, let $\mathcal{S}a$ be the aps whose point is a and which consists of all $b \in \mathcal{S}$ reachable from a in a finite number of transitions. □

[2] The reader is referred to Chapter 1 of [Kun 80]. ([End 77] is a good introduction to set theory.)

[3] In the usual terminology of process algebras, τ is an internal (invisible) event. However, its purpose in this paper is merely to ensure that Σ^+ is non-empty.

It is easily seen that any aps must be a small transition system.

Definition 2 Given any transition systems $\mathcal{S}$, $\mathcal{S}'$, a map $\mathcal{F} : \mathcal{S} \to \mathcal{S}'$ is a *morphism* iff:

- $a \xrightarrow{\delta} b \Rightarrow \mathcal{F}(a) \xrightarrow{\delta} \mathcal{F}(b)$, and
- $\mathcal{F}(a) \xrightarrow{\delta} b' \Rightarrow \exists b. a \xrightarrow{\delta} b \wedge \mathcal{F}(b) = b'$. $\Box$

Alternatively, that is equivalent to saying that $\mathcal{F}(a_{\mathcal{S},\delta}) = (\mathcal{F}(a))_{\mathcal{S}',\delta}$ for all $a \in \mathcal{S}$, $\delta \in \Sigma^+$. The idea is that a point and its image under a morphism cannot be told apart by an experimentor who can only observe the passing of events (both internal and external). A closely linked notion is that of a binary relation which relates pairs of points with the same behaviours (in the same sense).

Definition 3 A binary relation R on a transition system $\mathcal{S}$ is a *bisimulation* iff:

- $aRb \wedge a \xrightarrow{\delta} a' \Rightarrow \exists b'. b \xrightarrow{\delta} b' \wedge a'Rb'$, and
- $aRb \wedge b \xrightarrow{\delta} b' \Rightarrow \exists a'. a \xrightarrow{\delta} a' \wedge a'Rb'$. $\Box$[4]

A maximum bisimulation exists on any transition system $\mathcal{S}$. It is given by the union of all small bisimulations[5] on $\mathcal{S}$, and it is an equivalence relation. If $\mathcal{S}$ is small, the maximum bisimulation on $\mathcal{S}$ is also given by the set of all pairs $\langle a, b\rangle$ such that there is a morphism $\mathcal{F}$ with domain $\mathcal{S}$ with $\mathcal{F}(a) = \mathcal{F}(b)$.

Definition 4 A transition system $\mathcal{S}$ is *strongly extensional* iff the maximum bisimulation on $\mathcal{S}$ is the identity relation (i.e. the diagonal) on $\mathcal{S}$. $\Box$[6]

Lemma 1 *(a) Given a transition system $\mathcal{S}$, there exists a strongly extensional transition system $\tilde{\mathcal{S}}$ (its quotient) and a unique surjective morphism $\mathcal{F} : \mathcal{S} \to \tilde{\mathcal{S}}$.*

(b) If $\mathcal{G} : \mathcal{S} \to \mathcal{S}'$ is any surjective morphism, then there is a unique surjective morphism $\mathcal{H} : \mathcal{S}' \to \tilde{\mathcal{S}}$ such that $\mathcal{H} \circ \mathcal{G} = \mathcal{F}$. $\Box$

In (a), if $\mathcal{S}$ is a proper class, the formalization becomes non-trivial if we want to avoid using a stronger choice principle, since we do not assume the Axiom of Foundation. For any equivalence class $\mathcal{C}$, we need to consider the class of all aps's on well-founded sets which are isomorphic to $\mathcal{S}a$ for a point $a \in \mathcal{C}$, and then take its subset consisting of all of its elements which have minimal rank as the representation for $\mathcal{C}$ — see the proof of Lemma 2.17 in [Acz 88].

Lemma 2 *For a transition system $\mathcal{S}$, the following are equivalent:*

(a) $\mathcal{S}$ is strongly extensional.

(b) For any small transition system $\mathcal{S}'$, there is at most one morphism from $\mathcal{S}'$ into $\mathcal{S}$.

(c) For any transition system $\mathcal{S}'$, there is at most one morphism from $\mathcal{S}'$ into $\mathcal{S}$.

[4] These have sometimes been called 'strong bisimulations', because they treat internal and external events in the same way.

[5] A bisimulation is small iff it is a set.

[6] A transition system $\mathcal{S}$ is sometimes said to be 'weakly extensional' iff $a = b$ whenever $a_{\mathcal{S},\delta} = b_{\mathcal{S},\delta}$ for all $\delta \in \Sigma^+$.

(d) Any morphism with domain $\mathcal{S}$ is injective. $\square$

For more results and examples about morphisms, the reader can look at pages 215–230 of [Rosc 82]. It uses '.' instead of 'τ', and deals only with small transition systems, which it calls 'P,Q-spaces'. Lemma 1 for small transition systems is proved there. [Miln 89], as well as [Acz 88] (which proves Lemma 2 for $\Sigma = \emptyset$ — see Theorem 2.19 there), have more about bisimulations.

3 The spaces of canonical approximations

From now on, α, β, γ, ζ, η, θ, ν and ξ (and their variations) will always denote ordinals. γ will always be a limit ordinal. κ and λ will always be cardinals.

We also abbreviate 'transition system' to simply 'system' in the future. In this section and the next, we assume that all systems are small (i.e. 'system' will mean 'small system').

Given a point a in a system $\mathcal{S}$, we can think of all the sequences of transitions of length at most n that a can perform as determining an n-step approximation to a. We generalise this idea as follows.

Definition 5 Given a system $\mathcal{S}$, we define the following maps on $\mathcal{S}$ by transfinite recursion starting from 1:

- $\mathcal{H}_1^{\mathcal{S}}(a) = \emptyset$.
- $\mathcal{H}_{\alpha+1}^{\mathcal{S}}(a) = \{\langle \delta, \mathcal{H}_\alpha^{\mathcal{S}}(b)\rangle \mid a \xrightarrow{\delta} b\}$.
- $\mathcal{H}_\gamma^{\mathcal{S}}(a) = \langle \mathcal{H}_\alpha^{\mathcal{S}}(a) \mid 0 < \alpha < \gamma\rangle$. $\square$

Lemma 3 *If $\mathcal{F} : \mathcal{S} \to \mathcal{S}'$ is a morphism, then $\mathcal{H}_\alpha^{\mathcal{S}'} \circ \mathcal{F} = \mathcal{H}_\alpha^{\mathcal{S}}$ for all $\alpha > 0$.*

Proof The proof is by transfinite induction on α. The base case and the limit case are trivial. For the successor case, suppose $\mathcal{H}_\alpha^{\mathcal{S}'} \circ \mathcal{F} = \mathcal{H}_\alpha^{\mathcal{S}}$ for some $\alpha > 0$. Then, for any $a \in \mathcal{S}$, we have:

$$\begin{aligned}
\mathcal{H}_{\alpha+1}^{\mathcal{S}'}(\mathcal{F}(a)) &= \bigcup_{\delta\in\Sigma^+} \{\langle \delta, \mathcal{H}_\alpha^{\mathcal{S}'}(b')\rangle \mid \mathcal{F}(a) \xrightarrow{\delta} b'\} \\
&= \bigcup_{\delta\in\Sigma^+} \{\langle \delta, \mathcal{H}_\alpha^{\mathcal{S}'}(\mathcal{F}(b))\rangle \mid a \xrightarrow{\delta} b\} \\
&= \bigcup_{\delta\in\Sigma^+} \{\langle \delta, \mathcal{H}_\alpha^{\mathcal{S}}(b)\rangle \mid a \xrightarrow{\delta} b\} \\
&= \mathcal{H}_{\alpha+1}^{\mathcal{S}}(a). \square
\end{aligned}$$

Since a subset $\mathcal{S}'$ of a system $\mathcal{S}$ is a subsystem of $\mathcal{S}$ iff the identity map from $\mathcal{S}'$ into $\mathcal{S}$ is a morphism, an immediate corollary of Lemma 3 is that, if $\mathcal{S}'$ is a subsystem of $\mathcal{S}$, then $\mathcal{H}_\alpha^{\mathcal{S}} \restriction \mathcal{S}' = \mathcal{H}_\alpha^{\mathcal{S}'}$ for all $\alpha > 0$.

We can now define the spaces of canonical approximations as the ranges of the $\mathcal{H}_\alpha^{\mathcal{S}}$ maps.

Definition 6 For any $\alpha > 0$, let

$$\mathcal{T}_\alpha = \{\mathcal{H}_\alpha^{\mathcal{S}}(a) \mid a \textit{ is a point in a system } \mathcal{S}\}.^7$$

[7] Since $\mathcal{T}_1 = \{\emptyset\}$, $\mathcal{T}_{\alpha+1} \subseteq \mathcal{P}(\Sigma^+ \times \mathcal{T}_\alpha)$ and $\mathcal{T}_\gamma \subseteq (\bigcup_{0<\alpha<\gamma} \mathcal{T}_\alpha)^{\gamma\setminus\{0\}}$, it is easy to see (by transfinite induction on α) that $\mathcal{T}_\alpha$ is a set for all $\alpha > 0$.

We make each $\mathcal{T}_\alpha$ into a system as follows:

- $\emptyset \in \mathcal{T}_1$ has no transitions.
- $a, b \in \mathcal{T}_{\alpha+1} \Rightarrow (a \xrightarrow{\delta} b \Leftrightarrow \exists b'.\langle\delta, b'\rangle \in a \wedge b = \mathcal{H}^{\mathcal{T}_\alpha}_{\alpha+1}(b'))$.
- $\underline{a}, \underline{b} \in \mathcal{T}_\gamma \Rightarrow (\underline{a} \xrightarrow{\delta} \underline{b} \Leftrightarrow \forall 0 < \alpha < \gamma.\langle\delta, b_\alpha\rangle \in a_{\alpha+1})$.[8] $\square$

Example 1 Let $\mathcal{S} = \{a_n \mid n < \omega\} \cup \{a^*\}$ be a system whose transitions are given by $a_{n+1} \xrightarrow{\tau} a_n$ and $a^* \xrightarrow{\tau} a_n$ for all $n < \omega$. Then, for any non-zero $m < \omega$, we have:

$$\begin{array}{rcl} \mathcal{H}^{\mathcal{S}}_m(a_n) & = & \overbrace{\{\langle\tau, ...\{\langle\tau, \emptyset\rangle\}...\}}^{n \cap (m-1)} \quad (n < \omega) \\ \mathcal{H}^{\mathcal{S}}_1(a^*) & = & \emptyset \\ \mathcal{H}^{\mathcal{S}}_{m+1}(a^*) & = & \{\langle\tau, \mathcal{H}^{\mathcal{S}}_m(a_n)\rangle \mid n < \omega\} \\ & = & \{\langle\tau, \mathcal{H}^{\mathcal{S}}_m(a_n)\rangle \mid n \leq m-1\}. \end{array}$$

Hence, for any $n < \omega$, we have

$$\begin{array}{rcl} \mathcal{H}^{\mathcal{S}}_\omega(a_{n+1}) & = & \langle\emptyset\rangle\frown\langle\mathcal{H}^{\mathcal{S}}_{m+1}(a_{n+1}) \mid 0 < m < \omega\rangle \\ & = & \langle\emptyset\rangle\frown\langle\{\langle\tau, \mathcal{H}^{\mathcal{S}}_m(a_n)\rangle\} \mid 0 < m < \omega\rangle, \end{array}$$

so that $(\mathcal{H}^{\mathcal{S}}_\omega(a_{n+1}))_{\mathcal{T}_\omega,\tau} = \{\mathcal{H}^{\mathcal{S}}_\omega(a_n)\}$ (and $(\mathcal{H}^{\mathcal{S}}_\omega(a_0))_{\mathcal{T}_\omega,\tau} = \emptyset$).

Also, we have

$$\begin{array}{rcl} \mathcal{H}^{\mathcal{S}}_\omega(a^*) & = & \langle\emptyset\rangle\frown\langle\mathcal{H}^{\mathcal{S}}_{m+1}(a^*) \mid 0 < m < \omega\rangle \\ & = & \langle\emptyset\rangle\frown\langle\{\langle\tau, \mathcal{H}^{\mathcal{S}}_m(a_n)\rangle \mid n \leq m-1\} \mid 0 < m < \omega\rangle. \end{array}$$

Now, if $0 < m < \omega$, $n \leq m-1$ and $n' \leq m$, then $\mathcal{H}^{\mathcal{S}}_m(a_n) = \mathcal{H}^{\mathcal{T}_{m+1}}_m(\mathcal{H}^{\mathcal{S}}_{m+1}(a_{n'}))$ iff either $n = n'$ or $n = m-1$ and $n' = m$. Hence it follows that $(\mathcal{H}^{\mathcal{S}}_\omega(a^*))_{\mathcal{T}_\omega,\tau} = \{\mathcal{H}^{\mathcal{S}}_\omega(a_n) \mid n < \omega\} \cup \{\underline{b}\}$, where $\underline{b} = \langle\mathcal{H}^{\mathcal{S}}_m(a_{m-1}) \mid 0 < m < \omega\rangle$ is such that $\underline{b}_{\mathcal{T}_\omega,\tau} = \{\underline{b}\}$.

Since $\mathcal{H}^{\mathcal{S}}_\omega$ is a morphism on the subsystem $\{a_n \mid n < \omega\}$ of $\mathcal{S}$, it is not difficult to see using Lemma 3 that $\mathcal{H}^{\mathcal{S}}_{\omega+1}$ is a morphism (on $\mathcal{S}$). $\square$

Lemma 4 *(a) For any $\alpha > 0$, we have $\mathcal{H}^{\mathcal{T}_\alpha}_\alpha(a) = a$ for all $a \in \mathcal{T}_\alpha$.*

(b) If $0 < \beta \leq \alpha$, then $\mathcal{H}^{\mathcal{T}_\alpha}_\beta \circ \mathcal{H}^{\mathcal{S}}_\alpha = \mathcal{H}^{\mathcal{S}}_\beta$ for any system $\mathcal{S}$.

Proof We prove (a) and (b) simultaneously by transfinite induction on α.
Base case. Trivial.
Successor case. Suppose (a) and (b) hold for some $\alpha > 0$. For (b), we argue by transfinite induction on non-zero $\beta \leq \alpha + 1$. The base case and the limit case are trivial, so suppose $\mathcal{H}^{\mathcal{T}_{\alpha+1}}_\beta \circ \mathcal{H}^{\mathcal{S}}_{\alpha+1} = \mathcal{H}^{\mathcal{S}}_\beta$ for some $0 < \beta < \alpha + 1$ and all systems $\mathcal{S}$. Then, for any a in a system $\mathcal{S}$, we have:

$$\begin{array}{rcl} \mathcal{H}^{\mathcal{T}_{\alpha+1}}_{\beta+1}(\mathcal{H}^{\mathcal{S}}_{\alpha+1}(a)) & = & \{\langle\delta, \mathcal{H}^{\mathcal{T}_{\alpha+1}}_\beta(b)\rangle \mid \mathcal{H}^{\mathcal{S}}_{\alpha+1}(a) \xrightarrow{\delta} b\} \\ & = & \{\langle\delta, \mathcal{H}^{\mathcal{T}_{\alpha+1}}_\beta(\mathcal{H}^{\mathcal{T}_\alpha}_{\alpha+1}(\mathcal{H}^{\mathcal{S}}_\alpha(b')))\rangle \mid a \xrightarrow{\delta} b'\} \\ & = & \{\langle\delta, \mathcal{H}^{\mathcal{T}_\alpha}_\beta(\mathcal{H}^{\mathcal{S}}_\alpha(b'))\rangle \mid a \xrightarrow{\delta} b'\} \\ & = & \mathcal{H}^{\mathcal{S}}_{\beta+1}(a). \end{array}$$

[8] We use $\underline{a}$ (etc.) as an abbreviation for the sequence $\langle a_\alpha \mid 0 < \alpha < \gamma\rangle$.

For (a), if $a \in \mathcal{T}_{\alpha+1}$, we have:

$$\begin{aligned}
\mathcal{H}^{\mathcal{T}_{\alpha+1}}_{\alpha+1}(a) &= \{\langle \delta, \mathcal{H}^{\mathcal{T}_{\alpha+1}}_{\alpha}(b)\rangle \mid a \xrightarrow{\delta} b\} \\
&= \{\langle \delta, \mathcal{H}^{\mathcal{T}_{\alpha+1}}_{\alpha}(\mathcal{H}^{\mathcal{T}_{\alpha}}_{\alpha+1}(b'))\rangle \mid \langle \delta, b'\rangle \in a\} \\
&= \{\langle \delta, \mathcal{H}^{\mathcal{T}_{\alpha}}_{\alpha}(b')\rangle \mid \langle \delta, b'\rangle \in a\} \\
&= a.
\end{aligned}$$

Limit case. Suppose (a) and (b) hold for all non-zero $\alpha < \gamma$. We prove that $\mathcal{H}^{\mathcal{T}_{\gamma}}_{\alpha}(\underline{a}) = a_\alpha$ for all $\underline{a} \in \mathcal{T}_\gamma$, $0 < \alpha < \gamma$ by transfinite induction on α.

(i) The base case is trivial.

(ii) Suppose the claim holds for some non-zero $\alpha < \gamma$, and consider some $\underline{a} \in \mathcal{T}_\gamma$. Then:

$$\begin{aligned}
\mathcal{H}^{\mathcal{T}_{\gamma}}_{\alpha+1}(\underline{a}) &= \{\langle \delta, \mathcal{H}^{\mathcal{T}_{\gamma}}_{\alpha}(\underline{b})\rangle \mid \underline{a} \xrightarrow{\delta} \underline{b}\} \\
&= \{\langle \delta, b_\alpha\rangle \mid \underline{a} \xrightarrow{\delta} \underline{b}\}.
\end{aligned}$$

Now $\langle \delta, b_\alpha\rangle \in a_{\alpha+1}$ whenever $\underline{a} \xrightarrow{\delta} \underline{b}$, so it suffices to show that, if $\langle \delta, b^*\rangle \in a_{\alpha+1}$, then $b^* = b_\alpha$ for some $\underline{b} \in \underline{a}_{\mathcal{T}_\gamma,\delta}$. But we know that $\underline{a} = \mathcal{H}^{\mathcal{S}}_{\gamma}(a')$ for some point a' in a system $\mathcal{S}$, so if $\langle \delta, b^*\rangle \in a_{\alpha+1} = \mathcal{H}^{\mathcal{S}}_{\alpha+1}(a')$, then $b^* = \mathcal{H}^{\mathcal{S}}_{\alpha}(b')$ for some $b' \in a'_{\mathcal{S},\delta}$, which gives us what we want since $\underline{a} = \mathcal{H}^{\mathcal{S}}_{\gamma}(a') \xrightarrow{\delta} \mathcal{H}^{\mathcal{S}}_{\gamma}(b')$ (just observe that $\forall 0 < \beta < \gamma. \langle \delta, \mathcal{H}^{\mathcal{S}}_{\beta}(b')\rangle \in \mathcal{H}^{\mathcal{S}}_{\beta+1}(a')$).

(iii) Suppose the claim holds for all non-zero $\alpha < \gamma'$, where $\gamma' < \gamma$. If $\underline{a} \in \mathcal{T}_\gamma$, then:

$$\begin{aligned}
\mathcal{H}^{\mathcal{T}_{\gamma}}_{\gamma'}(\underline{a}) &= \langle \mathcal{H}^{\mathcal{T}_{\gamma}}_{\alpha}(\underline{a}) \mid 0 < \alpha < \gamma'\rangle \\
&= \langle a_\alpha \mid 0 < \alpha < \gamma'\rangle \\
&= a_{\gamma'}.
\end{aligned}$$

Now that the claim is established, (a) and (b) for γ in place of α follow at once. □

If A is any subset of $\Sigma^+ \times \mathcal{T}_\alpha$, consider a system $\mathcal{S} = \{a\} \cup \mathcal{T}_\alpha$ (where $a \notin \mathcal{T}_\alpha$) which inherits the transitions on $\mathcal{T}_\alpha$ from $\mathcal{T}_\alpha$ and such that $a_{\mathcal{S},\delta} = \{b \mid \langle \delta, b\rangle \in A\}$. Then $\mathcal{H}^{\mathcal{S}}_{\alpha+1}(a) = \{\langle \delta, \mathcal{H}^{\mathcal{S}}_{\alpha}(b)\rangle \mid \langle \delta, b\rangle \in A\} = A$ by the remark after Lemma 3 and by (a) of Lemma 4. Therefore, for any $\alpha > 0$, we have

$$\mathcal{T}_{\alpha+1} = \mathcal{P}(\Sigma^+ \times \mathcal{T}_\alpha).$$

Whenever $\underline{a} \in \mathcal{T}_\gamma$, (b) of Lemma 4 gives us

$$0 < \beta \leq \alpha < \gamma \Rightarrow a_\beta = \mathcal{H}^{\mathcal{T}_{\alpha}}_{\beta}(a_\alpha),$$

so that any $\mathcal{T}_\gamma$ is a subset of the inverse limit $\mathcal{T}^0_\gamma$ of the sets $\mathcal{T}_\alpha$ $(0 < \alpha < \gamma)$ with respect to the maps $\mathcal{H}^{\mathcal{T}_{\alpha}}_{\beta} : \mathcal{T}_\alpha \to \mathcal{T}_\beta$ $(0 < \beta \leq \alpha < \gamma)$. Transitions on $\mathcal{T}^0_\gamma$ (and any $\mathcal{S} \subseteq \mathcal{T}^0_\gamma$) are defined as in Definition 6.

If $\mathcal{F} : \mathcal{T}_\alpha \to \mathcal{S}$ is any morphism, then $\mathcal{H}^{\mathcal{S}}_{\alpha} \circ \mathcal{F} = \mathcal{H}^{\mathcal{T}_{\alpha}}_{\alpha}$ is the identity map on $\mathcal{T}_\alpha$, and so $\mathcal{F}$ is injective. The following theorem now follows from Lemma 2.

Theorem 5 *For any $\alpha > 0$, $\mathcal{T}_\alpha$ is strongly extensional.* □

Suppose $\mathcal{S} \subseteq \mathcal{T}_\gamma^0$ is such that $\mathcal{H}_\alpha^{\mathcal{S}}(\underline{a}) = a_\alpha$ for all $0 < \alpha < \gamma$, $\underline{a} \in \mathcal{S}$. Then trivially $\mathcal{H}_\gamma^{\mathcal{S}}(\underline{a}) = \underline{a}$ for all $\underline{a} \in \mathcal{S}$, and so $\mathcal{S} \subseteq \mathcal{T}_\gamma$ by Definition 6. We conclude that $\mathcal{T}_\gamma$ is the largest subset of $\mathcal{T}_\gamma^0$ on which the maps $\mathcal{H}_\alpha^{\mathcal{T}_\gamma}$ for $0 < \alpha < \gamma$ agree with the canonical inverse limit maps on $\mathcal{T}_\gamma^0$.

Given any $\mathcal{S} \subseteq \mathcal{T}_\gamma^0$, let

$$\Phi(\mathcal{S}) = \{\underline{a} \in \mathcal{S} \mid 0 < \alpha < \gamma \wedge \langle \delta, b^* \rangle \in a_{\alpha+1} \Rightarrow \exists \underline{b} \in \underline{a}_{\mathcal{S},\delta}. b^* = b_\alpha\}.$$

Then Φ is a monotonic map on the complete lattice $\langle \mathcal{P}(\mathcal{T}_\gamma^0), \subseteq \rangle$, and we have the following result.[9]

Theorem 6 *$\mathcal{T}_\gamma$ is the greatest fixed point of Φ.*

Proof We have already seen in (ii) in the proof of Lemma 4 that $\mathcal{T}_\gamma$ is a fixed point of Φ.

Suppose $\mathcal{S} \subseteq \mathcal{T}_\gamma^0$ is a fixed point of Φ. Then it follows as in the limit case of the same proof that $\mathcal{H}_\gamma^{\mathcal{S}}(\underline{a}) = \underline{a}$ for all $\underline{a} \in \mathcal{S}$, and hence $\mathcal{S} \subseteq \mathcal{T}_\gamma$. □

We now know by Knaster–Tarski Theorem that

$$\mathcal{T}_\gamma = \bigcup \{\mathcal{S} \subseteq \mathcal{T}_\gamma^0 \mid \mathcal{S} \subseteq \Phi(\mathcal{S})\}.$$

Alternatively, any complete lattice with its order reversed is a complete partial order. Hence, if we define by transfinite recursion:

- $\mathcal{T}_\gamma^{\alpha+1} = \Phi(\mathcal{T}_\gamma^\alpha)$,
- $\mathcal{T}_\gamma^{\gamma'} = \bigcap_{\alpha<\gamma'} \mathcal{T}_\gamma^\alpha$,

then the $\mathcal{T}_\gamma^\alpha$ form a decreasing chain of subsets of $\mathcal{T}_\gamma^0$ which becomes constant at some $\alpha^* < |\mathcal{T}_\gamma^0|^+$,[10] and we have that $\mathcal{T}_\gamma = \mathcal{T}_\gamma^{\alpha^*}$.

The definition of $\mathcal{T}_\gamma$ we gave can in principle be replaced by either of these, which are in a sense "more direct".

We will continue this kind of study of $\mathcal{T}_\gamma^0$ after we establish a few results of a different kind.

From now on, we will usually omit the superscript in $\mathcal{H}_\alpha^{\mathcal{S}}$, provided that does not introduce ambiguity which is not covered by Lemma 3.

Given a system $\mathcal{S}$, the decreasing chain of sets

$$U_\alpha = \{\langle a, b \rangle \in \mathcal{S} \times \mathcal{S} \mid \mathcal{H}_\alpha(a) = \mathcal{H}_\alpha(b)\}$$

for $0 < \alpha < \gamma$ forms a fundamental system of entourages of a uniformity $\mathcal{U}_\gamma^{\mathcal{S}}$ on $\mathcal{S}$.[11] In this way, for any γ, the maps $\mathcal{H}_\alpha$ $(0 < \alpha < \gamma)$ give rise to a notion of convergence of points in an arbitrary system $\mathcal{S}$.

Let $\mathcal{V}_\gamma$ be the uniformity on $\mathcal{T}_\gamma^0$ which is the inverse limit of the discrete uniformities on $\mathcal{T}_\alpha$ $(0 < \alpha < \gamma)$. Then $\mathcal{U}_\gamma^{\mathcal{T}_\gamma}$ is the subspace uniformity on $\mathcal{T}_\gamma$ induced by $\mathcal{V}_\gamma$.

[9] For partial orders, we refer the reader to [DPr 90]. Chapter 4 there concentrates on fixed points.

[10] Since $|\mathcal{T}_\gamma^0| \geq \omega$ (see e.g. Corollary 8), $|\mathcal{T}_\gamma^0|^+$ is an infinite regular cardinal. (Suppose κ is infinite and regular, and that X_α for $\alpha < \kappa$ form a decreasing sequence of sets such that $\forall \alpha < \kappa. |X_\alpha| < \kappa$. For any $x \in X_0' = X_0 \setminus \bigcap_{\alpha<\kappa} X_\alpha$, let $\alpha^x < \kappa$ be minimal such that $x \notin X_{\alpha^x}$. Then, letting $\alpha^* = \bigcup_{x \in X_0'} \alpha^x$, we have that the X_α for $\alpha^* \leq \alpha < \kappa$ are all identical.)

[11] For both uniform spaces and inverse limits, we refer the reader to [Bourb 66].

Whenever $\mathcal{F} : \mathcal{S} \to \mathcal{S}'$ is a morphism, $\mathcal{U}_\gamma^{\mathcal{S}}$ is by Lemma 3 the inverse image under $\mathcal{F}$ of $\mathcal{U}_\gamma^{\mathcal{F}(\mathcal{S})}$ (which is the subspace uniformity on $\mathcal{F}(\mathcal{S})$ induced by $\mathcal{U}_\gamma^{\mathcal{S}'}$, since $\mathcal{F}(\mathcal{S})$ is a subsystem of $\mathcal{S}'$). In particular, given any system $\mathcal{S}$, the uniformity $\mathcal{U}_\gamma^{\mathcal{S}}$ on it is an inverse image of the uniformity $\mathcal{U}_\gamma^{\widetilde{\mathcal{S}}}$ on its strongly extensional quotient.

Suppose now that $\mathcal{S}$ is an arbitrary system, and let

$$\begin{aligned}\Psi(R) \quad = \quad & \{\langle a, b\rangle \in \mathcal{S} \times \mathcal{S} \mid (a \xrightarrow{\delta} a' \Rightarrow \exists b'.b \xrightarrow{\delta} b' \wedge a'Rb') \wedge \\ & (b \xrightarrow{\delta} b' \Rightarrow \exists a'.a \xrightarrow{\delta} a' \wedge a'Rb')\}\end{aligned}$$

for any binary relation R on $\mathcal{S}$. Then R is a bisimulation iff $R \subseteq \Psi(R)$, and the maximum bisimulation $\sim$ on $\mathcal{S}$ is the greatest fixed point of Ψ. Hence $\sim$ is given by the eventual constant value of the following decreasing chain:

- $R^{(1)} = \mathcal{S} \times \mathcal{S}$.
- $R^{(\alpha+1)} = \Psi(R^{(\alpha)})$.
- $R^{(\gamma)} = \bigcap_{0<\alpha<\gamma} R^{(\alpha)}$.

But it is easily seen that $R^{(\alpha)} = \{\langle a, b\rangle \mid \mathcal{H}_\alpha(a) = \mathcal{H}_\alpha(b)\}$ for all $\alpha > 0$. Hence, if $\mathcal{S}$ is strongly extensional, we conclude that $\mathcal{U}_\gamma^{\mathcal{S}}$ is Hausdorff for large enough γ.

We now turn to the question of when are the maps $\mathcal{H}_\alpha^{\mathcal{S}} : \mathcal{S} \to \mathcal{T}_\alpha$ morphisms.

Theorem 7 *If $\mathcal{H}_\alpha$ is a morphism on a system $\mathcal{S}$ and $\beta \geq \alpha$, then $\mathcal{H}_\beta$ is also a morphism on $\mathcal{S}$.*

Proof The proof is by transfinite induction on $\beta \geq \alpha$.
Base case. Trivial.
Successor case. Suppose $\mathcal{H}_\beta$ is a morphism on $\mathcal{S}$ for some $\beta \geq \alpha$. Then, for any $a \in \mathcal{S}$, $\delta \in \Sigma^+$, we have:

$$\begin{aligned}(\mathcal{H}_{\beta+1}(a))_{\mathcal{T}_{\beta+1},\delta} &= \{\mathcal{H}_{\beta+1}(\mathcal{H}_\beta(b)) \mid a \xrightarrow{\delta} b\} \\ &= \{\mathcal{H}_{\beta+1}(b) \mid a \xrightarrow{\delta} b\} \\ &= \mathcal{H}_{\beta+1}(a_{\mathcal{S},\delta}).\end{aligned}$$

Limit case. Suppose $\mathcal{H}_\beta$ for all $\alpha \leq \beta < \gamma$ are morphisms on $\mathcal{S}$. Whenever $a \xrightarrow{\delta} b$ in $\mathcal{S}$, we trivially have $\mathcal{H}_\gamma(a) \xrightarrow{\delta} \mathcal{H}_\gamma(b)$.

Hence suppose $\mathcal{H}_\gamma(a) \xrightarrow{\delta} \underline{c}$. Then $\langle \delta, c_\alpha\rangle \in \mathcal{H}_{\alpha+1}(a)$, so $c_\alpha = \mathcal{H}_\alpha(b)$ for some $b \in a_{\mathcal{S},\delta}$. Now, given any $\alpha \leq \beta < \gamma$, we have $c_\beta = \mathcal{H}_\beta(b')$ for some $b' \in a_{\mathcal{S},\delta}$, and so

$$\mathcal{H}_\beta(b) = \mathcal{H}_\beta(\mathcal{H}_\alpha(b)) = \mathcal{H}_\beta(c_\alpha) = \mathcal{H}_\beta(\mathcal{H}_\alpha(c_\beta)) = \mathcal{H}_\beta(\mathcal{H}_\alpha(\mathcal{H}_\beta(b'))) = \mathcal{H}_\beta(\mathcal{H}_\alpha(b')) = \mathcal{H}_\beta(b') = c_\beta.$$

Therefore, $\underline{c} = \mathcal{H}_\gamma(b)$. □

Corollary 8 *If $0 < \alpha \leq \beta$, then $\mathcal{H}_\beta$ is a morphism on $\mathcal{T}_\alpha$.*

Proof $\mathcal{H}_\alpha$ is a morphism on $\mathcal{T}_\alpha$ by Lemma 4 (a). □

Definition 7 Given a system $\mathcal{S}$, let $i(\mathcal{S})$, the *index of non-determinism* of $\mathcal{S}$, be the smallest infinite regular cardinal which is strictly greater than $|a_{\mathcal{S},\delta}|$ for all $a \in \mathcal{S}$, $\delta \in \Sigma^+$. □

Theorem 9 *Suppose $\mathcal{S}$ is any system.*

(a) $\mathcal{H}_{i(\mathcal{S})}$ is a morphism on $\mathcal{S}$.

(b) If $\mathcal{S}$ is countable, then $\mathcal{H}_\alpha$ is a morphism on $\mathcal{S}$ for some countable α.

Proof

(a) Since $i(\mathcal{S})$ is a limit ordinal, $a \xrightarrow{\delta} b \Rightarrow \mathcal{H}_{i(\mathcal{S})}(a) \xrightarrow{\delta} \mathcal{H}_{i(\mathcal{S})}(b)$.

So suppose $\mathcal{H}_{i(\mathcal{S})}(a) \xrightarrow{\delta} \underline{c}$. For any $0 < \alpha < i(\mathcal{S})$, let

$$X_\alpha = \{b \in a_{\mathcal{S},\delta} \mid \mathcal{H}_\alpha(b) = c_\alpha\}.$$

Now $0 < \alpha < i(\mathcal{S}) \Rightarrow \langle \delta, c_\alpha \rangle \in \mathcal{H}_{\alpha+1}(a)$, so each X_α is non-empty. Also, if $\alpha \le \beta$ and $b \in X_\beta$, then $\mathcal{H}_\alpha(b) = \mathcal{H}_\alpha(\mathcal{H}_\beta(b)) = \mathcal{H}_\alpha(c_\beta) = c_\alpha$, so that $b \in X_\alpha$. Hence the X_α form a decreasing chain and so, since $|X_\alpha| \le |a_{\mathcal{S},\delta}| < i(\mathcal{S})$ for all α, we can pick a $b \in \bigcap_{0<\alpha<i(\mathcal{S})} X_\alpha$, for which we will have $\underline{c} = \mathcal{H}_{i(\mathcal{S})}(b)$.

(b) For any $a \in \mathcal{S}$, $0 < \alpha < \omega_1$, let

$$Y_\alpha^a = \{b \in \mathcal{S} \mid \mathcal{H}_\alpha(b) = \mathcal{H}_\alpha(a)\}.$$

Then each Y_α^a is countable (and trivially non-empty), and $\alpha \le \beta \Rightarrow Y_\alpha^a \supseteq Y_\beta^a$. Hence, given any $a \in \mathcal{S}$, there exists a non-zero $\alpha^a < \omega_1$ such that $\alpha^a \le \alpha < \omega_1 \Rightarrow Y_\alpha^a = Y_{\alpha^a}^a$.

Let $\alpha^* = \bigcup_{a\in\mathcal{S}} \alpha^a$, and pick a limit ordinal $\gamma > \alpha^*$ such that $\gamma < \omega_1$. We claim that $\mathcal{H}_\gamma$ is a morphism on $\mathcal{S}$.

It is again immediate that $a \xrightarrow{\delta} b \Rightarrow \mathcal{H}_\gamma(a) \xrightarrow{\delta} \mathcal{H}_\gamma(b)$, so suppose $\mathcal{H}_\gamma(a) \xrightarrow{\delta} \underline{c}$. Then $\langle \delta, c_{\alpha^*} \rangle \in \mathcal{H}_{\alpha^*+1}(a)$, so $c_{\alpha^*} = \mathcal{H}_{\alpha^*}(b)$ for some $b \in a_{\mathcal{S},\delta}$. But then, for any $\alpha^* \le \alpha < \omega_1$, there is some $b' \in a_{\mathcal{S},\delta}$ with $c_\alpha = \mathcal{H}_\alpha(b')$ for which we have $\mathcal{H}_{\alpha^*}(b') = \mathcal{H}_{\alpha^*}(\mathcal{H}_\alpha(b')) = \mathcal{H}_{\alpha^*}(c_\alpha) = c_{\alpha^*} = \mathcal{H}_{\alpha^*}(b)$ and so $b' \in Y_{\alpha^*}^b = Y_\alpha^b$, which gives us that $c_\alpha = \mathcal{H}_\alpha(b') = \mathcal{H}_\alpha(b)$. Hence $\underline{c} = \mathcal{H}_\gamma(b)$, which establishes the claim. □

Whenever $\mathcal{H}_\alpha$ is a morphism on $\mathcal{S}$, it is easy to see that $\{\langle a, b\rangle \in \mathcal{S} \times \mathcal{S} \mid \mathcal{H}_\alpha(a) = \mathcal{H}_\alpha(b)\}$ is a bisimulation.[12] Hence it is the maximum bisimulation on $\mathcal{S}$, being a fixed point of Ψ and so the eventual constant value of the $R^{(\alpha)}$ chain.

If $\mathcal{H}_\gamma$ is a morphism on a strongly extensional system $\mathcal{S}$, we know by Lemma 2 that $\mathcal{H}_\gamma$ is injective on $\mathcal{S}$, and so $\mathcal{S}$ is isomorphic to $\mathcal{H}_\gamma(\mathcal{S})$ which is a subsystem of $\mathcal{T}_\gamma$. Also, the uniformity $\mathcal{U}_\gamma^{\mathcal{S}}$ is isomorphic to the subspace uniformity on $\mathcal{H}_\gamma(\mathcal{S})$ induced by $\mathcal{U}_\gamma^{\mathcal{T}_\gamma}$. In this sense, the $\mathcal{T}_\alpha$ are canonical systems, and the $\mathcal{U}_\gamma^{\mathcal{T}_\gamma}$ are canonical uniformities.

Given an infinite regular cardinal κ, let

$$\mathcal{T}_\kappa^* = \{\mathcal{H}_\kappa^{\mathcal{S}}(a) \mid i(\mathcal{S}) \le \kappa \wedge a \in \mathcal{S}\}.$$

Then $\mathcal{T}_\kappa^*$ is a subsystem of $\mathcal{T}_\kappa$, $i(\mathcal{T}_\kappa^*) = \kappa$ and, given any system $\mathcal{S}$ with $i(\mathcal{S}) \le \kappa$, there is a unique morphism from $\mathcal{S}$ into $\mathcal{T}_\kappa^*$. (The systems $\mathcal{T}_\kappa^*$ are important to the study of operational semantics because they are closed under naturally defined CSP operators.)

[12] The converse is not true in general — consider $\mathcal{H}_1$ on $\mathcal{S} = \{a, b\}$ with transitions given by $a \xrightarrow{\tau} b$ and $b \xrightarrow{\tau} a$.

One important case is that where $|\Sigma^+| \leq \omega$ and $\kappa = \omega_1$ (so that we allow only countable branching). Then, given a point a in a system $\mathcal{S}$ with $i(\mathcal{S}) \leq \omega_1$, we have that $\mathcal{S}a$ is countable, and so $\mathcal{H}_\alpha$ is a morphism on $\mathcal{S}a$ for some countable α. Hence $\mathcal{H}^{\mathcal{S}}_{\omega_1}(a) = \mathcal{H}^{\mathcal{S}a}_{\omega_1}(a) = \mathcal{H}^{\mathcal{T}_\alpha}_{\omega_1}(\mathcal{H}^{\mathcal{S}a}_\alpha(a))$, and we conclude that $\mathcal{T}^*_{\omega_1} \subseteq \mathcal{T}^c_{\omega_1}$, where we define

$$\mathcal{T}^c_\gamma = \{\mathcal{H}^{\mathcal{T}_\alpha}_\gamma(a) \mid 0 < \alpha < \gamma \wedge a \in \mathcal{T}_\alpha\}.$$

Each $\mathcal{T}^c_\gamma$ is a subsystem of $\mathcal{T}_\gamma$. (The systems $\mathcal{T}^c_\gamma$ are closed under some useful forms of sums.)

From now on, unless stated otherwise, we will assume that, for any γ, the uniformity on $\mathcal{T}_\gamma$ is $\mathcal{U}^{\mathcal{T}_\gamma}_\gamma$, and the uniformity on $\mathcal{T}^0_\gamma$ is $\mathcal{V}_\gamma$.

If $\underline{a} \in \mathcal{T}^0_\gamma$ and $\delta \in \Sigma^+$, then $\underline{a}_{\mathcal{T}^0_\gamma,\delta}$ consists of all sequences in the inverse limit of the sets $\{b \mid \langle \delta, b\rangle \in a_{\alpha+1}\}$ (with respect to the maps $\mathcal{H}^{\mathcal{T}_\beta}_\alpha$, for $0 < \alpha \leq \beta < \gamma$). If $\underline{a} \in \mathcal{T}_\gamma$, then $\underline{a}_{\mathcal{T}_\gamma,\delta}$ is the set of all sequences from the same inverse limit which are also in $\mathcal{T}_\gamma$.

Lemma 10 *Suppose a is a point in a system $\mathcal{S}$ and $\delta \in \Sigma^+$.*

(a) $(\mathcal{H}_\gamma(a))_{\mathcal{T}^0_\gamma,\delta} = \overline{\mathcal{H}_\gamma(a_{\mathcal{S},\delta})}^{\mathcal{T}^0_\gamma}$.

(b) $(\mathcal{H}_\gamma(a))_{\mathcal{T}_\gamma,\delta} = \overline{\mathcal{H}_\gamma(a_{\mathcal{S},\delta})}^{\mathcal{T}_\gamma}$.

Proof By the remarks above, it suffices to observe that

$$\{b_\alpha \mid \underline{b} \in \mathcal{H}_\gamma(a_{\mathcal{S},\delta})\} = \mathcal{H}_\alpha(a_{\mathcal{S},\delta}) = \{b' \mid \langle \delta, b'\rangle \in a_{\alpha+1}\}$$

for all $0 < \alpha < \gamma$.[13] $\square$

Theorem 11 *(a) If $X_\delta \subseteq \mathcal{T}^0_\gamma$ for each $\delta \in \Sigma^+$, then there exists an $\underline{a} \in \mathcal{T}^0_\gamma$ such that $\forall \delta \in \Sigma^+ . \underline{a}_{\mathcal{T}^0_\gamma,\delta} = X_\delta$ iff each X_δ is closed.*

(b) If $X_\delta \subseteq \mathcal{T}_\gamma$ for each $\delta \in \Sigma^+$, then there exists an $\underline{a} \in \mathcal{T}_\gamma$ such that $\forall \delta \in \Sigma^+ . \underline{a}_{\mathcal{T}_\gamma,\delta} = X_\delta$ iff each X_δ is closed in $\mathcal{T}_\gamma$.

Proof For (a), suppose we have such an $\underline{a} \in \mathcal{T}^0_\gamma$. Then, for any $\delta \in \Sigma^+$, $X_\delta = \underline{a}_{\mathcal{T}^0_\gamma,\delta}$ is the inverse limit of the sets $\{b_\alpha \; \underline{b} \in X_\delta\}$, and is hence closed.

Conversely, if each X_δ is closed, let

$$a_{\alpha+1} = \bigcup_{\delta \in \Sigma^+} \{\langle \delta, b_\alpha\rangle \mid \underline{b} \in X_\delta\}$$

for any $0 < \alpha < \gamma$. This gives us a unique $\underline{a} \in \mathcal{T}^0_\gamma$. For any $\delta \in \Sigma^+$, we have that $\underline{a}_{\mathcal{T}^0_\gamma,\delta}$ is the inverse limit of the sets $\{b' \mid \langle \delta, b'\rangle \in a_{\alpha+1}\} = \{b_\alpha \mid \underline{b} \in X_\delta\}$, which equals X_δ since X_δ is closed.

For (b), we can use Lemma 10 (b). If we have such an $\underline{a} \in \mathcal{T}_\gamma$, then

$$X_\delta = \underline{a}_{\mathcal{T}_\gamma,\delta} = (\mathcal{H}^{\mathcal{T}_\gamma}_\gamma(\underline{a}))_{\mathcal{T}_\gamma,\delta} = \overline{\mathcal{H}^{\mathcal{T}_\gamma}_\gamma(\underline{a}_{\mathcal{T}_\gamma,\delta})}^{\mathcal{T}_\gamma} = \overline{\underline{a}_{\mathcal{T}_\gamma,\delta}}^{\mathcal{T}_\gamma} = \overline{X_\delta}^{\mathcal{T}_\gamma}$$

for all $\delta \in \Sigma^+$.

[13] See Corollary to Proposition 9 of Chapter I, Section 4.4 in [Bourb 66].

Finally, if each X_δ is closed in $\mathcal{T}_\gamma$, then

$$\left(\mathcal{H}_\gamma\left(\sum_{\delta'\in\Sigma^+}\sum_{\underline{b}\in X_{\delta'}}\delta'.\underline{b}\right)\right)_{\mathcal{T}_\gamma,\delta} = \overline{\mathcal{H}_\gamma^{\mathcal{T}_\gamma}(X_\delta)}^{\mathcal{T}_\gamma} = \overline{X_\delta}^{\mathcal{T}_\gamma} = X_\delta$$ [14]

for all $\delta \in \Sigma^+$. $\square$

The strong extensionality of $\mathcal{T}_\gamma$ gives us that the correspondence between points in $\mathcal{T}_\gamma$ and Σ^+-tuples of closed subsets of $\mathcal{T}_\gamma$ in (b) of Theorem 11 is 1–1.

In their setting, Forti and Honsell take these ideas further, resulting in their theory of comprehension properties of hyperuniverses — see [FH 89].

Suppose $\kappa < cf(\gamma)$ and Y_ξ for $\xi < \kappa$ are closed subsets of $\mathcal{T}_\gamma^0$. If $\underline{a} \notin \bigcup_{\xi<\kappa} Y_\xi$, then, for each $\xi < \kappa$, there exists a non-zero $\alpha_\xi < \gamma$ such that $\{\underline{b} \in \mathcal{T}_\gamma^0 \mid b_{\alpha_\xi} = a_{\alpha_\xi}\} \cap Y_\xi = \emptyset$. Letting $\alpha^* = \bigcup_{\xi<\kappa} \alpha_\xi$, we have that $\{\underline{b} \in \mathcal{T}_\gamma^0 \mid b_{\alpha^*} = a_{\alpha^*}\} \cap \bigcup_{\xi<\kappa} Y_\xi = \emptyset$. Hence $\bigcup_{\xi<\kappa} Y_\xi$ is closed. We conclude that, in $\mathcal{T}_\gamma^0$ (and hence in $\mathcal{T}_\gamma$), unions of strictly less than $cf(\gamma)$ many closed sets are closed.

In particular, the X_δ in Theorem 11 are closed whenever $|X_\delta| < cf(\gamma)$ for all $\delta \in \Sigma^+$.

Definition 8 For any $0 < \alpha$, let $\mathcal{T}'_\alpha = \mathcal{T}_\alpha \setminus \bigcup_{0<\beta<\alpha} \mathcal{H}_\alpha(\mathcal{T}_\beta)$. $\square$

It seems very plausible to conjecture that

$$0 < \alpha < \beta \wedge a \in \mathcal{T}'_\beta \Rightarrow \mathcal{H}_\alpha(a) \in \mathcal{T}'_\alpha.$$

However, that is false, essentially because the statement

$$a \in \mathcal{T}'_{\alpha+1} \Leftrightarrow \exists \delta \in \Sigma^+.\exists b \in \mathcal{T}'_\alpha.\langle \delta, b\rangle \in a$$

fails whenever α is a limit ordinal, which is easily seen by Theorem 11, once we observe that $\mathcal{T}'_\alpha = \mathcal{T}_\alpha \setminus \mathcal{T}^c_\alpha$ for such α. In particular, if we fix γ and pick $a \in \mathcal{T}'_{\gamma+1}$ (recall that $|\mathcal{T}_{\gamma+1}| = |\mathcal{P}(\Sigma^+ \times \mathcal{T}_\gamma)| \geq 2^{|\mathcal{T}_\gamma|} > |\mathcal{T}_\gamma|$), it is not difficult to see that, for any $n < \omega$, we have

$$0 < m \leq n \Rightarrow \mathcal{H}_{\gamma+m}(\overbrace{\tau.\ldots.\tau}^{n}.a) = \mathcal{H}_{\gamma+m}(\overbrace{\tau.\ldots.\tau}^{n}.\mathcal{H}_\gamma(a)) \notin \mathcal{T}'_{\gamma+m},$$

in spite of the fact that $\mathcal{H}_{\gamma+n+1}(\overbrace{\tau.\ldots.\tau}^{n}.a) \in \mathcal{T}'_{\gamma+n+1}$.

To construct a counter-example to the statement 'if $\underline{a}' \in \mathcal{T}'_{\gamma'}$, then there is a non-zero $\alpha < \gamma'$ such that $\alpha \leq \beta < \gamma' \Rightarrow a'_\beta \in \mathcal{T}'_\beta$', first let $b_0 = a$ and $b_{n+1} = a + \tau.\tau.b_n$ for each $n < \omega$. Then we have

$$0 < m < \omega \Rightarrow (\mathcal{H}_{\gamma+m}(b_n) \in \mathcal{T}'_{\gamma+m} \Leftrightarrow (2 \not| m \wedge m \leq 2n+1)).$$

Hence, it follows that

$$0 < m < \omega \Rightarrow (\mathcal{H}_{\gamma+m+1}\left(\sum_{n<\omega}\tau.b_n\right) \in \mathcal{T}'_{\gamma+m+1} \Leftrightarrow 2 \not| m).$$

[14]Here $\sum_{\delta'\in\Sigma^+}\sum_{\underline{b}\in X_{\delta'}}\delta'.\underline{b}$ is a "new" point whose transitions are given by $\sum_{\delta'\in\Sigma^+}\sum_{\underline{b}\in X_{\delta'}}\delta'.\underline{b} \xrightarrow{\delta} \underline{b}$ (with subsequent behavious being that of $\underline{b} \in \mathcal{T}_\gamma$) for $\delta \in \Sigma^+$, $\underline{b} \in X_\delta$.

A counter-example to 'if $\underline{a'} \in \mathcal{T}^c_{\gamma'}$, then $\underline{a'}$ is an isolated point of $\mathcal{T}_{\gamma'}$' can be constructed as follows. For any $n < \omega$, let

$$c_n = (\overbrace{\tau. \dots .\tau}^{n+1}.a) + \left(\sum_{m \in \omega \setminus \{n\}} \overbrace{\tau. \dots .\tau}^{m+1}.\mathcal{H}_\gamma(a) \right),$$

and let

$$d = \sum_{m<\omega} \overbrace{\tau. \dots .\tau}^{m+1}.\mathcal{H}_\gamma(a).$$

Then $\mathcal{H}_{\gamma+n+1}(c_n) = \mathcal{H}_{\gamma+n+1}(d)$, but $\mathcal{H}_{\gamma+n+2}(c_n) \in \mathcal{T}'_{\gamma+n+2}$, so that $\mathcal{H}_{\gamma+n+2}(c_n) \neq \mathcal{H}_{\gamma+n+2}(d)$ (since $\mathcal{H}_{\gamma+n+2}(d) = \mathcal{H}_{\gamma+n+2}(\mathcal{H}_{\gamma+1}(d)))$. Hence $\mathcal{H}_{\gamma+\omega}(d)$ is not an isolated point of $\mathcal{T}_{\gamma+\omega}$ (although $\mathcal{H}_{\gamma+\omega}(d) \in \mathcal{T}^c_{\gamma+\omega}$).

Theorem 12 *Given $\underline{a} \in \mathcal{T}^c_{\gamma^*}$, let α^* be the smallest ordinal such that $0 < \alpha^* < \gamma^*$ and $\alpha^* \leq \alpha < \gamma^* \Rightarrow a_\alpha \notin \mathcal{T}'_\alpha$.*

(a) $\alpha^* = \alpha' + 1$ for some $\alpha' > 0$, and we have $\alpha' \leq \alpha < \gamma^* \Rightarrow a_\alpha = \mathcal{H}_\alpha(a_{\alpha'})$.

(b) $0 < n < \omega \wedge n < \alpha^* \Rightarrow a_n \in \mathcal{T}'_n$.

(c) $\gamma < \alpha^* \Rightarrow a_\gamma \in \mathcal{T}'_\gamma$.

(d) If $\gamma + \omega < \alpha^*$, then $a_{\gamma+n} \in \mathcal{T}'_{\gamma+n}$ for infinitely many $n < \omega$.

Proof We claim:

(i) If $0 < n < m < \omega$ and $a_m \in \mathcal{T}'_m$, then $a_n \in \mathcal{T}'_n$.

(ii) If $\beta < \beta' < \gamma^*$ are such that $\beta < \xi \leq \beta' \Rightarrow a_\xi \notin \mathcal{T}'_\xi$, then $a_{\beta'} = \mathcal{H}_{\beta'}(a_\beta)$.

(iii) If $\gamma < \gamma^*$, then $a_\gamma \in \mathcal{T}'_\gamma$ iff $\{\beta \mid 0 < \beta < \gamma \wedge a_\beta \in \mathcal{T}'_\beta\}$ is cofinal in γ.

(iv) If $n < \omega$, $\gamma < \gamma^*$ and $a_{\gamma+n} \in \mathcal{T}'_{\gamma+n}$, then $a_\gamma \in \mathcal{T}'_\gamma$.

It is easy to see that (a)–(d) follow from (i)–(iv), so it remains to prove (i)–(iv).

For (i), observe that, if $0 < k < \omega$ and $b \in \mathcal{T}_k$, then $b \in \mathcal{T}'_k$ iff b can perform $k-1$ consecutive transitions.

For (ii), we prove

$$\beta \leq \xi \leq \beta' \Rightarrow a_\xi = \mathcal{H}_\xi(a_\beta)$$

by transfinite induction on ξ.
Base case. Trivial.
Successor case. Suppose $\beta \leq \xi < \beta'$ and $\alpha_\xi = \mathcal{H}_\xi(a_\beta)$. Then, since $a_{\xi+1} \notin \mathcal{T}'_{\xi+1}$, we have that $a_{\xi+1} = \mathcal{H}_{\xi+1}(b)$ for some $b \in \mathcal{T}_\xi$. But then

$$a_\xi = \mathcal{H}_\xi(a_{\xi+1}) = \mathcal{H}_\xi(\mathcal{H}_{\xi+1}(b)) = \mathcal{H}_\xi(b) = b,$$

and hence

$$a_{\xi+1} = \mathcal{H}_{\xi+1}(a_\xi) = \mathcal{H}_{\xi+1}(\mathcal{H}_\xi(a_\beta)) = \mathcal{H}_{\xi+1}(a_\beta).$$

Limit case. If $\beta < \gamma \leq \beta'$ is such that $\beta \leq \xi < \gamma \Rightarrow a_\xi = \mathcal{H}_\xi(a_\beta)$, then

$$a_\gamma = \langle a_\xi \mid 0 < \xi < \gamma \rangle = \langle \mathcal{H}_\xi(a_\alpha) \mid 0 < \xi < \gamma \rangle = \mathcal{H}_\gamma(a_\alpha).$$

The 'if' part of (iii) is trivial, and the 'only if' part follows at once from (ii). (Recall that $\mathcal{T}'_\gamma = \mathcal{T}_\gamma \setminus \mathcal{T}^c_\gamma$.)

To prove (iv), consider first $a_{\gamma+n} \in \mathcal{T}'_{\gamma+n}$ with $n \geq 2$. Then $a_{\gamma+n} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_1} \mathcal{H}_{\gamma+n}(b)$ for some $b \in \mathcal{T}'_{\gamma+1}$ (since $a' \in \mathcal{T}'_{\gamma+m+1} \Leftrightarrow \exists \delta \in \Sigma^+ . \exists b' \in \mathcal{T}'_{\gamma+m}.\langle \delta, b' \rangle \in a'$ whenever $m \geq 1$), and so $a_\gamma = \mathcal{H}_\gamma(a_{\gamma+n}) \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_1} \mathcal{H}_\gamma(\mathcal{H}_{\gamma+n}(b)) = \mathcal{H}_\gamma(b)$.

Now $\mathcal{T}^c_\gamma$ is a subsystem of $\mathcal{T}_\gamma$, and hence it suffices to show that $a_{\gamma+1} \in \mathcal{T}'_{\gamma+1} \Rightarrow a_\gamma \in \mathcal{T}'_\gamma$. This will in turn follow once we establish that $\mathcal{H}_\gamma$ is a morphism on $\mathcal{X} = (\mathcal{H}_\gamma^{\mathcal{T}_{\gamma+1}})^{-1}(\mathcal{T}^c_\gamma)$ (which is a subsystem of $\mathcal{T}_{\gamma+1}$), because $\mathcal{H}_\gamma$ must then be injective on $\mathcal{X}$, so that $\mathcal{X} = \mathcal{H}_{\gamma+1}(\mathcal{T}^c_\gamma)$, and hence $a_\gamma \notin \mathcal{T}'_\gamma \Rightarrow a_{\gamma+1} \in \mathcal{X} \subseteq \mathcal{H}_{\gamma+1}(\mathcal{T}_\gamma)$.

We claim that, for any $0 < \beta < \gamma$, $\mathcal{H}_\gamma(\mathcal{T}_\beta)$ is a closed subset of $\mathcal{T}_\gamma$. Suppose $\underline{b} \in \mathcal{T}_\gamma \setminus \mathcal{H}_\gamma(\mathcal{T}_\beta)$. Then $b_{\beta'} \neq \mathcal{H}_{\beta'}(b_\beta)$ for some $\beta < \beta' < \gamma$, which gives us that $\{\underline{b'} \in \mathcal{T}_\gamma \mid b'_{\beta'} = b_{\beta'}\} \cap \mathcal{H}_\gamma(\mathcal{T}_\beta) = \emptyset$, and the claim is established. Now, if $c \in \mathcal{X}$, then $\mathcal{H}_\gamma(c) \in \mathcal{H}_\gamma(\mathcal{T}_\beta)$ for some non-zero $\beta < \gamma$, and hence

$$(\mathcal{H}_\gamma(c))_{\mathcal{T}_\gamma,\delta} = \overline{\mathcal{H}_\gamma(c_{\mathcal{X},\delta})}^{\mathcal{T}_\gamma} = \overline{\mathcal{H}_\gamma(c_{\mathcal{X},\delta})}^{\mathcal{H}_\gamma(\mathcal{T}_\beta)} = \mathcal{H}_\gamma(c_{\mathcal{X},\delta})$$

for each $\delta \in \Sigma^+$ by Lemma 10 (b) (observe that $\mathcal{H}_\gamma(\mathcal{T}_\beta)$ is a discrete subspace of $\mathcal{T}_\gamma$). □

If $\underline{a}$ is an isolated point in $\mathcal{T}_\gamma$, then $\{\underline{a'} \in \mathcal{T}_\gamma \mid a'_\alpha = a_\alpha\} = \{\underline{a}\}$ for some non-zero $\alpha < \gamma$, so that $\underline{a} = \mathcal{H}_\gamma(a_\alpha) \in \mathcal{T}^c_\gamma$. In the other direction, if $\underline{a} \in \mathcal{T}^c_\gamma$ and γ is not of the form $\gamma' + \omega$, then it is an easy consequence of Theorem 12 that $\underline{a}$ is isolated.

Let t_α for ordinals α be points such that the transitions of any t_α are given by $t_\alpha \xrightarrow{\tau} t_\beta$ for all $\beta < \alpha$. Then $\mathcal{O}_\alpha = \{t_\beta \mid \beta < \alpha\}$ is a system for any α.

Theorem 13 *For any $\alpha > 0$, $\mathcal{H}_\alpha$ is an injective morphism on $\mathcal{O}_\alpha$, but not a morphism on $\mathcal{O}_{\alpha+1}$.*[15]

Proof It easily follows by transfinite inductions that every $\mathcal{O}_\alpha$ is strongly extensional, and that, for any $\alpha > 0$, $\mathcal{H}_\alpha$ is a morphism on $\mathcal{O}_\alpha$. We claim that:

(i) $(\mathcal{H}_{\alpha+1}(t_{\alpha+1}))_{\mathcal{T}_{\alpha+1},\tau} = \mathcal{H}_{\alpha+1}(\{t_\beta \mid \beta < \alpha\} \cup \{\mathcal{H}_\alpha(t_\alpha)\})$ for any $\alpha > 0$.

(ii) $(\mathcal{H}_\gamma(t_\gamma))_{\mathcal{T}_\gamma,\tau} = \{\mathcal{H}_\gamma(t_\beta) \mid \beta < \gamma\} \cup \{\mathcal{H}_\gamma(t_\gamma)\}$ for any γ.

We will then have that $\mathcal{H}_\alpha(t_\alpha)$ can perform an infinite sequence of $\xrightarrow{\tau}$ transitions whenever $\alpha \geq \omega$. Also, for any non-zero $n < \omega$, t_n can perform n consecutive transitions, whereas any $a \in \mathcal{T}_n$ can perform at most $n-1$. Hence we will have that no $\mathcal{H}_\alpha$ for $\alpha > 0$ is a morphism on $\mathcal{O}_{\alpha+1}$ (observe that no t_α can perform infinitely many consecutive transitions). Therefore, it suffices to prove (i) and (ii).

Now (i) is trivial, since we know that $\mathcal{H}_\alpha$ is a morphism on $\mathcal{O}_\alpha$.

For (ii), we first prove

$$0 < \alpha \leq \beta \Rightarrow \mathcal{H}_\alpha(t_\beta) = \mathcal{H}_\alpha(t_\alpha)$$

by transfinite induction on α.

Base case. Trivial.

[15] A version of this result was known to Forti and Honsell — see Remark 1.5 in [FH 89].

Successor case. Suppose $\beta \geq \alpha \Rightarrow \mathcal{H}_\alpha(t_\beta) = \mathcal{H}_\alpha(t_\alpha)$ for some $\alpha > 0$. Then, for any $\beta \geq \alpha + 1$, we have:

$$\begin{aligned} \mathcal{H}_{\alpha+1}(t_\beta) &= \{\langle \tau, \mathcal{H}_\alpha(t_{\beta'}) \rangle \mid \beta' < \beta\} \\ &= \{\langle \tau, \mathcal{H}_\alpha(t_{\beta'}) \rangle \mid \beta' < \alpha\} \cup \{\langle \tau, \mathcal{H}_\alpha(t_\alpha) \rangle\} \\ &= \{\langle \tau, \mathcal{H}_\alpha(t_{\beta'}) \rangle \mid \beta' \leq \alpha\} \\ &= \mathcal{H}_{\alpha+1}(t_{\alpha+1}). \end{aligned}$$

Limit case. Suppose $\beta \geq \alpha \Rightarrow \mathcal{H}_\alpha(t_\beta) = \mathcal{H}_\alpha(t_\alpha)$ for all non-zero $\alpha < \gamma$. Then, for any $\beta \geq \gamma$, we have:

$$\begin{aligned} \mathcal{H}_\gamma(t_\beta) &= \langle \mathcal{H}_\alpha(t_\beta) \mid 0 < \alpha < \gamma \rangle \\ &= \langle \mathcal{H}_\alpha(t_\alpha) \mid 0 < \alpha < \gamma \rangle \\ &= \mathcal{H}_\gamma(t_\gamma). \end{aligned}$$

Now, Lemma 10 (b) gives us that $(\mathcal{H}_\gamma(t_\gamma))_{\mathcal{T}_\gamma, \tau} = \overline{\{\mathcal{H}_\gamma(t_\beta) \mid \beta < \gamma\}}^{\mathcal{T}_\gamma}$. For any $\beta < \gamma$, we have $\mathcal{H}_\gamma(t_\beta) = \langle \mathcal{H}_{\beta'}(t_\beta) \mid 0 < \beta' < \gamma \rangle$, and we know that $\mathcal{H}_{\beta'}(t_\beta) = \mathcal{H}_{\beta'}(t_{\beta'})$ whenever $0 < \beta' \leq \beta$. Also $\mathcal{H}_{\beta'}(t_\beta) \neq \mathcal{H}_{\beta'}(t_{\beta'})$ whenever $\beta + 1 < \beta' < \gamma$, since if $\beta' < \omega$, then $\mathcal{H}_{\beta'}(t_{\beta'}) \in \mathcal{T}'_{\beta'}$ and $\mathcal{H}_{\beta'}(t_\beta) = \mathcal{H}_{\beta'}(\mathcal{H}_{\beta+1}(t_\beta)) \notin \mathcal{T}'_{\beta'}$, and if $\beta' \geq \omega$, then $\mathcal{H}_{\beta'}(t_{\beta'})$ can perform infinitely many consecutive transitions. Hence, recalling that $\mathcal{H}_\gamma(t_\gamma) = \langle \mathcal{H}_\beta(t_\beta) \mid 0 < \beta < \gamma \rangle$, it is easy to see that $\overline{\{\mathcal{H}_\gamma(t_\beta) \mid \beta < \gamma\}}^{\mathcal{T}_\gamma} = \{\mathcal{H}_\gamma(t_\beta) \mid \beta < \gamma\} \cup \{\mathcal{H}_\gamma(t_\gamma)\}$. $\square$

We observed that $\mathcal{H}_n(t_n) \in \mathcal{T}'_n$ whenever $0 < n < \omega$. Consider any γ. Then $\mathcal{H}_\gamma(t_\gamma) \in \mathcal{T}'_\gamma$, but $\mathcal{H}_{\gamma+1}(t_{\gamma+1}) = \mathcal{H}_{\gamma+1}(\mathcal{H}_\gamma(t_\gamma)) \notin \mathcal{T}'_{\gamma+1}$. Whenever $2 \leq n < \omega$, we again have $\mathcal{H}_{\gamma+n}(t_{\gamma+n}) \in \mathcal{T}'_{\gamma+n}$ (since $\langle \tau, \mathcal{H}_{\gamma+n-1}(t_{\gamma+n-2}) \rangle \in \mathcal{H}_{\gamma+n}(t_{\gamma+n})$ and $\mathcal{H}_{\gamma+n-1}(t_{\gamma+n-2}) \in \mathcal{T}'_{\gamma+n-1}$).

For any γ, we know that

$$|\mathcal{T}_\gamma| \leq |\mathcal{T}^0_\gamma| \leq |\mathcal{T}^c_\gamma|^{|\gamma|} \leq 2^{|\mathcal{T}^c_\gamma| \times |\gamma|} = 2^{|\mathcal{T}^c_\gamma|}$$

($|\mathcal{T}^c_\gamma| \geq |\gamma|$ by Theorem 13). If $\mathcal{X}, \mathcal{X}' \subseteq \mathcal{T}^c_\gamma$ are distinct, it is not difficult to see that

$$\mathcal{H}_\gamma\left(\sum_{0<\alpha<\gamma \wedge a \in \mathcal{X} \cap \mathcal{T}'_\alpha} \prec t_\alpha, a \succ\right) \neq \mathcal{H}_\gamma\left(\sum_{0<\alpha<\gamma \wedge a' \in \mathcal{X}' \cap \mathcal{T}'_\alpha} \prec t_\alpha, a' \succ\right),$$

where $\prec d, e \succ$ is an abbreviation for $(\tau.\tau.d) + \tau.(\tau.d + \tau.e)$[16]. Hence in fact $|\mathcal{T}_\gamma| = 2^{|\mathcal{T}^c_\gamma|}$.

Definition 9 We say that a tree $\langle W, \leq \rangle$[17] is a γ-*special tree* iff:

[16] Note how this expression corresponds to $\{\{d\}, \{d, e\}\}$, which is the standard Kuratowski's set-theoretic definition of an ordered pair $\langle d, e \rangle$.

[17] A tree is a partial order $\langle W, \leq \rangle$ such that $\{y \in W \mid y < x\}$ is well-ordered by $<$ for each $x \in W$. For any $x \in W$, we write $ht(x, W)$ for $type(\langle \{y \in W \mid y < x\}, < \rangle)$. For any α, $Lev_\alpha(W) = \{x \in W \mid ht(x, W) = \alpha\}$, and we take $ht(W)$ to be the smallest α with $Lev_\alpha(W) = \emptyset$. A subtree of $\langle W, \leq \rangle$ is a downwards-closed $W' \subseteq W$ with the order induced by $\leq$.

For any maximal chain $C \subseteq W$, let $h(C) \leq ht(W)$ be such that C contains exactly one element of $Lev_\alpha(W)$ for $\alpha < h(C)$, and no elements of $Lev_\alpha(W)$ for $h(C) \leq \alpha < ht(W)$. A path through W is a maximal chain $C \subseteq W$ with $h(C) = ht(W)$.

It will sometimes be convenient to "relabel" the indices so that $W = \bigcup_{0<\alpha<ht(W)} Lev_\alpha(W)$ and $\forall x \in Lev_\alpha(W). ht(x, W) = \alpha$ whenever $0 < \alpha < ht(W)$.

If κ is regular, a κ-Aronszajn tree is a tree $\langle W, \leq \rangle$ such that $ht(W) = \kappa$, $\forall \alpha < \kappa. |Lev_\alpha(W)| < \kappa$, and there are no paths through W.

For an introductory account of trees, see e.g. Chapter 2 of [Kun 80].

(a) $ht(W) = \gamma$,

(b) $(x \in W \wedge ht(x, W) < \alpha < \gamma) \Rightarrow \exists y \in Lev_\alpha(W). x \leq y$,

(c) There exists a strictly increasing sequence $\alpha \mapsto \eta_\alpha : \gamma \to \gamma \setminus \{0\}$ such that $\forall \alpha < \gamma. |Lev_\alpha(W)| \leq |\mathcal{T}_{\eta_\alpha}|$, and

(d) There are no paths through W. $\square$

We say that γ is *ω-like* iff either $cf(\gamma) = \omega$ or γ is a weakly compact cardinal[18].

In Section 2 of [FH 89], Forti and Honsell essentially establish the following. (It is obvious that if $\gamma' > \gamma$, $cf(\gamma') = cf(\gamma)$ and a γ-special tree exists, then a γ'-special tree exists.)

Theorem 14 *There are no γ-special trees iff γ is ω-like.* $\square$

Given any γ,

$$\underline{a} \leftrightarrow \mathcal{K}_{\underline{a}} = \{X \subseteq \mathcal{T}_\gamma \mid \exists \alpha. 0 < \alpha < \gamma \wedge \{\underline{b} \in \mathcal{T}_\gamma \mid b_\alpha = a_\alpha\} \subseteq X\}$$

gives a 1–1 correspondence between points of $\mathcal{T}^0_\gamma$ and minimal Cauchy filters on $\mathcal{T}_\gamma$, such that $\mathcal{K}_{\underline{a}}$ converges iff $\underline{a} \in \mathcal{T}_\gamma$, and in that case the limit point of $\mathcal{K}_{\underline{a}}$ is $\underline{a}$. Hence $\mathcal{T}^0_\gamma$ with the identity mapping from $\mathcal{T}_\gamma$ into $\mathcal{T}^0_\gamma$ is a completion of $\mathcal{T}_\gamma$. In particular, $\mathcal{T}_\gamma$ is complete iff $\mathcal{T}_\gamma = \mathcal{T}^0_\gamma$.

Theorem 15 *$\mathcal{T}_\gamma$ is complete iff there are no γ-special trees.*[19]

Proof For the 'if' part, suppose $\mathcal{T}_\gamma$ is not complete, and pick $\underline{a^0} \in \mathcal{T}^0_\gamma \setminus \mathcal{T}_\gamma$. By Theorem 6, $\underline{a^0} \in \mathcal{T}^{\beta_0}_\gamma \setminus \mathcal{T}^{\beta_0+1}_\gamma$ for some $\beta_0 < |\mathcal{T}^0_\gamma|^+$. Therefore, for some $0 < \alpha^* < \gamma$ and $\langle \delta_0, b^* \rangle \in a^0_{\alpha^*+1}$, there is no $\underline{c} \in \underline{a^0}_{\mathcal{T}^{\beta_0}_\gamma, \delta_0}$ such that $b^* = c_{\alpha^*}$. If for some $\underline{a^1} \in \underline{a^0}_{\mathcal{T}^0_\gamma, \delta_0}$ we have $b^* = a^1_{\alpha^*}$, then there exists $\beta_1 < \beta_0$ such that $\underline{a^1} \in \mathcal{T}^{\beta_1}_\gamma \setminus \mathcal{T}^{\beta_1+1}_\gamma$, and we proceed as with $\underline{a^0}$.

After finitely many steps, we will arrive at some $\underline{a^n} \in \mathcal{T}^{\beta_n}_\gamma \setminus \mathcal{T}^{\beta_n+1}_\gamma$ such that, for some non-zero $\alpha^* < \gamma$ and $\langle \delta_n, b^* \rangle \in a^n_{\alpha^*+1}$, there is no $\underline{c} \in \underline{a^n}_{\mathcal{T}^0_\gamma, \delta_n}$ such that $b^* = c_{\alpha^*}$. Then the tree $W = \bigcup_{0<\alpha<\gamma} Lev_\alpha(W)$ given by

$$\begin{array}{rcl} Lev_\alpha(W) & = & \{\langle \alpha, \mathcal{H}_\alpha(b^*) \rangle\} \quad (0 < \alpha \leq \alpha^*) \\ Lev_\alpha(W) & = & \{\langle \alpha, b \rangle \mid \langle \delta_n, b \rangle \in a^n_{\alpha+1} \wedge b^* = \mathcal{H}_{\alpha^*}(b)\} \quad (\alpha^* < \alpha < \gamma), \end{array}$$

and with the order induced by the maps $\mathcal{H}^{\mathcal{T}_{\alpha'}}_\alpha$ $(0 < \alpha < \alpha' < \gamma)$ is a γ-special tree.[20]

For the 'only if' part, suppose that $\langle W, \leq \rangle$ is a γ-special tree. Suppose also that $\underline{u} \in \mathcal{T}^0_\gamma$ and $n^* < \omega$ are such that $\gamma' < \gamma \wedge n \geq n^* \Rightarrow u_{\gamma'+n} \in \mathcal{T}'_{\gamma'+n}$. (We can take $\underline{u} = \mathcal{H}_\gamma(t_\gamma)$ and $n^* = 2$.) We fix a mapping $\alpha \mapsto \eta_\alpha : \gamma \to \gamma \setminus \{0\}$ associated to W such that $\forall \alpha < \gamma. \neg(\exists \gamma' < \gamma. \exists n. 0 \leq n < n^* \wedge \eta_\alpha = \gamma' + n)$,[21] and then we fix an injective mapping F with domain W such that $F(Lev_\alpha(W)) \subseteq \mathcal{T}'_{\eta_\alpha+1}$ for all $\alpha < \gamma$.

[18] κ is weakly inaccessible iff κ is a regular limit cardinal.

κ is strongly inaccessible iff $\kappa > \omega$, κ is a regular cardinal, and $2^\lambda < \kappa$ whenever $\lambda < \kappa$. (In particular, κ is then weakly inaccessible.)

κ is weakly compact iff κ is strongly inaccessible and there are no κ-Aronszajn trees.

[19] This result is essentially established in [FH 89]. We give an alternative proof which provides us with some additional information to be used later in the paper.

[20] The pairing with α ensures explicitly that $\alpha \neq \alpha' \Rightarrow Lev_\alpha(W) \cap Lev_{\alpha'}(W) = \emptyset$.

[21] It is trivial to obtain such a mapping from any mapping associated with W.

For any $x \in W$ and $c \in \mathcal{X} = \bigcup_{0<\alpha<\gamma} \mathcal{T}'_{\eta_\alpha+1}$, we define an ordinal $\zeta_{x,c}$ as follows. If $c = F(x')$ for some $x' \leq x$, then let $\zeta_{x,c} = \eta_{ht(x',W)}$. Otherwise, let $\zeta_{x,c} = \eta_{ht(x,W)}+1$. Now, for any $x \in W$, let

$$G_{\underline{u}}(x) = \mathcal{H}_{\eta_{ht(x,W)}+4}\left(\sum_{c\in\mathcal{X}} \tau. \prec c, u_{\zeta_{x,c}} \succ\right).$$

It is not difficult to see the following:

(i) $(ht(x,W) = ht(y,W) \wedge x \neq y) \Rightarrow G_{\underline{u}}(x) \neq G_{\underline{u}}(y)$.

(ii) $x \leq y \Rightarrow \mathcal{H}_{\eta_{ht(x,W)}+4}(G_{\underline{u}}(y)) = G_{\underline{u}}(x)$.

Let W' be the tree of height γ given by

$$Lev_\alpha(W') = \{\langle \alpha, G_{\underline{u}}(x)\rangle \mid x \in Lev_\alpha(W)\}$$

for all $\alpha < \gamma$, with the order induced by the maps $\mathcal{H}^{\mathcal{T}_{\eta_{\alpha'}+4}}_{\eta_\alpha+4}$ $(\alpha < \alpha' < \gamma)$. (i) and (ii) give us that $ht(x,W) \leq ht(y,W) \wedge \mathcal{H}_{\eta_{ht(x,W)}+4}(G_{\underline{u}}(y)) = G_{\underline{u}}(x) \Rightarrow x \leq y$, and so we have that $x \mapsto \langle ht(x,W), G_{\underline{u}}(x)\rangle$ is an isomorphism between W and W'.

Now, for any $\alpha < \gamma$, let

$$a_{\eta_\alpha+5} = \{\langle \tau, G_{\underline{u}}(x)\rangle \mid x \in Lev_\alpha(W)\}.$$

Then $\alpha < \alpha' \Rightarrow \mathcal{H}_{\eta_\alpha+5}(a_{\eta_{\alpha'}+5}) = a_{\eta_\alpha+5}$ (by (i) and Definition 9 (b)), and hence the $a_{\eta_\alpha+5}$ can be uniquely extended to an $\underline{a} \in \mathcal{T}^0_\gamma$. To see that $\underline{a} \notin \mathcal{T}_\gamma$, pick an $x \in Lev_0(W)$. Then $\langle \tau, G_{\underline{u}}(x)\rangle \in a_{\eta_0+5}$, but any $\underline{b} \in \underline{a}_{\mathcal{T}^0_\gamma,\tau}$ would clearly yield a path $\{\langle \alpha, b_{\eta_\alpha+4}\rangle \mid \alpha < \gamma\}$ through W', which is a contradiction. Hence $\underline{a} \notin \mathcal{T}^1_\gamma \supseteq \mathcal{T}_\gamma$. □

In particular, as long as $cf(\gamma) = \omega$, all the $\mathcal{T}_\gamma$ are complete. In fact, in [Rosc 82], $\mathcal{T}_\gamma = \mathcal{T}^0_\gamma$ served as the definition of $\mathcal{T}_\gamma$ for such γ.

For any γ, let:

$$\begin{aligned}
\mathcal{Q}_\gamma &= \{\underline{a}^0 \in \mathcal{T}^0_\gamma \mid n < \omega \wedge \underline{a}^0 \xrightarrow{\delta_0} \ldots \xrightarrow{\delta_{n-1}} \underline{a}^n \wedge \langle \delta_n, b^*\rangle \in a^n_{\alpha+1} \Rightarrow \\
&\quad \exists \underline{b} \in \underline{a}^n{}_{\mathcal{T}^0_\gamma,\delta_n}.b^* = b_\alpha\},\\
\mathcal{Q}'_\gamma &= \{\underline{a} \in \mathcal{T}_\gamma \mid \mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{a}) = \underline{a}\},\\
\mathcal{Q}''_\gamma &= \mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\mathcal{T}^0_\gamma).
\end{aligned}$$

Theorem 16 *(a) $\mathcal{Q}_\gamma$ is the largest subset of $\mathcal{T}_\gamma$ which is a subsystem of $\mathcal{T}^0_\gamma$.*

(b) $\mathcal{Q}'_\gamma$ is the largest subset of $\mathcal{T}^0_\gamma$ containing $\mathcal{T}^c_\gamma$ on which $\mathcal{V}_\gamma$ and $\mathcal{U}^{\mathcal{T}^0_\gamma}_\gamma$ induce the same topology.

(c) If γ is ω-like, then $\mathcal{T}^c_\gamma \subset \mathcal{Q}_\gamma = \mathcal{Q}'_\gamma = \mathcal{Q}''_\gamma = \mathcal{T}_\gamma = \mathcal{T}^0_\gamma$.

(d) If γ is not ω-like, then $\mathcal{T}^c_\gamma \subset \mathcal{Q}_\gamma \subset \mathcal{Q}'_\gamma \subset \mathcal{Q}''_\gamma \subseteq \mathcal{T}_\gamma \subset \mathcal{T}^0_\gamma$.

Proof For (a), observe first that $\underline{a}^0 \in \mathcal{Q}_\gamma \wedge \underline{a}^0 \xrightarrow{\delta_0} \underline{a}^1 \Rightarrow \underline{a}^1 \in \mathcal{Q}_\gamma$, so that $\mathcal{Q}_\gamma$ is a subsystem of $\mathcal{T}^0_\gamma$. Then $\mathcal{Q}_\gamma$ is a fixed point of Φ, and hence $\mathcal{Q}_\gamma \subseteq \mathcal{T}_\gamma$ by Theorem 6. If $\mathcal{S} \subseteq \mathcal{T}_\gamma$ is a subsystem of $\mathcal{T}^0_\gamma$, then $\mathcal{S}$ is a fixed point of Φ since $\mathcal{T}_\gamma$ is, and so $\mathcal{S} \subseteq \mathcal{Q}_\gamma$ by the definition of $\mathcal{Q}_\gamma$.

Now, we claim that $\mathcal{T}^c_\gamma \subset \mathcal{Q}_\gamma \subseteq \mathcal{Q}'_\gamma \subseteq \mathcal{Q}''_\gamma \subseteq \mathcal{T}_\gamma \subseteq \mathcal{T}^0_\gamma$ for any γ. Since $\mathcal{T}^c_\gamma$ is a subsystem of $\mathcal{T}^0_\gamma$, we have $\mathcal{T}^c_\gamma \subseteq \mathcal{Q}_\gamma$, and e.g. $\mathcal{H}_\gamma(t_\gamma) \in \mathcal{Q}_\gamma \setminus \mathcal{T}^c_\gamma$. Also, $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma \restriction \mathcal{Q}_\gamma = \mathcal{H}^{\mathcal{T}_\gamma}_\gamma \restriction \mathcal{Q}_\gamma$, and so $\mathcal{Q}_\gamma \subseteq \mathcal{Q}'_\gamma$. The remaining inclusions are trivial.

Since $a_\alpha = \mathcal{H}^{\mathcal{T}^0_\gamma}_\alpha(\underline{a})$ for all $\underline{a} \in \mathcal{Q}'_\gamma$, $0 < \alpha < \gamma$, the uniformities (and hence the topologies) on $\mathcal{Q}'_\gamma$ induced by $\mathcal{V}_\gamma$ and $\mathcal{U}^{\mathcal{T}^0_\gamma}_\gamma$ are the same. So suppose that $\mathcal{T}^c_\gamma \subseteq \mathcal{S} \subseteq \mathcal{T}^0_\gamma$ and that $\mathcal{V}_\gamma$ and $\mathcal{U}^{\mathcal{T}^0_\gamma}_\gamma$ induce the same topology on $\mathcal{S}$. Consider any $\underline{a} \in \mathcal{S}$, and let $\mathcal{X} = \{\mathcal{H}_\gamma(\mathcal{H}^{\mathcal{T}^0_\gamma}_\alpha(\underline{a})) \mid 0 < \alpha < \gamma\}$. Then any $\mathcal{O} \subseteq \mathcal{S}$ which is open with respect to $\mathcal{U}^{\mathcal{T}^0_\gamma}_\gamma$ and such that $\underline{a} \in \mathcal{O}$ intersects $\mathcal{X}$. Hence any $\mathcal{O} \subseteq \mathcal{S}$ open with respect to $\mathcal{V}_\gamma$ such that $\underline{a} \in \mathcal{O}$ intersects $\mathcal{X}$, and so $\underline{a} = \mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{a})$. Therefore, $\mathcal{S} \subseteq \mathcal{Q}'_\gamma$, which establishes (b).

If γ is ω-like, we know that $\mathcal{T}_\gamma = \mathcal{T}^0_\gamma$, and so (c) follows at once from (a).

For (d), suppose γ is not ω-like. Then we can pick $\underline{a} \in \mathcal{T}^0_\gamma \setminus \mathcal{T}_\gamma$ such that $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{a}) = \mathcal{H}_\gamma(\emptyset)$ (such an $\underline{a}$ is given by the proof of Theorem 15), and let $\underline{b} = \mathcal{H}_\gamma(\sum_{0<\alpha<\gamma} \tau.a_\alpha)$. By Lemma 10 (a), we have

$$\underline{b}_{\mathcal{T}^0_\gamma,\tau} = \overline{\{\mathcal{H}_\gamma(a_\alpha) \mid 0 < \alpha < \gamma\}}^{\mathcal{T}^0_\gamma} = \{\mathcal{H}_\gamma(a_\alpha) \mid 0 < \alpha < \gamma\} \cup \{\underline{a}\},$$

and so $\underline{b} \notin \mathcal{Q}_\gamma$. Also, since $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{a}) = \mathcal{H}_\gamma(\emptyset) = \mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\mathcal{H}_\gamma(a_1))$, it follows that $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{b}) = \underline{b}$, and hence $\underline{b} \in \mathcal{Q}'_\gamma \setminus \mathcal{Q}_\gamma$.

It remains to prove that $\mathcal{Q}''_\gamma \setminus \mathcal{Q}'_\gamma \neq \emptyset$. Let $\langle W, \leq\rangle$ be a γ-special tree. We take $n^* = 4$, and fix $\langle \eta_\alpha \mid \alpha < \gamma\rangle$ as in the proof of Theorem 15. Recalling Definition 9, there exist maximal chains $C_\xi \subseteq W$ for $\xi < cf(\gamma)$ such that $\langle h(C_\xi) \mid \xi < cf(\gamma)\rangle$ is a strictly increasing sequence of limit ordinals cofinal in γ.

For any $\xi < cf(\gamma)$, we add a new point x_ξ to $Lev_{h(C_\xi)}(W)$ such that $\forall y \in C_\xi . y \leq x_\xi$. This gives us a tree $\langle W', \leq\rangle$, which clearly still satisfies $\forall \alpha < \gamma . |Lev_\alpha(W')| \leq |\mathcal{T}_{\eta_\alpha}|$. In the same way that $G_{\underline{u}}$ was defined in the proof of Theorem 15, we can find $a_\xi \in \mathcal{T}_{\eta_{h(C_\xi)}+4}$ for $\xi < cf(\gamma)$ such that the tree Z given by

$$Lev_\alpha(Z) = \{\mathcal{H}_{\eta_\alpha+4}(a_\xi) \mid \xi < cf(\gamma) \wedge h(C_\xi) \geq \alpha\}$$

(for all $\alpha < \gamma$) with the order induced by the maps $\mathcal{H}^{\mathcal{T}_{\eta_{\alpha'}+4}}_{\eta_\alpha+4}$ (for $\alpha < \alpha' < \gamma$) is isomorphic to the subtree $\bigcup_{\xi<cf(\gamma)} C_\xi \cup \{x_\xi\}$ of W' (the isomorphism being given by mapping any $y \in C_\xi \cup \{x_\xi\}$ to $\mathcal{H}_{\eta_{ht(y,W')}+4}(a_\xi)$).

We now fix a mapping F with domain W as in the proof of Theorem 15. For any $\xi < cf(\gamma)$, let $\underline{b^\xi} \in \mathcal{T}^0_\gamma \setminus \mathcal{T}_\gamma$ be constructed as in that proof, with $\underline{u}$ replaced by $\underline{u^\xi} = \mathcal{H}_\gamma(\prec t_\gamma, a_\xi \succ)$. Then, given any ξ, we have $\langle \tau, G_{\underline{u^\xi}}(x)\rangle \in b^\xi_{\eta_{h(C_\xi)+5}+5}$ for some $x \in Lev_{h(C_\xi)+5}(W)$ such that

$$\langle \tau, \mathcal{H}_{\eta_{h(C_\xi)+5}+3}(\prec c, \mathcal{H}_{\eta_{h(C_\xi)+5}+1}(\prec t_\gamma, a_\xi \succ) \succ)\rangle \in G_{\underline{u^\xi}}(x)$$

for some c. Observing that $\eta_{h(C_\xi)+5} + 1 \geq \eta_{h(C_\xi)} + 6$, it follows that $b^\xi_{\eta_{h(C_\xi)+5}+5} \neq b^{\xi'}_{\eta_{h(C_\xi)+5}+5}$ for all $\xi' \neq \xi$. Also, $\{\underline{b^\xi} \mid \xi < cf(\gamma)\}$ is closed in $\mathcal{T}^0_\gamma$.

Let $\underline{c} \in \mathcal{T}^0_\gamma \setminus \mathcal{T}_\gamma$ be constructed with $\underline{u}$ replaced by $\underline{u^*} = \mathcal{H}_\gamma(\prec \emptyset, t_\gamma \succ)$ (F is still fixed). For any $\xi < cf(\gamma)$, let $\eta'_\xi = \eta_{h(C_\xi)+5} + 5$ and let $\underline{d^\xi} \in \mathcal{T}^0_\gamma$ be given by

$$d^\xi_{\beta+1} = b^\xi_{\beta+1} \cup \mathcal{H}_{\beta+1}(c_{\eta'_\xi})$$

for all non-zero $\beta < \gamma$. The way we chose $\underline{u^\xi}$ and $\underline{u^*}$ ensures that $b^\xi_{\eta_\alpha+5}$ and $c_{\eta_\alpha+5}$ are disjoint whenever $\eta_\alpha + 1 \geq 4$ (for all $\xi < cf(\gamma)$). It is then not difficult to see that $\{\underline{d^\xi} \mid \xi < cf(\gamma)\}$ is closed in $\mathcal{T}^0_\gamma$. Also, we have that $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{d^\xi}) = \mathcal{H}_\gamma(c_{\eta'_\xi})$ for all ξ.

By Theorem 11 (a), let $\underline{e} \in \mathcal{T}^0_\gamma$ be such that $\underline{e}_{\mathcal{T}^0_\gamma,\tau} = \{\underline{d^\xi} \mid \xi < cf(\gamma)\}$ and $\underline{e}_{\mathcal{T}^0_\gamma,\delta} = \emptyset$ for all $\delta \in \Sigma$. Then Lemma 10 (a) gives us that

$$
\begin{aligned}
(\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{e}))_{\mathcal{T}^0_\gamma,\tau} &= \overline{\{\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{d^\xi}) \mid \xi < cf(\gamma)\}}^{\mathcal{T}^0_\gamma} \\
&= \overline{\{\mathcal{H}_\gamma(c_{\eta'_\xi}) \mid \xi < cf(\gamma)\}}^{\mathcal{T}^0_\gamma} \\
&= \{\mathcal{H}_\gamma(c_{\eta'_\xi}) \mid \xi < cf(\gamma)\} \cup \{\underline{c}\}.
\end{aligned}
$$

Hence we have $(\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{e})))_{\mathcal{T}^0_\gamma,\tau} \ni \mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{c}) = \mathcal{H}_\gamma(\emptyset) \notin (\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{e}))_{\mathcal{T}^0_\gamma,\tau}$, and so $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{e})) \neq \mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{e})$, so that $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{e}) \in \mathcal{Q}''_\gamma \setminus \mathcal{Q}'_\gamma$. $\square$

If γ is not ω-like, Theorem 16 leaves open the question of whether and when is the inclusion $\mathcal{Q}''_\gamma \subseteq \mathcal{T}_\gamma$ proper. By definition, $\mathcal{T}_\gamma$ consists of all the images under $\mathcal{H}_\gamma$ of points in arbitrary transition systems. It is quite plausible to expect that we do not need to look for these points further from $\mathcal{T}^0_\gamma$, in other words that $\mathcal{T}^0_\gamma$ is rich enough so that the image under $\mathcal{H}_\gamma$ of an arbitrary point is the image under $\mathcal{H}_\gamma$ of a point in $\mathcal{T}^0_\gamma$, which is just saying that $\mathcal{Q}''_\gamma = \mathcal{T}_\gamma$. On the other hand, after a closer examination, this perhaps seems unlikely, since any $\underline{a} \in \mathcal{T}^0_\gamma \setminus \mathcal{T}_\gamma$ gives rise to some $\underline{b} = \mathcal{H}_\gamma(\sum_{\alpha\in\gamma\setminus\alpha^*} \tau.a_\alpha) \in \mathcal{T}_\gamma$ such that $\underline{b} \xrightarrow{\tau} \underline{a}$, but $\underline{b} \not\xrightarrow{\tau} \mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{a})$, and points in $\mathcal{T}^0_\gamma$ of this kind look as if they are not particularly likely to be images under $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma$ of points in $\mathcal{T}^0_\gamma$.

Fortunately, the method used in the proof of $\mathcal{Q}'_\gamma \subset \mathcal{Q}''_\gamma$ above gives us a lot of hope. Namely, suppose that we have a point $\underline{c} \in \mathcal{T}_\gamma \setminus \mathcal{Q}''_\gamma$. To obtain a contradiction, it would suffice to construct a point $\underline{c'} \in \mathcal{T}^0_\gamma$ which is bisimilar to $\underline{c}$, where we consider $\underline{c}$ as an element of $\mathcal{T}_\gamma$ (i.e. ignore all transitions to points in $\mathcal{T}^0_\gamma \setminus \mathcal{T}_\gamma$), for then we would have that $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{c'}) = \mathcal{H}^{\mathcal{T}_\gamma}_\gamma(\underline{c}) = \underline{c}$. Such a $\underline{c'}$ would be constructed by modifying $\underline{c}$ by replacing some of the points reachable from $\underline{c}$ by their "unions" with carefully chosen points in $\mathcal{T}^0_\gamma \setminus \mathcal{T}_\gamma$ which have no transitions, so that we eliminate in $\underline{c'}$ all the transitions corresponding to transitions of $\underline{c}$ leading to points $\underline{d} \in \mathcal{T}^0_\gamma \setminus \mathcal{T}_\gamma$ for which a transition to $\mathcal{H}^{\mathcal{T}^0_\gamma}_\gamma(\underline{d})$ is not present at the appropriate level (if $n < \omega$, let "the n^{th} level of $\underline{c}$" consist of all the points in $\mathcal{T}_\gamma$ which are reachable from $\underline{c}$ in n transitions, but no less). However, this task presents us with several difficulties:

- The "disjointness" requirement in the method above may be very hard to meet, especially when a point in $\mathcal{T}_\gamma$ reachable from $\underline{c}$ has a large (e.g. of cardinality $|\gamma|$) number of transitions labelled by the same action.
- When forming the "unions", we have to avoid introducing new unwanted transitions (which could be either to a point in $\mathcal{T}_\gamma$ or to a point in $\mathcal{T}^0_\gamma \setminus \mathcal{T}_\gamma$).
- We have to take care of all of the ω-many levels of $\underline{c}$, and find a way to put together all the results to obtain $\underline{c'}$.

4 Fixed points

In a variety of mathematical treatments of process algebras, the notions of *non-destructiveness* and *constructiveness* have been important in the study of fixed points (which are used to model recursion). If we have a family of restriction maps $\downarrow n$ for $n < \omega$ on a set X so that:

- $x \downarrow 0 = y \downarrow 0$,
- $(x \downarrow n) \downarrow m = x \downarrow (n \cap m)$, and
- $(\forall n < \omega . x \downarrow n = y \downarrow n) \Rightarrow x = y$,

then a function is said to be non-destructive iff $(f(x)) \downarrow n = (f(x \downarrow n)) \downarrow n$, and constructive iff $(f(x)) \downarrow n+1 = (f(x \downarrow n)) \downarrow n+1$ (for all $x \in X$, $n < \omega$). If we define a metric d on X by $d(x,y) = inf\{2^{-n} \mid x \downarrow n = y \downarrow n\}$ (which is usually complete), then the non-destructive maps are precisely the non-expanding maps, and the constructive maps are just the contraction maps (to which the Banach Fixed Point Theorem applies when the space is complete).

We can generalise these ideas as follows.

Definition 10 • For any $\underline{a} \in \mathcal{T}_\gamma$, $0 < \alpha < \gamma$, let $\underline{a} \downarrow \alpha = \mathcal{H}_\gamma(a_\alpha)$.

- A function $f : \mathcal{T}_\gamma \to \mathcal{T}_\gamma$ is *non-destructive* iff $(f(\underline{a})) \downarrow \alpha = (f(\underline{a} \downarrow \alpha)) \downarrow \alpha$ for all $\underline{a} \in \mathcal{T}_\gamma$, $0 < \alpha < \gamma$.

- A function $f : \mathcal{T}_\gamma \to \mathcal{T}_\gamma$ is *constructive* iff $(f(\underline{a})) \downarrow \alpha+1 = (f(\underline{a} \downarrow \alpha)) \downarrow \alpha+1$ for all $\underline{a} \in \mathcal{T}_\gamma$, $0 < \alpha < \gamma$. $\Box$

Theorem 17 *Suppose $f : \mathcal{T}_\gamma \to \mathcal{T}_\gamma$ is constructive.*

(a) f has at most one fixed point in $\mathcal{T}_\gamma$.

(b) If γ is countable, then f has a unique fixed point in $\mathcal{T}_\gamma$.

Proof We construct $\underline{a} \downarrow \alpha$ by transfinite recursion on non-zero $\alpha < \gamma$, where $\underline{a}$ is a hypothetical fixed point.
Base case. There is only one possible value for $\underline{a} \downarrow 1$.
Successor case. $\underline{a} \downarrow \alpha+1 = (f(\underline{a} \downarrow \alpha)) \downarrow \alpha+1$.
Limit case. $\underline{a} \downarrow \gamma' = \mathcal{H}_{\gamma'}(\underline{b})$, where $\underline{b} = \langle \mathcal{H}_\alpha(\underline{a} \downarrow \alpha) \mid 0 < \alpha < \gamma' \rangle$, if $\underline{b} \in \mathcal{T}_{\gamma'}$. (If γ' is countable, we always have $\underline{b} \in \mathcal{T}_{\gamma'}$.)

It is easy to see, by transfinite induction, that if $\underline{a'}$ is a fixed point of f, then $\underline{a'} \downarrow \alpha = \underline{a} \downarrow \alpha$ for all non-zero $\alpha < \gamma$ (and in that case all the $\underline{a} \downarrow \alpha$ are well-defined), so that f always has at most one fixed point.

If γ is countable, then all the $\underline{a} \downarrow \alpha$ are well-defined. Letting $\underline{a} = \langle \mathcal{H}_\alpha(\underline{a} \downarrow \alpha) \mid 0 < \alpha < \gamma \rangle \in \mathcal{T}_\gamma$, it follows by another straightforward transfinite induction (showing that $(f(\underline{a})) \downarrow \alpha = \underline{a} \downarrow \alpha$ for all α) that $\underline{a}$ is a fixed point of f. $\Box$

If γ is uncountable, one is naturally interested in identifying types of functions on $\mathcal{T}_\gamma$ which always have a fixed point. (The fixed points needed in the work on CSP can be shown to exist by different means, by working with the spaces $\mathcal{T}^*_\kappa$.)

Definition 11 Say a predicate R on $\mathcal{T}_\gamma$ is *continuous* iff, whenever $\neg R(\underline{a})$, there is a maximal non-zero $\alpha < \gamma$ such that $\exists \underline{b} \in \mathcal{T}_\gamma . \underline{b} \downarrow \alpha = \underline{a} \downarrow \alpha \wedge R(\underline{b})$. $\Box$

If R is a predicate on $\mathcal{T}_\gamma$, it is easy to see that R is continuous iff $\mathcal{X} = \{\underline{a} \in \mathcal{T}_\gamma \mid R(\underline{a})\}$ is non-empty and $\mathcal{H}_{\gamma'}(\mathcal{X})$ is closed in $\mathcal{T}_{\gamma'}$ for all $\gamma' \leq \gamma$. Non-empty finite disjunctions preserve continuity, but arbitrary conjunctions (even consistent pairwise conjunctions) do not seem to. This is an interesting topic for future research.

Theorem 18 *If $f : \mathcal{T}_\gamma \to \mathcal{T}_\gamma$ is constructive with fixed point $\underline{a}$ and R is a continuous predicate such that $\forall \underline{b} \in \mathcal{T}_\gamma . R(\underline{b}) \Rightarrow R(f(\underline{b}))$, then $R(\underline{a})$ holds.*

Proof If $R(\underline{a})$ fails, then there is a maximal non-zero $\alpha < \gamma$ such that there exists $\underline{b} \in \mathcal{T}_\gamma$ with $\underline{b} \downarrow \alpha = \underline{a} \downarrow \alpha$ and $R(\underline{b})$. But then $R(f(\underline{b}))$ holds and

$$(f(\underline{b})) \downarrow \alpha + 1 = (f(\underline{b} \downarrow \alpha)) \downarrow \alpha + 1 = (f(\underline{a} \downarrow \alpha)) \downarrow \alpha + 1 = (f(\underline{a})) \downarrow \alpha + 1 = \underline{a} \downarrow \alpha + 1,$$

which is a contradiction. □

Forti, Honsell and Lenisa have studied fixed points in the context of hyperuniverses — see Section 4 of [FHL 94].

5 Transition to non-well-founded sets

From now on, we assume that $\Sigma = \emptyset$ (so that $\Sigma^+ = \{\tau\}$), and we drop the assumption that all systems are small. We simplify some of the notation as follows:

- A system is now a class $\mathcal{S}$ with a binary relation $\to$ on $\mathcal{S}$ such that $a_\mathcal{S} = \{b \in \mathcal{S} \mid a \to b\}$ is a set for all $a \in \mathcal{S}$.
- For any $\alpha > 0$, we now have $\mathcal{H}^\mathcal{S}_{\alpha+1}(a) = \{\mathcal{H}^\mathcal{S}_\alpha(b) \mid a \to b\}$ for all $a \in \mathcal{S}$, and so $\mathcal{T}_{\alpha+1} = \mathcal{P}(\mathcal{T}_\alpha)$. (This gives rise to the obvious changes in the definitions of transitions on the $\mathcal{T}_\alpha$, and elsewhere — essentially, any $\langle \delta, b \rangle$ with $\delta \in \Sigma^+$ is replaced by just b.)

For any system $\mathcal{S}$, we can regard $\langle \mathcal{S}, \leftarrow \rangle$[22] as a model for the language of set theory (the first-order language with equality having only the binary predicate symbol $\in$). This observation provides the link between the study of transition systems and set theory.

Definition 12 A system $\mathcal{S}$ is *universal* iff, for any small system $\mathcal{S}'$, there exists a unique morphism $\mathcal{S}' \to \mathcal{S}$. □

In particular, any universal system is strongly extensional. Also, a system $\mathcal{S}$ is universal iff a unique morphism $\mathcal{S}' \to \mathcal{S}$ exists for *any* system $\mathcal{S}'$.

Anti-Foundation Axiom (AFA) $\langle V, \ni \rangle$[23] is a universal system. □

Definition 13 Let $\mathcal{T} = \bigcup_{\alpha > 0} \mathcal{T}'_\alpha$ and, for any $a \in \mathcal{T}$, let θ_a be the unique $\theta > 0$ such that $a \in \mathcal{T}'_\theta$. If $a, b \in \mathcal{T}$, let $a \to b$ iff $\mathcal{H}_{\theta_a \cup \theta_b}(a) \to \mathcal{H}_{\theta_a \cup \theta_b}(b)$. □

Theorem 19 *$\mathcal{T}$ is universal.*

Proof Suppose $\mathcal{S}$ is a small system. By Theorem 9, $\mathcal{H}_{i(\mathcal{S})}$ is a morphism on $\mathcal{S}$. For any $a \in \mathcal{T}_{i(\mathcal{S})}$, let $\alpha_a > 0$ be minimal such that $a \in \mathcal{H}_{i(\mathcal{S})}(\mathcal{T}_{\alpha_a})$. Then $\forall a \in \mathcal{T}_{i(\mathcal{S})}.\mathcal{H}_{\alpha_a}(a) \in \mathcal{T}'_{\alpha_a}$, so it follows that $\mathcal{F} : a \mapsto \mathcal{H}_{\alpha_a}(a)$ is an isomorphism $\mathcal{T}_{i(\mathcal{S})} \to \bigcup_{0 < \alpha \leq i(\mathcal{S})} \mathcal{T}'_\alpha \subseteq \mathcal{T}$. Hence $\mathcal{F} \circ \mathcal{H}_{i(\mathcal{S})} : \mathcal{S} \to \mathcal{T}$ is a morphism.

By Lemma 2, it remains to prove that any morphism $\mathcal{G}$ of $\mathcal{T}$ is injective. Suppose not, and let $a, b \in \mathcal{T}$ be such that $a \neq b$ and $\mathcal{G}(a) = \mathcal{G}(b)$. Let $\theta = \theta_a \cup \theta_b$. Then $\mathcal{G} \restriction \mathcal{X} (= \bigcup_{0 < \alpha \leq \theta} \mathcal{T}'_\alpha)$ is a morphism (observe that $\mathcal{X}$ is certainly a subsystem of $\mathcal{T}$) of $\mathcal{X}$ which is not injective. But, as above, $\mathcal{X}$ is isomorphic to $\mathcal{T}_\theta$, contradicting Theorem 5. □

Hence any system "can be found" inside $\mathcal{T}$ uniquely. If we consider the systems $\mathcal{T}_\alpha$ as having been defined recursively, this means that the construction (which built the $\mathcal{T}_\alpha$'s from scratch) "reaches" every system without introducing any unwanted garbage on the way.

[22] When $\mathcal{S}$ is a proper class, this is a metatheoretic abuse of notation.

[23] Here $V = \{x \mid x = x\}$ is the universal class.

Definition 14 A system $\mathcal{S}$ is *full* iff, for any set $X \subseteq \mathcal{S}$, there exists a unique $a \in \mathcal{S}$ such that $a_{\mathcal{S}} = X$. $\square$

Lemma 20 *Any universal system is full.* $\square$

The following is Rieger's Theorem — for a proof see either [Rieg 57] or Appendix B of [Acz 88].

Theorem 21 *Suppose* $\langle \mathcal{S}, \rightarrow \rangle$ *is a full system. Then* $\langle \mathcal{S}, \leftarrow \rangle$ *is a model of* ZFC^-. $\square$

Suppose we have a point in a universal system $\mathcal{S}$ which is a small system "encoded" within $\mathcal{S}$ (e.g. in the way corresponding to how small systems are usually encoded as sets, that is as ordered pairs consisting of a set and a binary relation on it). That point then gives us a small system (encoded within our universe), whose underlying set is a subset of $\mathcal{S}$, and then we have a unique morphism from this small system into $\mathcal{S}$ (since $\mathcal{S}$ is universal). Finally, we can encode that morphism back into $\mathcal{S}$, so that it is represented by a point in $\mathcal{S}$. It follows that any universal system is a model of AFA. (See the proof of Theorem 3.8 in [Acz 88].)

Theorem 22 $\langle \mathcal{T}, \leftarrow \rangle$ *is a model of* $ZFC^- + AFA$. $\square$[24]

6 Structural results

It will often be both more convenient and more intuitive to use AFA as if it is a basic axiom of the theory within which we are working (so far, this has been pure ZFC^-), in the sense to become apparent in the following definition. (It will be clear when exactly are we resorting to this technique.)

Definition 15 ($ZFC^- + AFA$) For any aps $\mathcal{S}a$, let $\widehat{\mathcal{S}a}$ be the image of a under the unique morphism $\langle \mathcal{S}a, \rightarrow \rangle \rightarrow \langle V, \ni \rangle$. For any system $\mathcal{S}$, let $\hat{\hat{\mathcal{S}}} = \{\widehat{\mathcal{S}a} \mid a \in \mathcal{S}\}$.

In particular, let $T_0 = \emptyset$ and $T_\alpha = \widehat{\widehat{\mathcal{T}_\alpha}}$ for each $\alpha > 0$. $\square$

The following are easy to prove:

(a) For any aps $\mathcal{S}a$, $\widehat{\mathcal{S}a} = \{\widehat{\mathcal{S}b} \mid a \rightarrow b\}$.

(b) $T_{\alpha+1} = \mathcal{P}(T_\alpha)$ for all α.

(c) $\bigcup_{\alpha<\gamma} T_\alpha \subseteq T_\gamma$ for all γ.

(d) T_α is transitive for all α.

(e) $\alpha \leq \beta \Rightarrow T_\alpha \subseteq T_\beta$.

(f) $\hat{\hat{\mathcal{T}}} = \bigcup_{\alpha \geq 0} T_\alpha$.

[24] Observe that, by Gödel's 2^{nd} Incompleteness Theorem, this result is not formalizable as a theorem of ZFC^-. Instead, in the same way as Rieger's Theorem, it is a collection of assertions in the metatheory that, for any axiom of $ZFC^- + AFA$, we can prove that it holds in $\langle \mathcal{T}, \leftarrow \rangle$, i.e. that its relativization to $\langle \mathcal{T}, \leftarrow \rangle$ is a theorem of ZFC^-. For further discussion of these and related points, see Chapter 4 of [Kun 80].

Given a cardinal κ, let $exp_0(\kappa) = \kappa$, $exp_{\alpha+1}(\kappa) = 2^{exp_\alpha(\kappa)}$, and $exp_\gamma(\kappa) = \bigcup_{\alpha<\gamma} exp_\alpha(\kappa)$. Then it follows that $|T_n| = exp_n(0)$ for all $n < \omega$, and that $|T_\alpha| = exp_{\alpha+1}(0)$ for all $\alpha \geq \omega$ (recall the remarks after Theorem 13).

If the R_α form the von Neumann hierarchy (so that $R_0 = \emptyset$, $R_{\alpha+1} = \mathcal{P}(R_\alpha)$, and $R_\gamma = \bigcup_{\alpha<\gamma} R_\alpha$), then it is immediate that $R_\alpha \subseteq T_\alpha$ for all α. But also $|R_\alpha| = exp_\alpha(0)$ for all α, so that in fact $R_n = T_n$ if $n < \omega$, and $R_\alpha \subset T_\alpha$ if $\alpha \geq \omega$.

Letting

$$\nu_\alpha = \bigcup_{x \in T_\alpha \cap WF} (rank(x) + 1),$$[25]

we know that $\nu_\alpha \geq \alpha$ for all α. Theorem 13 gives us that $\forall \alpha . \alpha \in T_{\alpha+1} \setminus T_\alpha$, and so it seems plausible to conjecture that $\forall \alpha . \nu_\alpha = \alpha$. In fact, Forti and Honsell show that this is the case whenever either $\alpha = \omega$ or α is weakly compact — see [FH 92]. Otherwise, we have the following.

Theorem 23 *($ZFC^- + AFA$)*

(a) $\nu_0 = 0$.

(b) $\nu_{\alpha+1} = \nu_\alpha + 1$ for all α.

(c) If $\gamma \neq \omega$ and γ is not weakly compact, then $exp_\gamma(0)^+ \leq \nu_\gamma \leq exp_{\gamma+1}(0)^+$.

Proof (a) and (b) are obvious (observe that $T_{\alpha+1} \cap WF = \mathcal{P}(T_\alpha \cap WF)$). If we had $\nu_\gamma > exp_{\gamma+1}(0)^+ = |T_\gamma|^+$, we would have an $x \in T_\gamma$ such that $rank(x) = |T_\gamma|^+$, so that $|x| = |T_\gamma|^+$, which would contradict the transitivity of T_γ. Hence it remains to prove the first inequality in (c).

First, we claim that:

(i) If $\gamma \neq \omega$ and γ is not strongly inaccessible, then $exp_\gamma(0)$ is singular.

(ii) If $\gamma = \omega$ or γ is strongly inaccessible, then $exp_\gamma(0) = \gamma$.

For (i), suppose $exp_\gamma(0)$ is regular. Then $\gamma \leq exp_\gamma(0) = cf(exp_\gamma(0)) = cf(\gamma) \leq \gamma$, so γ is regular and $2^\lambda \leq exp_{\lambda+1}(0) < \gamma$ whenever $\lambda < \gamma$, so that either $\gamma = \omega$ or γ is strongly inaccessible. (ii) is easy to see.

Now, we construct an $x_\alpha \in T_\gamma \cap WF$ such that $rank(x_\alpha) = \alpha$ by transfinite recursion on $\alpha < exp_\gamma(0)^+$. The base case and the successor case are obvious (for the latter, recall the remarks after Theorem 11). For the limit case, suppose we have constructed such x_α for all $\alpha < \gamma'$, where $\gamma' < exp_\gamma(0)^+$, and let $\langle \alpha_\xi \mid \xi < cf(\gamma') \rangle$ be cofinal in γ'.

If $\gamma \neq \omega$ and γ is not strongly inaccessible, then $cf(\gamma') < exp_\gamma(0)$ by (i), and so $|R_\beta| \geq cf(\gamma')$ for some $\beta < \gamma$. Pick an injective mapping $\xi \mapsto b_\xi : cf(\gamma') \to R_\beta$. Then Lemma 10 (b) gives us that

$$x_{\gamma'} = \{\langle b_\xi, x_{\alpha_\xi} \rangle \mid \xi < cf(\gamma')\} \in T_\gamma \cap WF$$

(recall that $R_\beta \subseteq T_\beta$, and see the last paragraph in the proof of Theorem 12), and clearly $rank(x_{\gamma'}) = \gamma'$.

For the remaining case, suppose γ is strongly inaccessible, but not weakly compact. Then $exp_\gamma(0) = \gamma$ by (ii). If $\{\mathcal{H}_\gamma(x_{\alpha_\xi}) \mid \xi < cf(\gamma')\}$[26] is closed in $\mathcal{T}_\gamma$ (which is always the case if

[25] Here $WF = \bigcup_{\alpha \geq 0} R_\alpha = \{x \mid x \textit{ is a well-founded set}\}$. For any $x \in WF$, $rank(x)$ is the smallest α such that $x \subseteq R_\alpha$.

[26] If x is a set and $\alpha > 0$, we write $\mathcal{H}_\alpha(x)$ for $\mathcal{H}_\alpha^{tc(\{x\})}(x)$, where $tc(\{x\})$ is the transitive closure of $\{x\}$, so that x is a point in the system $\langle tc(\{x\}), \ni \rangle$.

$cf(\gamma') < \gamma$, by the remarks after Theorem 11), then we can take $x_{\gamma'} = \{x_{\alpha_\xi} \mid \xi < cf(\gamma')\}$ by Lemma 10 (b).

Otherwise, $cf(\gamma') = \gamma$, and we have a $\underline{b} \in \overline{\{\mathcal{H}_\gamma(x_{\alpha_\xi}) \mid \xi < \gamma\}}^{\mathcal{T}_\gamma} \setminus \{\mathcal{H}_\gamma(x_{\alpha_\xi}) \mid \xi < \gamma\}$. For any non-zero $\beta < \gamma$, let $\xi_\beta < \gamma$ be such that $b_\beta = \mathcal{H}_\beta(x_{\alpha_{\xi_\beta}})$, and let $x'_\beta = x_{\alpha_{\xi_\beta}}$. Then $\beta \mapsto \xi_\beta$ is cofinal in γ, so that $\beta \mapsto \alpha_{\xi_\beta} = rank(x'_\beta)$ is cofinal in γ'.

Let $\langle W, \leq \rangle$ be a γ-Aronszajn tree. Since γ is strongly inaccessible, we can construct an injective mapping $y \mapsto \beta_y : W \to \gamma \setminus \{0\}$ such that $ht(y, W) < ht(y', W) \Rightarrow \beta_y < \beta_{y'}$. For any $\zeta < \gamma$, let $\beta^*_\zeta = \bigcap_{y \in Lev_\zeta(W)} \beta_y$.

Also, there exist maximal chains $C_\xi \subseteq W$ for $\xi < \gamma$ such that $\langle h(C_\xi) \mid \xi < \gamma \rangle$ is strictly increasing and cofinal in γ. Given any $\xi < \gamma$, let

$$X_\xi = \{\langle \beta, x'_{\beta^*_{h(C_\xi)}} \rangle \mid \beta \in \{\beta_y \mid y \in C_\xi\} \cup \{\beta^*_{h(C_\xi)}\}\}.$$

Then it is not difficult to see that the tree Z given by

$$\begin{aligned} Lev_\zeta(Z) &= \{\mathcal{H}_{\beta^*_{\zeta+1}+3}\left(\sum_{x'' \in X_\xi} \tau.x''\right) \mid \xi < \gamma \wedge h(C_\xi) > \zeta\} \\ &= \{\mathcal{H}_{\beta^*_{\zeta+1}+3}\left(\sum_{\beta \in \{\beta_y \mid y \in C_\xi\} \cup \{\beta^*_{h(C_\xi)}\}} \tau. \prec \mathcal{H}_{\beta^*_{\zeta+1}}(t_\beta), b_{\beta^*_{\zeta+1}} \succ\right) \mid \xi < \gamma \wedge h(C_\xi) > \zeta\} \end{aligned}$$

(for $\zeta < \gamma$) with the order induced by the maps $\mathcal{H}_{\beta^*_{\zeta+1}+3}^{\mathcal{T}_{\beta^*_{\zeta'+1}+3}}$ (for $\zeta < \zeta' < \gamma$) is isomorphic to the subtree $\bigcup_{\xi<\gamma} C_\xi$ of W. Hence it follows that $\{\mathcal{H}_\gamma(\sum_{x'' \in X_\xi} \tau.x'') \mid \xi < \gamma\}$ is closed in $\mathcal{T}_\gamma$, so that $x_{\gamma'} = \{X_\xi \mid \xi < \gamma\} \in T_\gamma \cap WF$. Finally, since both $\langle h(C_\xi) \mid \xi < \gamma \rangle$ and $\langle \beta^*_\zeta \mid \zeta < \gamma \rangle$ are cofinal in γ, so is $\langle \beta^*_{h(C_\xi)} \mid \xi < \gamma \rangle$, and hence $rank(x_{\gamma'}) = \gamma'$. $\Box$

7 Abstract results

If $\phi(x_1, ..., x_n)$ is a formula of the language of set theory and E is a binary relation on a class M, we write $\phi(x_1, ..., x_n)^{M,E}$ for the relativization of $\phi(x_1, ..., x_n)$ to $\langle M, E \rangle$.

If $N \subseteq M$, then $\phi(x_1, ..., x_n)$ is absolute for N, M, E iff

$$\forall x_1, ..., x_n \in N.\phi(x_1, ..., x_n)^{N, E \cap (N \times N)} \Leftrightarrow \phi(x_1, ..., x_n)^{M,E}$$

is provable in ZFC^-. Also, $\phi(x_1, ..., x_n)$ is absolute for N iff it is absolute for N, V, $\in$. A function $F(x_1, ..., x_n)$ defined by a formula $F(x_1, ..., x_n) = x_{n+1}$ is absolute for N, M, E (N, respectively) iff the formula $F(x_1, ..., x_n) = x_{n+1}$ is.[27]

Lemma 24 *If $\phi(x_1, ..., x_n)$, $F(x_1, ..., x_n)$ and $G_i(y_1, ..., y_m)$ for each $1 \leq i \leq n$ are a formula and functions (respectively) absolute for N, M, E, then $\phi(G_1(y_1, ..., y_m), ..., G_n(y_1, ..., y_m))$ and $F(G_1(y_1, ..., y_m), ..., G_n(y_1, ..., y_m))$ are a formula and a function (respectively) absolute for N, M, E.* $\Box$

[27] Here we assume that $\forall x_1, ..., x_n.\exists! x_{n+1}.F(x_1, ..., x_n) = x_{n+1}$ holds in both $\langle N, E \cap (N \times N) \rangle$ and $\langle M, E \rangle$ (i.e. that the appropriate relativizations are provable).

For an account of relativization and absoluteness, see Chapter 4 of [Kun 80]. (The proof Lemma 24 can be found there.)

In [Acz 88], Aczel shows that any two full[28] models of $ZFC^- + AFA$ are isomorphic.

Theorem 25 *(a) (ZFC)* $\langle V, \ni\rangle$ *and* $\langle WF^{\mathcal{T},\leftarrow}, \rightarrow\rangle$ *are isomorphic.*

(b) $(ZFC^- + AFA)$ *WF is a model of ZFC, the formulae* $a \in \mathcal{T}$ *and* $a, b \in \mathcal{T} \wedge a \rightarrow b$ *are absolute for WF, and* $\langle V, \ni\rangle$ *and* $\langle \mathcal{T}, \rightarrow\rangle$ *are isomorphic.*

Proof For (a), we work within ZFC, and first show that $\langle V, \ni\rangle$ is strongly extensional. Suppose $\mathcal{G}$ is a morphism from $\langle V, \ni\rangle$ into a system $\langle \mathcal{S}, \rightarrow\rangle$, and suppose $x, y \in V$ are such that $\mathcal{G}(x) = \mathcal{G}(y)$. Then $\leftarrow$ is well-founded and set-like on $\mathcal{S}\mathcal{G}(x)$, so we can recursively define a function $\mathcal{G}' : \mathcal{S}\mathcal{G}(x) \rightarrow V$ by $\forall a \in \mathcal{S}\mathcal{G}(x).\mathcal{G}'(a) = \{\mathcal{G}'(b) \mid b \leftarrow a\}$. Now $\mathcal{G}' \circ (\mathcal{G} \restriction (tc(\{x\})))$ and $\mathcal{G}' \circ (\mathcal{G} \restriction (tc(\{y\})))$ are morphisms, so it follows by $\in$-induction on $tc(\{x\})$ that $x = \mathcal{G}'(\mathcal{G}(x))$, and by $\in$-induction on $tc(\{y\})$ that $y = \mathcal{G}'(\mathcal{G}(y))$, so that $x = y$. Hence $\mathcal{G}$ is injective.

For any $x \in V$, let $\mathcal{F}(x) \in \mathcal{T}$ be the image of x under the unique morphism from $\langle tc(\{x\}), \ni\rangle$ into $\langle \mathcal{T}, \rightarrow\rangle$. Then, for any x, $\mathcal{F} \restriction (tc(\{x\}))$ is the unique morphism from $\langle tc(\{x\}), \ni\rangle$ into $\langle \mathcal{T}, \rightarrow\rangle$, so that $\mathcal{F}$ is a morphism on $\langle V, \ni\rangle$, which must be injective by the strong extensionality of $\langle V, \ni\rangle$. Suppose $a \in \mathcal{T}$ is such that $\leftarrow$ is well-founded on $\mathcal{T}a$. Since $\leftarrow$ is set-like on $\mathcal{T}a$, we can recursively define a morphism $\mathcal{G} : \langle \mathcal{T}a, \rightarrow\rangle \rightarrow \langle V, \ni\rangle$ as above. By Mostowski's Collapsing Theorem (observe that $\leftarrow$ is extensional on $\mathcal{T}a$), $\mathcal{G} : \langle \mathcal{T}a, \rightarrow\rangle \rightarrow \langle V\mathcal{G}(a), \ni\rangle$ is an isomorphism. Hence $\mathcal{F}(\mathcal{G}(a)) = a$, so that $\mathcal{F}$ is a required isomorphism.

The first two assertions in (b) are easy to see, and $\langle V, \ni\rangle$ and $\langle \mathcal{T}, \rightarrow\rangle$ are isomorphic since they are both full models of $ZFC^- + AFA$. (An isomorphism is given by mapping any x to the image of x under the unique morphism from $\langle tc(\{x\}), \ni\rangle$ into $\langle \mathcal{T}, \rightarrow\rangle$.) □

Also, using Theorem 25 (b), it is not difficult to see (in ZFC^-) that if $\langle \mathcal{S}, \leftarrow\rangle$ and $\langle \mathcal{S}', \leftarrow\rangle$ are models of $ZFC^- + AFA$ such that $\langle WF^{\mathcal{S},\leftarrow}, \rightarrow\rangle$ and $\langle WF^{\mathcal{S}',\leftarrow}, \rightarrow\rangle$ are isomorphic, then $\langle \mathcal{S}, \rightarrow\rangle$ and $\langle \mathcal{S}', \rightarrow\rangle$ are isomorphic — this is essentially the content of Theorem 3 in [FH 87, Part I].

It follows from Theorem 25 (a) that if ϕ is a sentence which is absolute for WF and consistent with (respectively, independent of) ZFC^-, then ϕ is consistent with (independent of) $ZFC^- + AFA$. (In particular, observe that MA, $\diamond$, CH and GCH are absolute for WF, and $V = L$ is not.)

It is well-known that $ZF^- + \neg AC$[29] is consistent provided ZF^- is.[30] It is not difficult to see that, working as above within a model of $ZF^- + \neg AC$, we can obtain a version of Theorem 22 which states that $\langle \mathcal{T}, \leftarrow\rangle$ is a model of $ZF^- + AFA$. (Rieger's Theorem uses AC only in order to establish that AC holds in a given full system.) Since AC is not used in the proof of Theorem 25 (a), it follows that $\langle \mathcal{T}, \leftarrow\rangle$ so constructed is in fact a model of $ZF^- + \neg AC + AFA$. Hence AC is independent of the rest of $ZFC^- + AFA$ (provided ZF^- is consistent).[31]

Definition 16 For any infinite κ, let:

- $H_\kappa = \{x \mid |tc(\{x\})| < \kappa\}$,[32] and

[28] See Definition 14.

[29] Here $ZF^- = ZFC^- - AC$.

[30] See the exercises for Chapters 4 and 7 in [Kun 80].

[31] [FH 87] is a study of the relationships between various axioms (including AFA) contradicting the Axiom of Foundation and various choice principles.

[32] No confusion with $\mathcal{H}_\kappa$ should occur.

- $B_\kappa = \{x \mid \forall y \in tc(\{x\}).|y| < \kappa\}$. $\Box$

Lemma 26 *(ZFC$^-$ + AFA) Suppose κ is infinite. Then:*

(a) $H_\kappa \subseteq B_\kappa$.

(b) $H_\kappa = B_\kappa$ *iff* $\kappa > \omega$ *and* κ *is regular.*

(c) $B_\kappa \subseteq T_\kappa$ *iff* κ *is regular.*

(d) $H_\kappa \subseteq T_\kappa$.

(e) $H_\kappa = \bigcup_{\alpha<\kappa} T_\alpha$ *whenever* κ *is strongly inaccessible.*

Proof (a) is trivial.

For (b), suppose $\kappa > \omega$, κ is regular, and $x \in B_\kappa$. Letting $y_0 = \{x\}$ and $y_{n+1} = \bigcup y_n$ for each $n < \omega$, it follows inductively that $\forall n < \omega.|y_n| < \kappa$, and so $|tc(\{x\})| = |\bigcup_{n<\omega} y_n| < \kappa$. If $\kappa = \omega$, then $\{0, \{1, ...\}\} \in B_\kappa \setminus H_\kappa$, and if κ is singular, then $\{\alpha_\xi \mid \xi < cf(\kappa)\} \in B_\kappa \setminus H_\kappa$, where $\langle\alpha_\xi \mid \xi < cf(\kappa)\rangle$ is a cofinal sequence in κ.

The 'if' part of (c) is immediate by Theorem 9. If κ is singular and $\langle\alpha_\xi \mid \xi < cf(\kappa)\rangle$ is cofinal in κ, then $\{\alpha_\xi \mid \xi < cf(\kappa)\} \in B_\kappa \setminus T_\kappa$ (recall the proof of Theorem 13).

(d) follows from (a) and (c). (If κ is singular, then it is a limit cardinal.)

For (e), suppose κ is strongly inaccessible. Then κ is a limit cardinal, so that $H_\kappa \subseteq \bigcup_{\alpha<\kappa} T_\alpha$ by (d). Hence it suffices to show that $|T_\alpha| < \kappa$ for all $\alpha < \kappa$ (recall that each T_α is transitive), which follows by a straightforward transfinite induction. $\Box$

Theorem 27 *(ZFC$^-$+AFA) If κ is strongly inaccessible, then $\bigcup_{\alpha<\kappa} T_\alpha$ is a model of ZFC$^-$+ AFA.*

Proof Since κ is strongly inaccessible, Lemma 26 (e) gives us that $\bigcup_{\alpha<\kappa} T_\alpha = H_\kappa$, and so it follows as when working in ZFC that $\bigcup_{\alpha<\kappa} T_\alpha$ is a model of $ZFC^- - P$.[33]

If $x \in T_\alpha$ for some $\alpha < \kappa$, then $\mathcal{P}(x) \in T_{\alpha+2}$, and hence P holds in $\bigcup_{\alpha<\kappa} T_\alpha$.

Suppose $\langle\mathcal{S}, \rightarrow\rangle \in H_\kappa$ is a small system, and let $\mathcal{F}$ be the unique morphism $\langle\mathcal{S}, \rightarrow\rangle \rightarrow \langle V, \ni\rangle$. Then $\mathcal{F}$ is in fact a morphism $\langle\mathcal{S}, \rightarrow\rangle \rightarrow \langle H_\kappa, \ni\rangle$ and $\mathcal{F} = \{\langle a, \mathcal{F}(a)\rangle \mid a \in \mathcal{S}\} \in H_\kappa$. Hence AFA holds in $\bigcup_{\alpha<\kappa} T_\alpha$.

For an alternative proof, observe first that $\bigcup_{\alpha<\kappa} R_\alpha = H_\kappa^{WF}$ is a model of ZFC, and so $\langle\mathcal{T}, \leftarrow\rangle^{\bigcup_{\alpha<\kappa} R_\alpha}$ is a model of $ZFC^- + AFA$. It is not difficult to see that $\langle\mathcal{T}, \rightarrow\rangle^{\bigcup_{\alpha<\kappa} R_\alpha}$ is in fact isomorphic to $\langle\bigcup_{\alpha<\kappa} T_\alpha, \ni\rangle$. $\Box$

When κ is the smallest strong inaccessible, Theorem 27 gives us that ($\neg\exists\kappa'.\kappa'$ *is strongly inaccessible*) is consistent with $ZFC^- + AFA$, which is also immediate from the remarks after Theorem 25.

Lemma 28 *(ZFC$^-$+AFA)* $\{\widehat{\mathcal{S}a} \mid a$ is a point in a small system $\langle\mathcal{S}, \rightarrow\rangle \in \bigcup_{\alpha<\gamma} T_\alpha\} = H_{exp_\gamma(0)}$.

Proof The inclusion '$\subseteq$' is straightforward.

Suppose $x \in H_{exp_\gamma(0)}$. Then $|T_\alpha| \geq |tc(\{x\})|$ for some $\alpha < \gamma$. Pick $\mathcal{S} \subseteq T_\alpha$ with $|\mathcal{S}| = |tc(\{x\})|$, and let $\rightarrow$ be such that $\langle\mathcal{S}a, \rightarrow\rangle$ is isomorphic to $\langle tc(\{x\}), \ni\rangle$ for some $a \in \mathcal{S}$. It remains to observe that $\langle\mathcal{S}, \rightarrow\rangle \in T_{\alpha+5}$ and that $\widehat{\mathcal{S}a} = x$. $\Box$

[33] P stands for the Power Set Axiom.

Theorem 29 *Each of Inf, P and Repl*[34] *are independent of the rest of $ZFC^- + AFA$ (provided ZF^- is consistent).*

Proof We work in $ZFC^- + AFA$.[35] Letting $Th_1 = ZFC^- - Inf + \neg Inf + AFA$, $Th_2 = ZFC^- - P + \neg P + AFA$, and $Th_3 = ZFC^- - Repl + \neg Repl + AFA$, it will suffice to establish the following (which show a bit more):

(i) If κ is infinite, then H_κ is a model of Th_1 iff $\kappa = \omega$.

(ii) If κ is infinite and regular, then H_κ is a model of Th_2 iff $\exists\alpha. \exp_\alpha(\omega) < \kappa \leq \exp_{\alpha+1}(\omega)$.

(iii) Suppose $\langle \alpha_\beta \mid \beta < \gamma \rangle$ is strictly increasing, and let $\alpha^* = \bigcup_{\beta<\gamma} \alpha_\beta$. Suppose further that $\gamma < \exp_{\alpha^*}(\omega)$ and that $\langle \alpha_\beta \mid \beta < \gamma \rangle$ is absolute for $H_{\exp_{\alpha^*}(\omega)}$.[36] Then $H_{\exp_{\alpha^*}(\omega)}$ is a model of Th_3.

(iv) Suppose $\langle \alpha_\beta \mid \beta < \gamma \rangle$ is strictly increasing, $\gamma < \alpha^* = exp_{\alpha^*}(0)$, and $\langle \alpha_\beta \mid \beta < \gamma \rangle$ is absolute for $\bigcup_{\beta<\gamma} T_{\alpha_\beta}$ (where α^* is as in (iii)). Then $\bigcup_{\beta<\gamma} T_{\alpha_\beta}$ is a model of Th_3.

(To obtain a sequence as in (iv), we can start from some $\langle \alpha_{0,\beta} \mid \beta < \gamma \rangle$ with $\gamma < \alpha_0^* = \bigcup_{\beta<\gamma} \alpha_{0,\beta}$ which is absolute for $\bigcup_{\beta<\alpha'} T_\beta$ for all $\alpha' \geq \alpha_0^*$. If we let $\alpha_{n+1,\beta} = exp_{\alpha_{n,\beta}}(0)$ for all $\beta < \gamma$, $n < \omega$, then the same properties are satisfied by all the sequences $\langle \alpha_{n,\beta} \mid \beta < \gamma \rangle$ for $n < \omega$ (recall Lemma 24). Now, if $\alpha_n^* = \bigcup_{\beta<\gamma} \alpha_{n,\beta}$ for all $n < \omega$, then $\forall n < \omega. \alpha_n^* \leq exp_{\alpha_n^*}(0) = \alpha_{n+1}^*$. Hence, letting $\alpha^* = \bigcup_{n<\omega} \alpha_n^*$, we have

$$exp_{\alpha^*}(0) = \bigcup_{n<\omega} exp_{\alpha_n^*}(0) = \bigcup_{n<\omega} \alpha_{n+1}^* = \alpha^*,$$

so that $\langle \bigcup_{n<\omega} \alpha_{n,\beta} \mid \beta < \gamma \rangle$ will have the required properties. Observe also that we can take $\langle \omega + n \mid n < \omega \rangle$ for $\langle \alpha_{0,\beta} \mid \beta < \gamma \rangle$.)

For (i), observe first that H_ω is a model of $ZFC^- - Inf + AFA$ (which follows as in the proof of Theorem 27), and clearly Inf fails in H_ω. If $\kappa > \omega$, then Inf holds in H_κ, and so H_κ is not a model of Th_1.

If κ is infinite and regular, we have that H_κ is a model of $ZFC^- - Inf - P + AFA$ (as before). Hence H_κ is a model of Th_2 iff Inf holds and P fails in H_κ, i.e. iff $\kappa > \omega$ and $2^\lambda \geq \kappa$ for some $\lambda < \kappa$, i.e. iff neither $\kappa = \omega$ nor $\kappa = \exp_\gamma(\omega)$ for some γ, i.e. iff $\exists\alpha. \exp_\alpha(\omega) < \kappa \leq \exp_{\alpha+1}(\omega)$.

Suppose $\langle \alpha_\beta \mid \beta < \gamma \rangle$ and α^* are as in (iii). Then $\kappa = \exp_{\alpha^*}(\omega) > \omega$ and $\lambda < \kappa \Rightarrow 2^\lambda < \kappa$, so it follows as before that H_κ is a model of $ZFC^- - Repl + AFA$. Now $\gamma \in H_\kappa$ and $\exp_{\alpha_\beta}(\omega) \in H_\kappa$ for all $\beta < \gamma$. Also, $\langle \exp_{\alpha_\beta}(\omega) \mid \beta < \gamma \rangle$ is absolute for H_κ by Lemma 24, and $\{\exp_{\alpha_\beta}(\omega) \mid \beta < \gamma\} \notin H_\kappa$. Hence *Repl* fails in H_κ, so that H_κ is a model of Th_3.

Suppose $\langle \alpha_\beta \mid \beta < \gamma \rangle$ and α^* are as in (iv). Then $\bigcup_{\beta<\gamma} T_{\alpha_\beta} = \bigcup_{\beta<\alpha^*} T_\beta$ is transitive and $\alpha^* > \omega$ is a limit ordinal, so it follows that $\bigcup_{\beta<\gamma} T_{\alpha_\beta}$ is a model of $ZFC^- - Repl$. To show that AFA holds in $\bigcup_{\beta<\gamma} T_{\alpha_\beta}$, it suffices by Lemma 28 to show that $H_{\alpha^*} = H_{exp_{\alpha^*}(0)} \subseteq \bigcup_{\beta<\gamma} T_{\alpha_\beta}$, so suppose $x \in H_{\alpha^*}$. Then, since α^* is a limit cardinal, we in fact have that $x \in H_\kappa$ for some infinite $\kappa < \alpha^*$, and hence $x \in T_\kappa \subseteq \bigcup_{\beta<\gamma} T_{\alpha_\beta}$ by Lemma 26 (d). It now remains, by considering the sequence $\langle \alpha_\beta \mid \beta < \gamma \rangle$, to observe as in the proof of (iii) that *Repl* fails in $\bigcup_{\beta<\gamma} T_{\alpha_\beta}$. □

[34] Inf stands for the Axiom of Infinity, and *Repl* for the Replacement Scheme.

[35] If ZF^- is consistent, then so is ZFC^-, and hence also $ZFC^- + AFA$.

[36] What we mean is that the function $\beta \mapsto \alpha_\beta$ is given by a formula $\phi(x, y)(\Leftrightarrow (x < \gamma \wedge y = \alpha_x))$ which is absolute for $H_{\exp_{\alpha^*}(\omega)}$.

A lot of the results above suggest that, in $ZFC^- + AFA$, the smallest α such that $x \subseteq T_\alpha$ should be regarded as the rank of a set x (which is possibly non-well-founded). However, as Theorem 23 shows, this does not always coincide with $rank(x)$ for $x \in WF$.[37]

Appendix

The following are some thoughts about forcing in the presence of AFA. Suppose that, working within $ZFC^- + AFA$, we have a countable transitive model M of ZFC.[38] Also, suppose $\langle P, \leq, \top \rangle \in M$ is a partial order, where $\leq$ is reflexive and transitive, but not necessarily anti-symmetric, and $\top$ is a top element. We call a filter $G \subseteq P$ *generic* iff G intersects every $D \in M$ which is a dense subset of P (typically, we have that $G \notin M$). Let a set τ be a P-*name* iff τ represents an aps with arcs labelled by elements of P, in the sense that τ is a set of ordered pairs such that $\forall \langle \sigma, p \rangle \in \tau . \sigma \textit{ is a } P\textit{-name} \wedge p \in P$.

For any generic G, we define $M[G]$ to be the set of all sets which are obtained by restricting a P-name $\tau \in M$ to G and then removing the labels, i.e. we have that $M[G] = \{\tau_G \mid \tau \in M \textit{ is a } P\textit{-name}\}$, where $\tau_G = \{\sigma_G \mid \exists p \in G . \langle \sigma, p \rangle \in \tau\}$ for any P-name $\tau \in M$. (Since M satisfies ZFC, observe that any P-name $\tau \in M$ represents a well-founded labelled aps.) Given a $p \in P$, a formula $\phi(x_1, ..., x_n)$ and P-names $\tau_1, ..., \tau_n \in M$, we write $p \Vdash \phi(\tau_1, ..., \tau_n)$ iff, for every generic G with $p \in G$, we have that $\phi(\tau_{1_G}, ..., \tau_{n_G})$ holds in $M[G]$. It turns out that, for any generic G, $M[G]$ is a model of ZFC. The crucial step in establishing this fact consists of defining a relation $\Vdash^*$ and proving that, given a formula $\phi(x_1, ..., x_n)$ and P-names $\tau_1, ..., \tau_n \in M$ as above, we have:

- $\forall p \in P . p \Vdash \phi(\tau_1, ..., \tau_n) \Leftrightarrow (p \Vdash^* \phi(\tau_1, ..., \tau_n))^M$, and
- $\phi(\tau_{1_G}, ..., \tau_{n_G})^{M[G]} \Leftrightarrow \exists p \in G . p \Vdash \phi(\tau_1, ..., \tau_n)$

for every generic G.

Abusing the notation, let T^M be the model of $ZFC^- + AFA$ such that $M = WF^{T^M}$, so that we can think of T^M as being obtained by constructing $\langle \mathcal{T}, \rightarrow \rangle$ starting from M. Then T^M is countable and transitive. Hence, for any generic G,[39] we can define $T^M[G]$ by restricting every P-name in T^M (which now doesn't necessarily represent a well-founded labelled aps) to G, removing the labels, and then taking the image of the point of the resulting unlabelled aps under the unique morphism into $\langle V, \ni \rangle$. (Since any countable transitive model of $ZFC^- + AFA$ is of the form T^M for some countable transitive model M of ZFC, this effectively defines the forcing construction starting from an arbitrary countable transitive model of $ZFC^- + AFA$.) We would expect that any such $T^M[G]$ satisfies $ZFC^- + AFA$. Furthermore, we would hope to establish the commutativity of the rectangular diagram which leads to the conclusion that $T^M[G] = T^{M[G]}$ for any generic G.

In order to achieve these aims, we seem to require a definition of a relation $p \Vdash^* \phi(\tau_1, ..., \tau_n)$, where $p \in P$, $\phi(x_1, ..., x_n)$ is a formula and $\tau_1, ..., \tau_n \in T^M$ are P-names, which satisfies the appropriate analogues of the properties above. In the well-founded case (i.e. when we restrict our attention to P-names $\tau_1, ..., \tau_n \in M$), the key part of this definition, when $\phi(x_1, ..., x_n)$ is

[37] In Section 2 of [FH 87, Part I], Forti and Honsell define V_α to be the union of all (in our notation) $\widehat{\widehat{\mathcal{S}}}$, where $\mathcal{S}$ is a small system such that $\mathcal{S} \subseteq R_\alpha$ and $a_{\mathcal{S}} = a$ whenever $\mathcal{S}a$ is well-founded, and then use the resulting hierarchy $\langle V_\alpha \mid \alpha \geq 0 \rangle$ to define, in the obvious way, a rank function which extends the von Neumann one.

[38] For an account of the metamathematical difficulties involved here (which are again related to Gödel's 2^{nd} Incompleteness Theorem), and of the ways of overcoming them, see Chapter 7 of [Kun 80].

[39] Observe that a filter $G \subseteq P$ is generic with respect to T^M iff it is generic with respect to M.

$x_1 = x_2$, proceeds by defining recursively, for any P-names $\tau_1, \tau_2 \in M$ (which then represent well-founded labelled aps's), the set of all $p \in P$ such that $p \Vdash^* \tau_1 = \tau_2$. Given arbitrary P-names $\tau_1, \tau_2 \in T^M$, we might attempt to take the maximum assignment (under the pointwise inclusion order) of a subset of P to every pair of P-names $\pi_1, \pi_2 \in T^M$ such that π_i represents a labelled sub-aps of τ_i for $i = 1, 2$, which satisfies the 'only if' part of the recursive definition mentioned above. Alternatively, it would be very pleasing if we could, instead of recursion on the structure of P-names $\tau_1, \tau_2 \in M$, use recursion on the rank of P-names $\tau_1, \tau_2 \in T^M$ suggested above.

Acknowledgements

We would like to thank Stephen Blamey, who spotted the connection between the study of transition systems and non-well-founded set theory well before the latter became fashionable.

We are very grateful to Robin Knight, Angus Macintyre and an anonymous referee for making a lot of useful comments on earlier drafts of the paper.

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