

# *SRIQ* and *SROIQ* are Harder than *SHOIQ*\*

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**Abstract.** We identify the complexity of (finite model) reasoning in the DL *SROIQ* to be N2ExpTime-complete. We also prove that (finite model) reasoning in the DL *SR*—a fragment of *SROIQ* without nominals, number restrictions, and inverse roles—is 2ExpTime-hard.

## 1 *RIQ*, *SRIQ*, *SROIQ* and *SHOIQ*

In this paper we study computational complexity of reasoning in the DL *SROIQ*—the logic chosen as a candidate for OWL 1.1<sup>1</sup>. *SROIQ* has been introduced in [1] as an extension of *SRIQ* which itself was introduced previously in [2] as an extension of *RIQ* [3]. These papers present tableau-decision procedures for the respective DLs and prove their soundness, completeness and termination.

In contrast to sub-languages of *SHOIQ* whose computational complexities are currently well understood [4], almost nothing was known, up until now, about the complexity of *SROIQ*, *SRIQ* and *RIQ* except for the hardness results inherited from their sub-languages: *SROIQ* is NExpTime-hard as an extension of *SHOIQ*, *SRIQ* and *RIQ* are ExpTime-hard as extensions of *SHIQ*. The difficulty was caused by generalized role inclusion axioms  $R_1 \circ \dots \circ R_n \sqsubseteq R$ , which cause exponential blowup in the tableau procedure. In this paper we demonstrate that this blowup was essentially unavoidable by proving that reasoning in *SRIQ* and *SROIQ* is exponentially harder than in *SHIQ* and *SHOIQ*.

We assume that the reader is familiar with the DL *SHOIQ* [5]. A *SHOIQ* signature is a triple  $\Sigma = (C_\Sigma, R_\Sigma, I_\Sigma)$  consisting of the sets of *atomic concepts*  $C_\Sigma$ , *atomic roles*  $R_\Sigma$  and *individuals*  $I_\Sigma$ . A *SHOIQ* interpretation is a pair  $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$  where  $\Delta^\mathcal{I}$  is a non-empty set called the *domain* of  $\mathcal{I}$ , and  $\cdot^\mathcal{I}$  is the *interpretation function* which assigns for every  $A \in C_\Sigma$  a subset  $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ , for every  $r \in R_\Sigma$  a relation  $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ , and for every  $a \in I_\Sigma$ , an element  $a^\mathcal{I} \in \Delta^\mathcal{I}$ . The interpretation  $\mathcal{I}$  is *finite* iff  $\Delta^\mathcal{I}$  is finite.

A *role* is either some  $r \in R_\Sigma$  or an *inverse role*  $r^-$ . For each  $r \in R_\Sigma$ , we set  $\text{Inv}(r) = r^-$  and  $\text{Inv}(r^-) = r$ . A *SHOIQ* *RBox* is a finite set  $\mathcal{R}$  of *role inclusion axioms* (RIA)  $R_1 \sqsubseteq R$ , *transitivity axioms*  $\text{Tra}(R)$  and *functionality axioms*  $\text{Fun}(R)$  where  $R_1$  and  $R$  are roles. Let  $\sqsubseteq_{\mathcal{R}}^*$  be the reflexive transitive closure of the relation  $\sqsubseteq_{\mathcal{R}}$  on roles defined by  $R_1 \sqsubseteq_{\mathcal{R}} R$  iff  $R_1 \sqsubseteq R \in \mathcal{R}$  or

\* Unless 2ExpTime = NExpTime, in which case just *SROIQ* is harder than *SHOIQ*

<sup>1</sup> <http://www.webont.org/owl/1.1>

$\text{Inv}(R_1) \sqsubseteq \text{Inv}(R) \in \mathcal{R}$ . A role  $S$  is called *simple* (w.r.t.  $\mathcal{R}$ ) if there is no role  $R$  such that  $R \sqsubseteq_{\mathcal{R}}^* S$  and  $\text{Tra}(R) \in \mathcal{R}$ .

Given an RBox  $\mathcal{R}$ , the set of *SHOIQ* concepts is the smallest set containing  $\top$ ,  $\perp$ ,  $A$ ,  $\{a\}$ ,  $\neg C$ ,  $C \sqcap D$ ,  $C \sqcup D$ ,  $\exists R.C$ ,  $\forall R.C$ ,  $\geq n.S.C$ , and  $\leq n.S.C$ , where  $A$  is an atomic concept,  $a$  an individual,  $C$  and  $D$  concepts,  $R$  a role,  $S$  a simple role w.r.t.  $\mathcal{R}$ , and  $n$  a non-negative integer. A *SHOIQ TBox* is a finite set  $\mathcal{T}$  of *general concept inclusion axioms* (GCIs)  $C \sqsubseteq D$  where  $C$  and  $D$  are concepts. We write  $C \equiv D$  as an abbreviation for  $C \sqsubseteq D$  and  $D \sqsubseteq C$ . A *SHOIQ ABox* is a finite set consisting of *concept assertions*  $C(a)$  and *role assertions*  $R(a, b)$ . A *SHOIQ ontology* is a triple  $\mathcal{O} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ , where  $\mathcal{R}$  a *SHOIQ* RBox,  $\mathcal{T}$  is a *SHOIQ* TBox for  $\mathcal{R}$ , and  $\mathcal{A}$  is a *SHOIQ* ABox.

The interpretation  $\mathcal{I}$  is extended to complex role, complex concepts, axioms, and assertions in the usual way. We say that  $\mathcal{I}$  is a *model* of a *SHOIQ* ontology  $\mathcal{O}$ , if every axiom and assertion in  $\mathcal{O}$  is satisfied in  $\mathcal{I}$ . A concept  $C$  is (*finitely*) *satisfiable* w.r.t.  $\mathcal{O}$  if  $C^{\mathcal{I}} \neq \emptyset$  for some (finite) model  $\mathcal{I}$  of  $\mathcal{O}$ . It is well-known [6, 4] that the problem of concept satisfiability for *SHOIQ* is NExpTime-complete.

*SROIQ* [1] extends *SHOIQ* in several ways. (1) It provides for the *universal role*  $U$  which is interpreted as the total relation:  $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ . (2) It allows for *negative role assertions*  $\neg R(a, b)$ . (3) It introduces a concept constructor  $\exists R.\text{Self}$  which is interpreted as  $\{x \in \Delta^{\mathcal{I}} \mid \langle x, x \rangle \in R^{\mathcal{I}}\}$ . (4) It allows for new role axioms  $\text{Sym}(R)$ ,  $\text{Ref}(R)$ ,  $\text{Asy}(S)$ ,  $\text{Irr}(S)$ ,  $\text{Disj}(S_1, S_2)$  where  $S_{(i)}$  are simple roles, which restrict  $R^{\mathcal{I}}$  to be *symmetric* or *reflexive*,  $S^{\mathcal{I}}$  to be *asymmetric* or *irreflexive*, or  $S_1^{\mathcal{I}}$  and  $S_2^{\mathcal{I}}$  to be *disjoint*. (5) Finally, it allows for *generalised role inclusion axioms* of the form  $R_1 \circ \dots \circ R_n \sqsubseteq R$  which require that  $R_1^{\mathcal{I}} \circ \dots \circ R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$  where  $\circ$  is the usual composition of binary relation. The notion of simple roles is adjusted to make sure that no simple role can be implied by a role composition. *SRIQ* [2] is the fragment of *SROIQ* without nominals.

The constructors (1)–(4) do not introduce too many difficulties in *SROIQ*—the existing tableau procedure for *SHOIQ* [5] can be relatively easily adapted to support the new constructors. Dealing with complex role inclusion axioms in DLs turned out to be more difficult. First, with an exception of the DL  $\mathcal{EL}$  [7], the unrestricted usage of complex RIAs, easily leads to undecidability of modal and description logics [8, 3]. Therefore special syntactic restrictions have been introduced in *SROIQ* to regain decidability. A *regular order on roles* is an irreflexive transitive binary relation  $\prec$  on roles such that  $R_1 \prec R_2$  iff  $\text{Inv}(R_1) \prec R_2$ . A RIA  $R_1 \circ \dots \circ R_n \sqsubseteq R$  is said to be  *$\prec$ -regular*, if it does not contain the universal role  $U$  and either: (i)  $n = 2$  and  $R_1 = R_2 = R$ , or (ii)  $n = 1$  and  $R_1 = \text{Inv}(R)$ , or (iii)  $R_i \prec R$  for  $1 \leq i \leq n$ , or (iv)  $R_1 = R$  and  $R_i \prec R$  for  $1 < i \leq n$ , or (v)  $R_n = R$  and  $R_i \prec R$  for  $1 \leq i < n$ .

*Example 1.* Consider the complex role inclusion axiom (1). This RIA is not  $\succ$ -regular regardless of the choice for the ordering  $\succ$ . Indeed, (1) does not satisfy (i)–(ii) since  $n = 3$ , and does not satisfy (iii)–(iv) since  $v = R_2 \not\prec R = v$ .

$$r \circ v \circ r \sqsubseteq v \tag{1}$$

$$v_i \circ v_i \sqsubseteq v_{i+1}, \quad 0 \leq i < n \tag{2}$$

As an example of  $\prec$ -regular complex RIAs, consider axioms (2) over the atomic roles  $v_0, \dots, v_n$ . It is easy to see that these axioms satisfy condition (iii) of  $\prec$ -regularity for every ordering  $\prec$  such that  $v_i \prec v_j$ , for every  $0 \leq i < j \leq n$ .

Although Example 1 does not demonstrate the usage of the conditions (i), (ii), (iv) and (v) for  $\prec$ -regularity of RIAs, as will be shown soon, already axioms that satisfy just the condition (iii) make reasoning in *SRIOQ* hard.

The syntactic restrictions on the set of RIAs of an RBox  $\mathcal{R}$  ensure that  $\mathcal{R}$  is regular in the following sense. Given a role  $R$ , let  $L_{\mathcal{R}}(R)$  be the language consisting of the words over roles defined by:

$$L_{\mathcal{R}}(R) := \{R_1 R_2 \dots R_n \mid \mathcal{R} \models (R_1 \circ \dots \circ R_n \sqsubseteq R)\}$$

It has been shown in [3] that if the RIAs of  $\mathcal{R}$  are  $\prec$ -regular for some ordering  $\prec$ , then for every role  $R$ , the language  $L_{\mathcal{R}}(R)$  is regular. The tableau procedure for *SRIOQ* presented in [1], utilizes the non-deterministic finite automata (NFA) corresponding to  $L_{\mathcal{R}}(R)$  to ensure that only finitely many states are produced by tableau expansion rules. Unfortunately, the NFA for  $L_{\mathcal{R}}(R)$  can be exponentially large in the size of  $\mathcal{R}$  which results in exponential blowup in the number of states produced in the worst case by the procedure for *SRIOQ* compared to the procedure for *SHOIQ*. It was conjectured in [1] that such blowup (for the size of the automaton) is unavoidable. In Example 2, we demonstrate that the minimal automata for axioms (2) are indeed exponentially large.

*Example 2 (Example 1 continued).* Let  $\mathcal{R}$  be an RBox consisting of the single axiom (1). It is easy to see that  $L_{\mathcal{R}}(s) = \{r^i v r^i \mid i \geq 0\}$ , where  $r^i$  denotes the word consisting of  $i$  letters  $r$ . The language  $L_{\mathcal{R}}(v)$  is non-regular which can be shown, e.g., by using the pumping lemma for regular languages (see, e.g., [9]).

On the other hand, the RBox  $\mathcal{R}$  consisting of the axioms (2) gives regular languages. It is easy to show by induction on  $i$  that  $L_{\mathcal{R}}(v_i)$  consist of finitely many words, and hence, are regular. Moreover, by induction on  $i$  it is easy to show that  $v_0^j \in L_{\mathcal{R}}(v_i)$  iff  $j = 2^i$ . Let  $Q(v_i)$  be a NFA for  $L_{\mathcal{R}}(v_i)$  and  $q_0, \dots, q_{2^i}$  a run of the automata accepting  $v_0^{2^i}$ . Then all the states in this run are different, since otherwise there is a cycle which means that  $A(v_i)$  accepts infinitely many words. Hence  $Q(v_i)$  has at least  $2^i + 1$  states.

## 2 The Lower Complexity Bounds

In this section, we prove that reasoning in *SROLF*—a fragment of *SRIOQ* that does not use number restrictions but functional roles—is **N2ExpTime**-hard. The proof is by reduction from the doubly exponential Domino tiling problem. We also demonstrate that reasoning in *SR*—a fragment of *SRIOQ* that does not use nominals  $\{a\}$ , counting and inverse roles—is **2ExpTime**-hard by reduction from the word problem for an exponential space alternating Turing machine.

The main idea of our reduction is to enforce double-exponentially long chains using *SR* axioms. Single-exponentially long chains can be enforced using a well-known integer counting technique [6]. A *counter*  $c(x)$  is an integer between 0 and

$2^n - 1$  which is assigned to an element  $x$  of the interpretation using  $n$  atomic concepts  $B_1, \dots, B_n$  as follows: the  $i$ -th bit of  $c(x)$  is equal to 1 iff  $B_i$  holds at  $x$ . It is easy to see that axioms (3)–(7) enforce an exponentially long chains by initializing the counter and incrementing it over a role.

$$Z \sqsubseteq \neg B_1 \sqcap \dots \sqcap \neg B_n \quad (3)$$

$$E \equiv B_1 \sqcap \dots \sqcap B_n \quad (4)$$

$$\neg E \equiv \exists r. \top \quad (5)$$

$$\top \equiv (B_1 \sqcap \forall r. \neg B_1) \sqcup (\neg B_1 \sqcap \forall r. B_1) \quad (6)$$

$$B_{i-1} \sqcap \forall r. \neg B_{i-1} \equiv (B_i \sqcap \forall r. \neg B_i) \sqcup (\neg B_i \sqcap \forall r. B_i), \quad 1 < i \leq n \quad (7)$$

Axiom (3) is responsible for initializing the counter to zero using the atomic concept  $Z$ . Axiom (4) can be used to detect whether the counter has reached the final value  $2^n - 1$ , by checking whether  $E$  holds. Thus, using axiom (5), we can express that an element has an  $r$ -successor if and only if its counter has not reached the final value. Axioms (6) and (7) express how the counter is incremented over  $r$ : axiom (6) expresses that the lowest bit of the counter is always flipped; axioms (7) express that any other bit of the counter is flipped if and only if the lower bit is changed from 1 to 0.

**Lemma 1.** *Let  $\mathcal{O}$  be an ontology containing axioms (3)–(7). Then for every model  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  of  $\mathcal{O}$  and  $x \in Z^{\mathcal{I}}$  there exist  $x_i \in \Delta^{\mathcal{I}}$  with  $0 \leq i < 2^n$  such that  $x = x_0$  and  $(x_{i-1}, x_i) \in r^{\mathcal{I}}$  for every  $i$  with  $1 \leq i < 2^n$ , and  $c(x_i) = i$ .*

Now we use similar ideas to enforce doubly-exponentially long chains in the model. This time, however, we cannot use just atomic concepts to encode the bits of the counter since there are exponentially many bits. Instead, the bits of the counter will be encoded using the values of one atomic concept  $X$  on the elements of exponentially long chains constructed using axioms (3)–(7): the  $i$ -th bit of the number corresponds to the value of  $X$  at the  $i$ -th element of the chain. In Figure 1(a) we have depicted a doubly exponential zig-zag-like chain that we are going to enforce using  $\mathcal{SR}$  axioms. The chain consists of  $2^{2^n}$   $r$ -chains, each having exactly  $2^n$  elements, that are joint together using a role  $v$ —the last element of every  $r$ -chain, except for the final chain, is  $v$ -connected to the first element of the next  $r$ -chain. The tricky part of the encoding is to ensure that the counters that correspond to  $r$ -chains are properly incremented. This is achieved by using regular role inclusion axioms (2) which allow us to propagate information using role  $v_n$  across chains of  $2^n$  roles. The structure in Figure 1(a) is enforced using axioms (8)–(15) in addition to axioms (2)–(7).

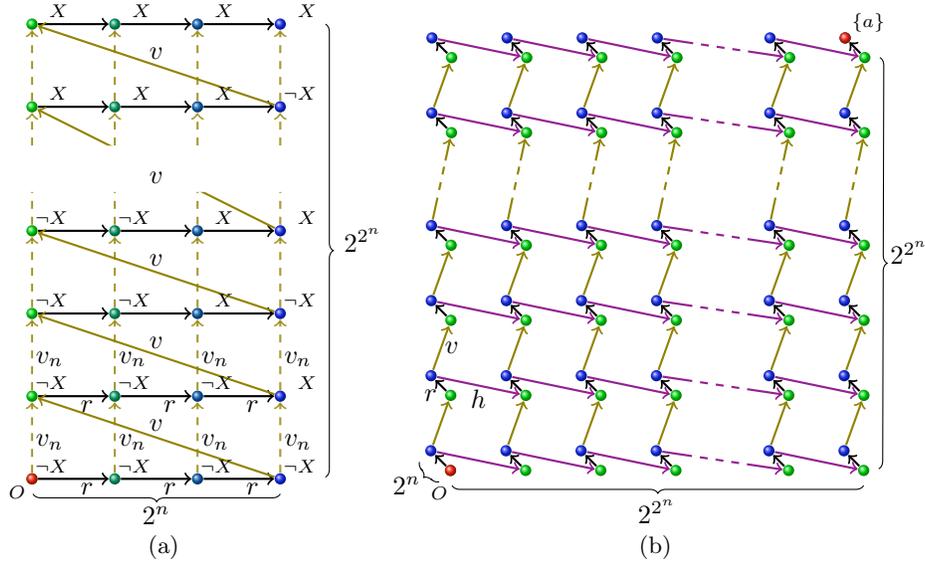
$$O \sqsubseteq Z \sqcap Z_v \sqcap E_v \quad (8)$$

$$\top \sqsubseteq \forall v. (Z \sqcap E_v) \quad (9)$$

$$Z_v \sqsubseteq \neg X \sqcap \forall r. Z_v \quad (10)$$

$$E_v \sqcap X \sqsubseteq \forall r. E_v \quad (11)$$

$$\neg E_v \sqsubseteq \forall r. \neg E_v \quad (12)$$



**Fig. 1.** (a) Using  $\mathcal{SR}$  axioms to encode double exponentially long chains; (b) Using  $\mathcal{SROIN}$  axioms to encode double exponentially large grids

$$E \sqcap \neg(E_v \sqcap X) \sqsubseteq \exists v. \top \quad (13)$$

$$r \sqsubseteq v_0, \quad v \sqsubseteq v_0 \quad (14)$$

$$\forall r. (X \sqcap \forall v_n. \neg X) \equiv (X \sqcap \forall v_n. \neg X) \sqcup (\neg X \sqcap \forall v_n. X) \quad (15)$$

The atomic concept  $O$  corresponds to the origin of our structure. Axioms (8) and (9) express that  $O$  and every  $v$ -successor start a new  $2^n$ -long  $r$ -chain because of the atomic concept  $Z$  and axioms (3)–(7). In addition, the  $r$ -chain starting from  $O$  should be initialized using  $Z_v$  and axiom (10). In order to identify the final chain, we use the atomic concept  $E_v$  which should hold on an element of an  $r$ -chain iff  $X$  holds on all the preceding elements of this  $r$ -chain. Axioms (8) and (9) say that  $E_v$  holds at the first element of every  $r$ -chain. Axioms (11) and (12) propagate the value of  $E_v$  over the elements of the  $r$ -chain. Now, axiom (13) says that the last elements of every non-final  $r$ -chains has a  $v$ -successor.

Axioms (14) and (15) together with axioms (2) are responsible for incrementing the counter between  $r$ -chains. Recall that axioms from (2) imply  $(v_0)^i \sqsubseteq v_n$  if and only if  $i = 2^n$ , where  $(v_0)^i$  denotes the composition of the role  $v_0$   $i$  times. Now, using axioms (14) we make sure that exactly the corresponding elements of the consequent  $r$ -chains are connected by the role  $v_n$ . Then axiom (15) expresses the transformation of bits in a similar way as axioms (6) and (7).

**Lemma 2.** *Let  $\mathcal{O}$  be an ontology containing axioms (2)–(15). Then for every model  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  of  $\mathcal{O}$  and  $x \in O^{\mathcal{I}}$  there are  $x_{(i,j)} \in \Delta^{\mathcal{I}}$  with  $0 \leq i < 2^n$*

and  $0 \leq j < 2^{2^n}$  such that  $x = x_{(0,0)}$ ,  $(x_{(i-1,j)}, x_{(i,j)}) \in r^{\mathcal{I}}$  for  $1 \leq i < 2^n$  and  $0 \leq j < 2^{2^n}$ , and  $(x_{(2^{n-1},j-1)}, x_{(0,j)}) \in v^{\mathcal{I}}$  for  $1 \leq j < 2^{2^n}$ .

Now we demonstrate that using *SRQIF* axioms one can express the grid-like structure in Figure 1(b). Our construction is similar to the one for *ALCQ* in [6] which uses a pair of counters to encode the coordinates of the grid elements and a nominal to join the elements with the same counters together. The only difference is that now we can use the counters up to  $2^{2^n}$  instead of just  $2^n$ .

The grid-like structure in Figure 1(b) consists of  $2^{2^n} \times 2^{2^n}$   $2^n$ -long  $r$ -chains which are joint vertically using the role  $v$  and horizontally using the role  $h$  in the same way as in Figure 1(a). Every  $r$ -chain stores information about two counters. The first counter uses the concept name  $X$  and corresponds to the vertical coordinate of the  $r$ -chain; the second counter uses  $Y$  and corresponds to the horizontal coordinate of the  $r$ -chain.

The axioms (2)–(15) are now used to express that the vertical counter for  $r$ -chains is initialized in  $O$  and is incremented over  $v$ . A copy of these axioms (16)–(24) expresses the analogous property for the horizontal counter.

$$O \sqsubseteq Z \sqcap Z_h \sqcap E_h \quad (16)$$

$$\top \sqsubseteq \forall v.(Z \sqcap E_h) \quad (17)$$

$$Z_h \sqsubseteq \neg Y \sqcap \forall r.Z_h \quad (18)$$

$$E_h \sqcap Y \sqsubseteq \forall r.E_h \quad (19)$$

$$\neg E_h \sqsubseteq \forall r.\neg E_h \quad (20)$$

$$E \sqcap \neg(E_h \sqcap Y) \sqsubseteq \exists v.\top \quad (21)$$

$$r \sqsubseteq h_0, \quad h \sqsubseteq h_0 \quad (22)$$

$$h_i \circ h_i \sqsubseteq h_{i+1}, \quad 0 \leq i < n \quad (23)$$

$$\forall r.(Y \sqcap \forall h_n.\neg Y) \equiv (Y \sqcap \forall h_n.\neg Y) \sqcup (\neg Y \sqcap \forall h_n.Y) \quad (24)$$

The grid structure in Figure 1(b) is now enforced by adding axioms (25)–(28).

$$\top \sqsubseteq (X \sqcap \forall h_n.X) \sqcup (\neg X \sqcap \forall h_n.\neg X) \quad (25)$$

$$\top \sqsubseteq (Y \sqcap \forall v_n.Y) \sqcup (\neg Y \sqcap \forall v_n.\neg Y) \quad (26)$$

$$E_v \sqcap X \sqcap E_h \sqcap Y \sqsubseteq \{a\} \quad (27)$$

$$\text{Fun}(r^-), \text{Fun}(h^-), \text{Fun}(v^-) \quad (28)$$

Axioms (25) and (26) express that the values of the vertical (horizontal) counters are copied across  $h$  (respectively  $v$ ). Axiom (27) expresses that the last element of the  $r$ -chain with the final coordinates is unique. Together with axiom (28) expressing that the roles  $r$ ,  $h$  and  $v$  are inverse functional, this ensures that no two different  $r$ -chains have the same coordinates. Note that the roles  $r$ ,  $h$  and  $v$  are simple since they do not occur on the right hand side of RIAs (2), (22), and (23). The following analogue of Lemmas 1 and 2 claims that the models of our axioms that satisfy  $O$  correspond to the grid in Figure 1(b).

**Lemma 3.** *Let  $\mathcal{O}$  be an ontology containing axioms (2)–(28). Then for every model  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  of  $\mathcal{O}$  and  $x \in O^{\mathcal{I}}$  there are  $x_{(i,j,k)} \in \Delta^{\mathcal{I}}$  with  $0 \leq i < 2^n$  and  $0 \leq j, k < 2^{2^n}$  such that  $x = x_{(0,0,0)}$ ,  $(x_{(i-1,j,k)}, x_{(i,j,k)}) \in r^{\mathcal{I}}$  for  $i, j, k$  with  $1 \leq i < 2^n$  and  $0 \leq j, k < 2^{2^n}$ ,  $(x_{(2^n-1,j-1,k)}, x_{(0,j,k)}) \in v^{\mathcal{I}}$  for  $1 \leq j < 2^{2^n}$ ,  $0 \leq k < 2^{2^n}$ , and  $(x_{(2^n-1,j,k-1)}, x_{(0,j,k)}) \in h^{\mathcal{I}}$  for  $0 \leq j < 2^{2^n}$ ,  $1 \leq k < 2^{2^n}$ .*

Our complexity result for *SROLF* is obtained by a reduction from the bounded domino tiling problem. A *domino system* is a triple  $\mathcal{D} = (T, H, V)$ , where  $T = \{1, \dots, k\}$  is a finite set of *tiles* and  $H, V \subseteq T \times T$  are *horizontal* and *vertical matching relations*. A *tiling* of  $m \times m$  for a domino system  $\mathcal{D}$  with *initial condition*  $c^0 = \langle t_1^0, \dots, t_n^0 \rangle$ ,  $t_i^0 \in T$ ,  $1 \leq i \leq n$ , is a mapping  $t : \{1, \dots, m\} \times \{1, \dots, m\} \rightarrow T$  such that  $\langle t(i-1, j), t(i, j) \rangle \in H$ ,  $1 < i \leq m$ ,  $1 \leq j \leq m$ ,  $\langle t(i, j-1), t(i, j) \rangle \in V$ ,  $1 < i \leq m$ ,  $1 \leq j \leq m$ , and  $t(i, 1) = t_i^0$ ,  $1 \leq i \leq n$ . It is a well known [10] that there exists a domino system  $\mathcal{D}_0$  which is *N2ExpTime*-complete for the following decision problem: given an initial condition  $c^0$  of the size  $n$ , check if  $\mathcal{D}_0$  admits the tiling of  $2^{2^n} \times 2^{2^n}$  for  $c^0$ . Axioms (29)–(34) in addition to axioms (2)–(28) provide a reduction from this problem to the problem of concept satisfiability in *SROLF*.

$$\top \sqsubseteq D_1 \sqcup \dots \sqcup D_k \quad (29)$$

$$D_i \sqcap D_j \sqsubseteq \perp, \quad 1 \leq i < j \leq k \quad (30)$$

$$D_i \sqsubseteq \forall r. D_i, \quad 1 \leq i \leq k \quad (31)$$

$$D_i \sqcap \forall h. D_j \sqsubseteq \perp, \quad \langle i, j \rangle \notin H \quad (32)$$

$$D_i \sqcap \forall v. D_j \sqsubseteq \perp, \quad \langle i, j \rangle \notin V \quad (33)$$

$$O \sqsubseteq D_{t_1} \sqcap \forall h_n. (D_{t_2} \sqcap \forall h_n. (D_{t_3} \sqcap \dots (\forall h_n. D_{t_n}) \dots)) \quad (34)$$

The atomic concepts  $D_1, \dots, D_k$  corresponds to the tiles of the domino system  $\mathcal{D}_0$ . Axioms (29) and (30) express that every element in the model is assigned with a unique tile  $D_i$ . Axiom (31) expresses that the elements of the same  $r$ -chain are assigned with the same tile. Axioms (32) and (33) express the horizontal and vertical matching properties. Finally, axiom (34) expresses the initial condition. It is easy to see that this reduction is polynomial in  $n$  ( $\mathcal{D}_0$  is fixed).

**Theorem 1.** *Let  $c^0$  be an initial condition of the size  $n$  for the domino system  $\mathcal{D}_0$  and  $\mathcal{O}$  an ontology consisting of the axioms (2)–(34). Then  $\mathcal{D}_0$  admits the tiling of  $2^{2^n} \times 2^{2^n}$  for  $c^0$  if and only if  $O$  is (finitely) satisfiable in  $\mathcal{O}$ .*

*Proof (sketch).* It is easy to show that if  $\mathcal{D}_0$  admits the tiling of  $2^{2^n} \times 2^{2^n}$  for  $c^0$  then the structure in Figure 1(b) (which finitely satisfies  $O$ ) can be expanded to a model of  $\mathcal{O}$  by interpreting  $D_i$  accordingly. On the other hand, using Lemma 3 one can demonstrate that any model of  $\mathcal{O}$  that satisfies  $O$  witnesses a tiling of  $2^{2^n} \times 2^{2^n}$  for  $c^0$ .  $\square$

**Corollary 1.** *The problem of (finite) concept satisfiability in the DL *SROLF* is *N2ExpTime*-hard (and so are all the standard reasoning problems).*

In the remainder of this section, we prove that (finite model) reasoning in the DL  $\mathcal{SR}$  is 2ExpTime-hard. The proof is by reduction from the word problem of an exponential space Turing machine. The main idea of our reduction is to use the zig-zag-like structures in Figure 1(a) to simulate a computation of an alternating Turing machine.

An *alternating Turing machine* (ATM) is a tuple  $M = (\Gamma, Q, \Sigma, q_0, \delta_1, \delta_2)$  where  $\Gamma$  is a finite *working alphabet* containing a *blank symbol*  $\square$ ;  $\Sigma \subseteq \Gamma$  is the *input alphabet*;  $Q = Q_{\exists} \uplus Q_{\forall} \uplus \{q_a\} \uplus \{q_r\}$  is a finite set of *states* partitioned into *existential states*  $Q_{\exists}$ , *universal states*  $Q_{\forall}$ , an *accepting state*  $q_a$  and a *rejecting state*  $q_r$ ;  $q_0 \in Q_{\exists}$  is the *starting state*, and  $\delta_1, \delta_2 : (Q_{\exists} \cup Q_{\forall}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, N\}$  are *transition functions*. A *configuration* of  $M$  is a word  $c = w_1qw_2$  where  $w_1, w_2 \in \Gamma^*$  and  $q \in Q$ . An *initial configuration* is a configuration  $c = q_0w$  where  $w \in \Sigma^*$ . The *size*  $|c|$  of a configuration  $c$  is the number of symbols in  $c$ . The *successor configurations*  $\delta_1(c)$  and  $\delta_2(c)$  of a configuration  $c = w_1qw_2$  with  $q \neq q_a, q_r$  over the transition functions  $\delta_1$  and  $\delta_2$  are defined like for deterministic Turing machines (see, e.g., [9]). The sets  $C_a(M)$  of *accepting configurations* and  $C_r(M)$  of *rejecting configurations* of  $M$  are the smallest sets such that (i)  $w_1q_aw_2 \in C_a(M)$  and  $w_1q_rw_2 \in C_r(M)$  for every  $w_1, w_2 \in \Gamma^*$ ; (ii)  $w_1qw_2 \in C_a(M)(C_r(M))$  if  $q \in Q_{\forall}(Q_{\exists})$  and  $q_1(c), q_2(c) \in C_a(M)(C_r(M))$  or  $q \in Q_{\exists}(Q_{\forall})$  and  $\delta_1(c)$  or  $\delta_2(c) \in C_a(M)(C_r(M))$ . The set of configurations of  $M$  *reachable from*  $c^0$  is the smallest set  $M(c^0)$  such that  $c^0 \in M(c^0)$  and  $\delta_1(c), \delta_2(c) \in M(c^0)$  for every  $c \in M(c^0)$ .  $M$  is  *$g(n)$  space bounded* if for every initial configuration  $c^0$ , we have that  $c^0 \in C_a(M) \cup C_r(M)$  and  $|c| \leq g(|c^0|)$  for every  $c \in M_0(c^0)$ . A classical result  $\text{AExpSpace} = \text{2ExpTime}$  (see, e.g., [11]) implies that there is a  $2^n$  space bounded ATM  $M_0$  for which the following decision problem is 2ExpTime-complete: given an initial configuration  $c^0$  decide whether  $c^0 \in C_a(M_0)$ .

Let  $c^0$  be an initial configuration of  $M_0$  and  $n = |c^0|$ . In order to decide whether  $c^0 \in C_a(M_0)$ , we try to build all the required accepting successor configurations of  $c^0$  from  $M_0(c^0)$ . We encode the configurations of  $M_0(c^0)$  on  $2^n$ -long  $r$ -chains. An  $r$ -chain corresponding to  $c$  is connected to  $r$ -chains corresponding to  $\delta_1(c)$  and  $\delta_2(c)$  via the roles  $r$  and  $h$  in a similar way as in Figure 1(a). It is a well-known property of the transition function of the Turing machines that the symbols  $c_i^1$  and  $c_i^2$  at the position  $i$  of  $\delta_1(c)$  and  $\delta_2(c)$  are uniquely determined by the symbols  $c_{i-1}, c_i$  and  $c_{i+1}$  of  $c$  at the positions  $i-1, i, i+1$ .<sup>2</sup> We assume that this correspondence is given by the (partial) functions  $\gamma_1$  and  $\gamma_2$  such that  $\gamma_1(c_{i-1}, c_i, c_{i+1}) = c_i^1$  and  $\gamma_2(c_{i-1}, c_i, c_{i+1}) = c_i^2$ . The computation of  $M_0$  from  $c^0$  can be encoded using axioms (35)–(48) in addition to axioms (2)–(24).

$$\top \sqsubseteq \bigsqcup_{s \in Q \cup \Gamma} A_s \quad (35)$$

$$A_{s_1} \sqcap A_{s_2} \sqsubseteq \perp, \quad s_1 \neq s_2 \quad (36)$$

<sup>2</sup> We assume w.l.o.g. that the transitions that depend on the tape content do not change the position of the head of the ATM; If  $i$  is the first or the last symbol of the configuration, we assume that  $c_{i-1}$ , respectively  $c_{i+1}$  is the blank symbol  $\square$

$$A_q \sqsubseteq V \sqcup H, \quad q \in Q_{\exists} \quad (37)$$

$$A_q \sqsubseteq V \sqcap H, \quad q \in Q_{\forall} \quad (38)$$

$$A_{q_r} \sqsubseteq \perp \quad (39)$$

$$V \sqsubseteq \forall r.V, \neg V \sqsubseteq \forall r.\neg V, H \sqsubseteq \forall r.H, \neg H \sqsubseteq \forall r.\neg H \quad (40)$$

$$V \sqcap S \sqcap A_{s_2} \sqcap \forall r.A_{s_3} \sqsubseteq \forall v_n.\gamma_1(A_{\square}, A_{s_2}, A_{s_3}) \quad (41)$$

$$H \sqcap S \sqcap A_{s_2} \sqcap \forall r.A_{s_3} \sqsubseteq \forall h_n.\gamma_2(A_{\square}, A_{s_2}, A_{s_3}) \quad (42)$$

$$V \sqcap A_{s_1} \sqcap \forall r.(A_{s_2} \sqcap E) \sqsubseteq \forall r.\forall v_n.\gamma_1(A_{s_1}, A_{s_2}, A_{\square}) \quad (43)$$

$$H \sqcap A_{s_1} \sqcap \forall r.(A_{s_2} \sqcap E) \sqsubseteq \forall r.\forall h_n.\gamma_2(A_{s_1}, A_{s_2}, A_{\square}) \quad (44)$$

$$V \sqcap A_{s_1} \sqcap \forall r.(A_{s_2} \sqcap \neg E \sqcap \forall r.A_{s_3}) \sqsubseteq \forall r.\forall v_n.\gamma_1(A_{s_1}, A_{s_2}, A_{s_3}) \quad (45)$$

$$H \sqcap A_{s_1} \sqcap \forall r.(A_{s_2} \sqcap \neg E \sqcap \forall r.A_{s_3}) \sqsubseteq \forall r.\forall h_n.\gamma_2(A_{s_1}, A_{s_2}, A_{s_3}) \quad (46)$$

$$O \sqsubseteq A_{c_1^0} \sqcap \forall r.(A_{c_2^0} \sqcap \forall r.(A_{c_3^0} \sqcap \dots (\forall r.A_{c_n^0} \sqcap \forall r.A_{\square}) \dots)) \quad (47)$$

$$A_{\square} \sqsubseteq \forall r.A_{\square} \quad (48)$$

We introduce an atomic concept  $A_s$  for every symbol  $s$  from the set of states  $Q$  and the working alphabet  $\Gamma$ . Axioms (35) and (36) express that to every element of the model a unique symbol  $s \in Q \cup \Gamma$  is assigned. Two atomic concepts  $V$  and  $H$  determine for a given accepting configuration  $c$  which of the successor configurations  $\delta_1(c)$  and  $\delta_2(c)$  are accepting: if  $V$  ( $H$ ) holds on the current  $r$ -chain then the  $r$ -chain accessible by  $v$  (respectively by  $h$ ) should be accepting. When the current configuration has an existential state, we need to pick either  $V$  or  $H$  (axiom (37)); when it has a universal state, we have to pick both (axiom (38)); when it has a rejecting state, we fail (axiom (39)). The values of  $V$  and  $H$  are propagated to all elements of the current  $r$ -chain using axioms (40). Axioms (41)–(46) express how the symbols of the accepting successor configurations are computed using the functions  $\gamma^1$  and  $\gamma^2$ . Finally, axioms (47) and (48) express the initialization of the starting configuration  $c^0$ .

**Theorem 2.** *Let  $c^0$  be the starting configuration for the ATM  $M_0$  and  $\mathcal{O}$  an ontology consisting of the axioms (2)–(24) and (35)–(48). Then  $c^0 \in C_a(M_0)$  if and only if  $\mathcal{O}$  is (finitely) satisfiable in  $\mathcal{O}$ .*

**Corollary 2.** *The problem of (finite) concept satisfiability in the DL  $\mathcal{SR}$  is 2ExpTime-hard (and so are all the standard reasoning problems).*

### 3 The Upper Complexity Bound

In this section we prove that complexity of  $\mathcal{SROIQ}$  is in N2ExpTime using an exponential time translation into the two variable fragment with counting  $\mathcal{C}^2$ .

Let  $\mathcal{O}$  be  $\mathcal{SROIQ}$  ontology for which we need to test satisfiability. By Theorem 9 from [1], w.l.o.g., we can assume that  $\mathcal{O}$  does not contain concept and role assertions, the universal role, and axioms of the form  $\text{lrr}(S)$ ,  $\text{Tra}(R)$  or  $\text{Sym}(R)$ . We also replace assertions  $\text{Ref}(R)$  with the axiom  $\top \sqsubseteq \exists R.\text{Self}$  and  $\text{Asy}(S)$  with  $\text{Disj}(S, \text{Inv}(S))$ . Next, it is possible to convert  $\mathcal{O}$  into the *simplified form* which

(1)	$A \sqsubseteq \forall r.B$	$\forall x.(A(x) \rightarrow \forall y.[r(x,y) \rightarrow B(y)])$
(2)	$A \sqsubseteq \geq n s.B$	$\forall x.(A(x) \rightarrow \exists^{\geq n} y.[s(x,y) \wedge B(y)])$
(3)	$A \sqsubseteq \leq n s.B$	$\forall x.(A(x) \rightarrow \exists^{\leq n} y.[s(x,y) \wedge B(y)])$
(4)	$A \equiv \exists s.\text{Self}$	$\forall x.(A(x) \leftrightarrow s(x,x))$
(5)	$A_a \equiv \{a\}$	$\exists^{\equiv 1} y.A_a(y)$
(6)	$\prod A_i \sqsubseteq \bigsqcup B_j$	$\forall x.(\bigwedge \neg A_i(x) \vee \bigvee B_j(x))$
(7)	$\text{Disj}(s_1, s_2)$	$\forall xy.(s_1(x,y) \wedge s_2(x,y) \rightarrow \perp)$
(8)	$s_1 \sqsubseteq s_2$	$\forall xy.(s_1(x,y) \rightarrow s_2(x,y))$
(9)	$s_1 \sqsubseteq s_2^-$	$\forall xy.(s_1(x,y) \rightarrow s_2(y,x))$
(10)	$r_1 \circ \dots \circ r_n \sqsubseteq v$	

**Table 1.** Translation of simplified  $\mathcal{SROIQ}$  axioms to  $\mathcal{C}^2$

contains only axioms of the form given in the first column of Table 1, where  $A_{(i)}$  and  $B_{(j)}$  are atomic concepts,  $r_{(i)}$  atomic roles,  $s_{(i)}$  simple atomic roles, and  $v$  a non-simple atomic role. The transformation can be done in polynomial time using the standard structural transformation which iteratively introduces definitions for compound sub-concept and sub-roles (see, e.g. [12]).

After the transformation, we eliminate RIAs of the form (10) using the correspondent automata in a similar way as in [13]. Axioms of the form (10) can cause unsatisfiability of  $\mathcal{O}$  only through axioms of the form (1), since other axioms do not contain non-simple roles. Given an axiom  $A \sqsubseteq \forall r.B$  of form (1) and an NFA for  $L_{\mathcal{R}}(r)$  with the set of states  $Q$ , starting state  $q_0 \in Q$ , accepting states  $F \subseteq Q$ , and a transition relation  $\delta \subseteq Q \times R_{\Sigma} \times Q$ , we replace this axiom with axioms (49)–(51) where  $A_q^r$  is a fresh atomic concept for every  $q \in Q$ .

$$A \sqsubseteq A_q^r, \quad q = q_0 \quad (49)$$

$$A_{q_1}^r \sqsubseteq \forall s.A_{q_2}^r, \quad (q_1, s, q_2) \in \delta \quad (50)$$

$$A_q^r \sqsubseteq B, \quad q \in F \quad (51)$$

**Lemma 4.** *Let  $\mathcal{O}$  be an ontology containing of axioms of the form (1)–(10) from Table 1, and  $\mathcal{O}'$  obtained from  $\mathcal{O}$  by replacing every axiom of the form (1) with axioms (49)–(51) and removing all axioms of form (10). Then (i) every model of  $\mathcal{O}$  can be expanded to a model of  $\mathcal{O}'$  by interpreting  $A_q^r$ , and (ii) every model of  $\mathcal{O}'$  can be expanded to a model of  $\mathcal{O}$  by interpreting the non-simple roles.*

**Theorem 3.** *(Finite) satisfiability of  $\mathcal{SROIQ}$  ontologies is in  $\text{N2ExpTime}$  (and so are all the standard reasoning problems).*

*Proof.* The input  $\mathcal{SROIQ}$  ontology  $\mathcal{O}$  can be translated in exponential time preserving (finite) satisfiability into a simplified ontology containing only axioms of the form (1)–(9) from Table 1 which can be translated into the two variable fragment with counting quantifiers  $\mathcal{C}^2$  according to the second column of Table 1. Since (finite) satisfiability of  $\mathcal{C}^2$  is  $\text{NExpTime}$ -complete [14], our reduction proves that satisfiability of  $\mathcal{SROIQ}$  is in  $\text{N2ExpTime}$ .  $\square$

## 4 Conclusions

In this paper we have identified the exact computational complexity of (finite model) reasoning in the DL  $\mathcal{SROIQ}$  to be  $\text{N2ExpTime}$ —that is, exponentially worse than for the DL  $\mathcal{SROIQ}$ . The complexity blowup is due to generalized role inclusion axioms, and in particular due to their ability to “chain” a fixed exponential number of roles. Indeed, the complexity blowup occurs already when no other complex constructors such as nominals, number restrictions and inverse roles are used:  $\mathcal{SR}$  and therefore  $\mathcal{SRIQ}$  is  $\text{2ExpTime}$ -hard, whereas  $\mathcal{SHIQ}$  is merely in  $\text{ExpTime}$ . Our complexity result proves that the exponential blowups in the tableau procedures for  $\mathcal{SRIQ}$  [2] and  $\mathcal{SROIQ}$  [1] are unavoidable.

Few open questions left for the future work. First, we did not obtain the upper complexity bound for the DL  $\mathcal{SR}$ . We think that a matching  $\text{2ExpTime}$  decision procedure can be obtained by an easy modification of the  $\text{ExpTime}$  automaton for  $\mathcal{SHIQ}$  [4]. Second, the question about the exact complexity of  $\mathcal{RIQ}$  [3] remains open, since  $\mathcal{RIQ}$  allows only for complex RIAs of the form  $R_1 \circ R_2 \sqsubseteq R_1$  or  $R_1 \circ R_2 \sqsubseteq R_2$  which do not capture our axioms (2).

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