

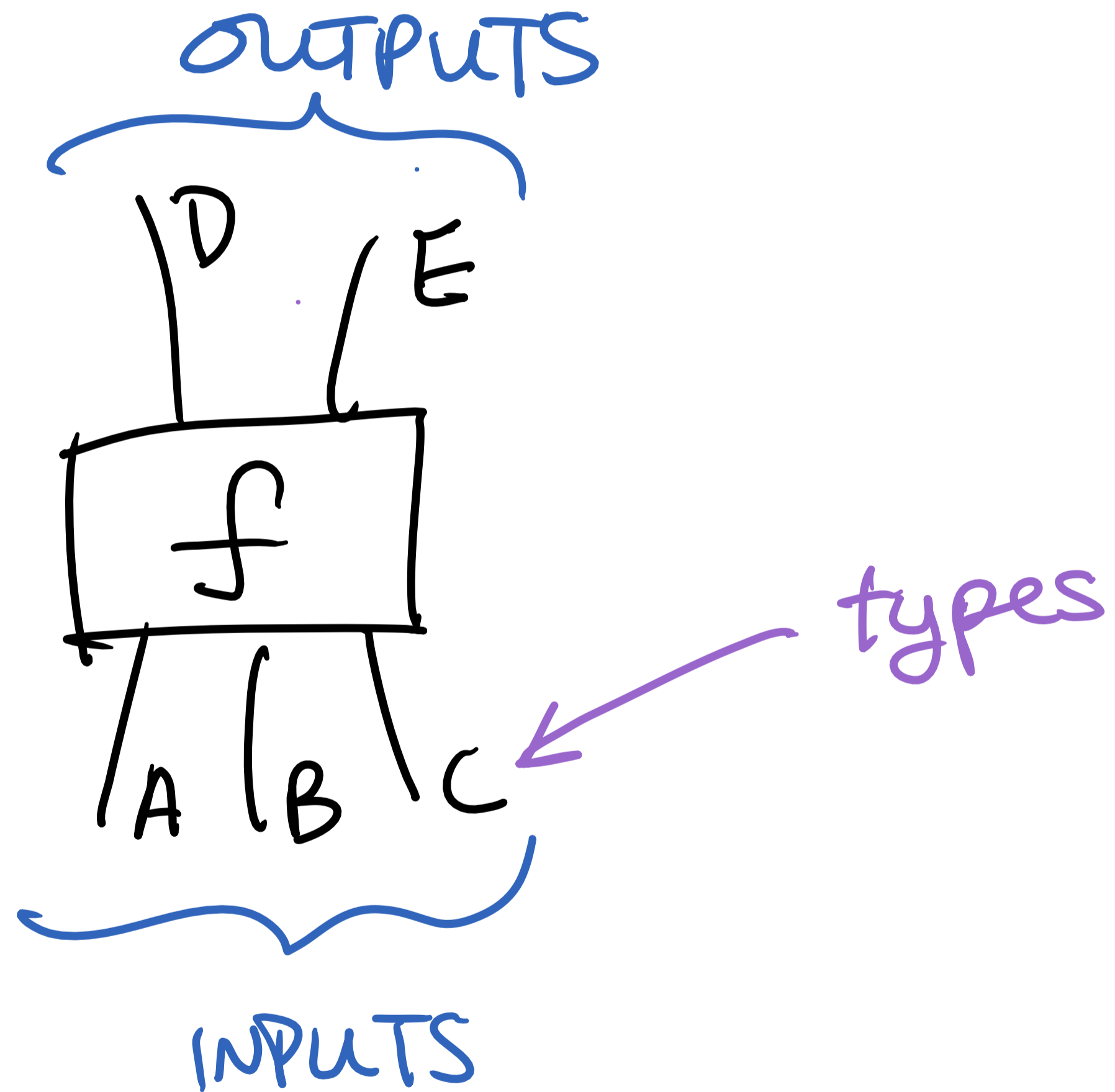
PICTURING QUANTUM PROCESSES I

ALEKS KISSINGER

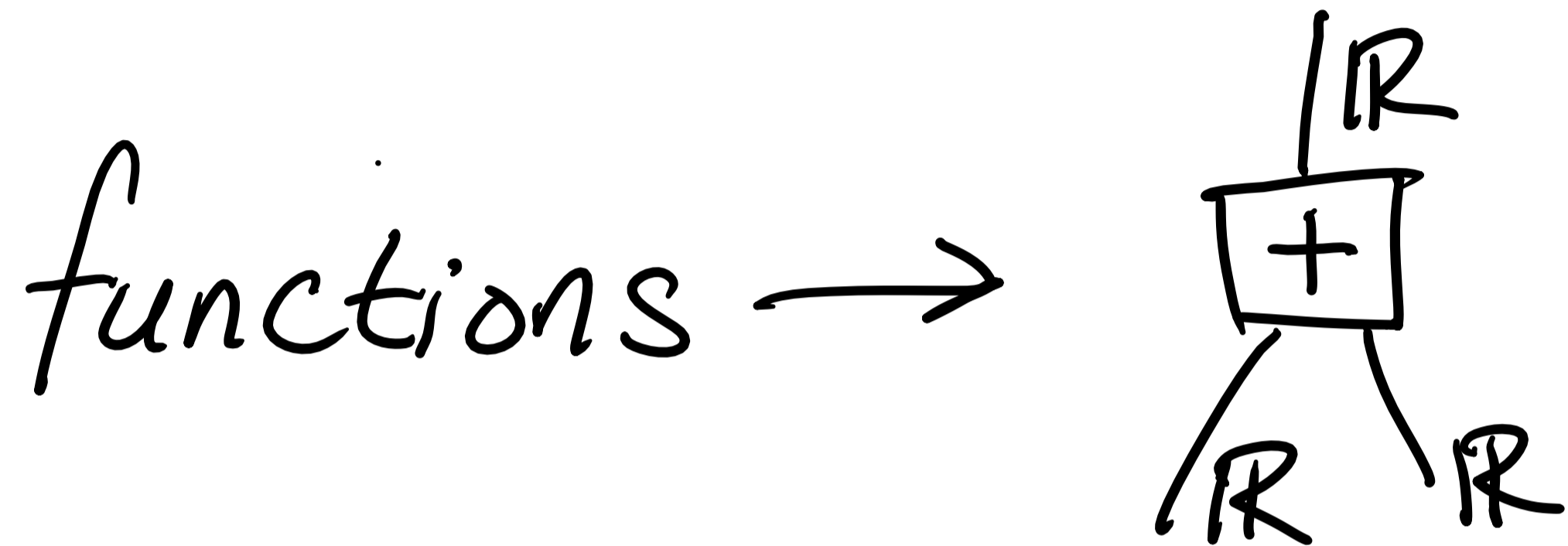
ISR 2019

Paris

PROCESSES

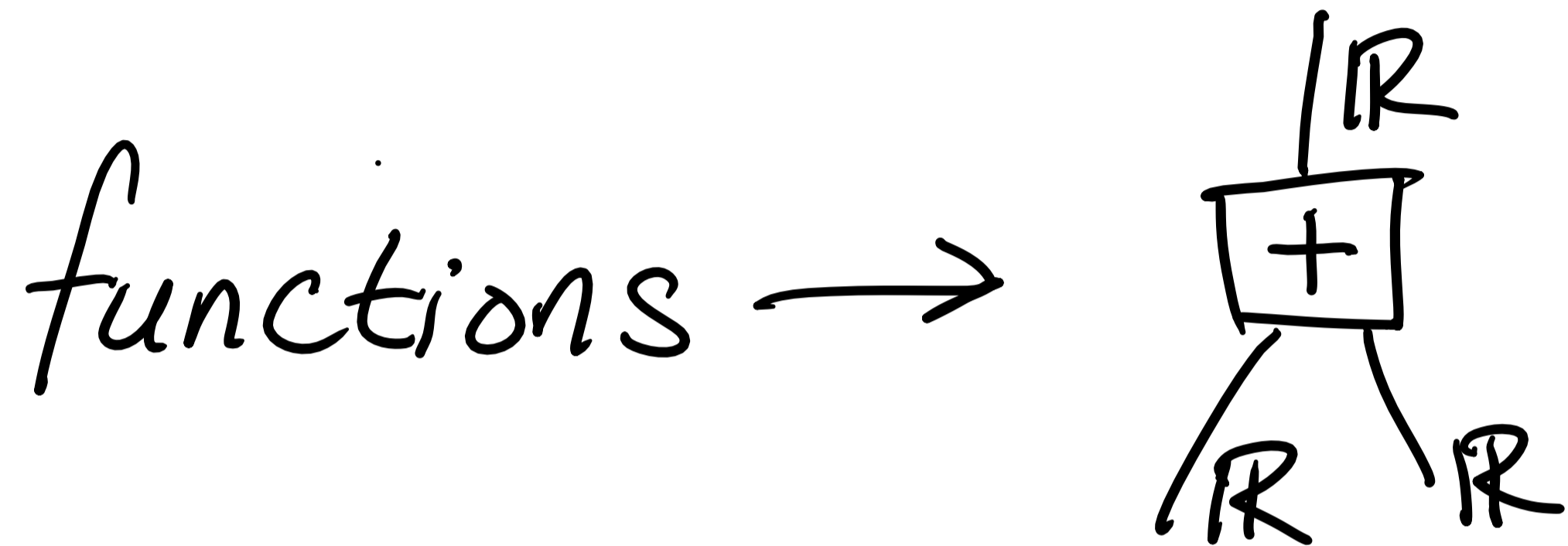


PROCESSES, e.g.

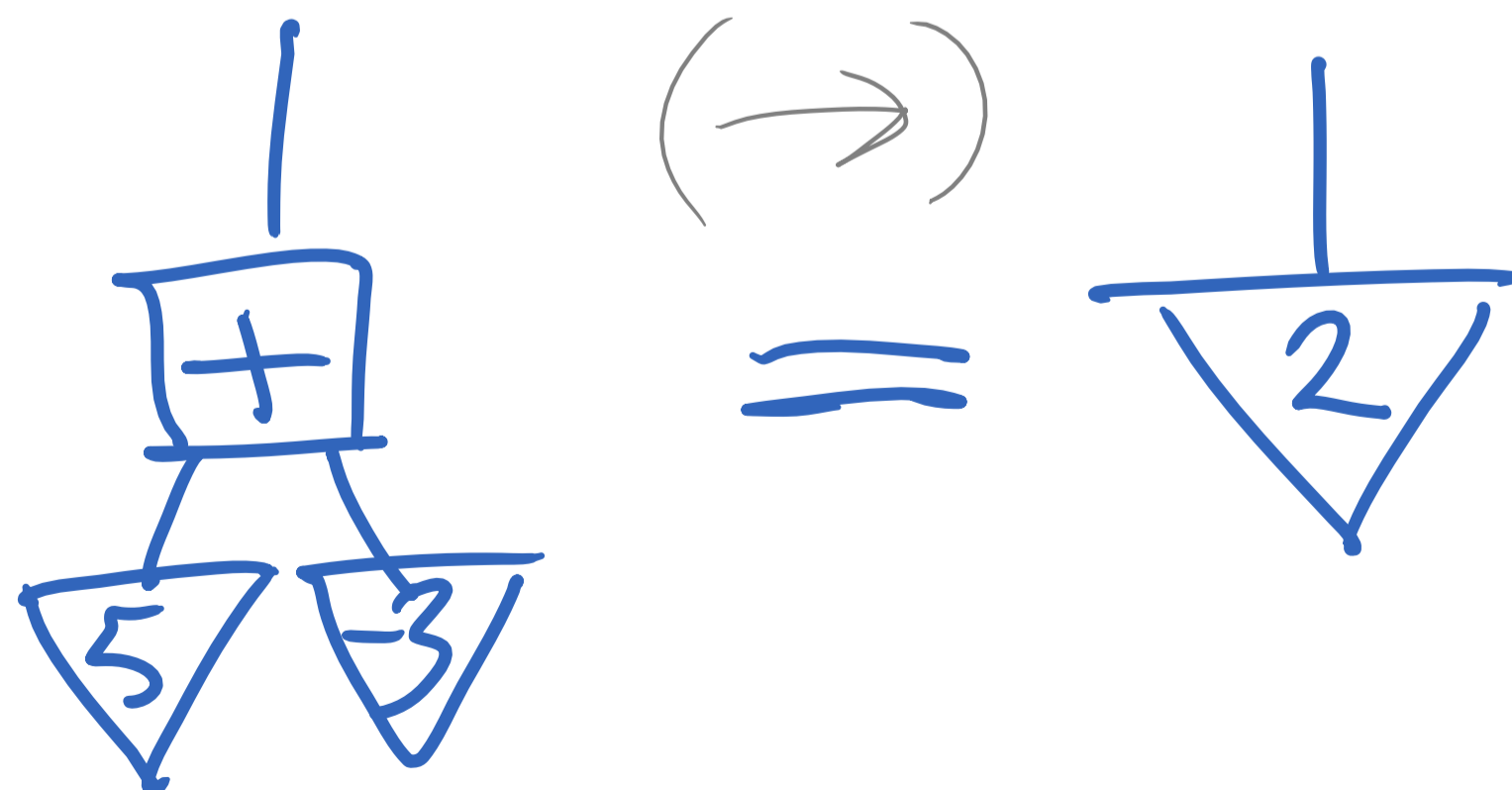


$$"+": \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

PROCESSES, e.g.

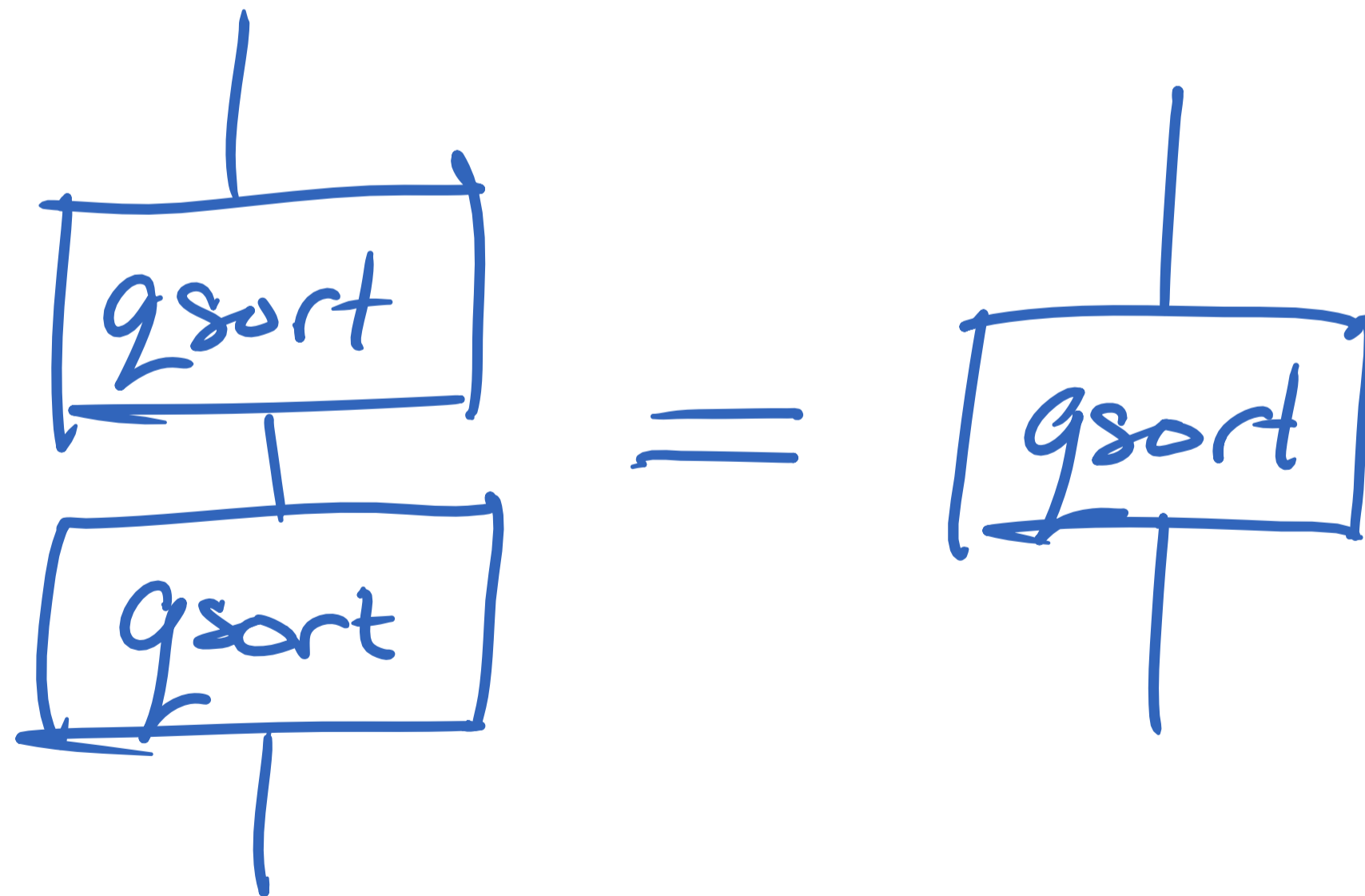
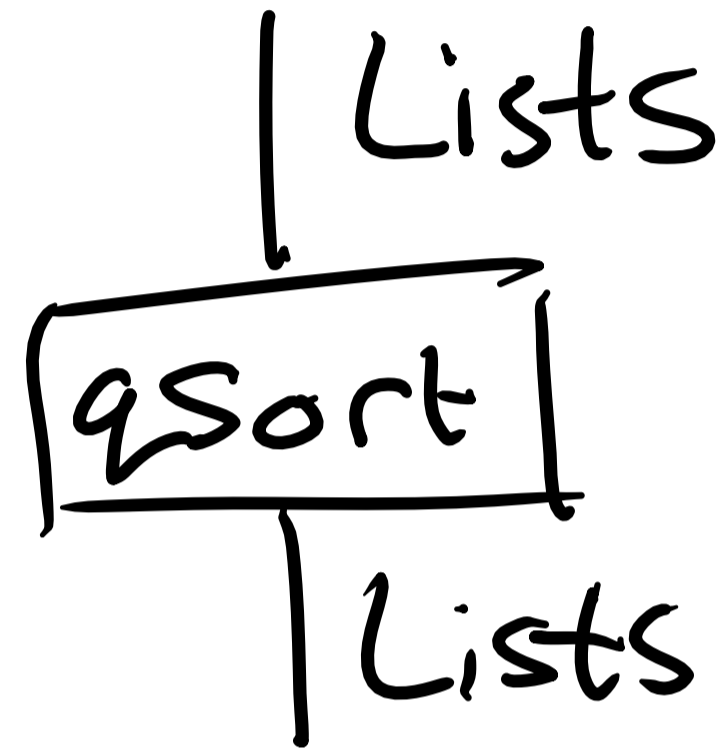


$$"+": \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$



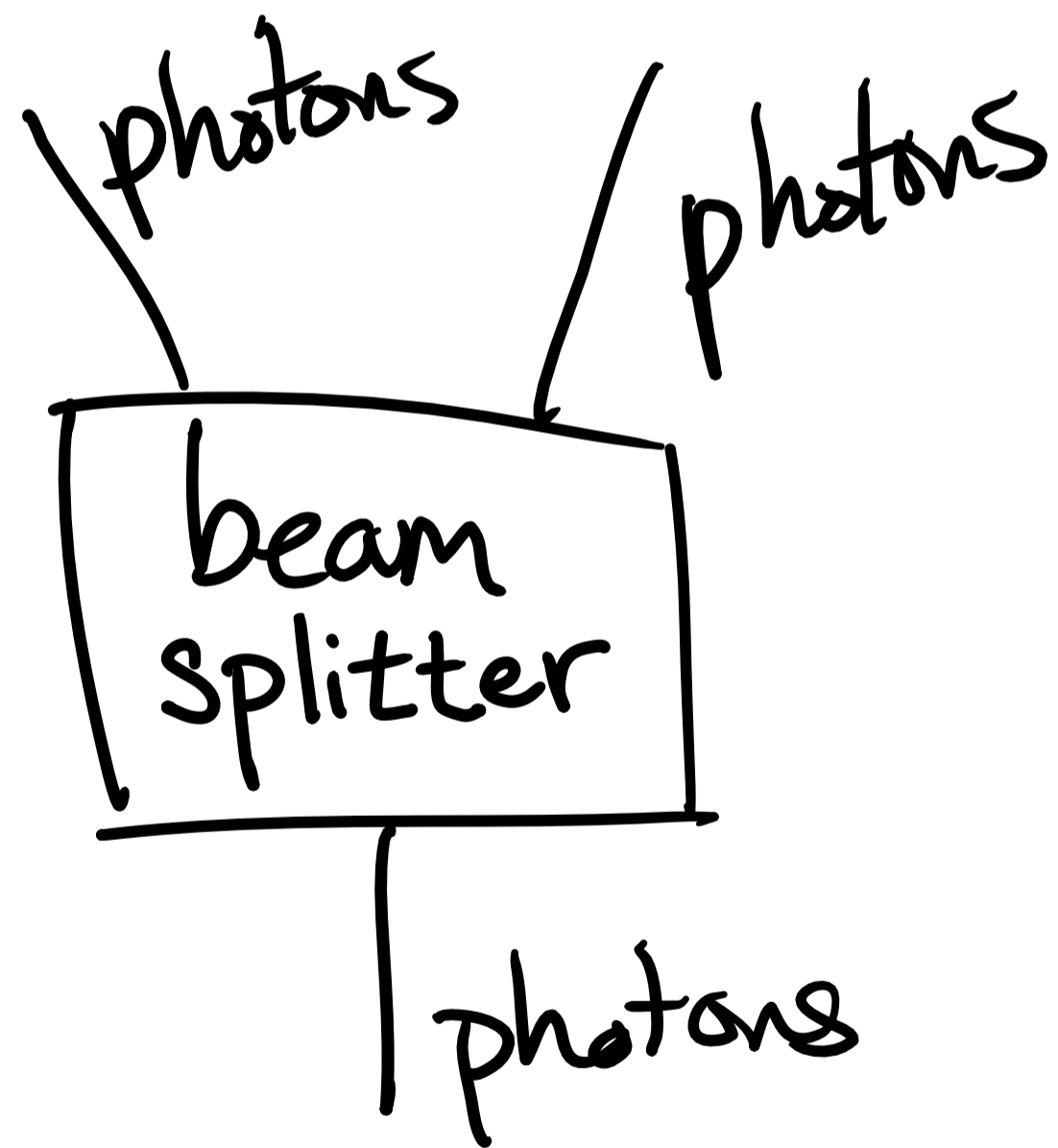
PROCESSES, e.g.

programs →



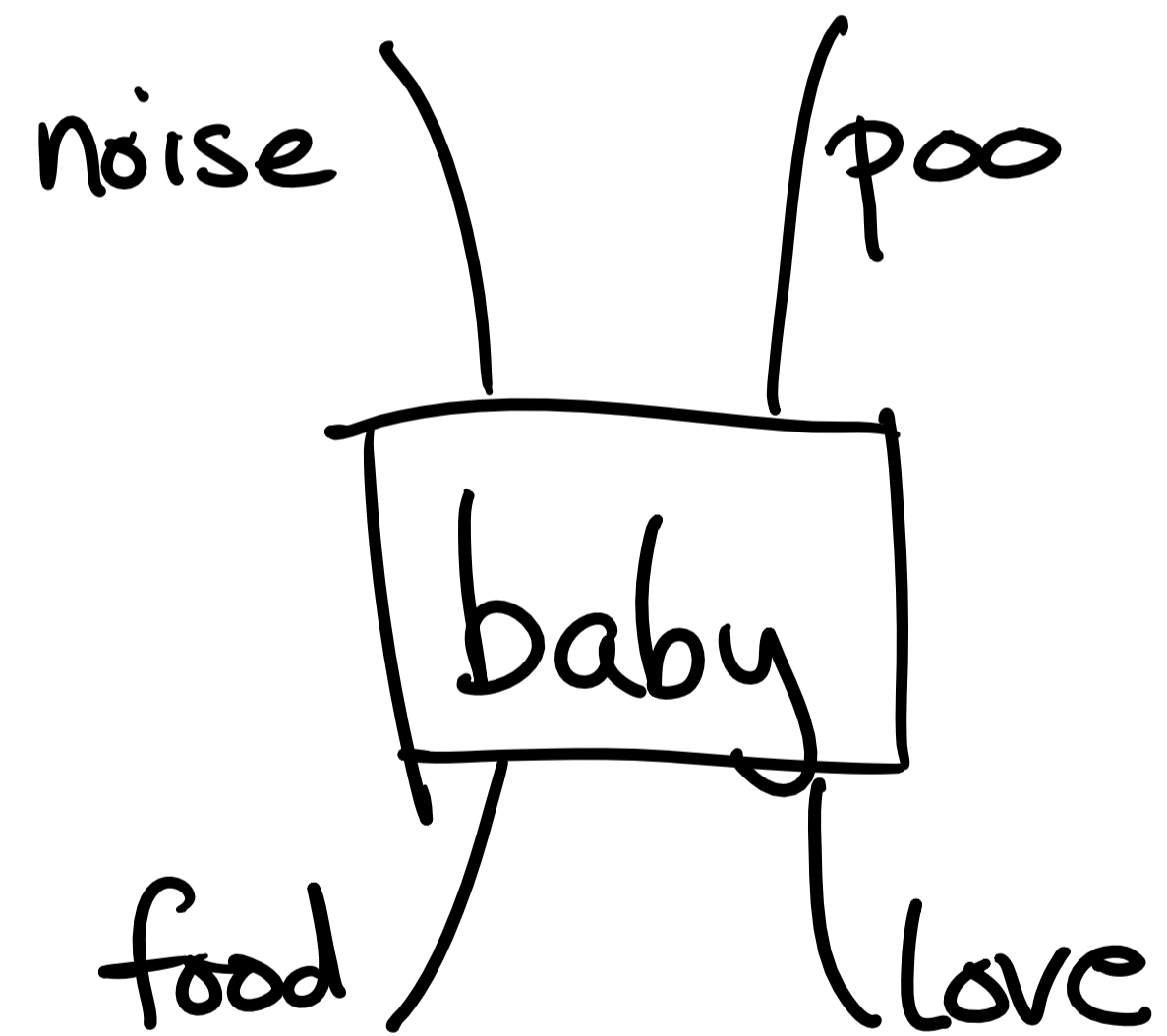
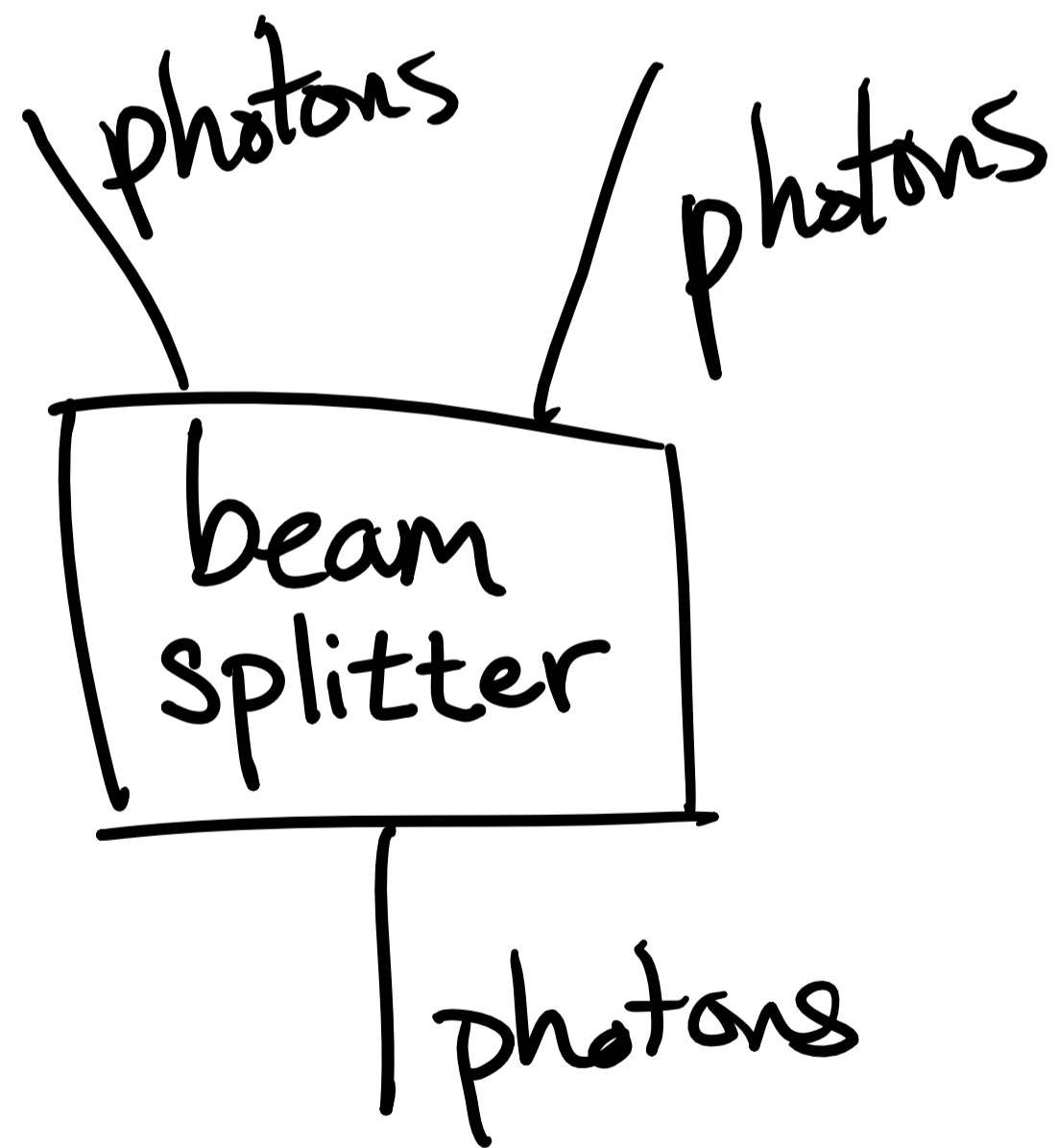
PROCESSES, e.g.

stuff →



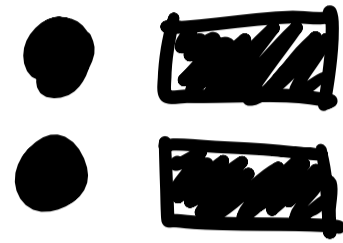
PROCESSES, e.g.

stuff →



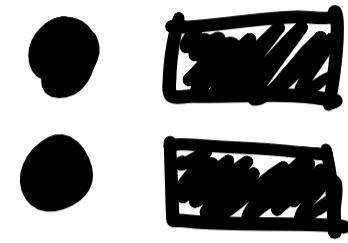
PROCESS THEORIES

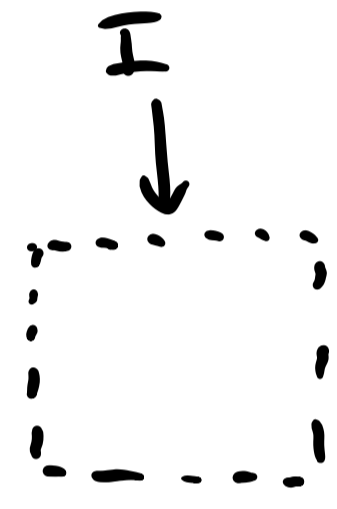
PROCESS THEORIES



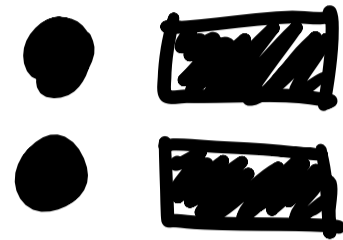
"a collection of processes
that makes sense to
plug together."

PROCESS THEORIES



1. a collection of types a , b , c , 

PROCESS THEORIES



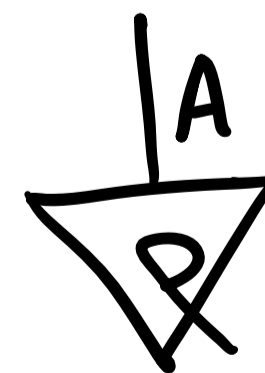
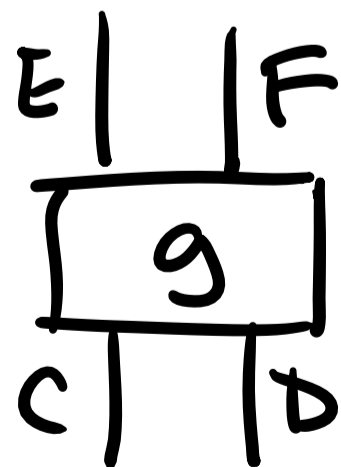
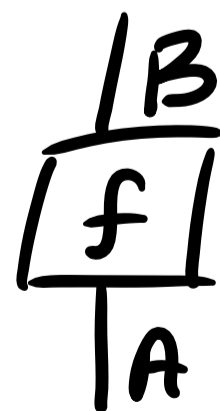
1. a collection of types $|A$, $|B$, $|C$, \square (dashed square) with an arrow F pointing down to it.

2. a collection of processes:

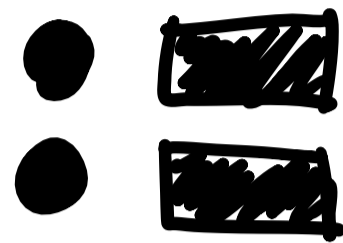
$$f: A \rightarrow B$$

$$g: C \otimes D \rightarrow E \otimes F$$

$$p: I \rightarrow A$$



PROCESS THEORIES

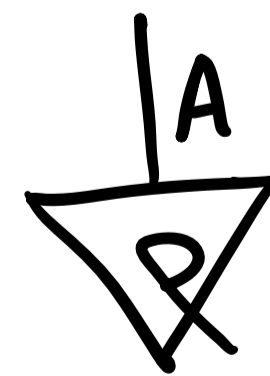
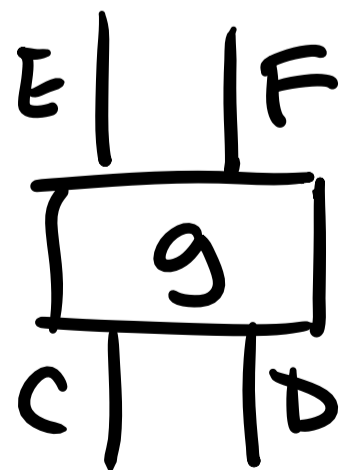
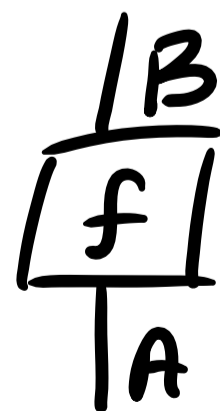


1. a collection of types $|A, |B, |C, \square$
2. a collection of processes:

$$f: A \rightarrow B$$

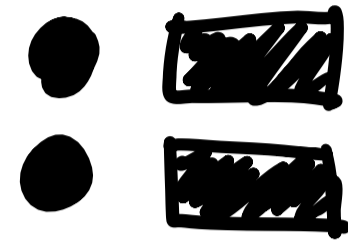
$$g: C \otimes D \rightarrow E \otimes F$$

$$p: I \rightarrow A$$

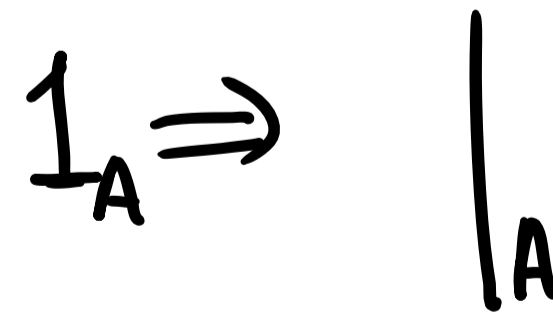
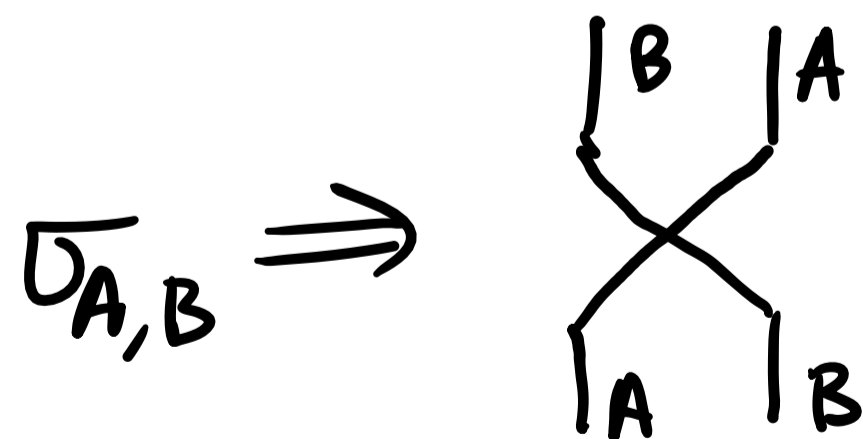
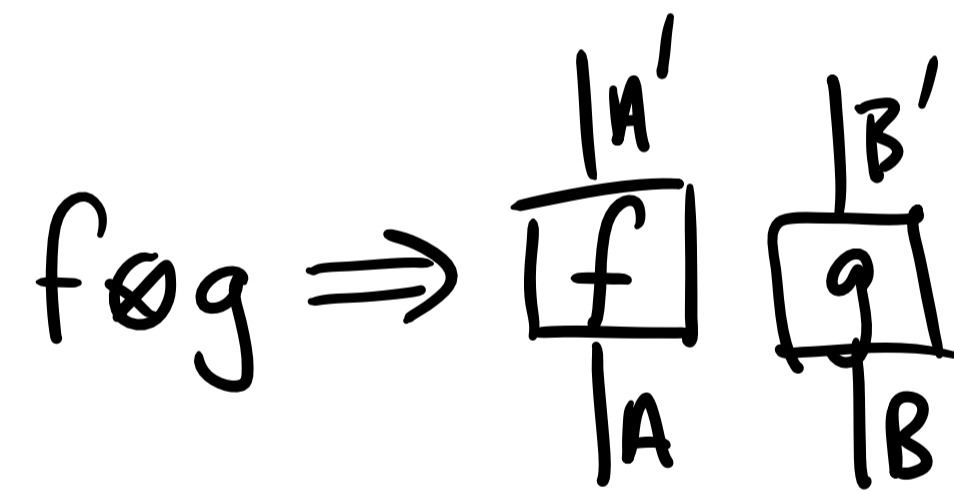
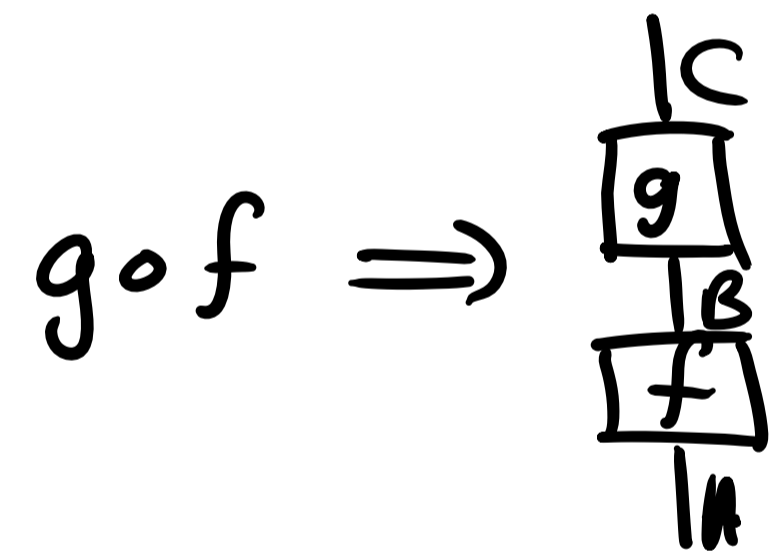


STATES

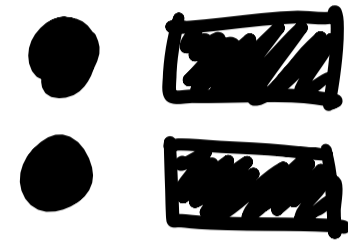
PROCESS THEORIES



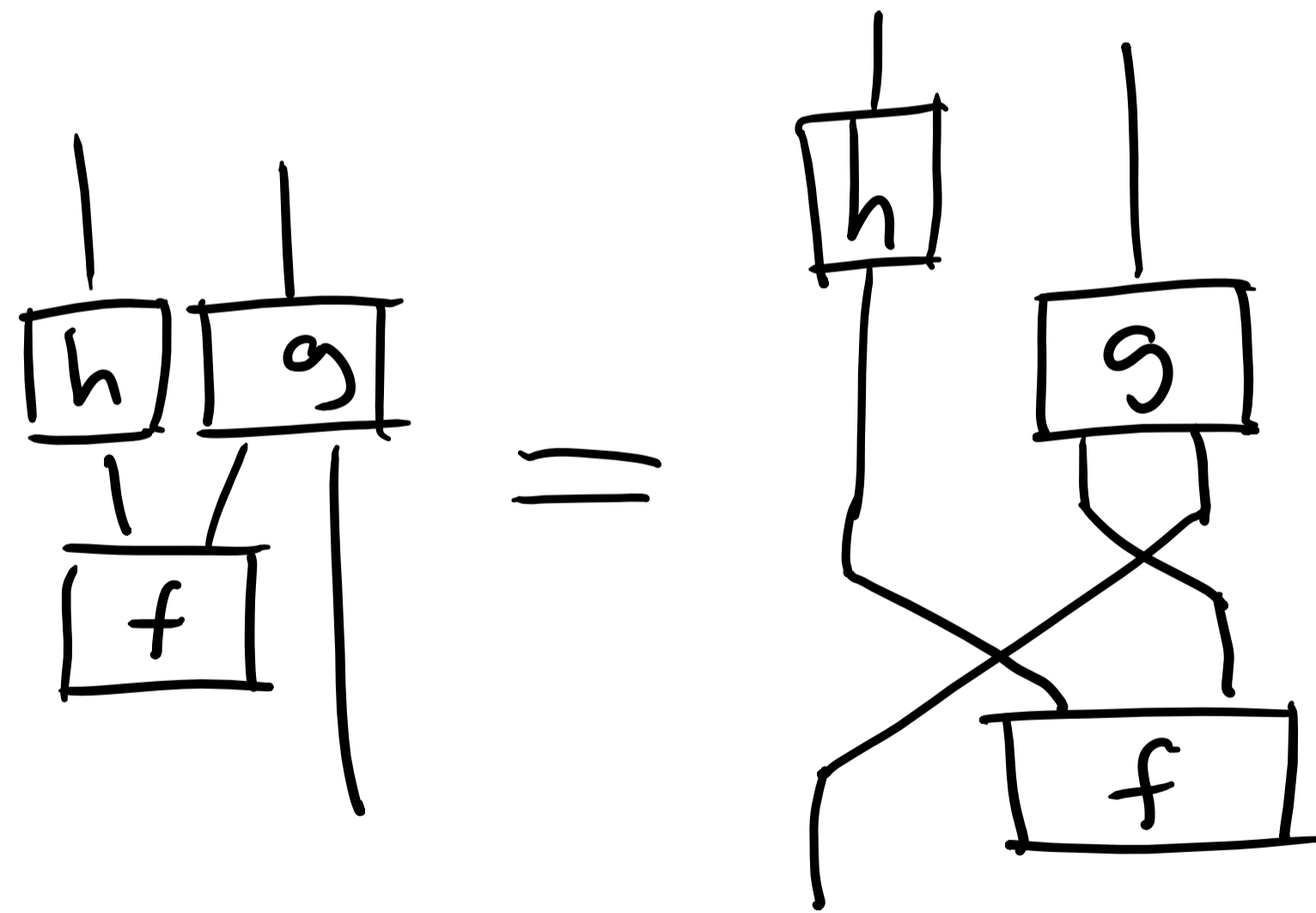
3. operations for building diagrams:



PROCESS THEORIES



4. some rules (only connectivity matters)



Ex: functions

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types: sets A, B, C, \dots

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processes: functions $\begin{array}{c} |B \\ \hline [f] \\ \hline |A \end{array} ::= f: A \rightarrow B$

Ex: functions

types: sets A, B, C, \dots

processes: functions $\begin{array}{c} D/E \\ \boxed{f} \\ A \quad B \quad C \end{array} ::= f : A \times B \times C \rightarrow D \times E$

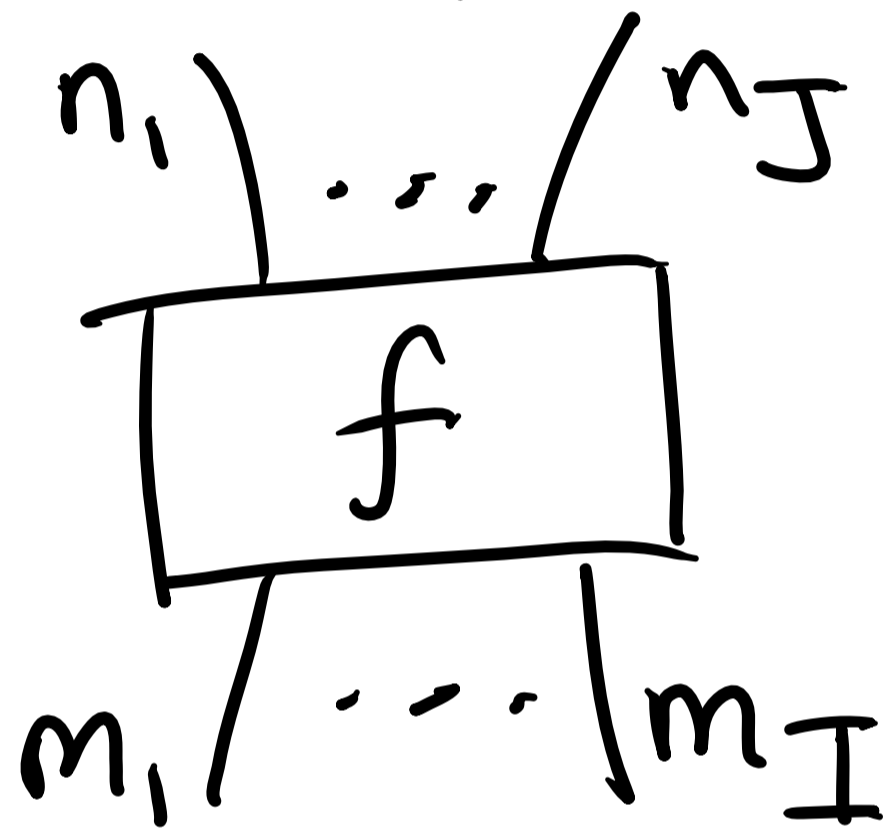
Ex: matrices

types: $n \in \mathbb{N}$ (dimension)

processes: $\begin{array}{c} |n \\ \boxed{f} \\ |m \end{array}$ $n \times m$ matrices

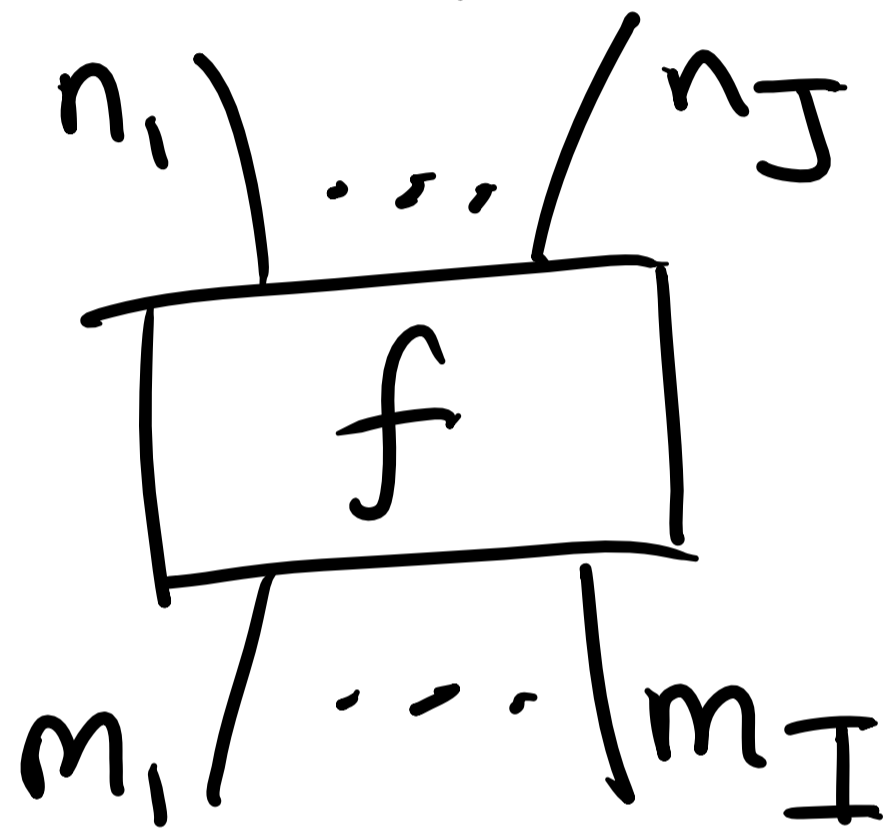
$$\begin{pmatrix} f_{11} & \dots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \dots & f_{nm} \end{pmatrix}$$

Ex: matrices

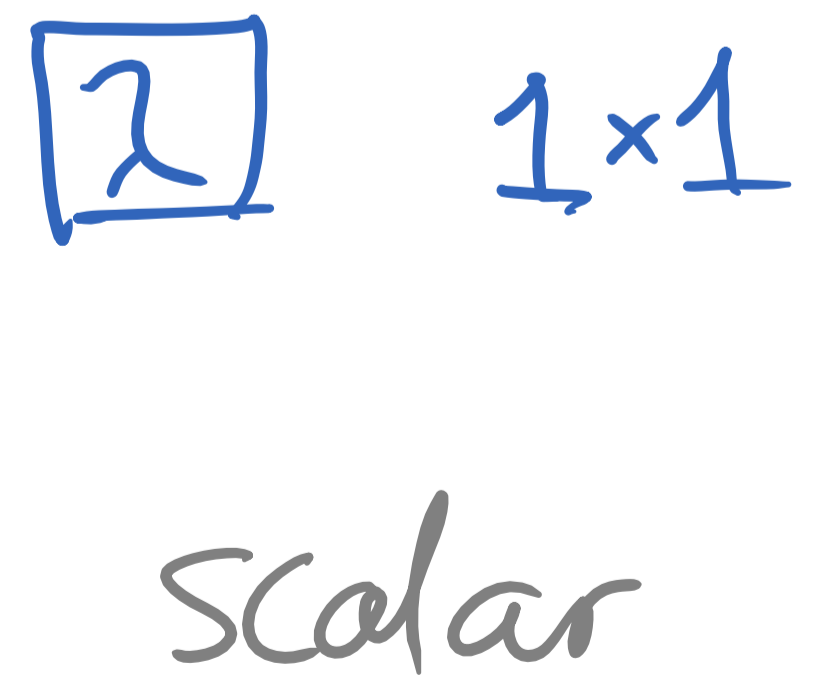
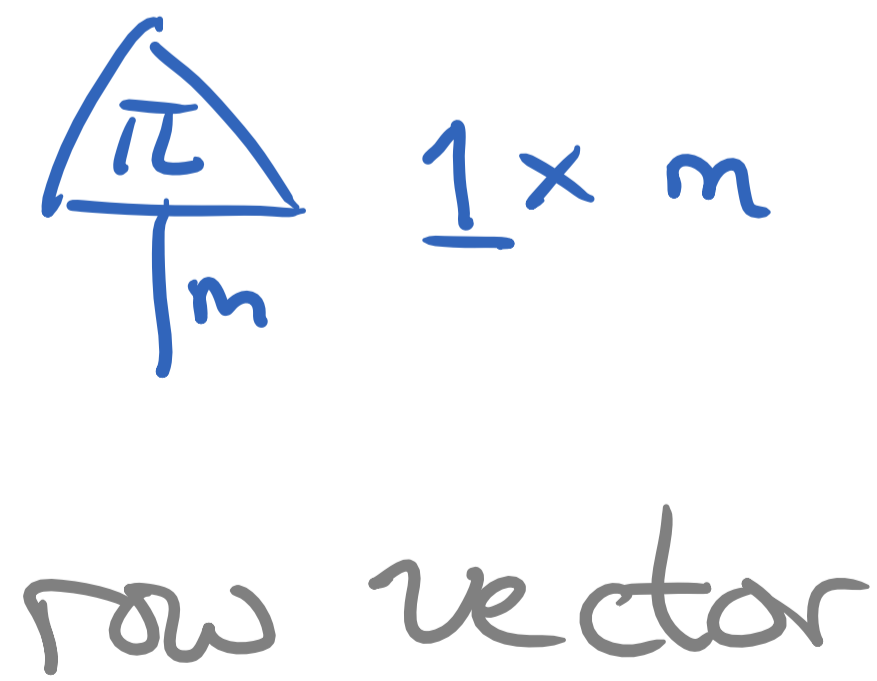
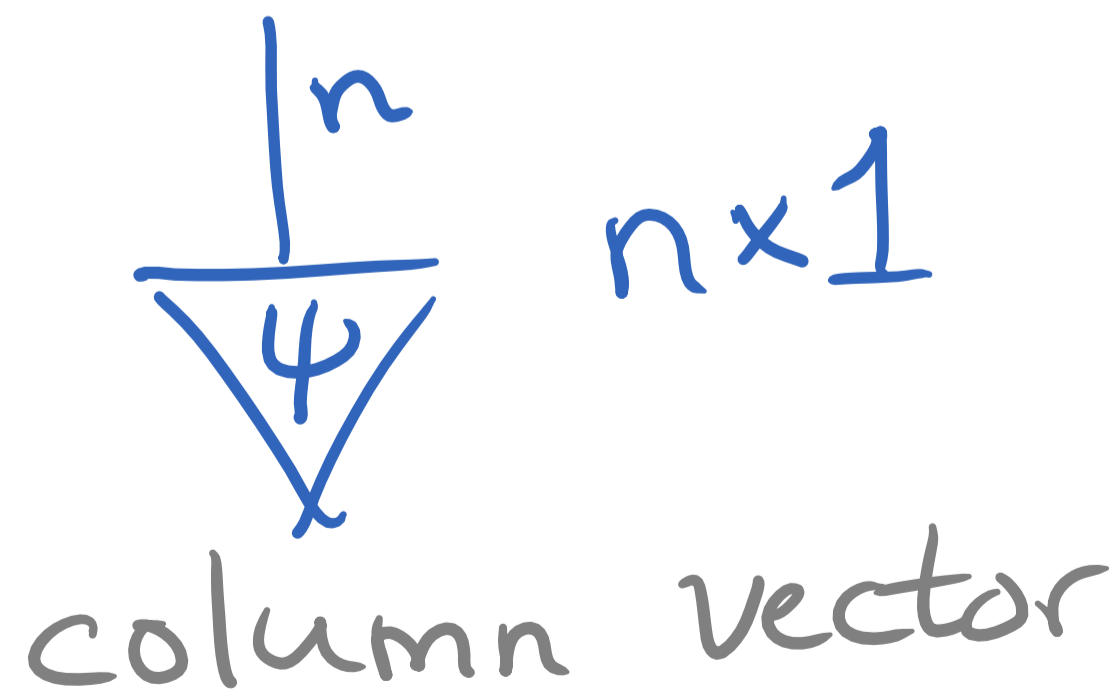


$(\prod n_j) \times (\prod m_i)$ matrices

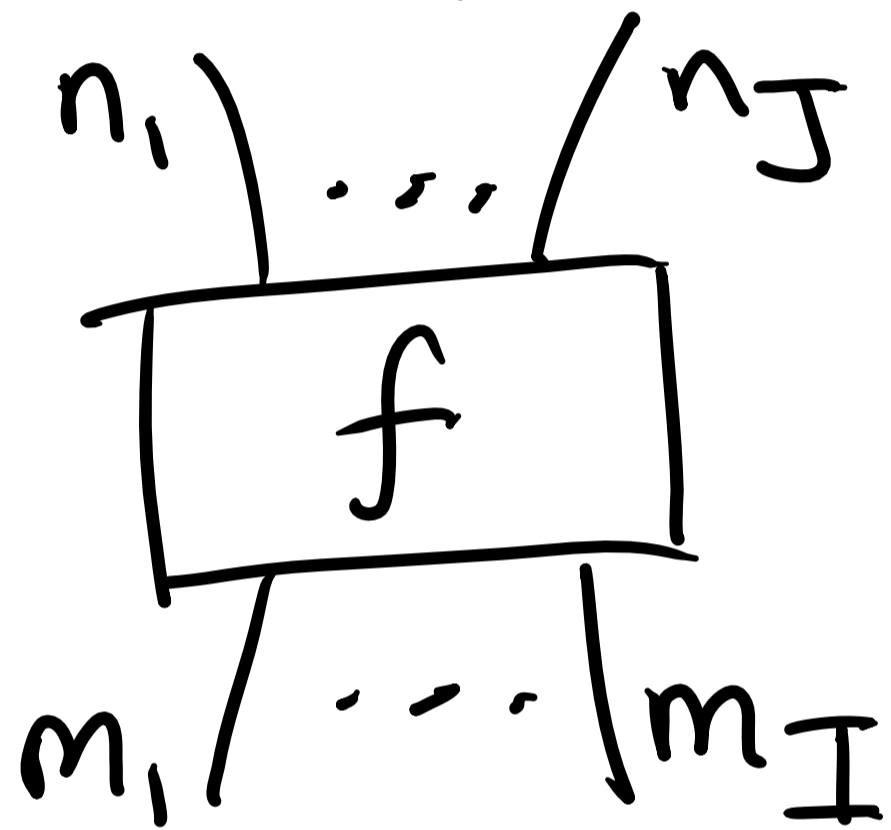
Ex: matrices



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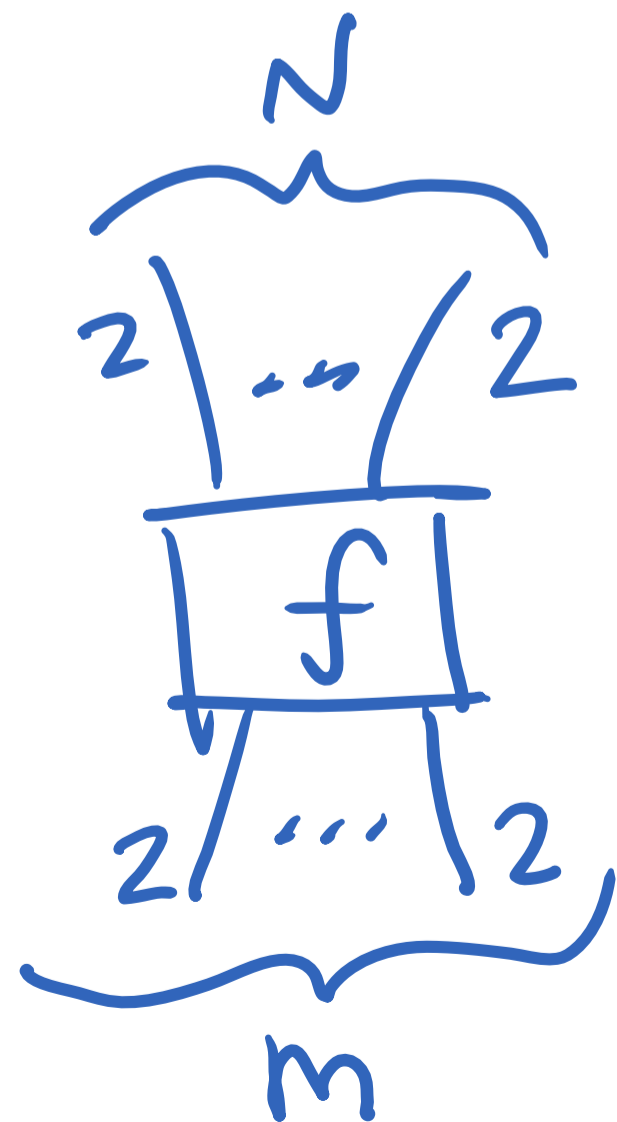


Ex: matrices



$$(\prod n_j) \times (\prod m_i)$$

matrices



$$2^N \times 2^M$$

COMPOSITION

$$\begin{array}{c} \boxed{g} \\ \downarrow \\ \boxed{f} \end{array} \begin{array}{c} \text{\scriptsize } l_n \\ \text{\scriptsize } m \\ \text{\scriptsize } l \end{array} = \underbrace{g \cdot f}_{n \times l}$$

The diagram illustrates the composition of two functions. On the left, a vertical stack of two boxes represents the composition. The top box contains the letter 'g' and is labeled with 'l_n' above it. A vertical line with an arrowhead pointing down connects the bottom of the 'g' box to the top of the bottom box, which contains the letter 'f'. This bottom box is labeled with 'm' above it and 'l' below it. To the right of this diagram is an equals sign followed by the expression 'g · f'. Above the 'g' is the label 'n × m' with a blue wavy underline. Above the 'f' is the label 'm × l' with a blue wavy underline. A large blue bracket underneath the entire 'g · f' expression is labeled 'n × l'.

COMPOSITION

$$\begin{array}{c} |n \\ \boxed{f} \\ |m \end{array} \begin{array}{c} |n' \\ \boxed{g} \\ |m' \end{array} \quad \stackrel{!}{=} \quad \underbrace{f}_{n \times m} \otimes \underbrace{g}_{n' \times m'} \quad \underbrace{\hspace{10em}}_{nn' \times mm'}$$

COMPOSITION

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

2x2

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

2x1

$$f \otimes g =$$

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$$f \otimes g = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

4×2

COMPOSITION

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$$f \otimes g = \begin{pmatrix} \phantom{f_{11}} & \phantom{f_{12}} \\ \phantom{f_{21}} & \phantom{f_{22}} \end{pmatrix}$$

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4x2

SWAP

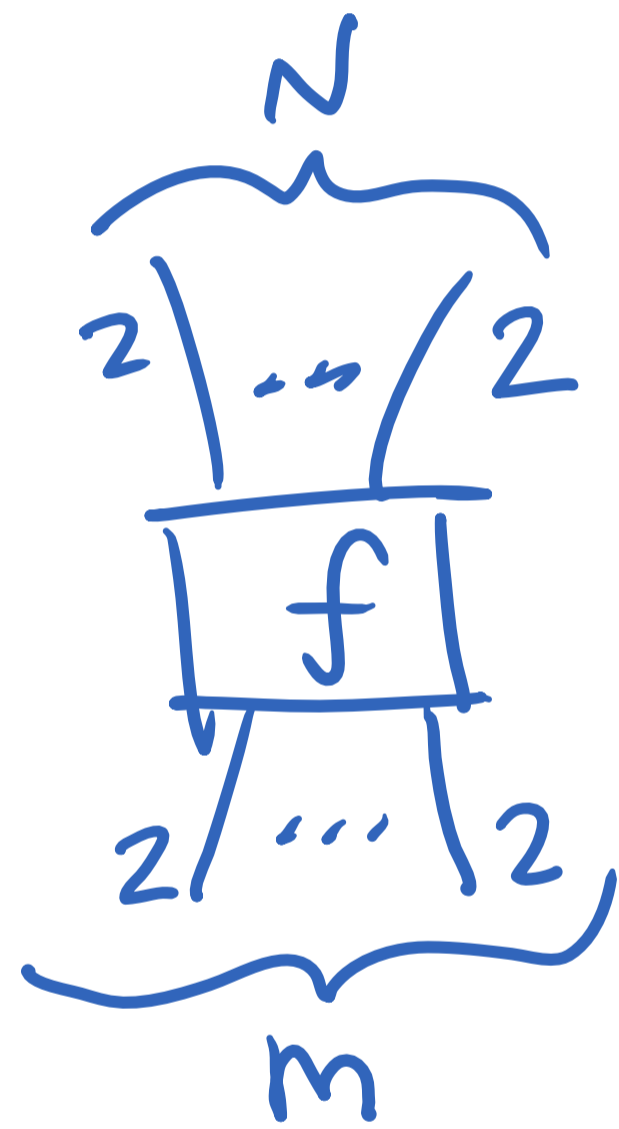
$$V \otimes \omega = \begin{pmatrix} v_1 \omega_1 \\ v_1 \omega_2 \\ v_2 \omega_1 \\ v_2 \omega_2 \end{pmatrix} \xrightarrow{?} \begin{pmatrix} \omega_1 v_1 \\ \omega_1 v_2 \\ \omega_2 v_1 \\ \omega_2 v_2 \end{pmatrix} = \omega \otimes V$$

SWAP

$$V \otimes \omega = \begin{pmatrix} v_1 \omega_1 \\ v_1 \omega_2 \\ v_2 \omega_1 \\ v_2 \omega_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \omega_1 v_1 \\ \omega_1 v_2 \\ \omega_2 v_1 \\ \omega_2 v_2 \end{pmatrix} = \omega \otimes V$$

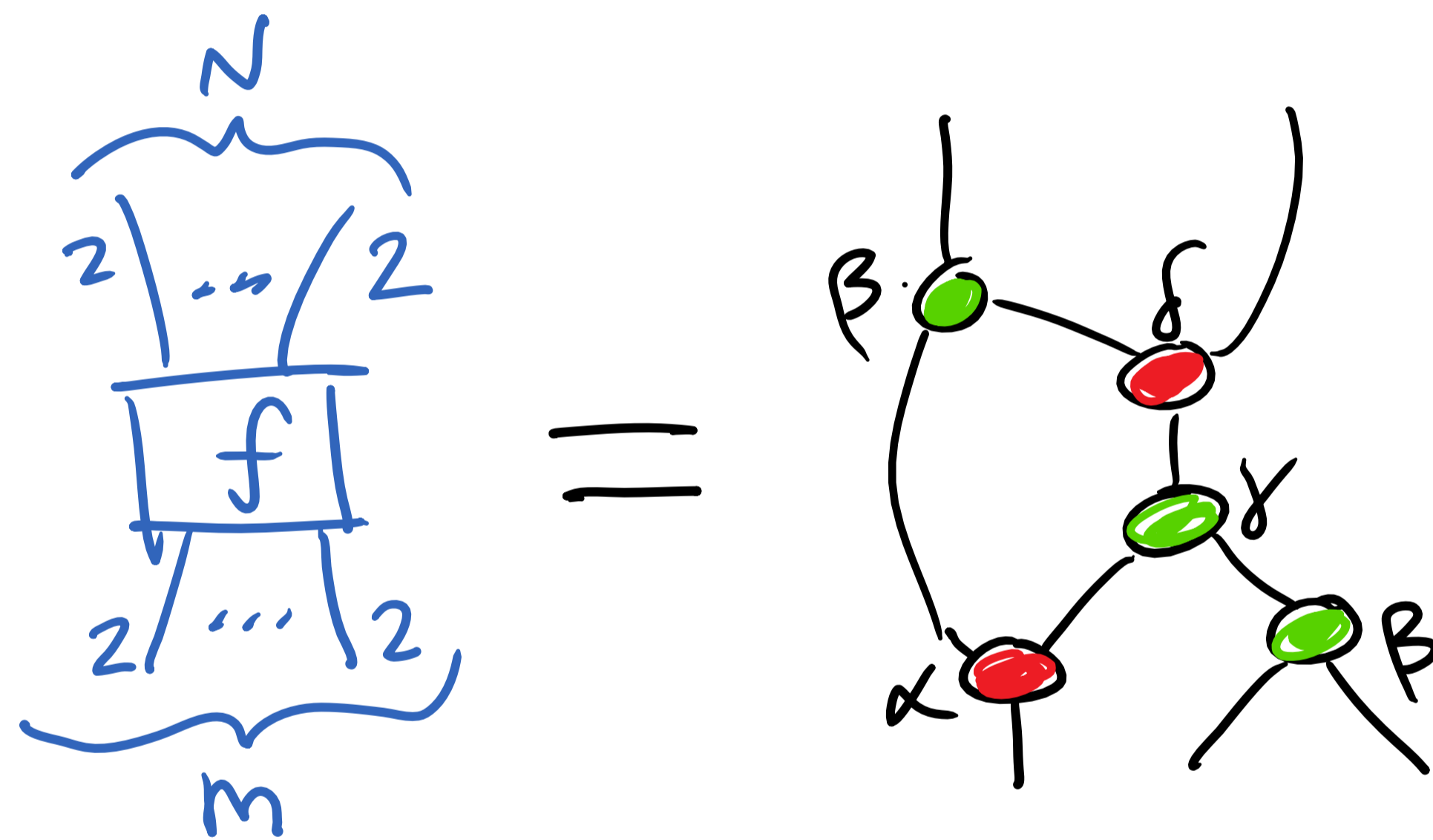
$$\text{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

PROBLEM: Matrices relevant for QT
are very large =



$$2^N \times 2^M$$

PROBLEM: Matrices relevant for QT
are very large =



SOLUTION: Decompose into smaller pieces
and use diagram rewrite rules.

ASIDE : COMPLEX NUMBERS

$$a + ib$$

ASIDE : COMPLEX NUMBERS

$$a + ib \rightsquigarrow r e^{i\alpha}$$

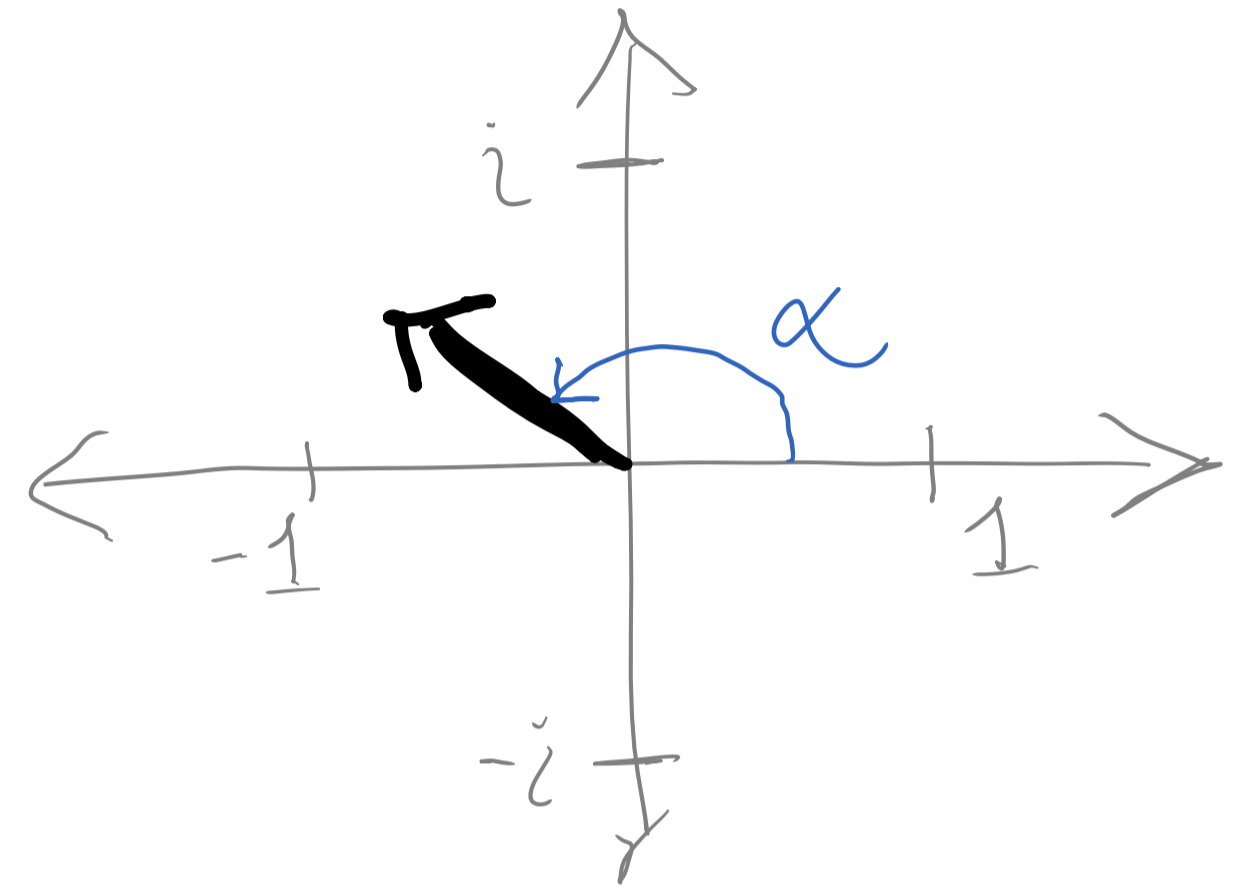
ASIDE : COMPLEX NUMBERS

$$a + ib \rightsquigarrow r e^{i\alpha}$$

magnitude \rightarrow r $e^{i\alpha}$ \leftarrow phase

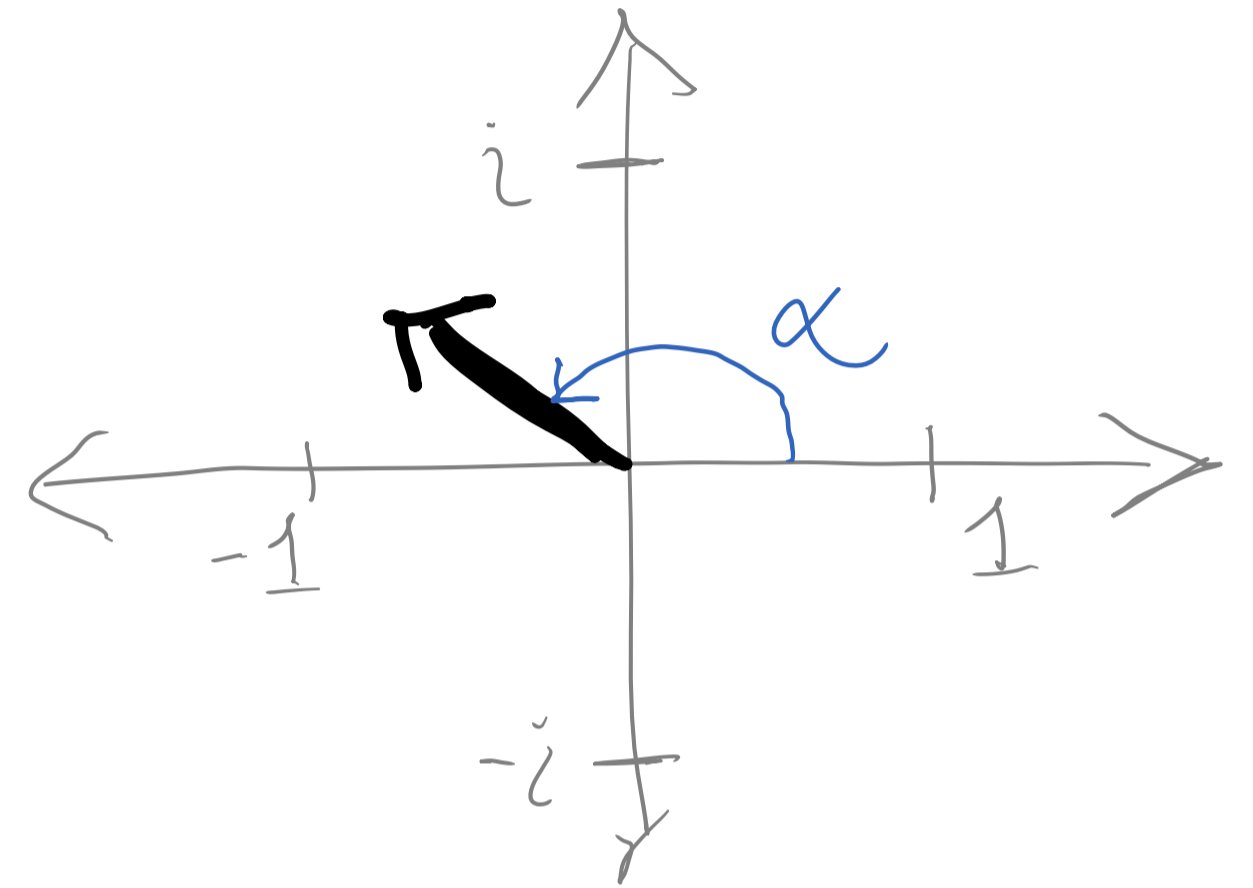
ASIDE : COMPLEX NUMBERS

$$e^{i\alpha} = \cos \alpha + i \sin \alpha$$



ASIDE : COMPLEX NUMBERS

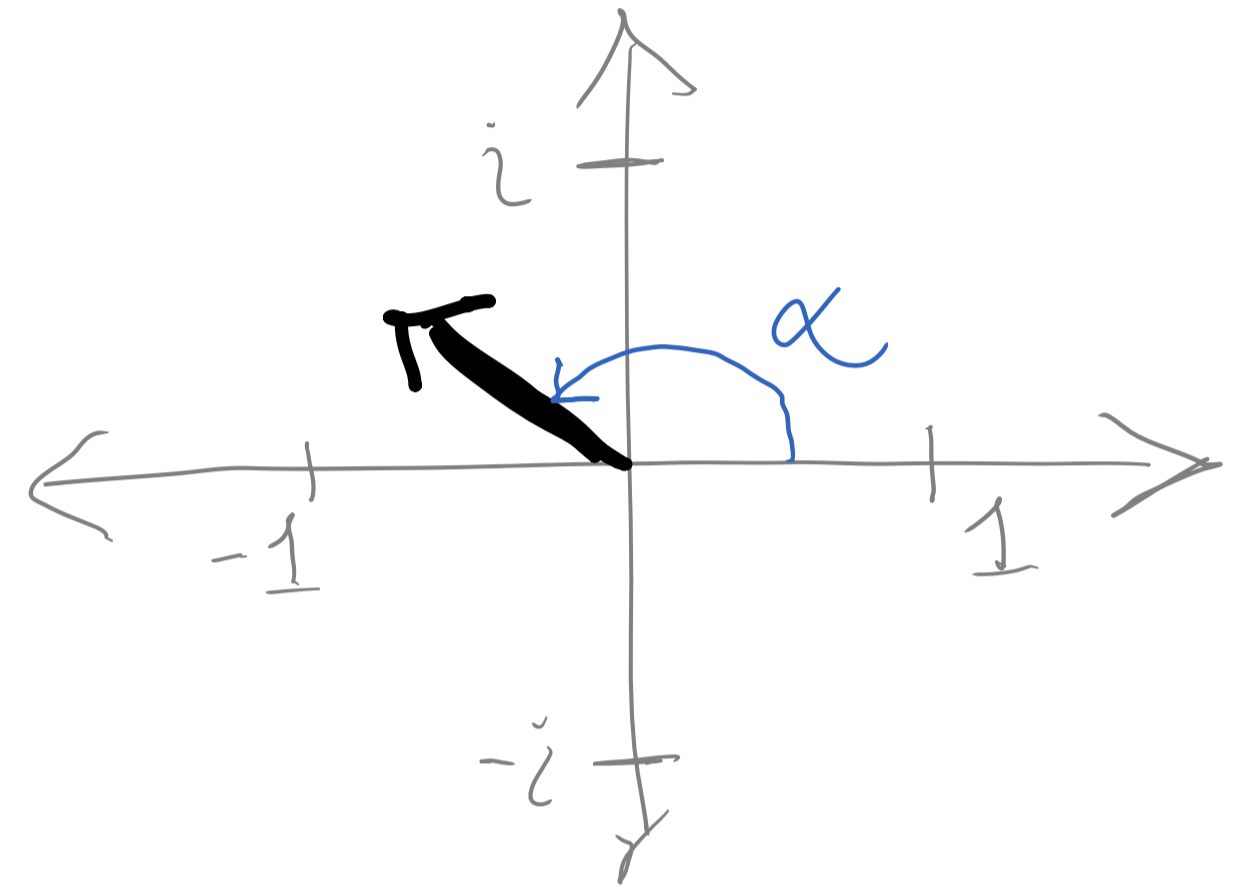
$$e^{i\alpha} = \cos \alpha + i \sin \alpha$$



$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$$

ASIDE : COMPLEX NUMBERS

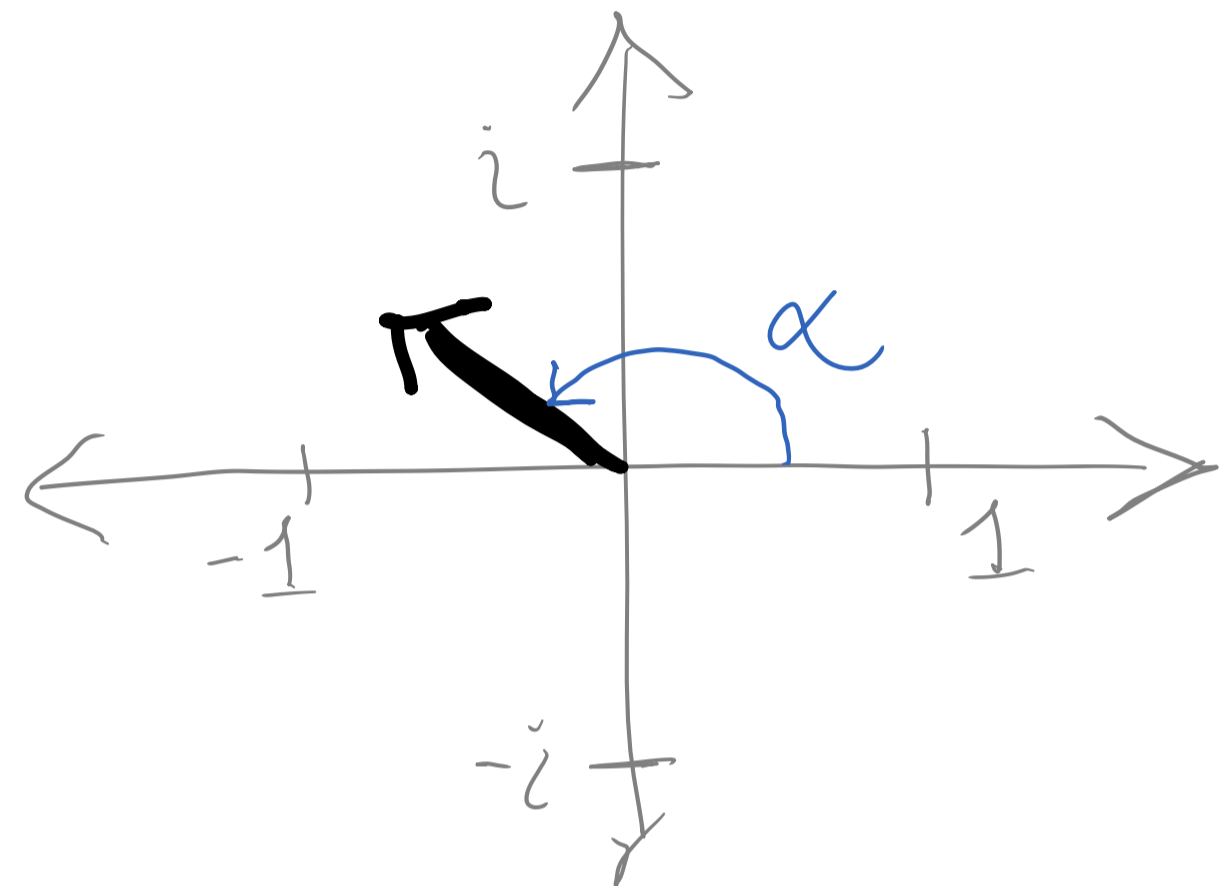
$$e^{i\alpha} = \cos \alpha + i \sin \alpha$$



$$e^{i\alpha} + e^{-i\alpha} = 2 \cos \alpha$$

ASIDE : COMPLEX NUMBERS

$$e^{i\alpha} = \cos \alpha + i \sin \alpha$$

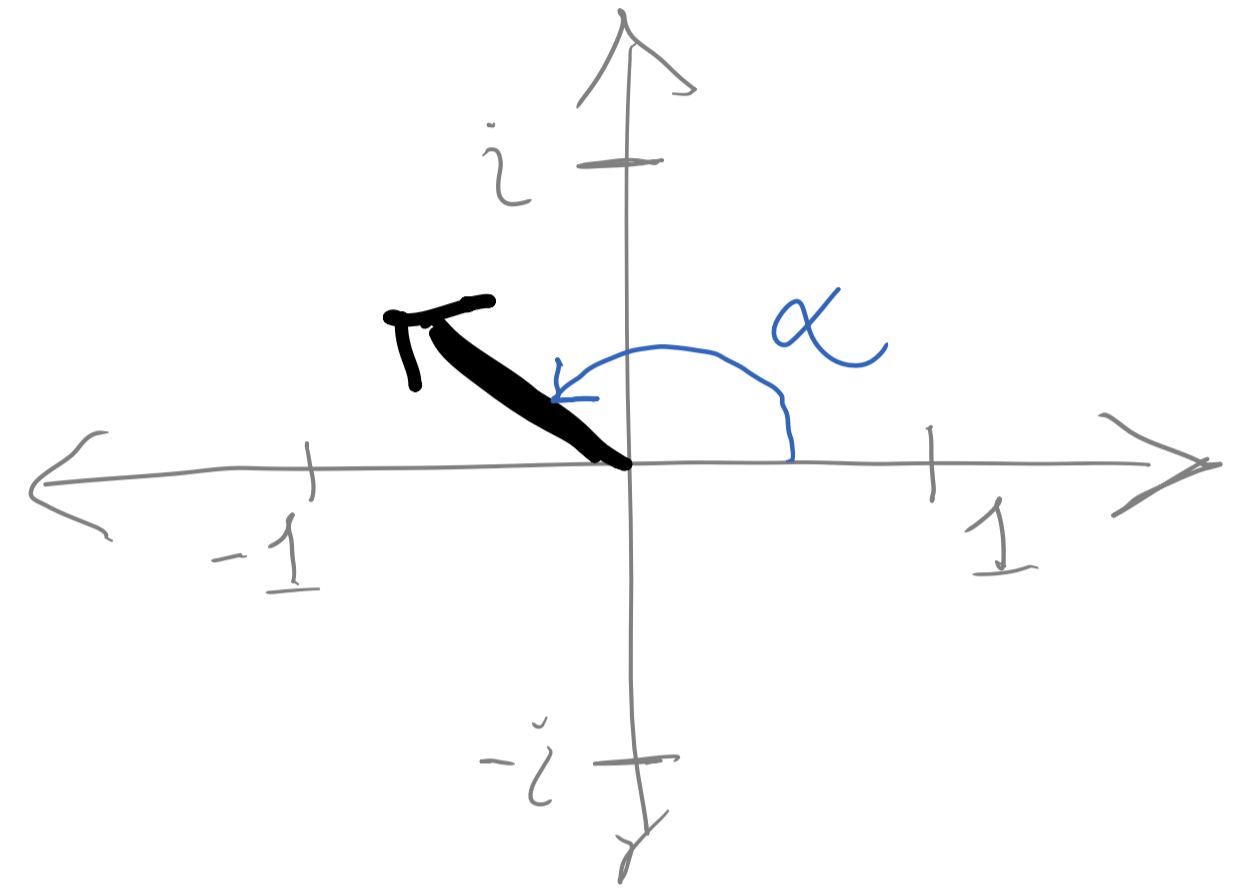


$$e^{i\alpha} + e^{i\alpha} = 2e^{i\alpha}$$

$$e^{i\alpha} + e^{i(\pi+\alpha)}$$

ASIDE : COMPLEX NUMBERS

$$e^{i\alpha} = \cos \alpha + i \sin \alpha$$

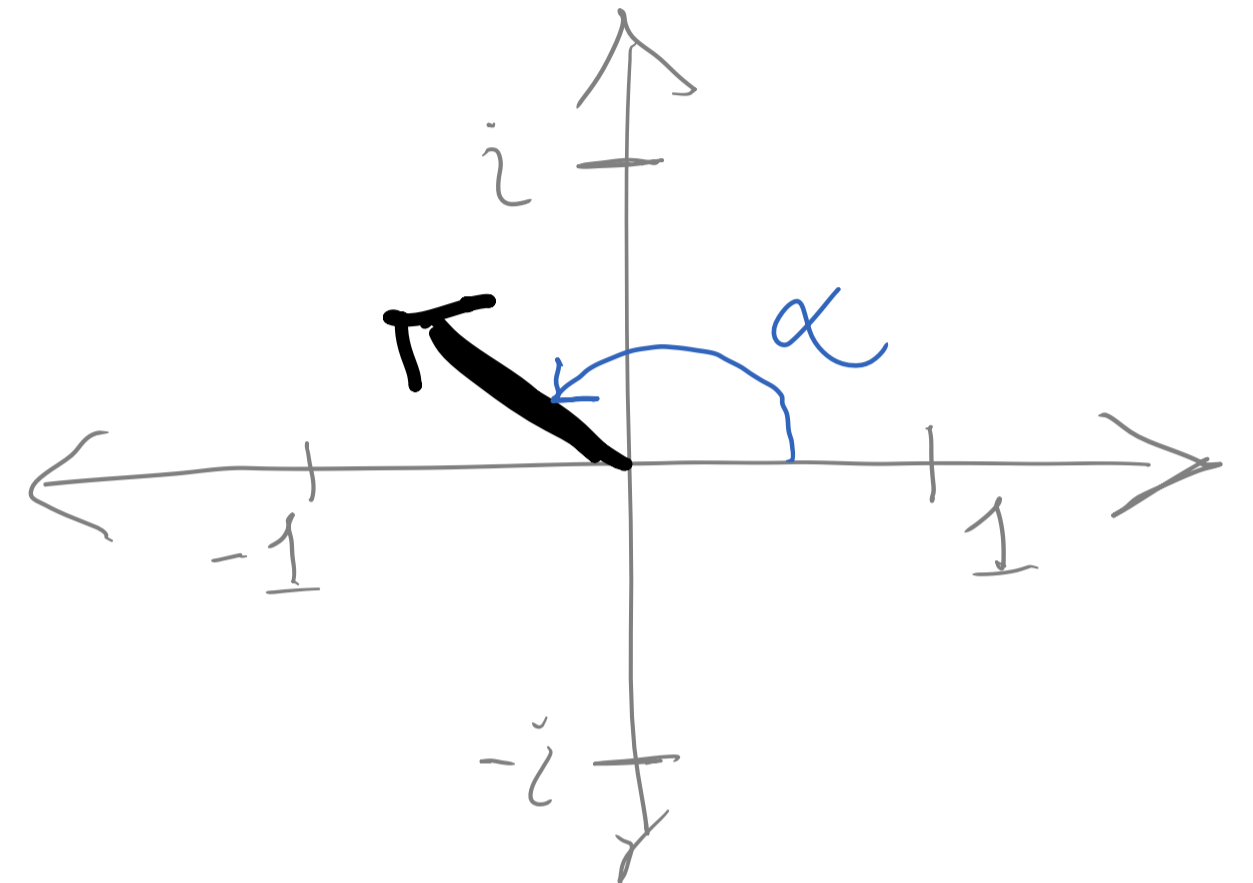


$$e^{i\alpha} + e^{i\alpha} = 2e^{i\alpha}$$

$$e^{i\alpha} + e^{i(\pi+\alpha)} = e^{i\alpha} + e^{i\pi} e^{i\alpha}$$

ASIDE : COMPLEX NUMBERS

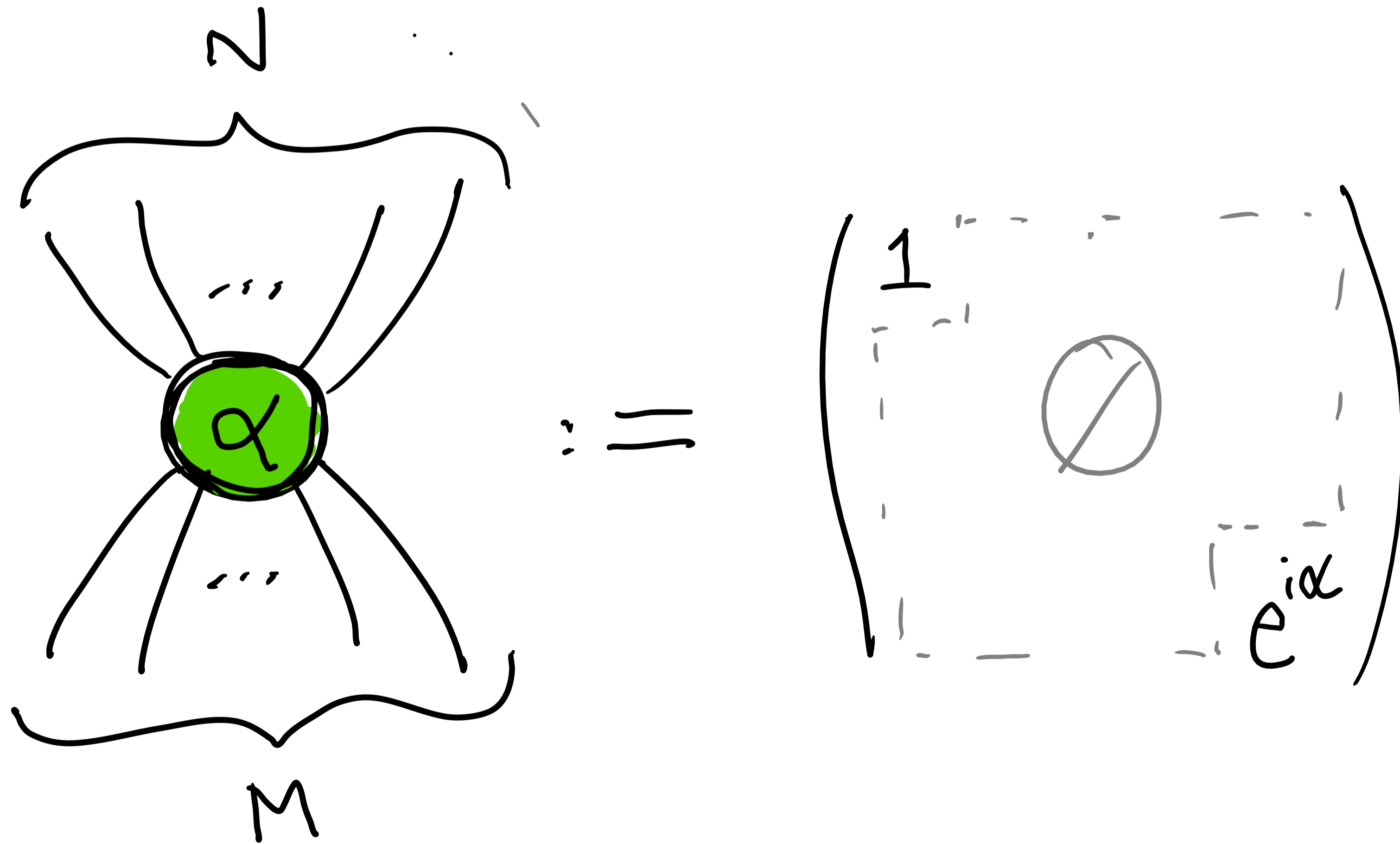
$$e^{i\alpha} = \cos \alpha + i \sin \alpha$$



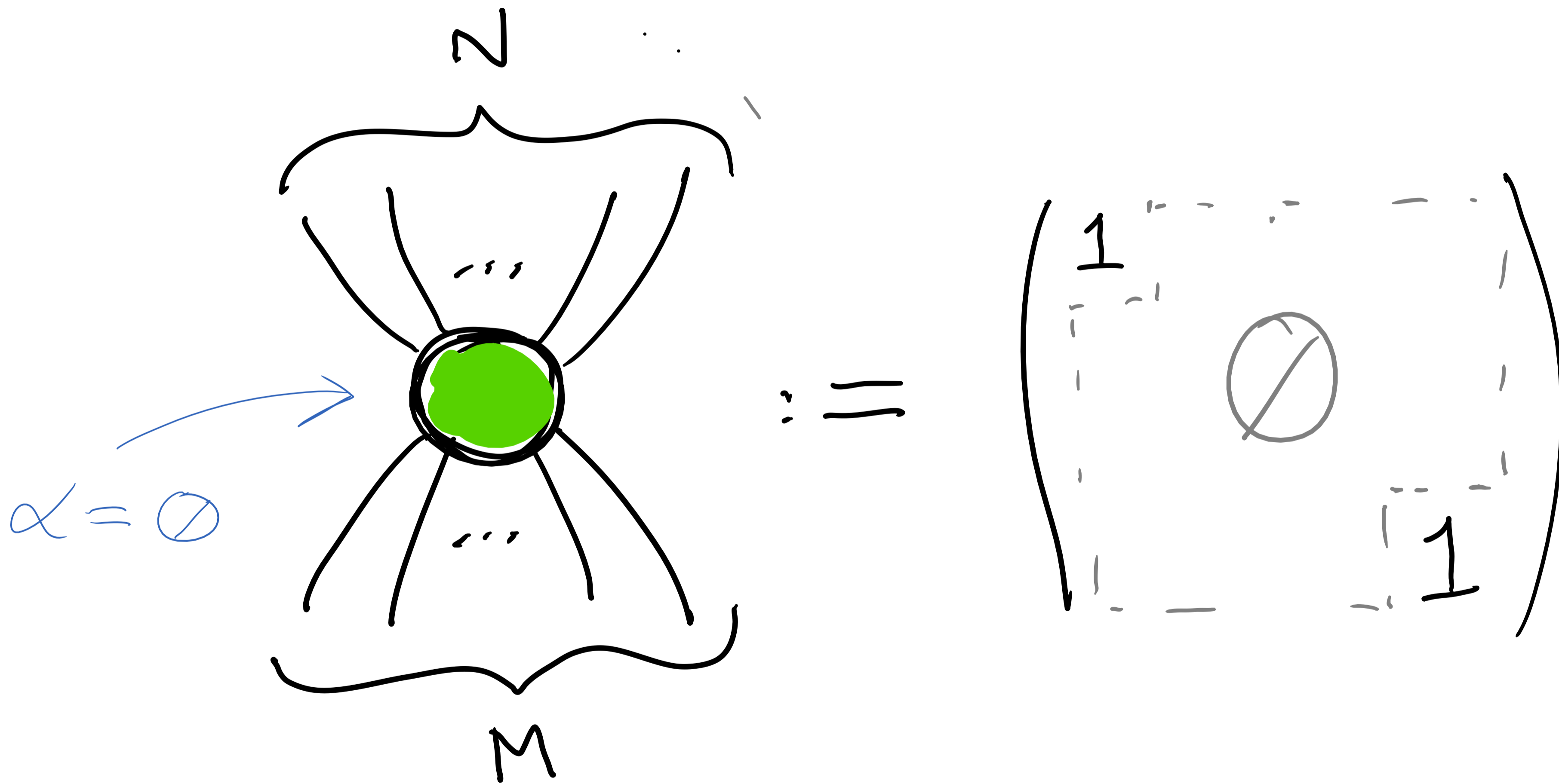
$$e^{i\alpha} + e^{i\alpha} = 2e^{i\alpha}$$

$$e^{i\alpha} + e^{i(\pi+\alpha)} = e^{i\alpha} + \cancel{e^{i\pi}} e^{i\alpha} = 0$$

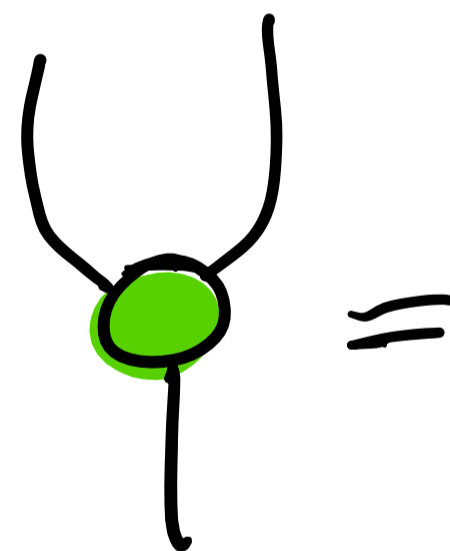
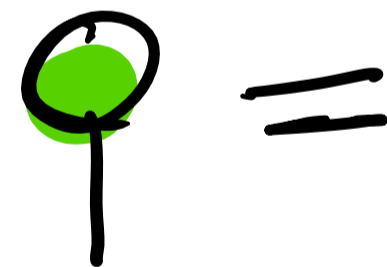
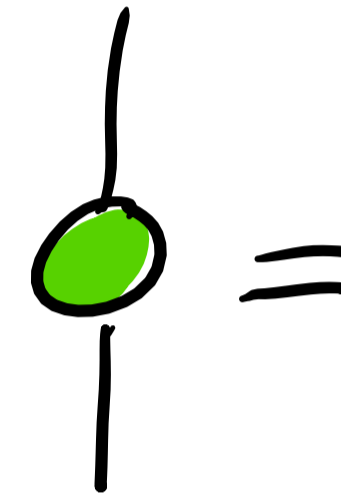
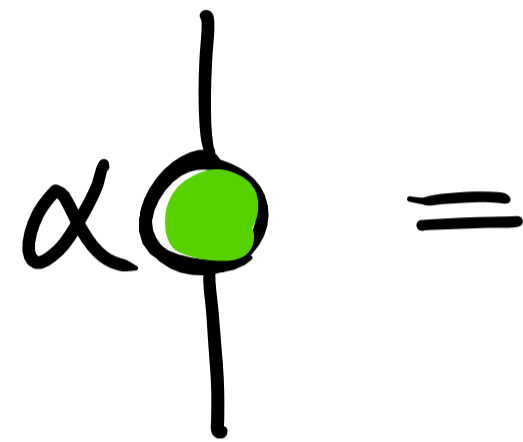
SPIDERS



SPIDERS



SPIDERS



SPIDERS

$$\alpha \text{ spider} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{i\alpha} \end{pmatrix}$$

$$\text{spider} =$$

$$\text{spider} =$$

$$\text{spider} =$$

$$\alpha \text{ spider} =$$

SPIDERS

$$\alpha \text{ spider} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

$$\text{spider} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\text{circle} =$$

$$\text{Y-spider} =$$

$$\alpha \text{ circle} =$$

SPIDERS

$$\alpha \text{ spider} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

$$\text{spider} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\text{circle} = (1 \ 1)$$

$$\text{Y-spider} =$$

$$\alpha \text{ circle} =$$

SPIDERS

$$\alpha \text{ spider} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha I \end{pmatrix}$$

$$\text{spider} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\text{pin} = (1 \ 1)$$

$$\text{Y-junction} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha \text{ circle} =$$

SPIDERS

$$\alpha \text{ spider} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

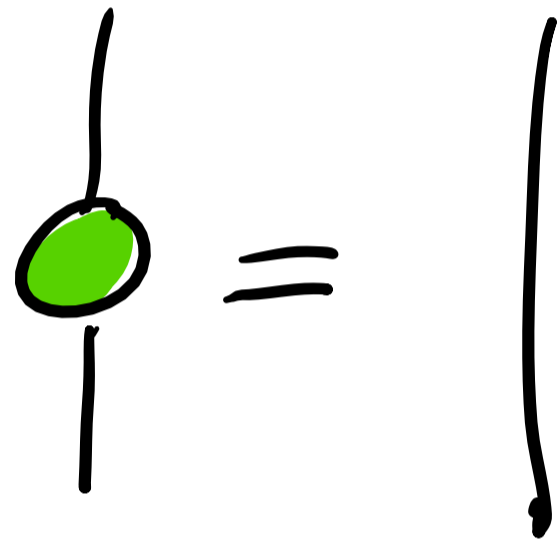
$$\text{spider} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\text{spider} = (1 \ 1)$$

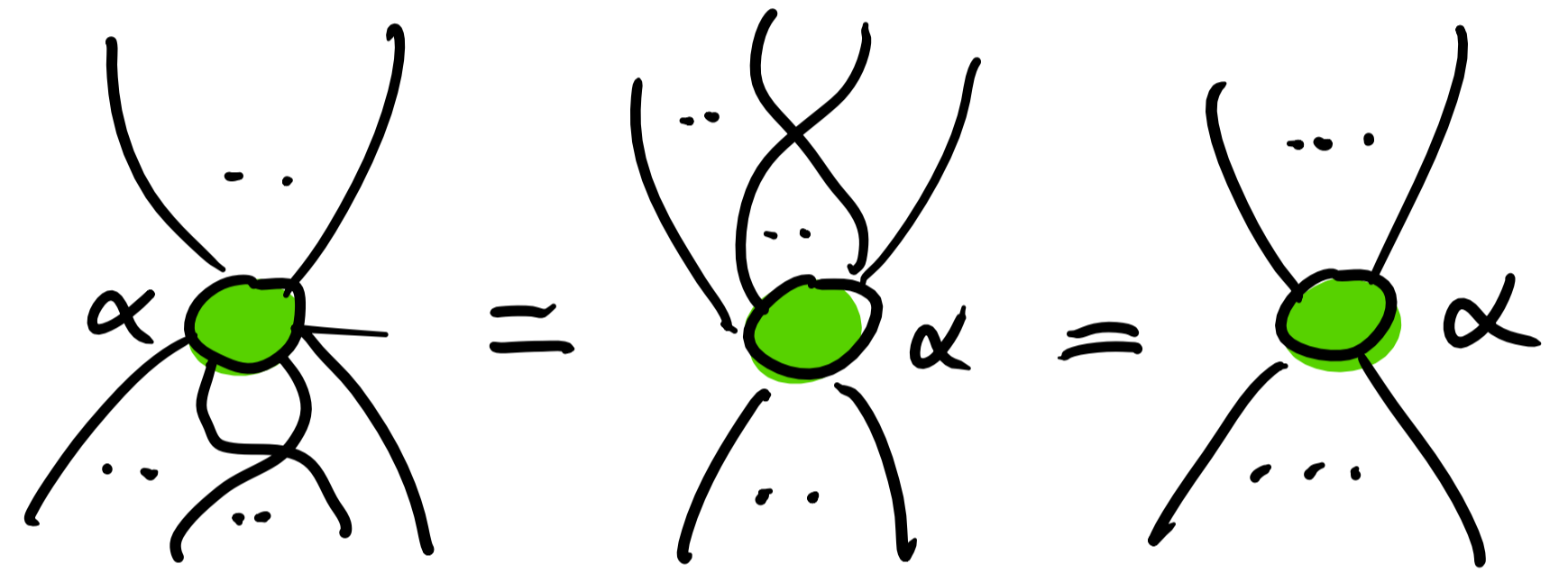
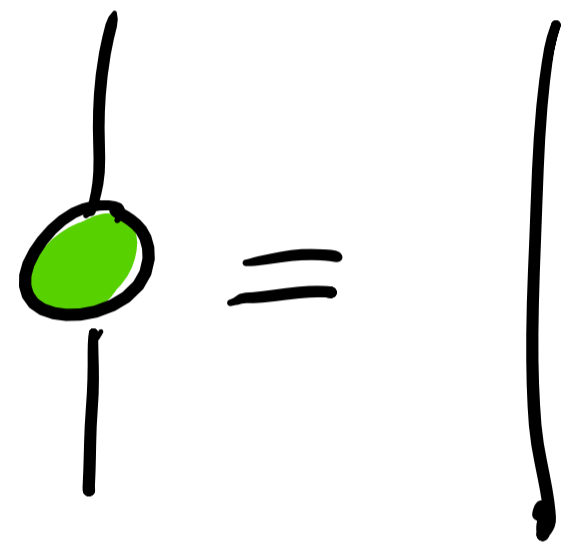
$$\text{spider} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha \text{ spider} = (1 + e^{i\alpha})$$

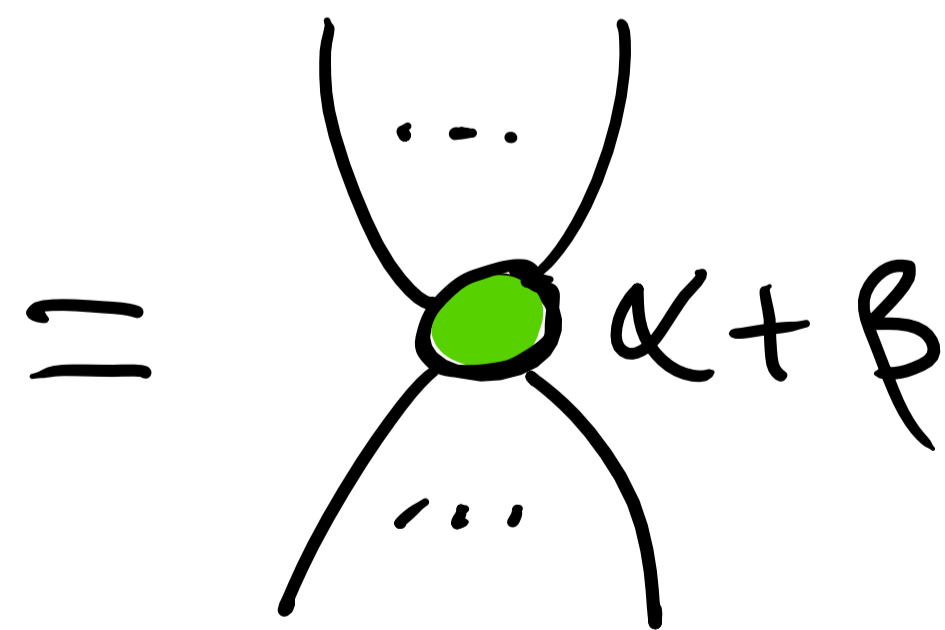
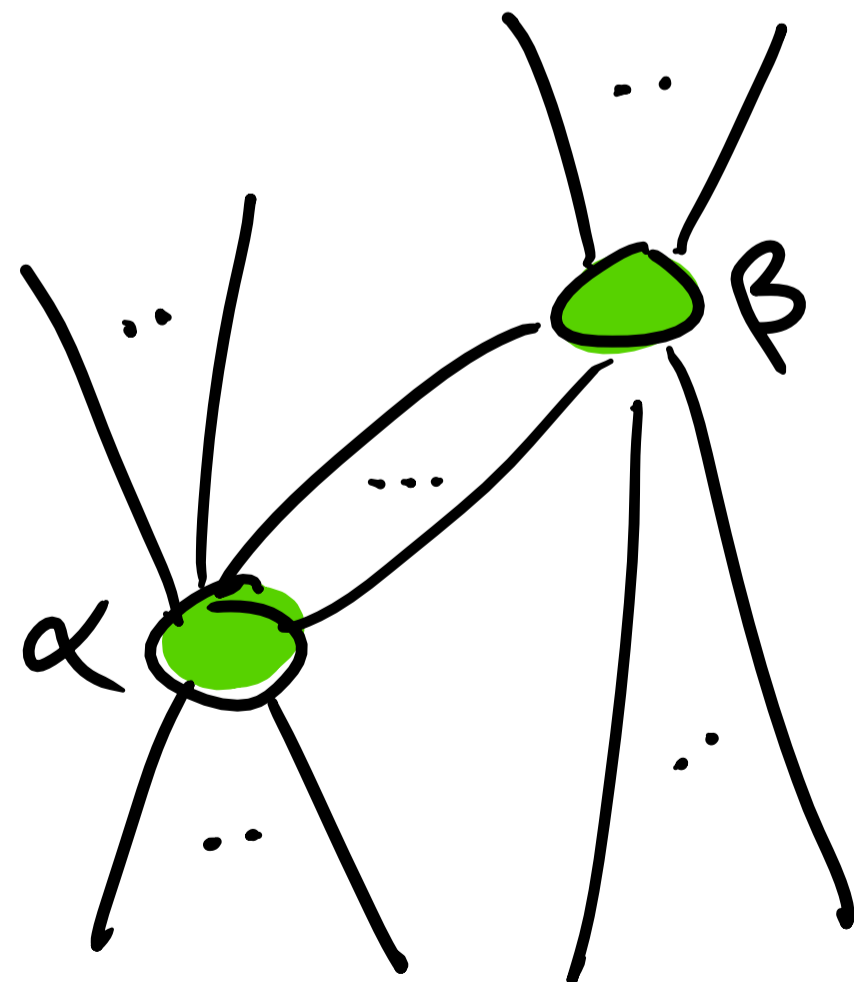
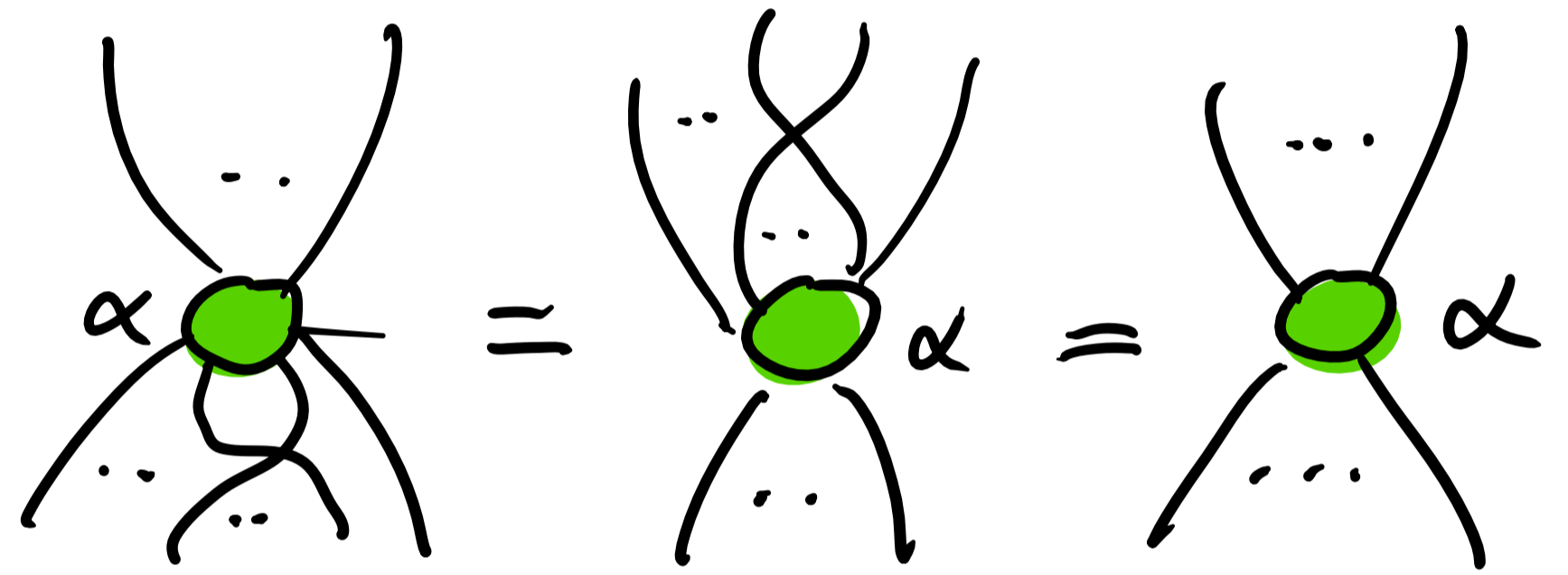
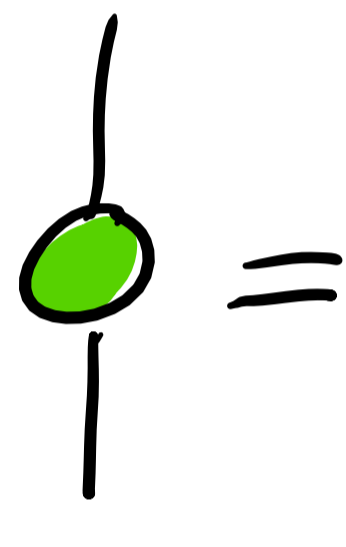
SPIDERS

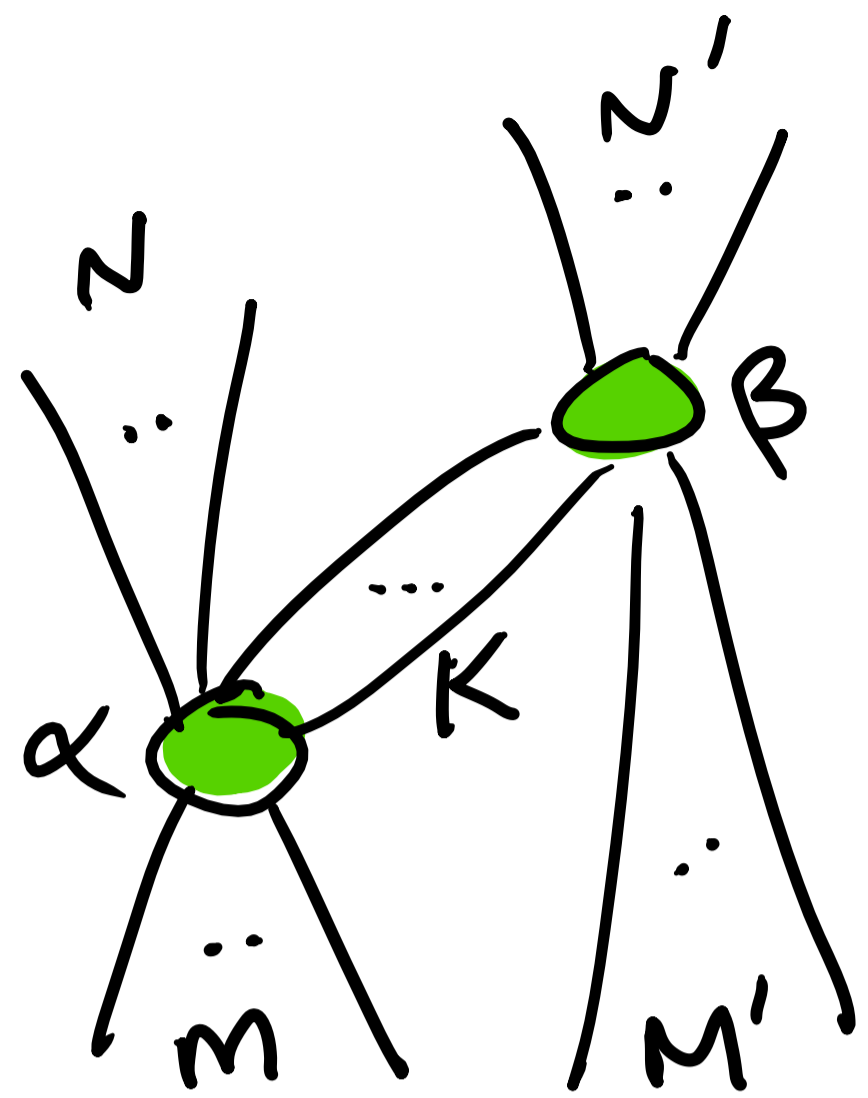


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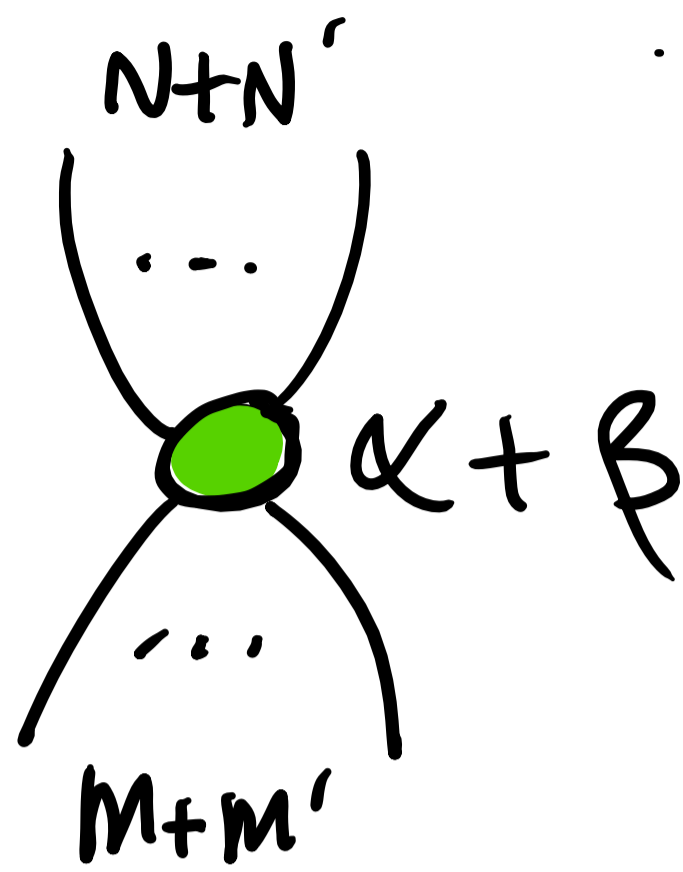


SPIDERS



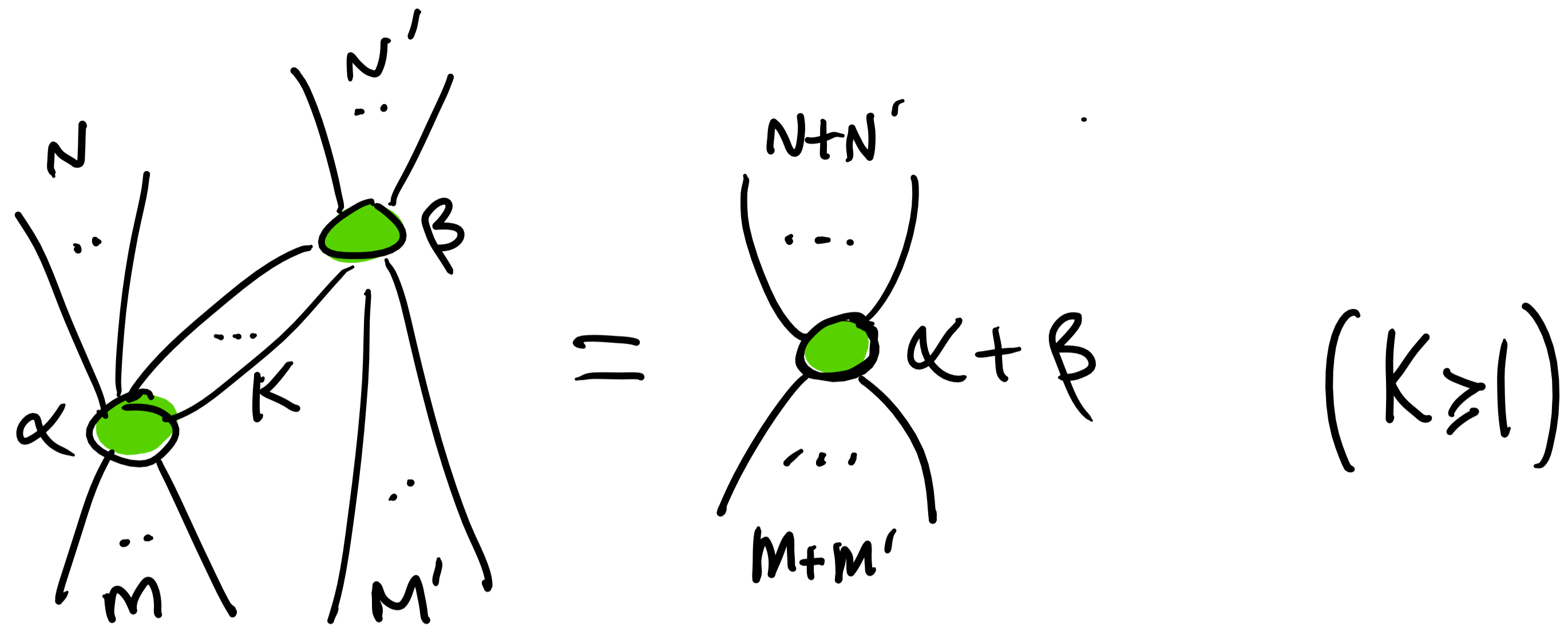


$=$



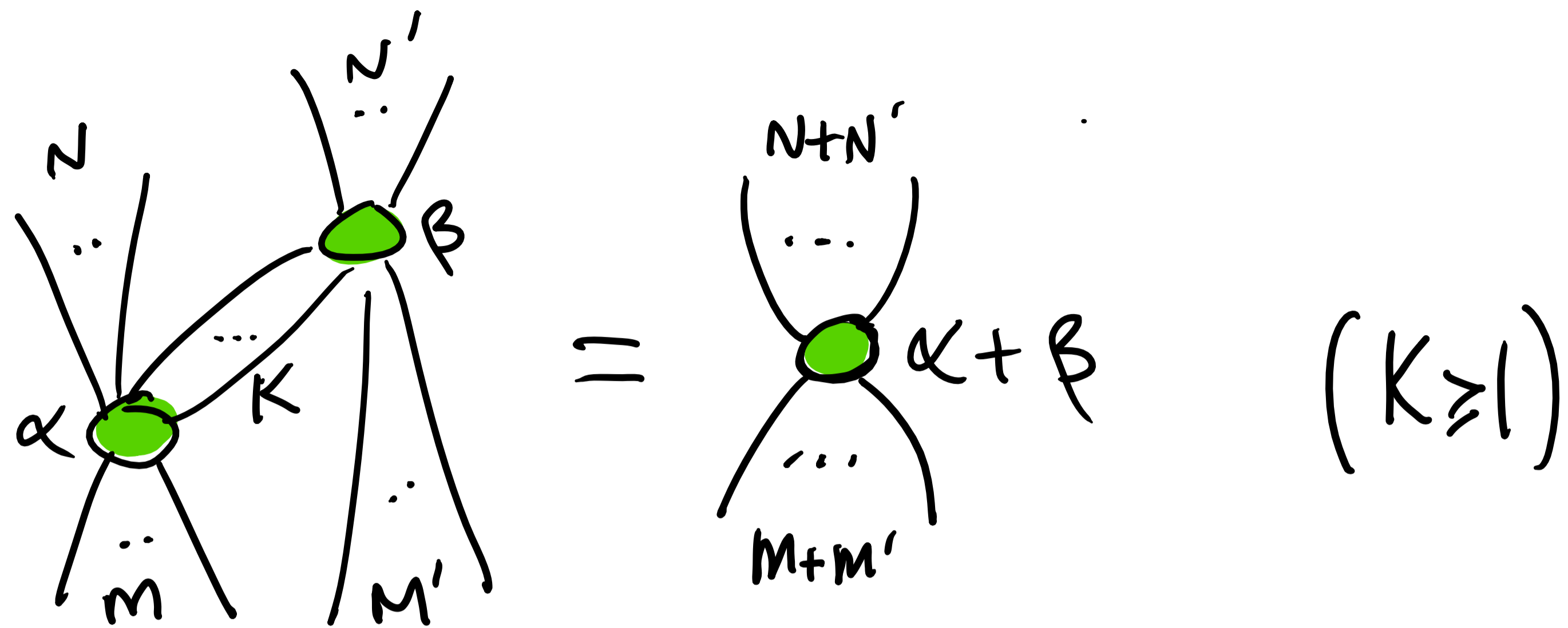
$(K \geq 1)$





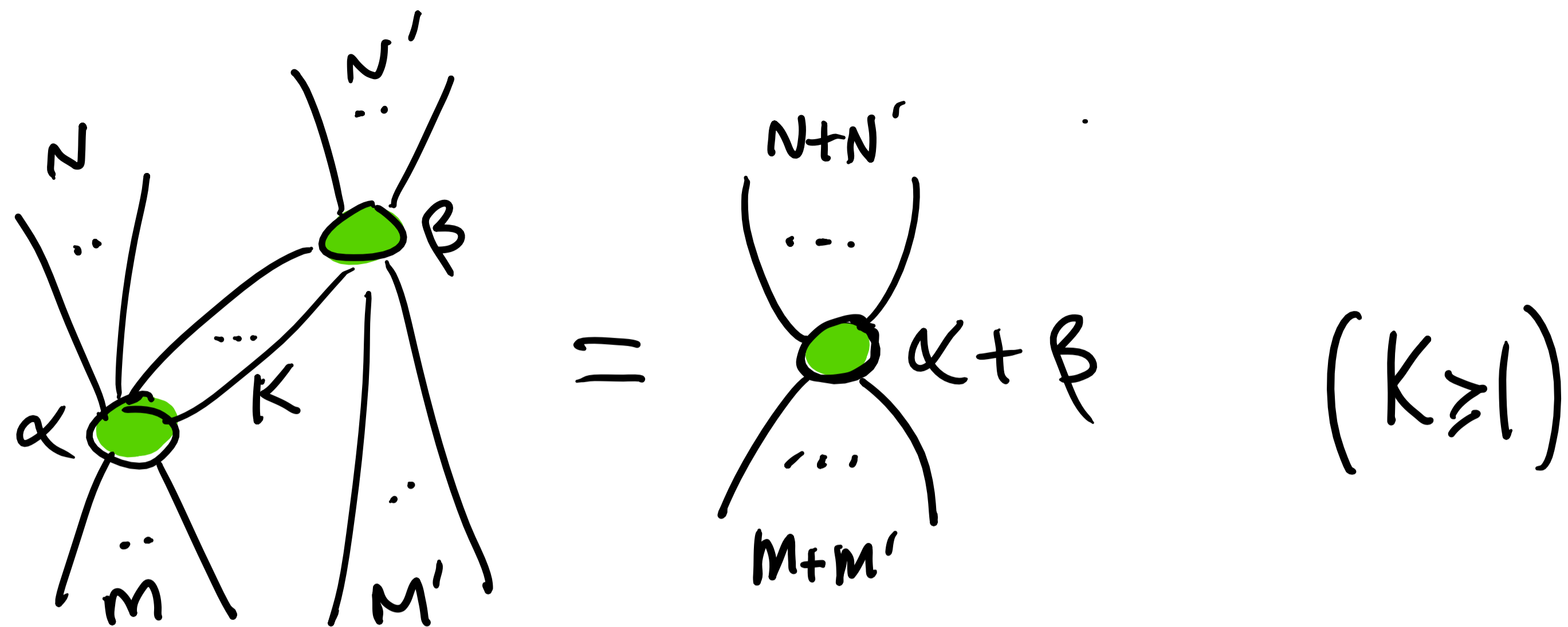
Diagrammatic equation for $K=1$. The left side shows two vertices α and β connected by a single vertical line. The right side shows the product of two matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

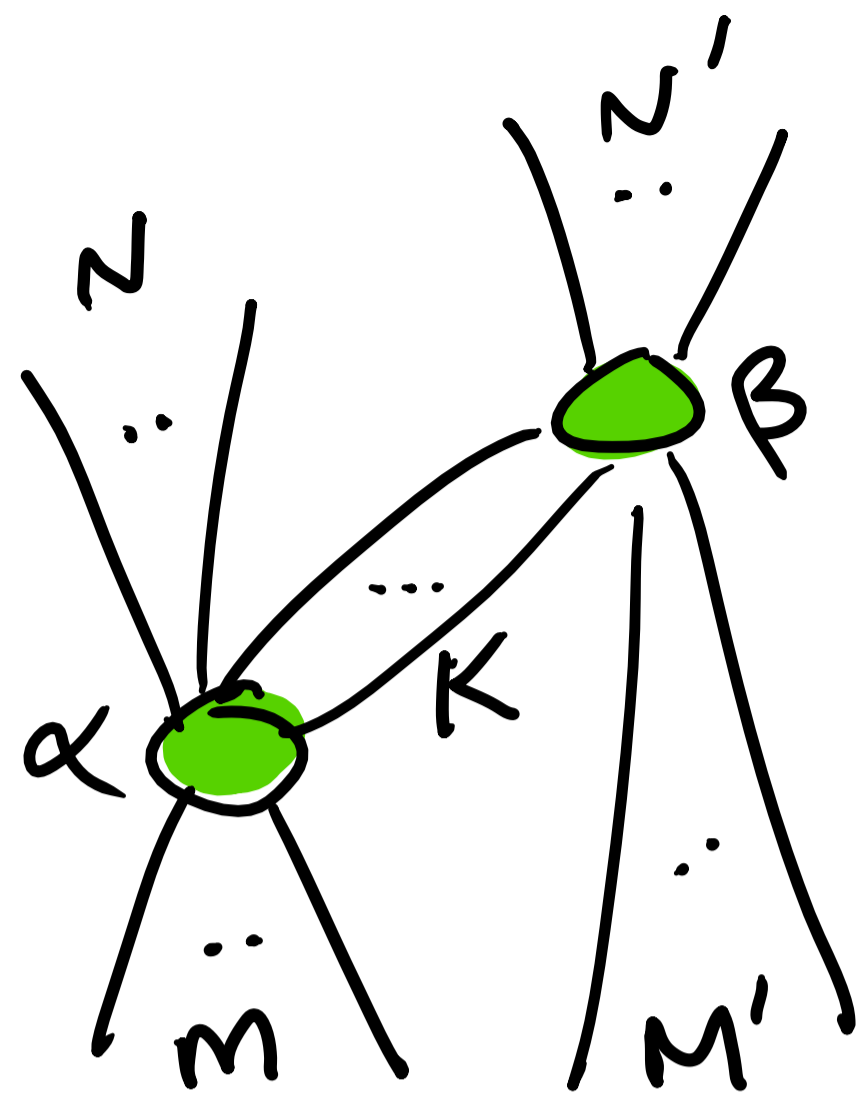


Diagrammatic equation for two vertices in series. The left side shows two vertices α and β connected by a single line. The right side shows the corresponding matrix multiplication of two 2×2 matrices, which simplifies to a single 2×2 matrix:

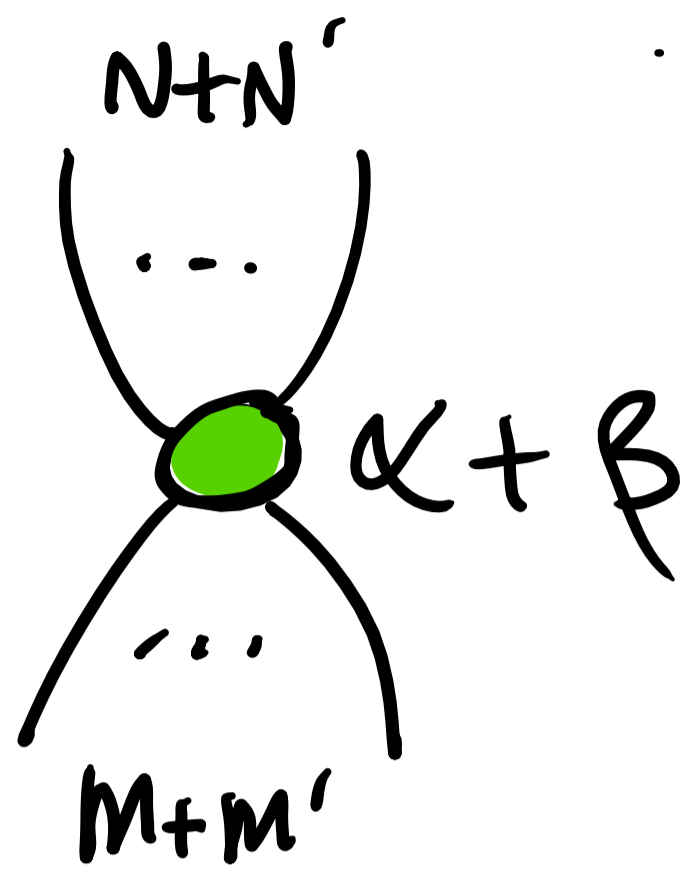
$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} e^{i\beta} \end{pmatrix}$$



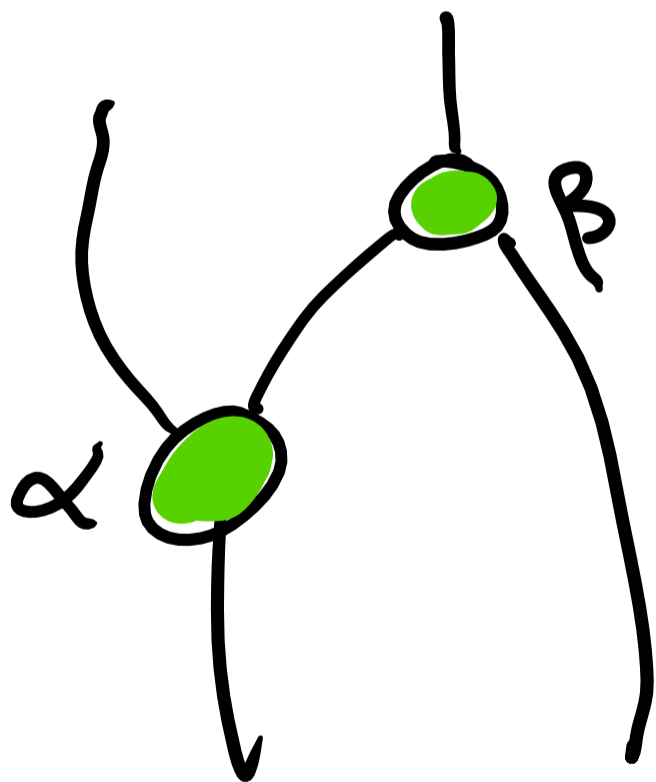
$$\begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \end{array} \begin{array}{l} \beta \\ \alpha \end{array} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} e^{i\beta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\alpha+\beta)} \end{pmatrix}$$

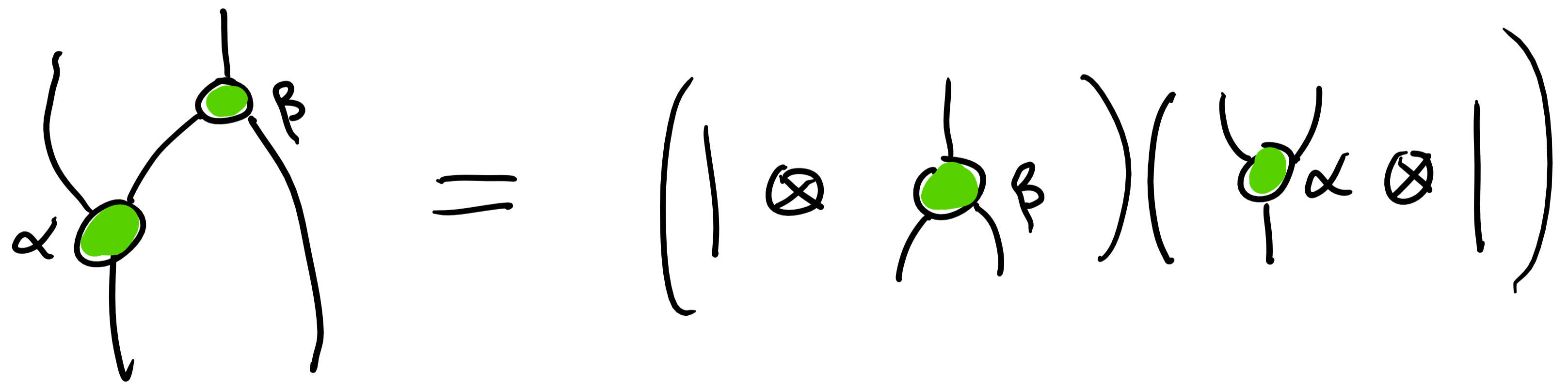
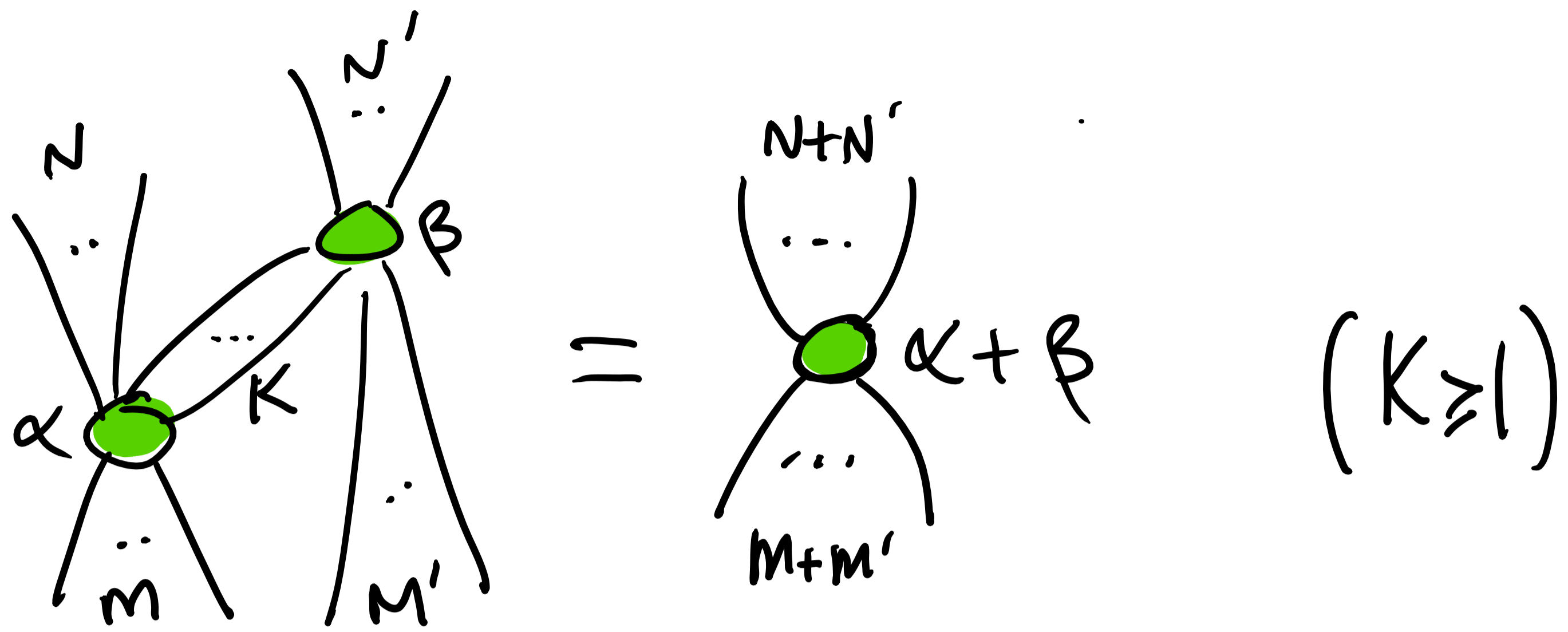


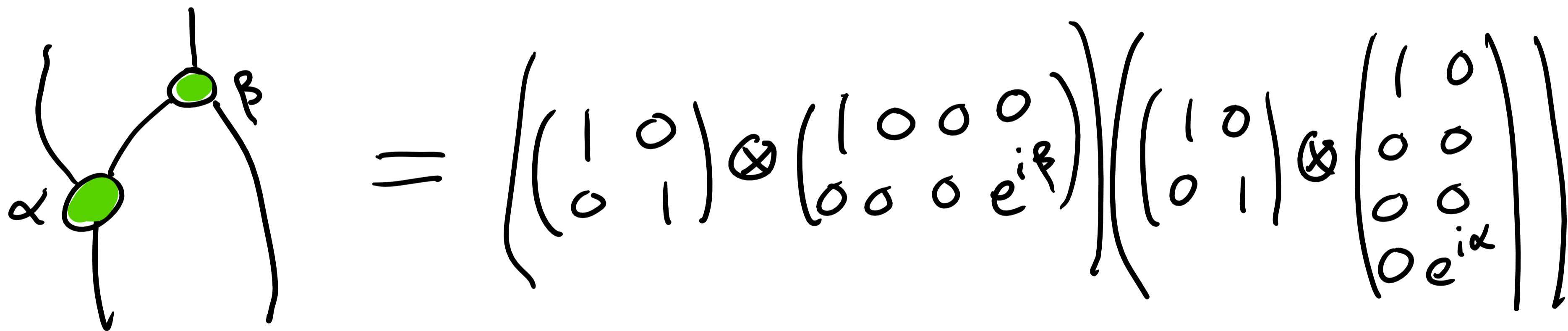
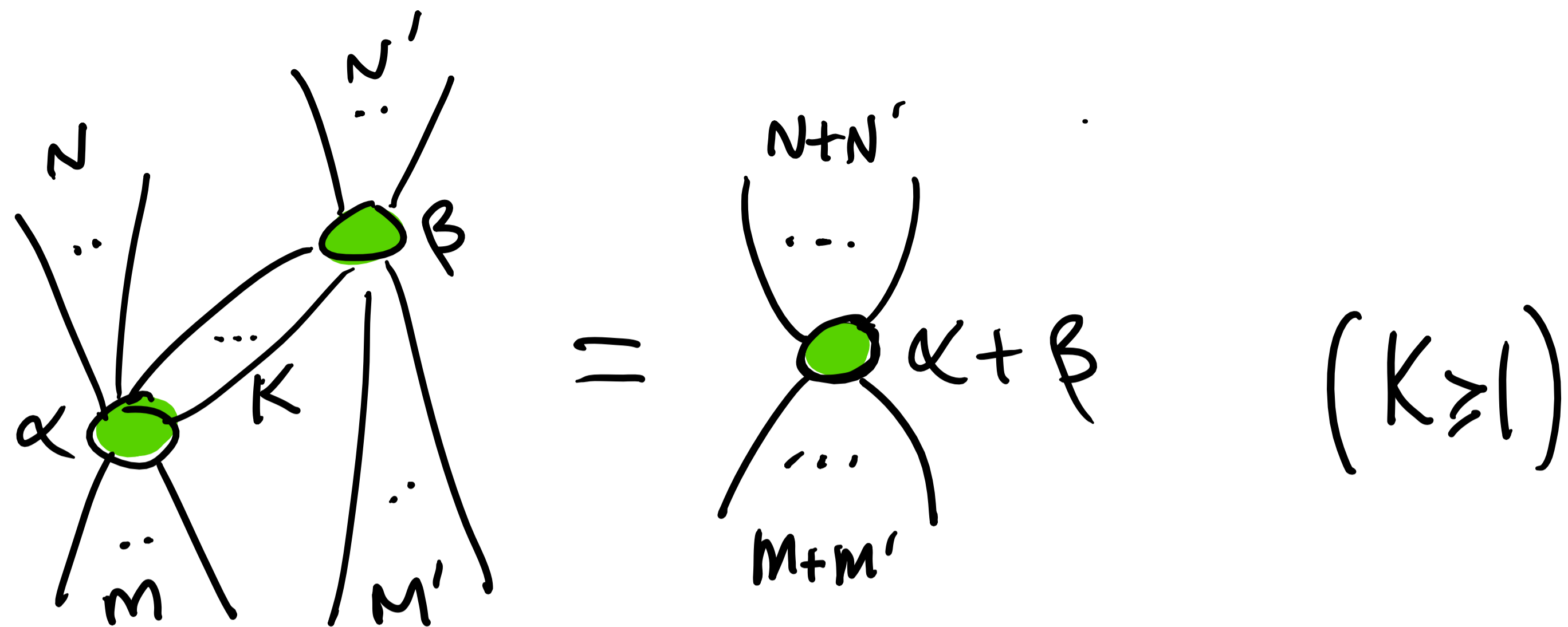
$=$

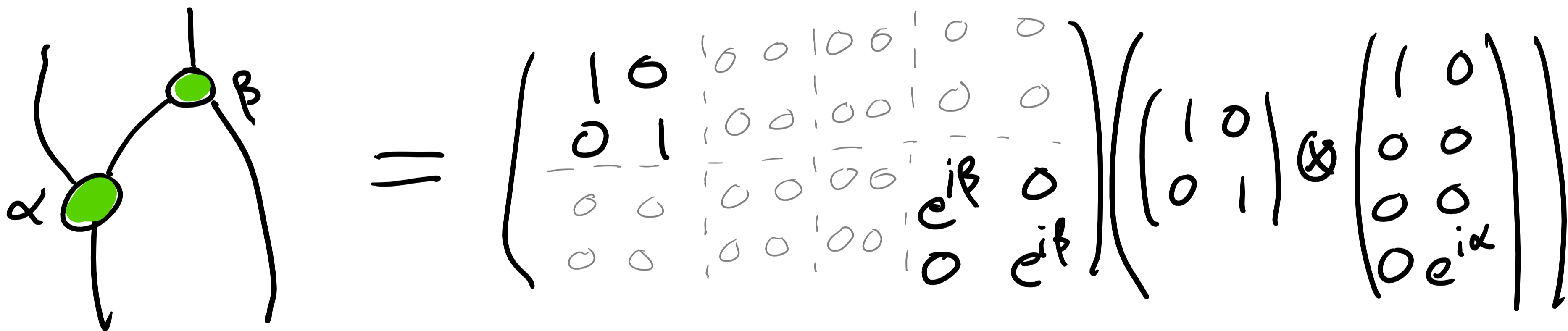
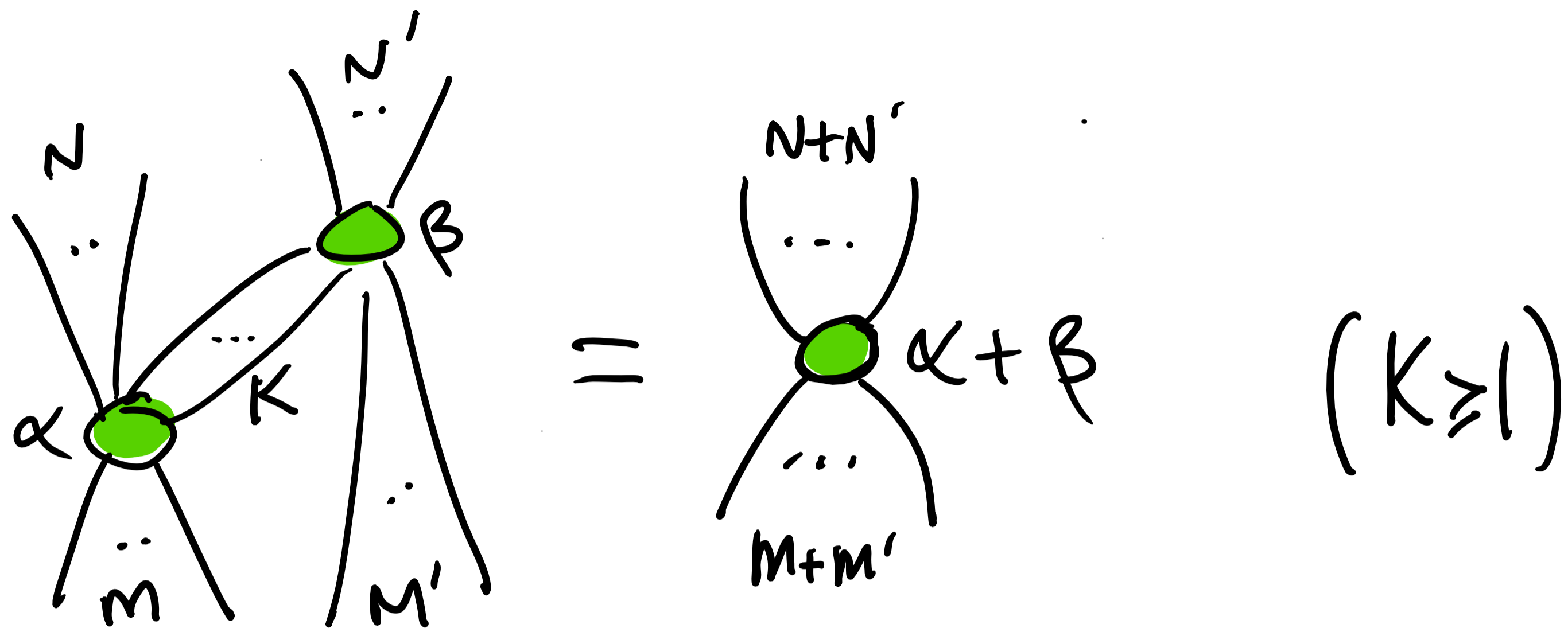


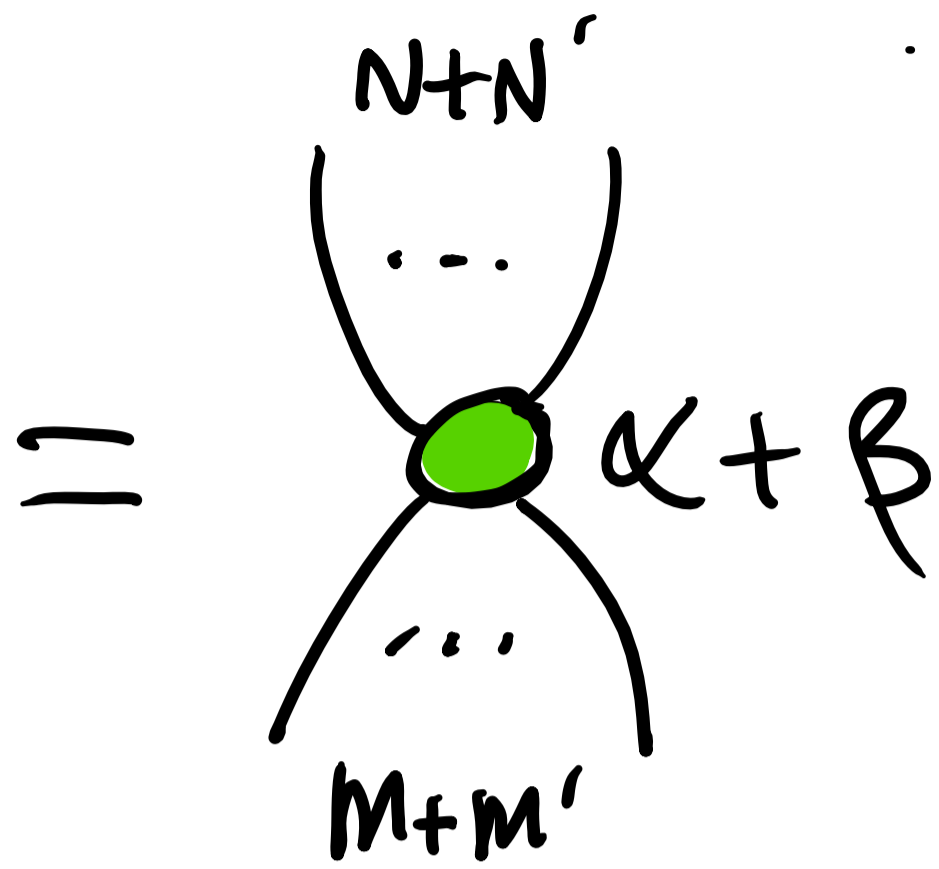
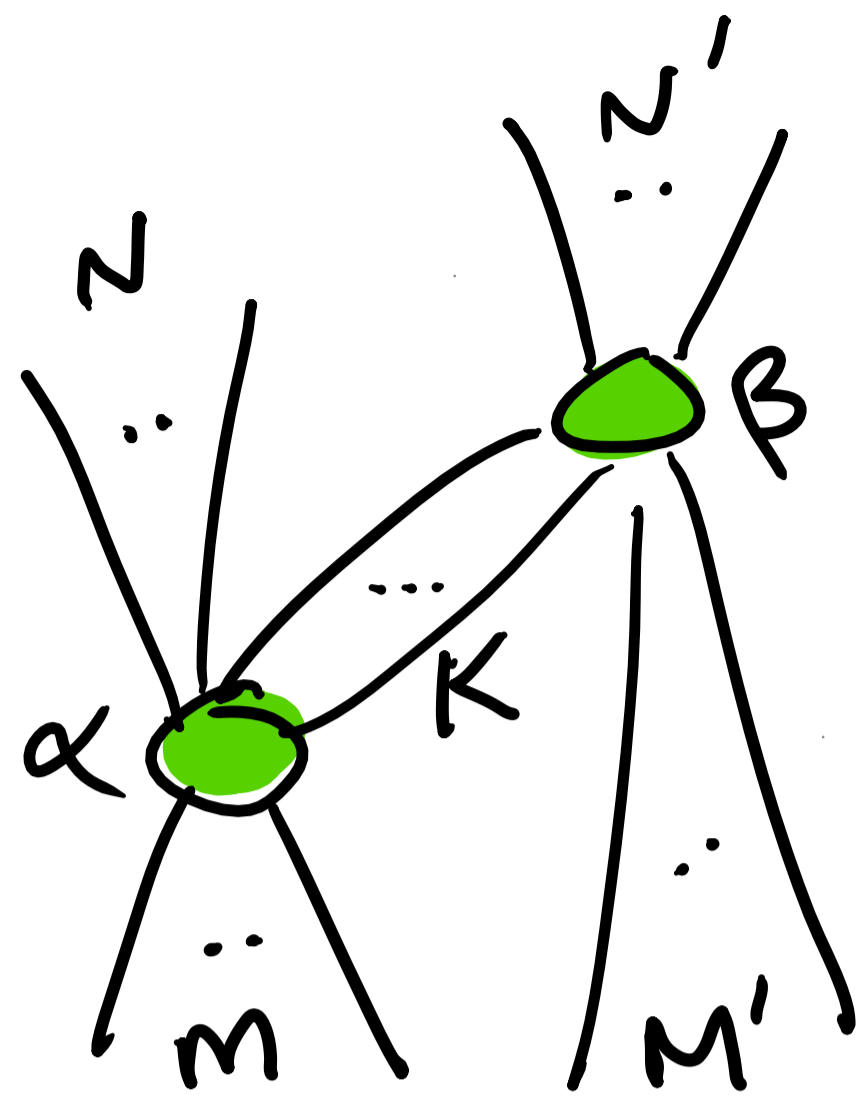
$(K \geq 1)$



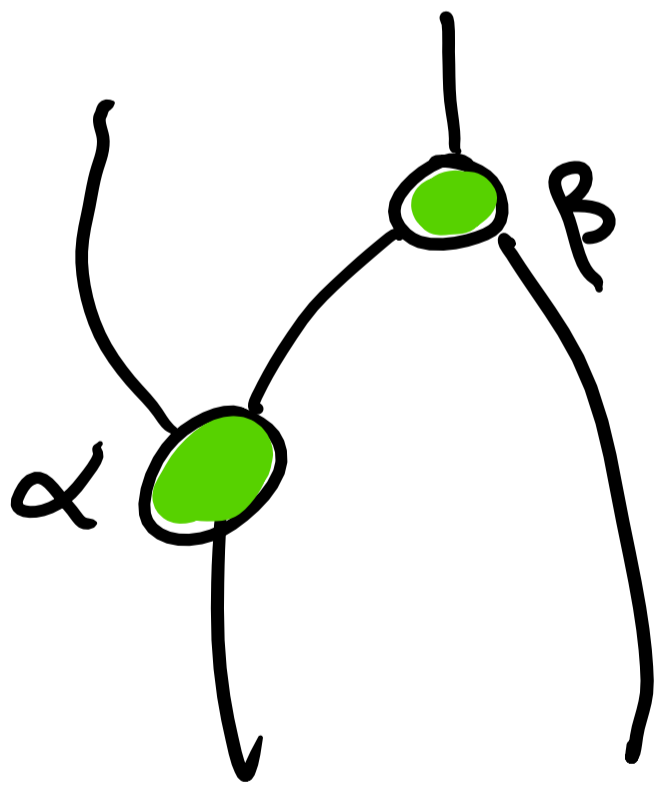






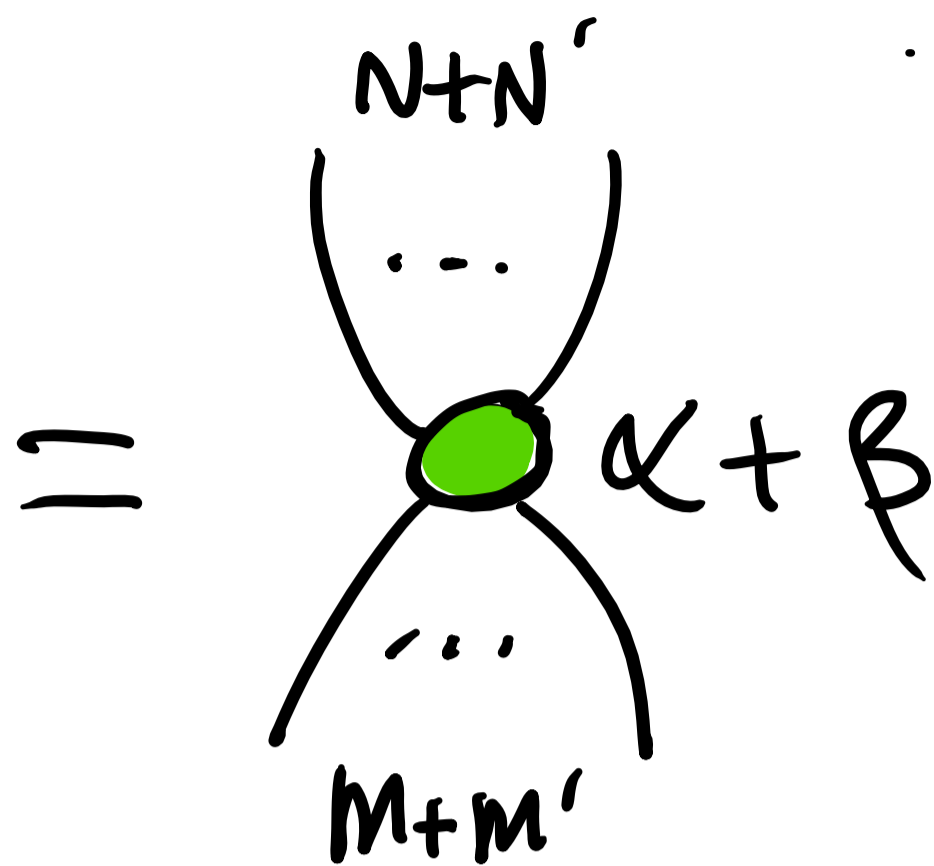
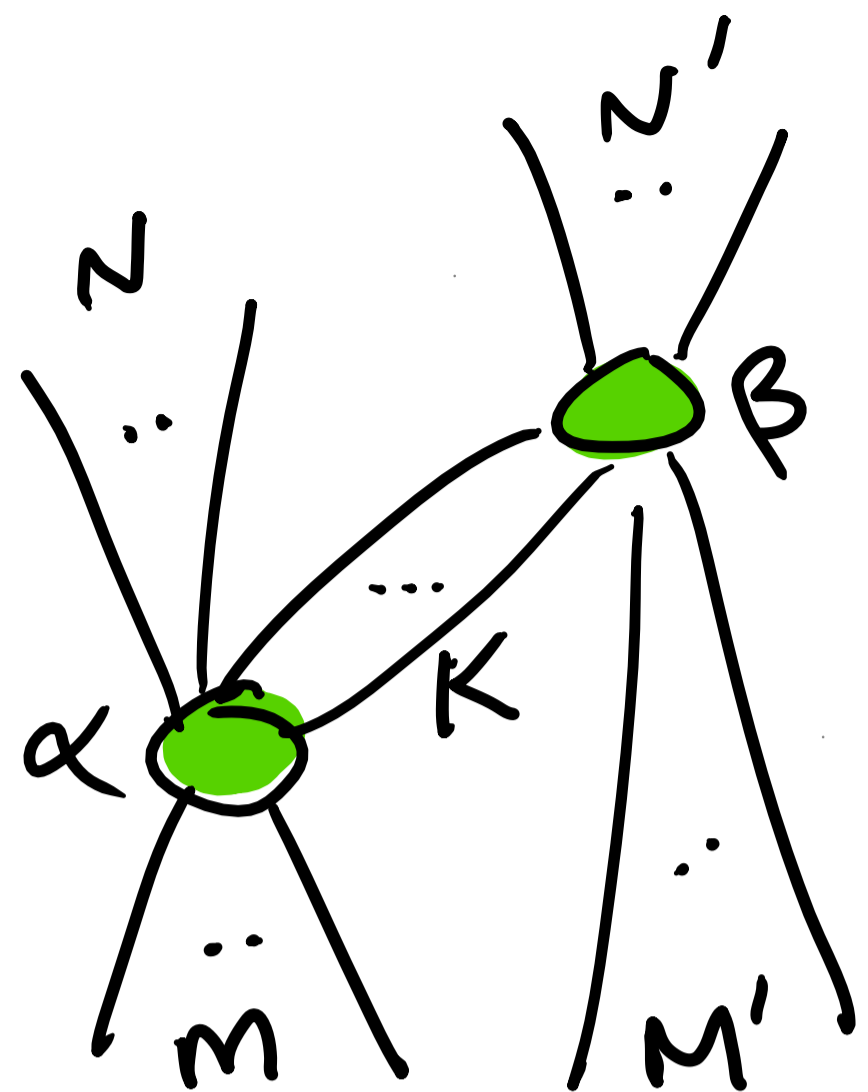


$$(K \geq 1)$$

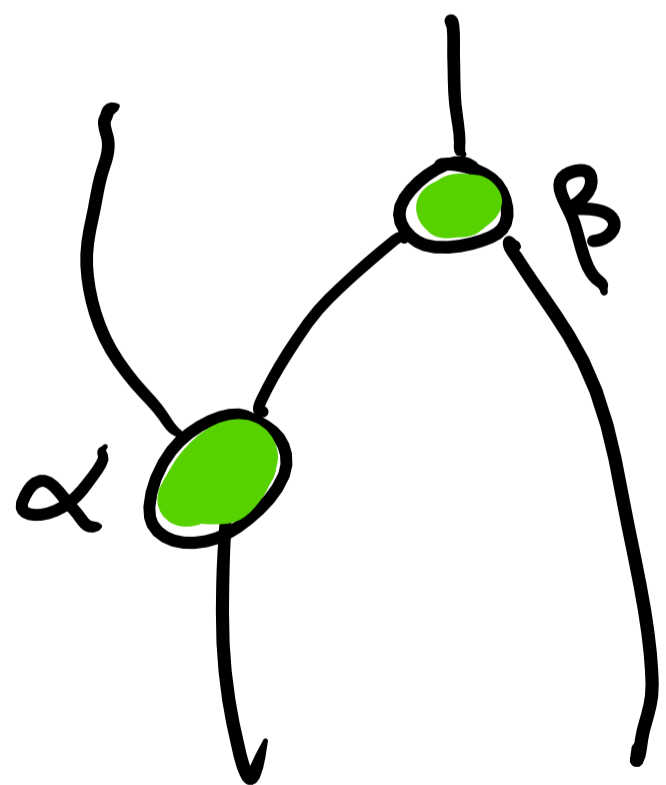


$$= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

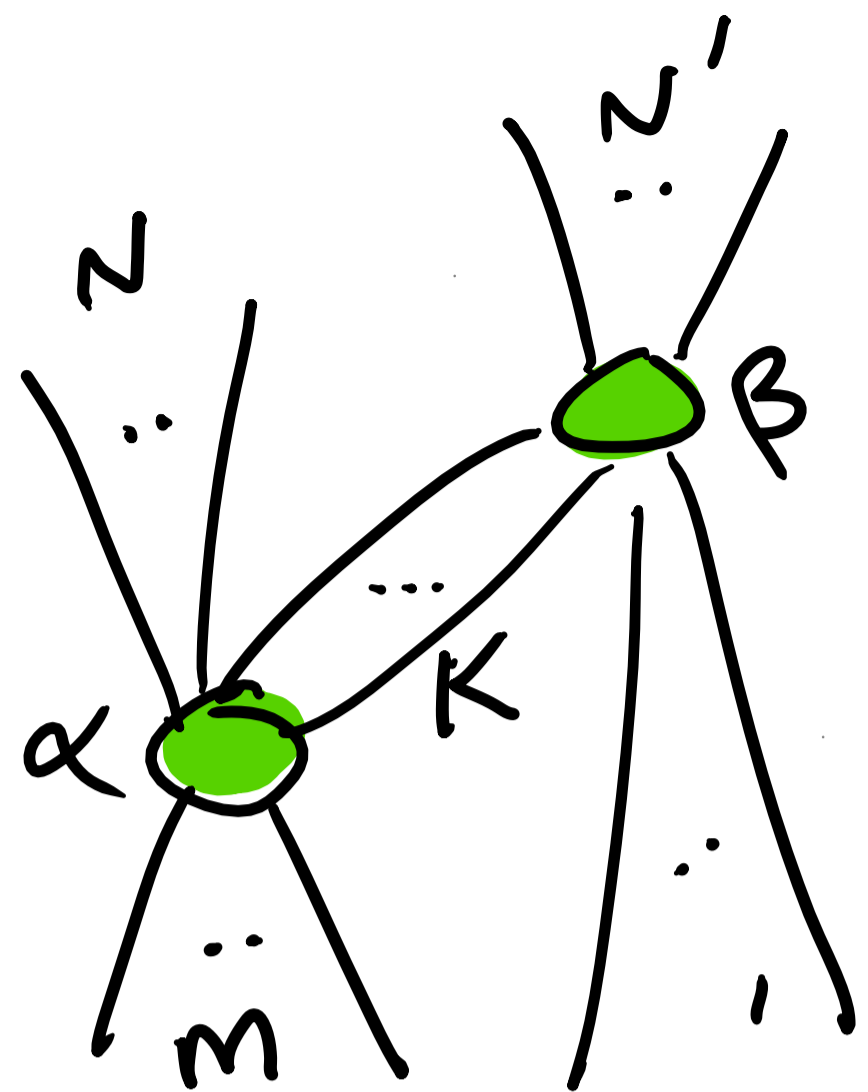
$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$



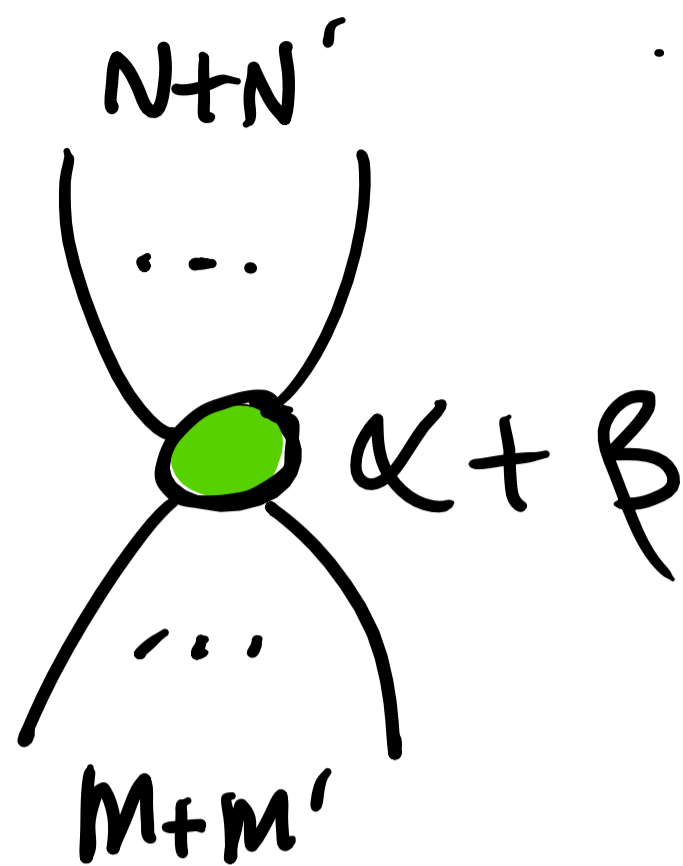
$(K \geq 1)$



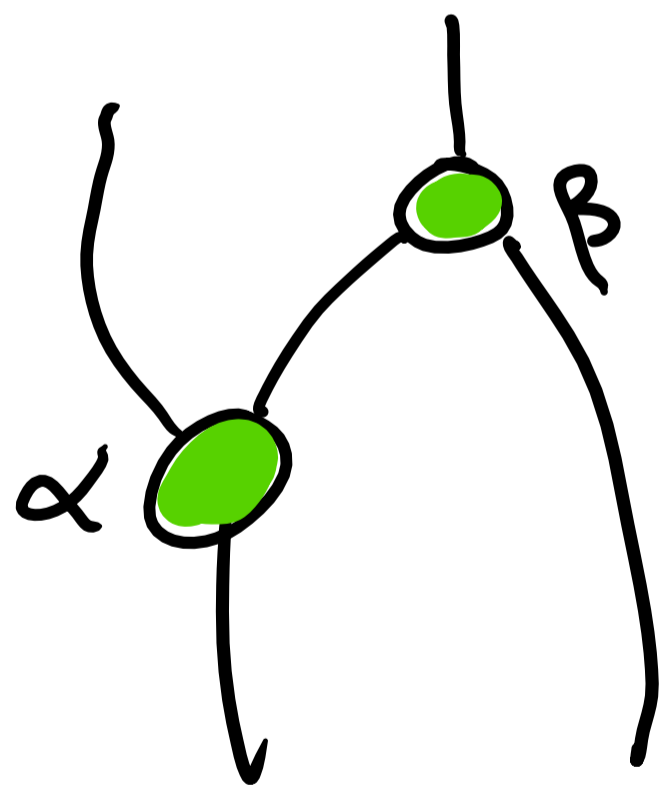
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha} e^{i\beta} \end{pmatrix}$$



$=$

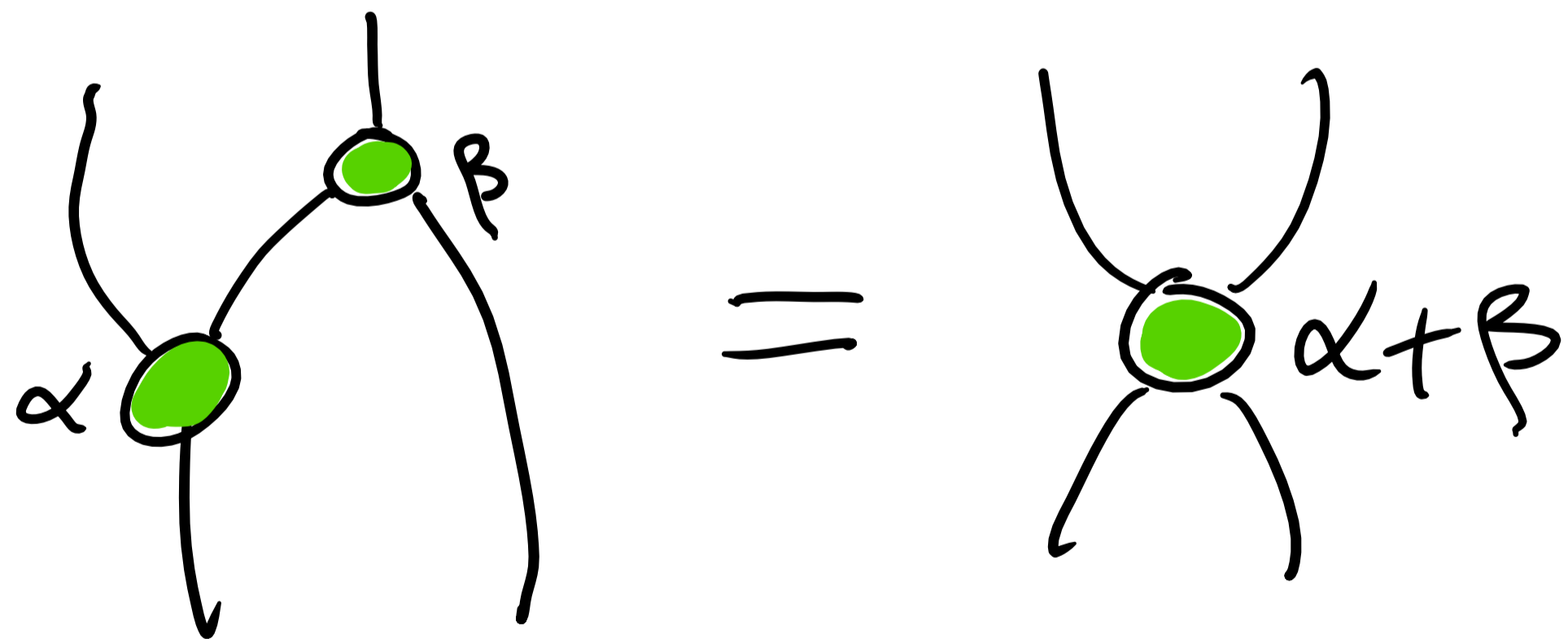
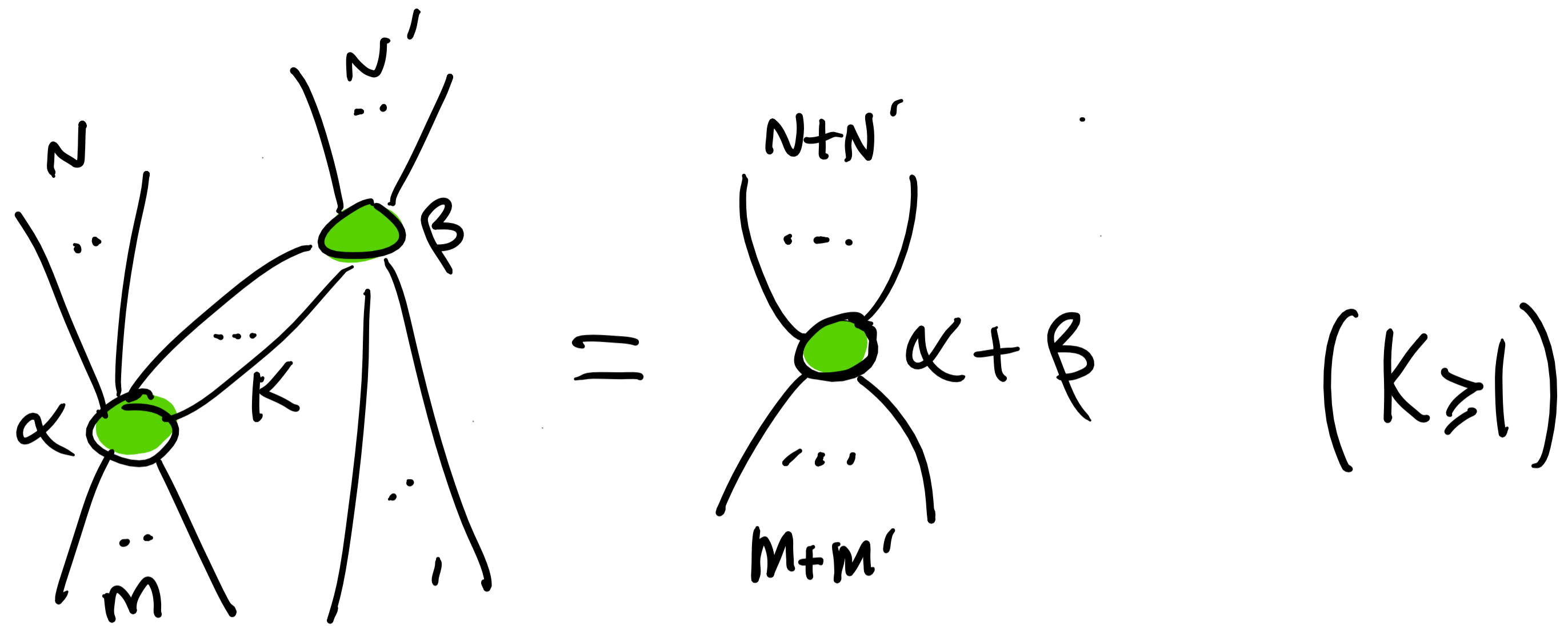


$(K \geq 1)$

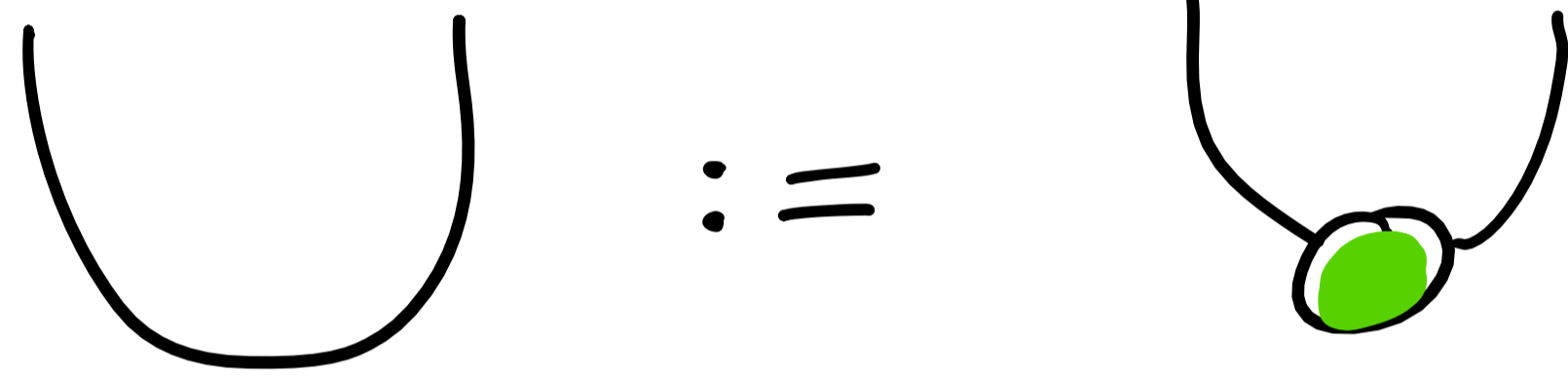


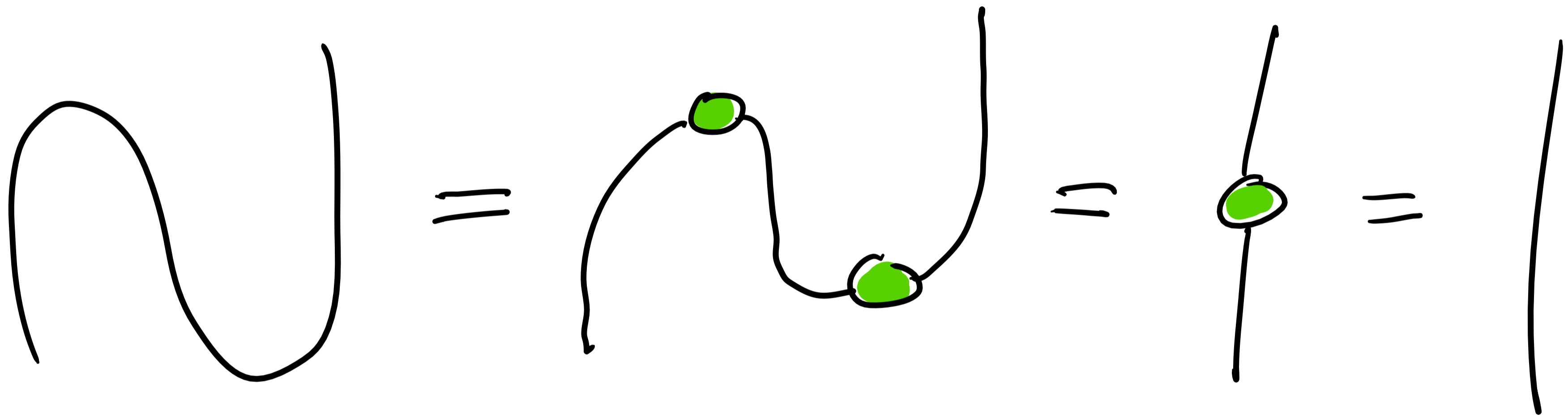
$=$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i(\alpha+\beta)} \end{pmatrix}$$



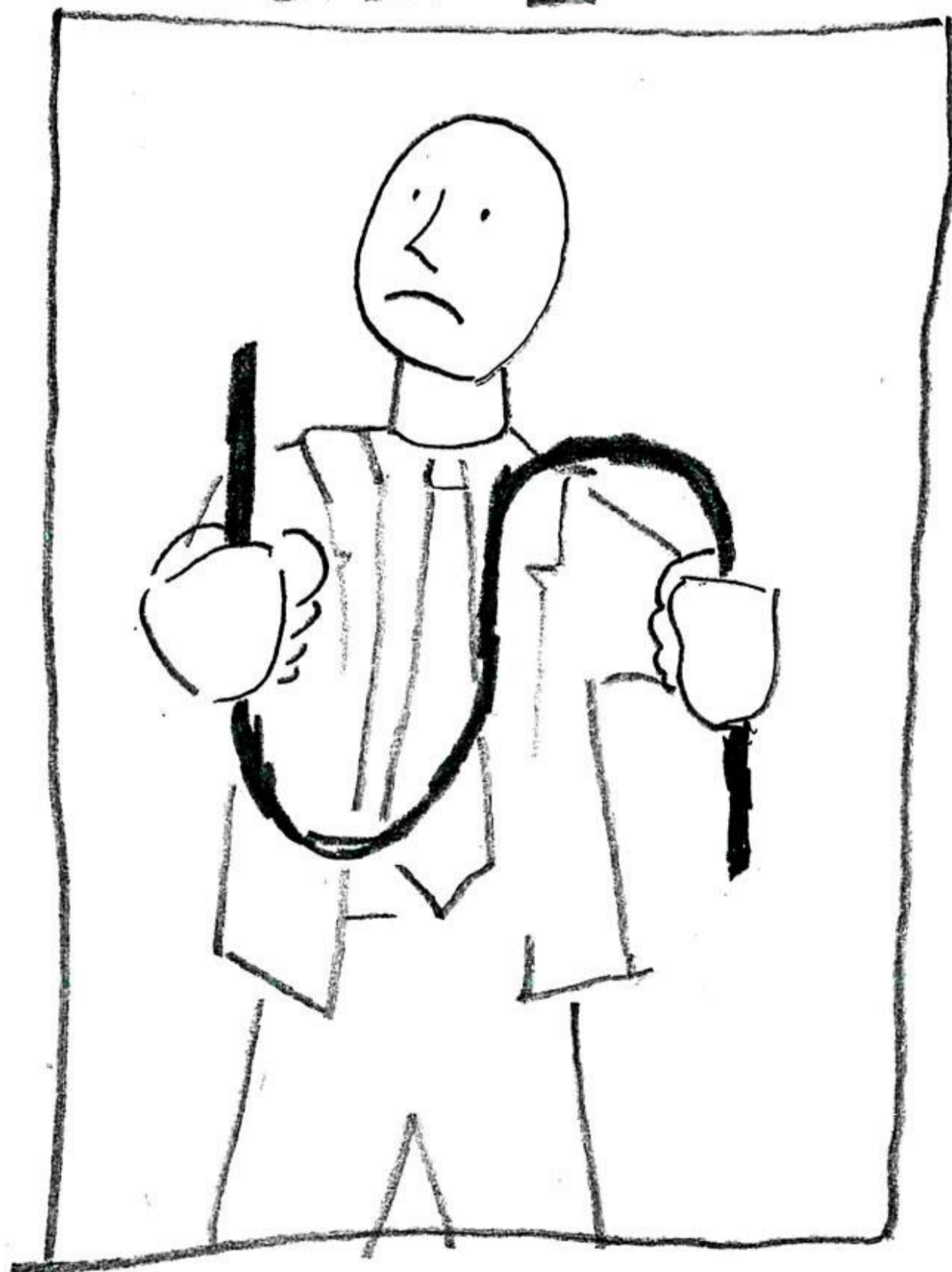
CUPS + CAPS





"THE FUNDAMENTALS OF CATEGORICAL QUANTUM MECHANICS"

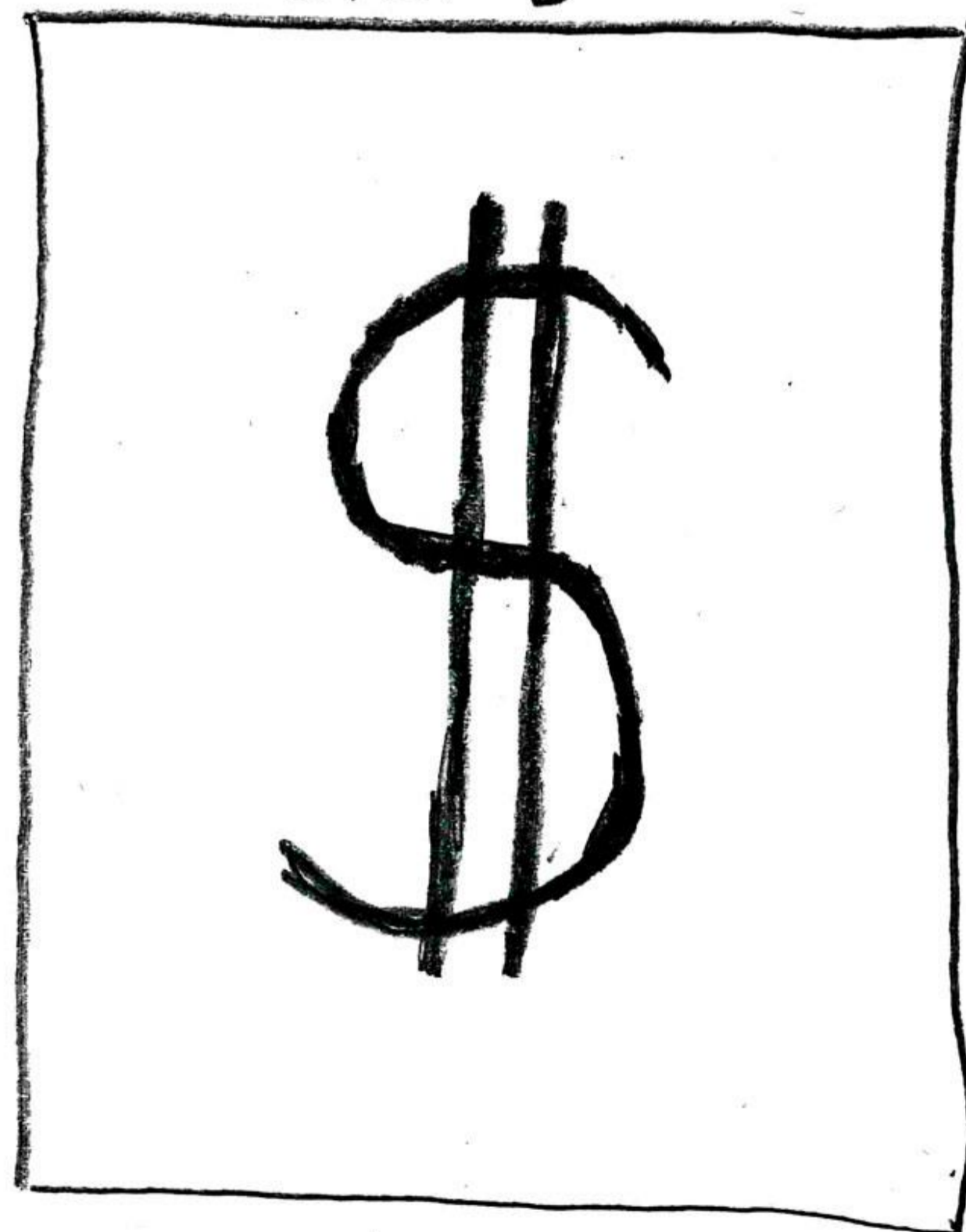
STEP 1



STEP 2

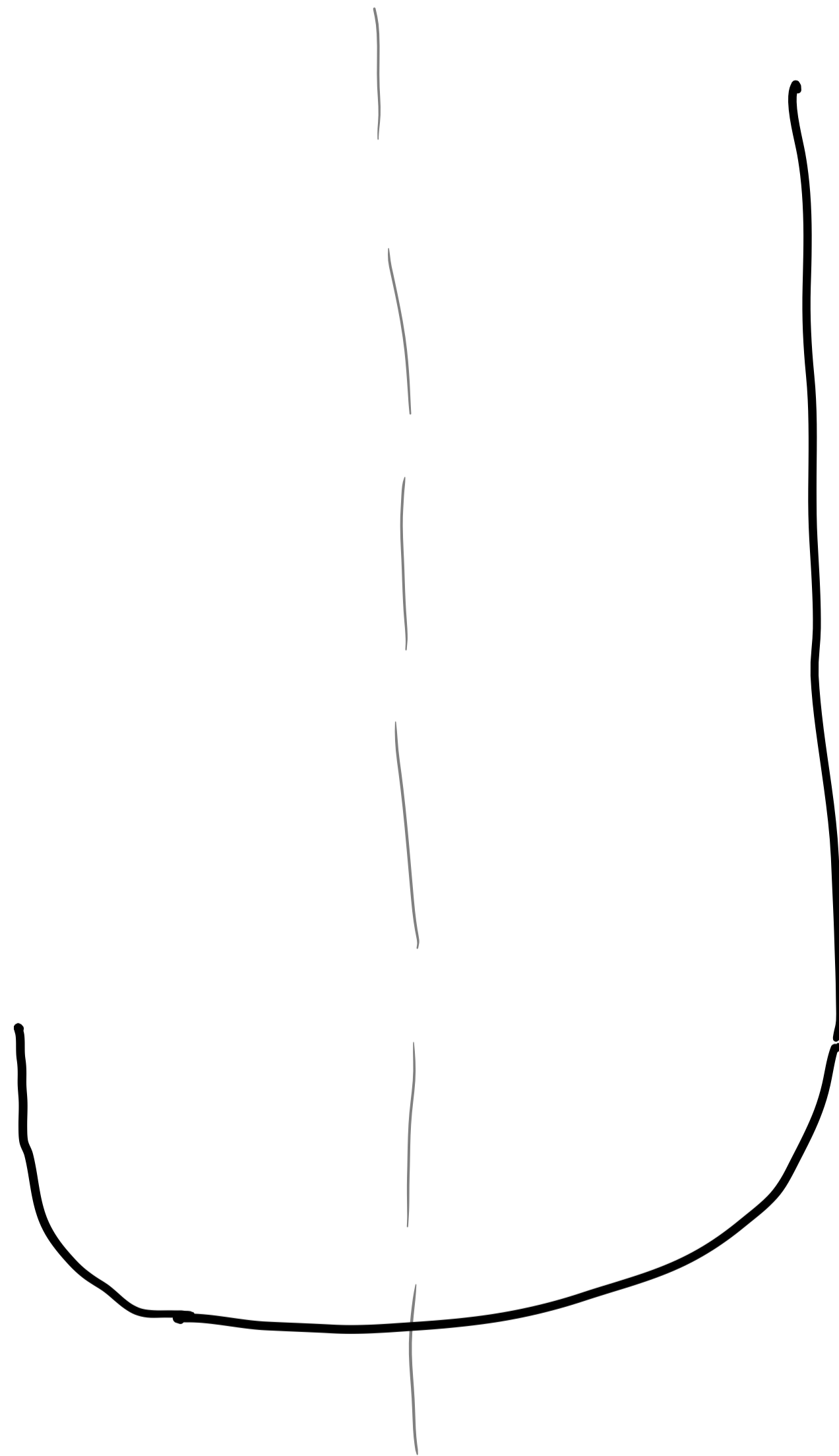
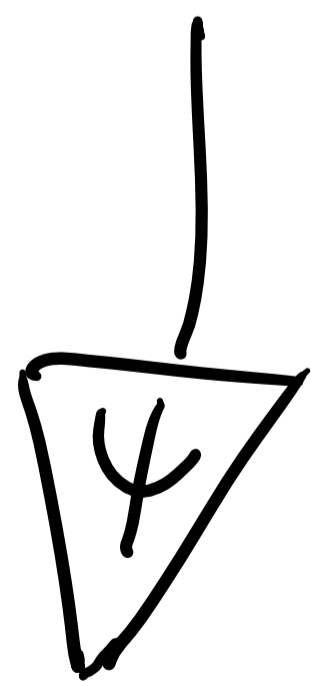


STEP 3



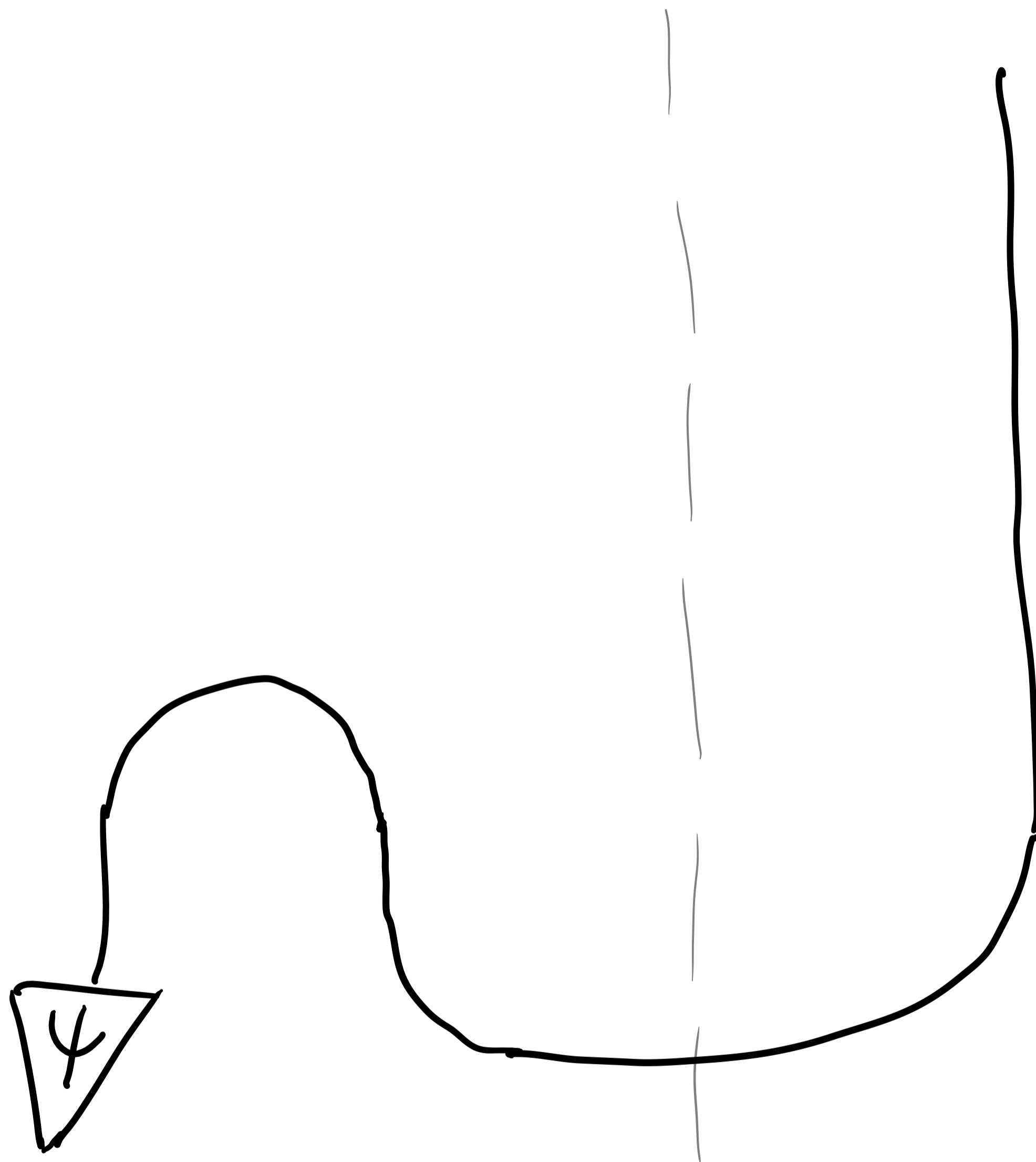
Alice

Bob



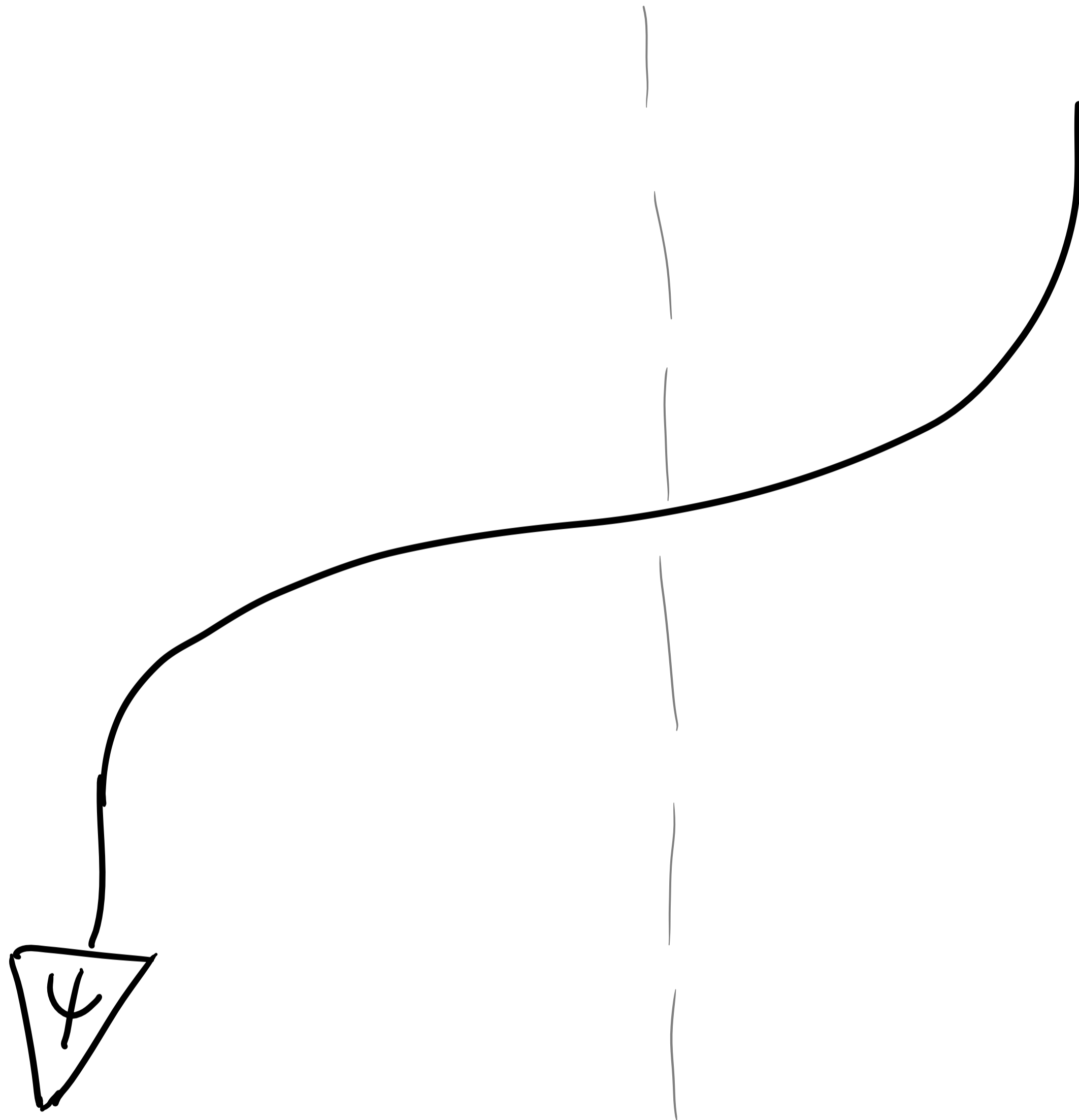
Alice

Bob



Alice

Bob

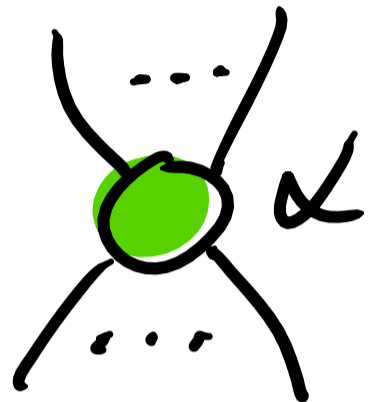


Alice

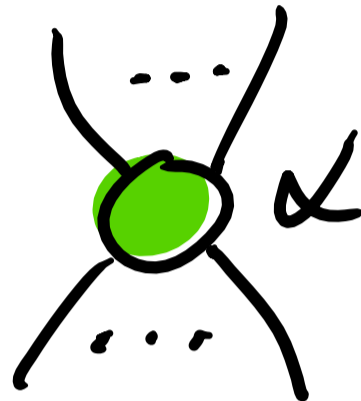
Bob

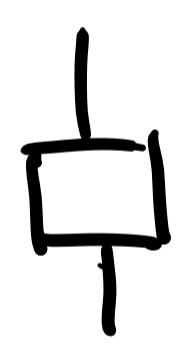


ZX-diagrams consist of:

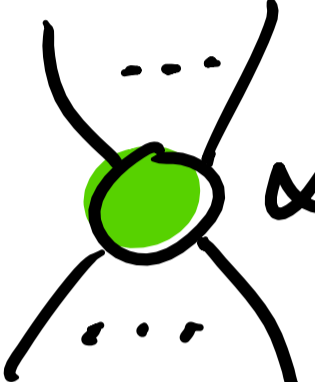
I. Z spiders  $\alpha := \begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix}$

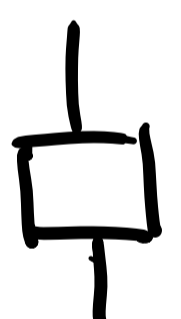
ZX-diagrams consist of:

I. Z spiders  $\alpha := \begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix}$

II. colour-changer  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

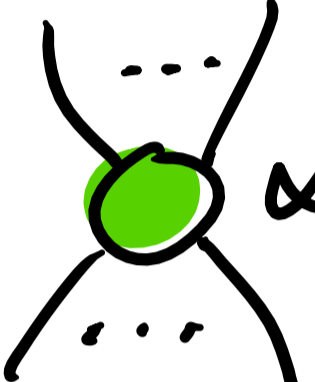
ZX-diagrams consist of:

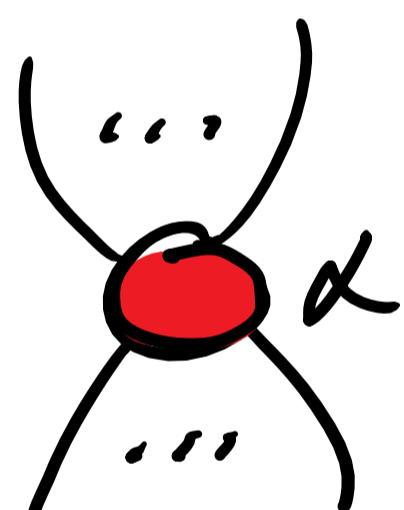
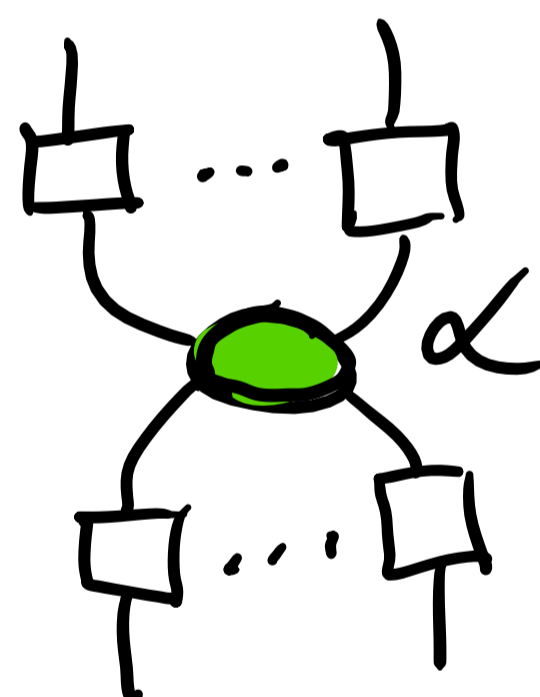
I. Z spiders.  $\alpha := \begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix}$

II. colour-changer  $:= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

III. identity & swap.

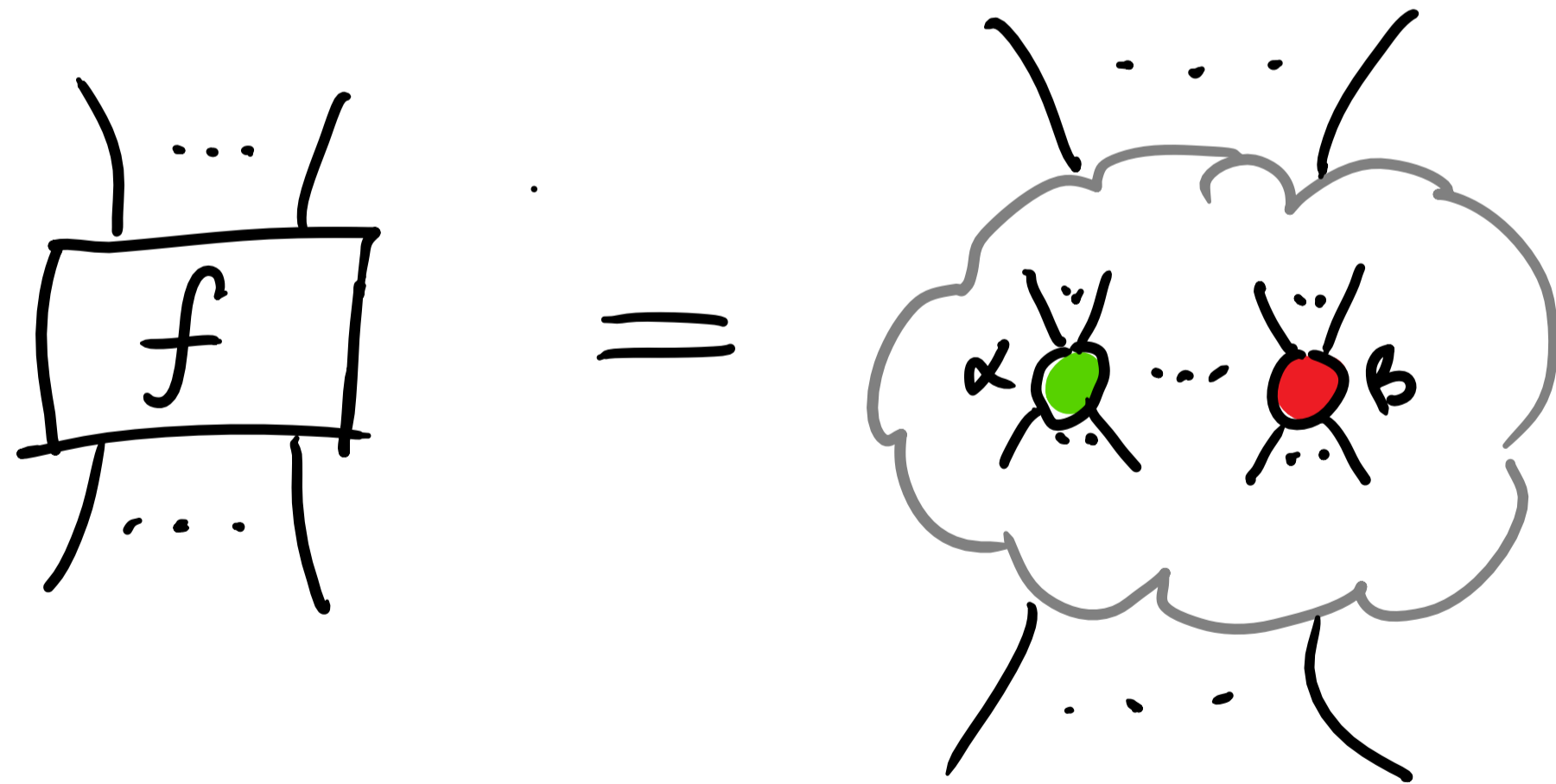
ZX-diagrams ^{equivalently} consist of:

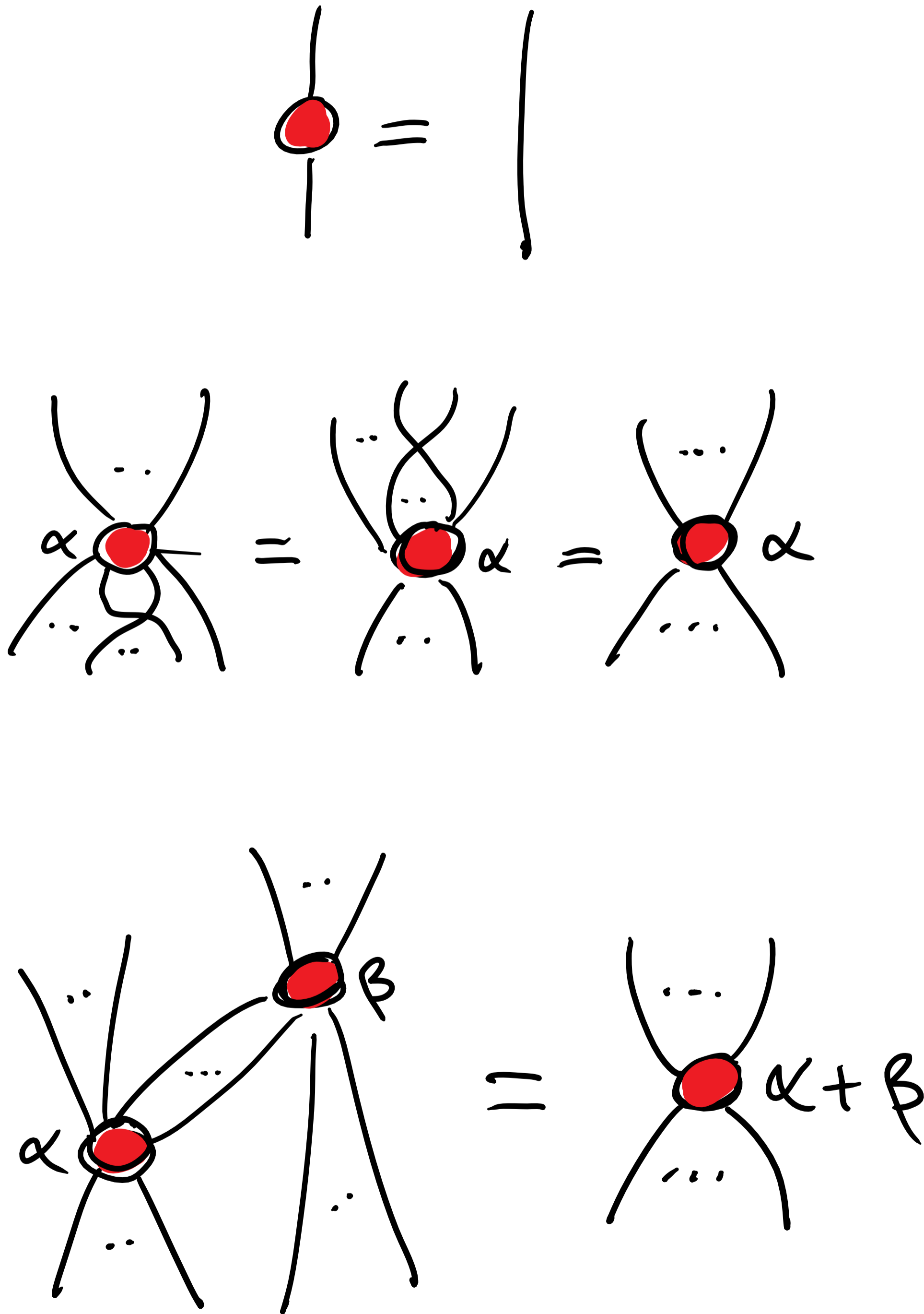
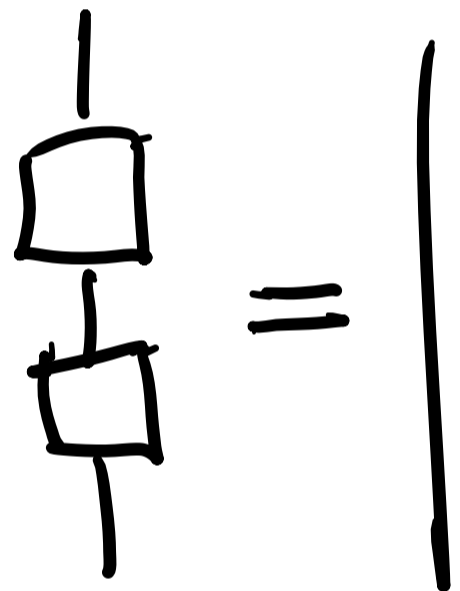
I. Z spiders.  $\alpha := \begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix}$

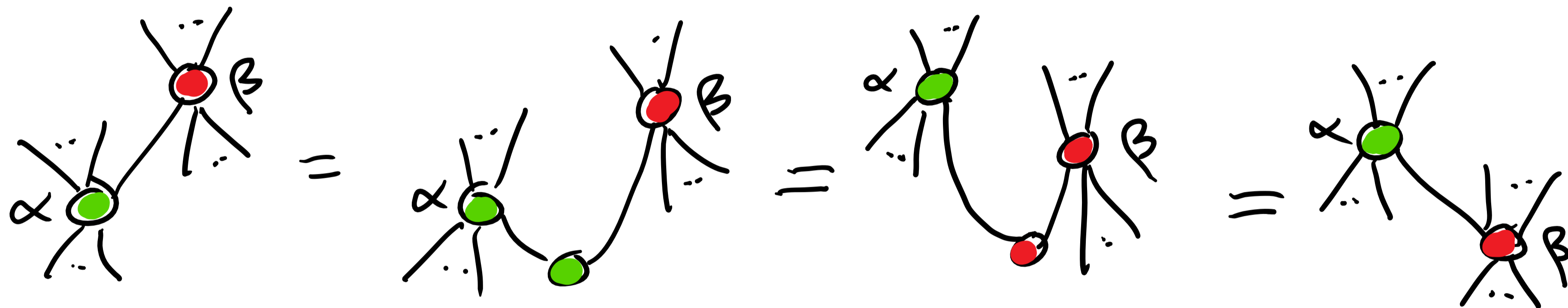
II. X spiders.  $\alpha :=$ 

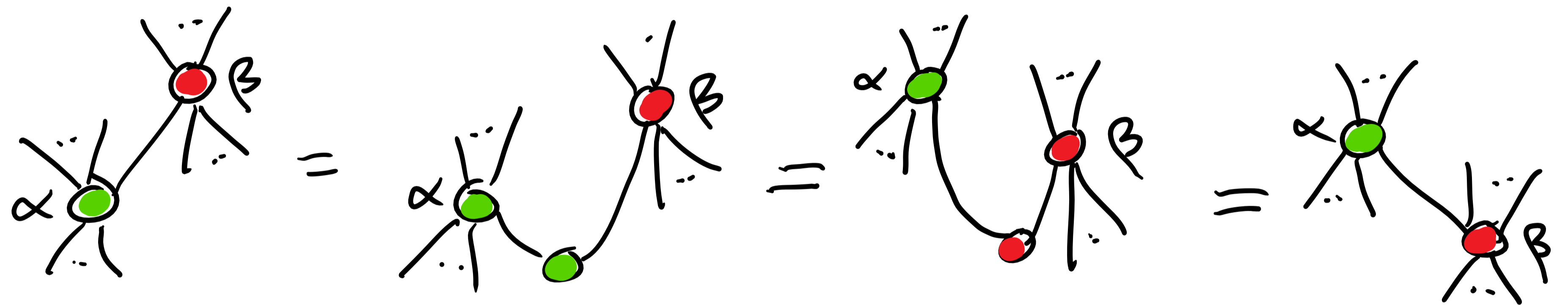
III. identity & swap.

THEOREM (Universality). Any $2^N \times 2^M$ matrix can be expressed as a ZX-diagram:





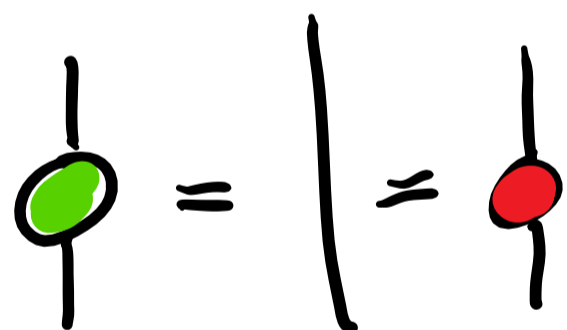
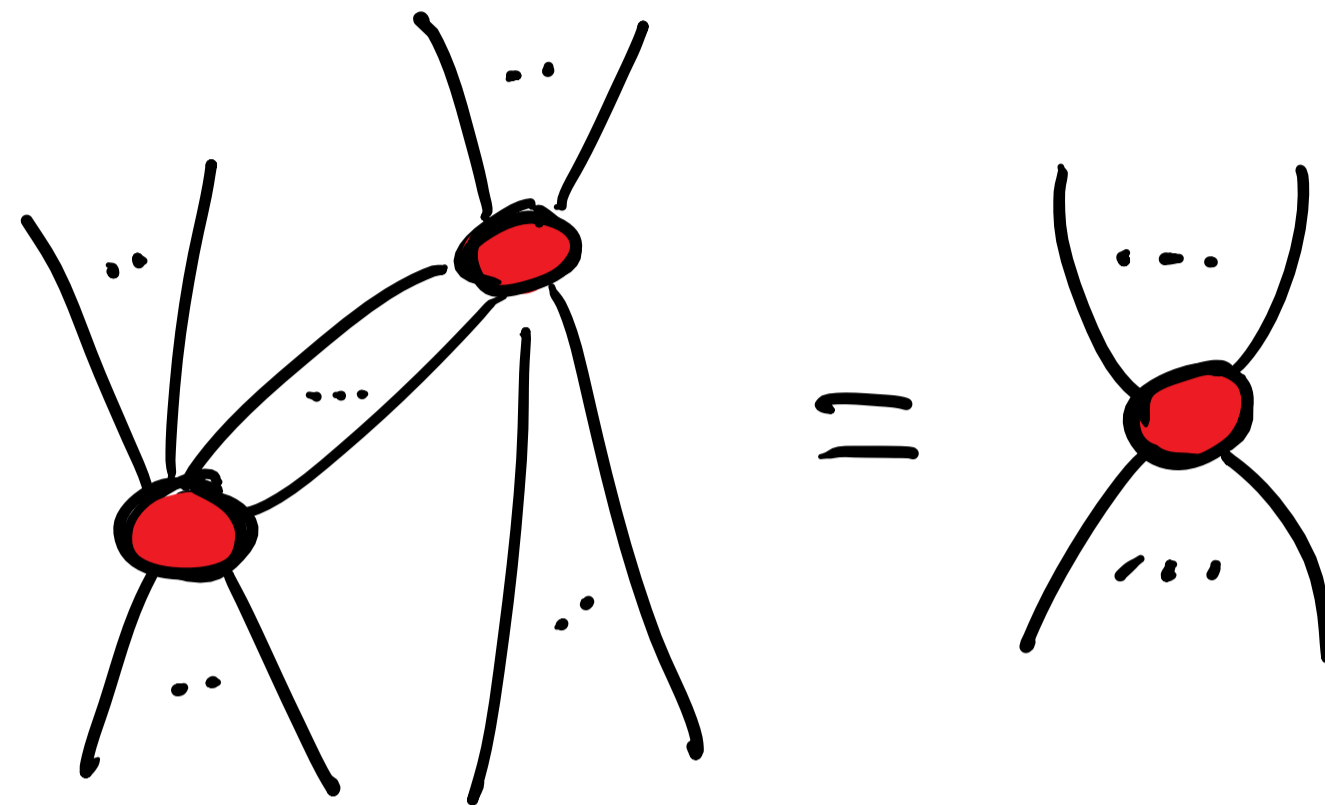
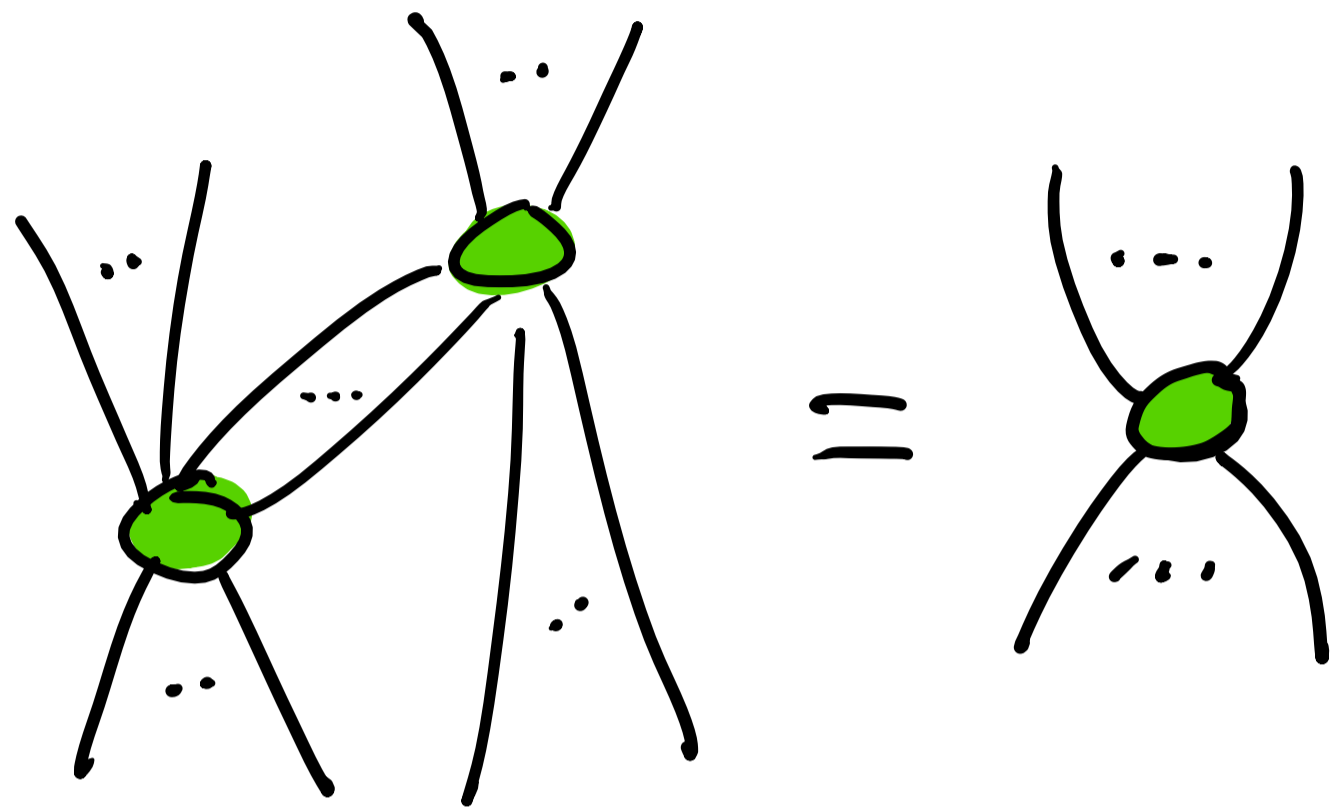




∴ ZX-diagrams are un-directed.

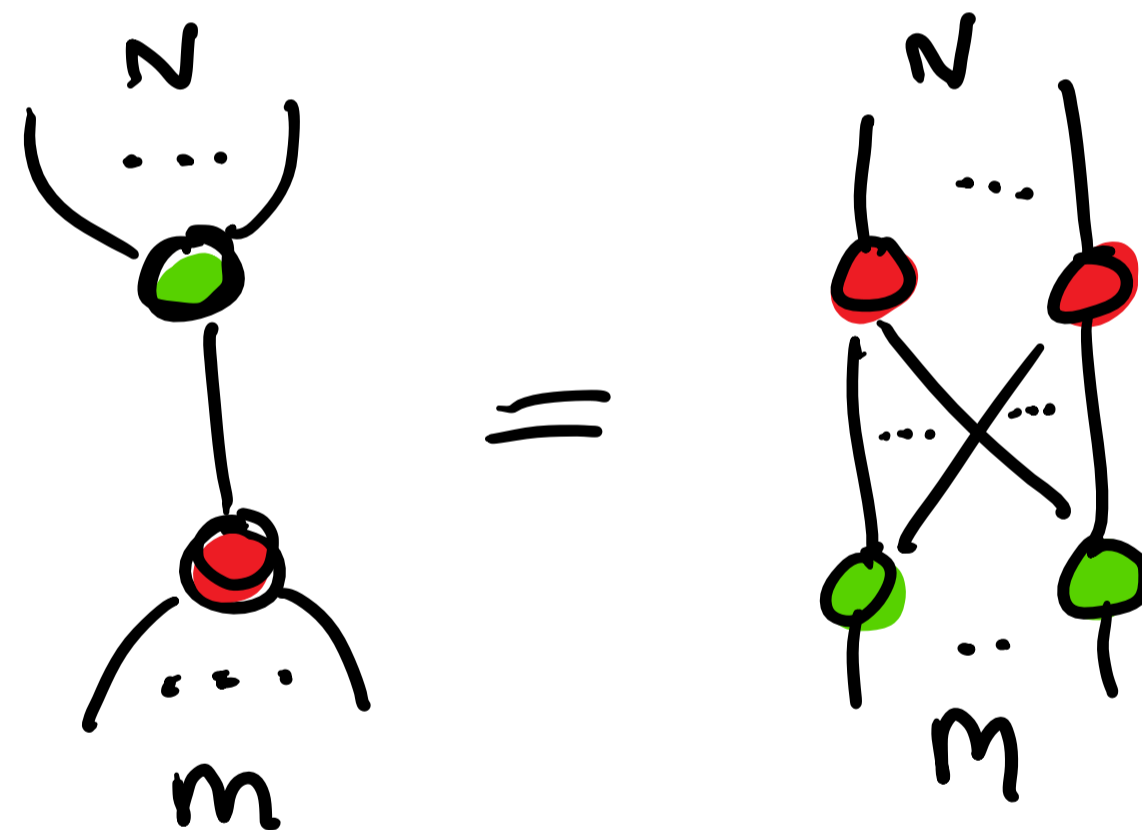
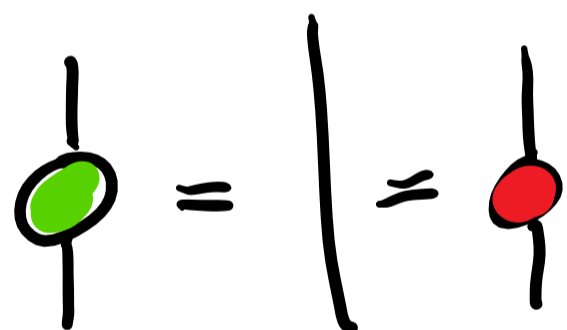
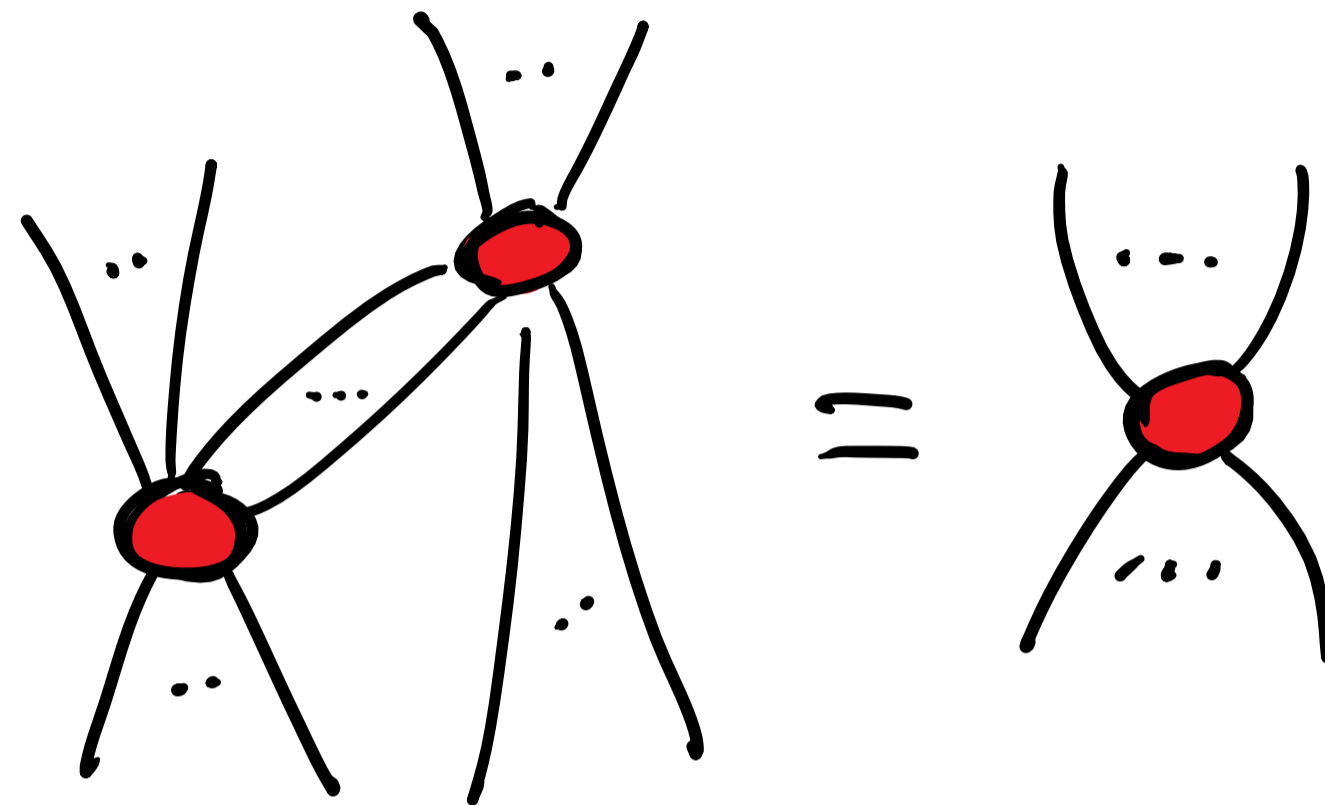
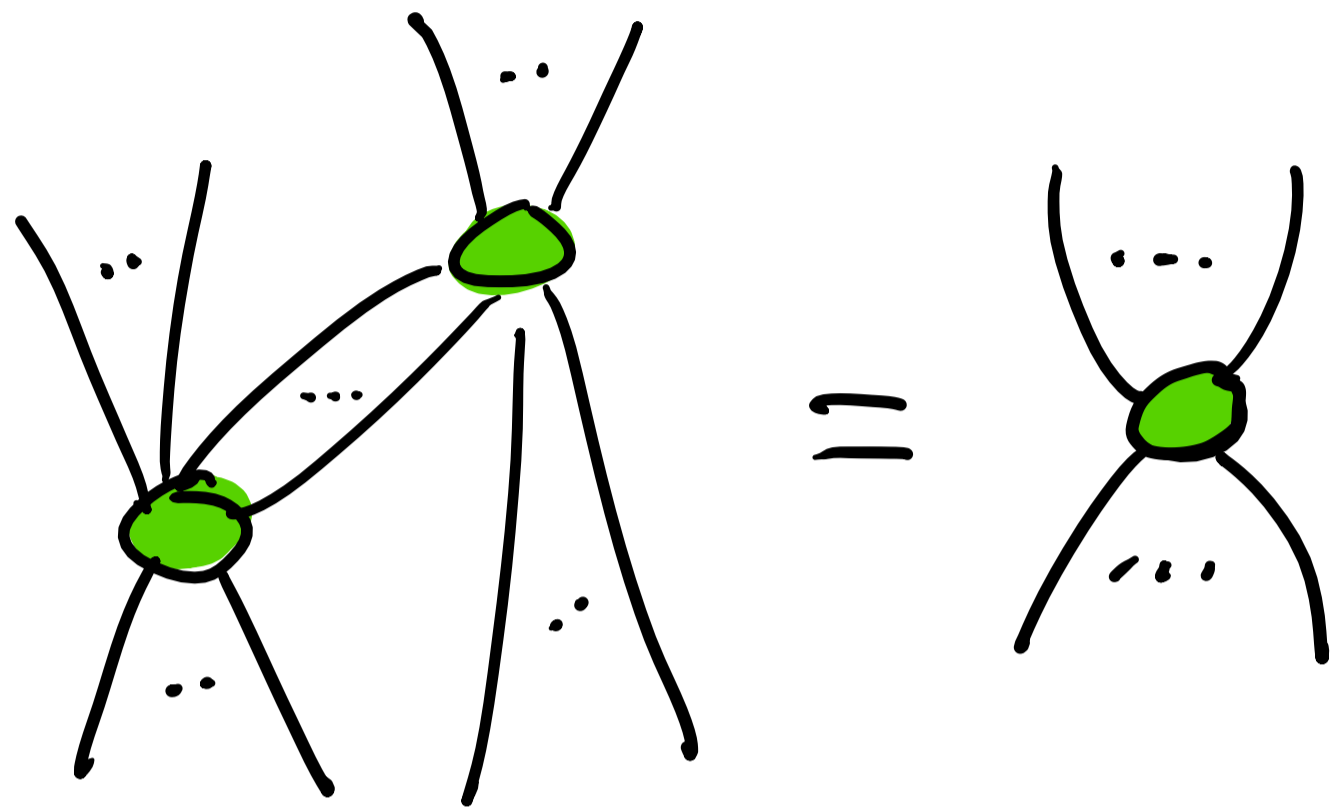
PHASE-FREE

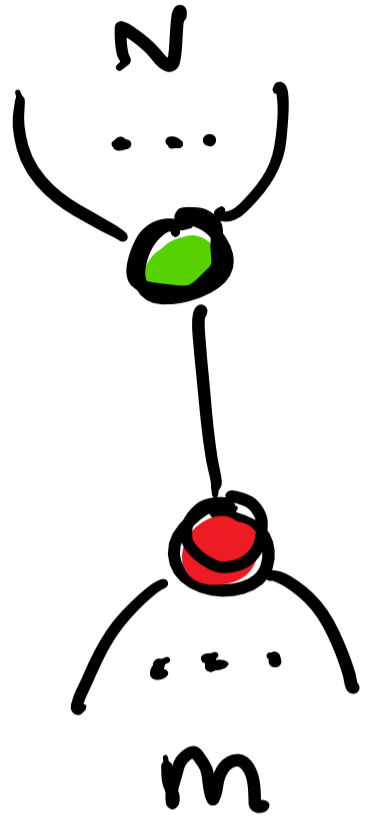
ZX-calculus



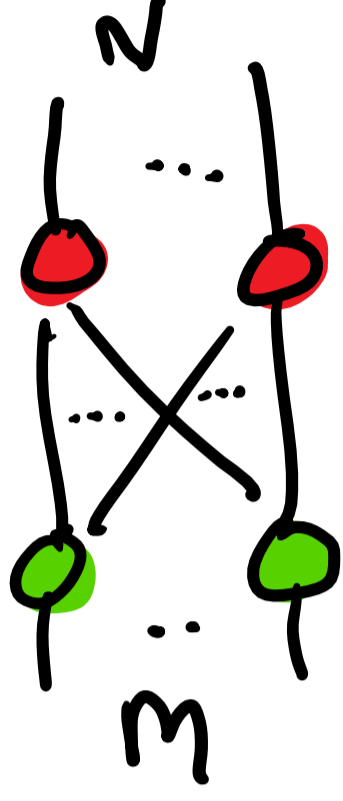
PHASE-FREE

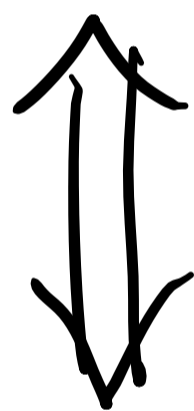
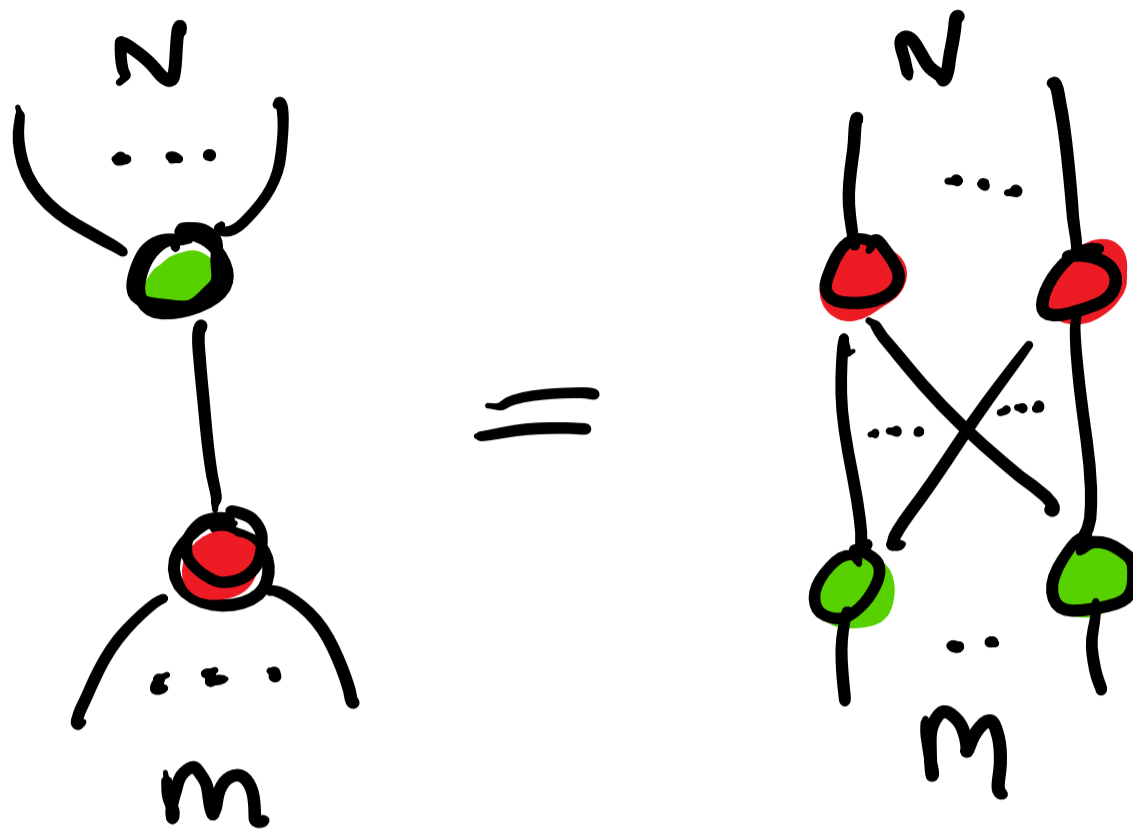
ZX-calculus





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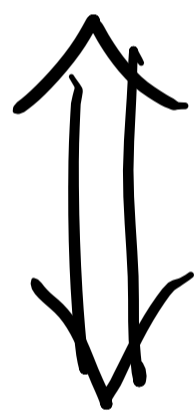
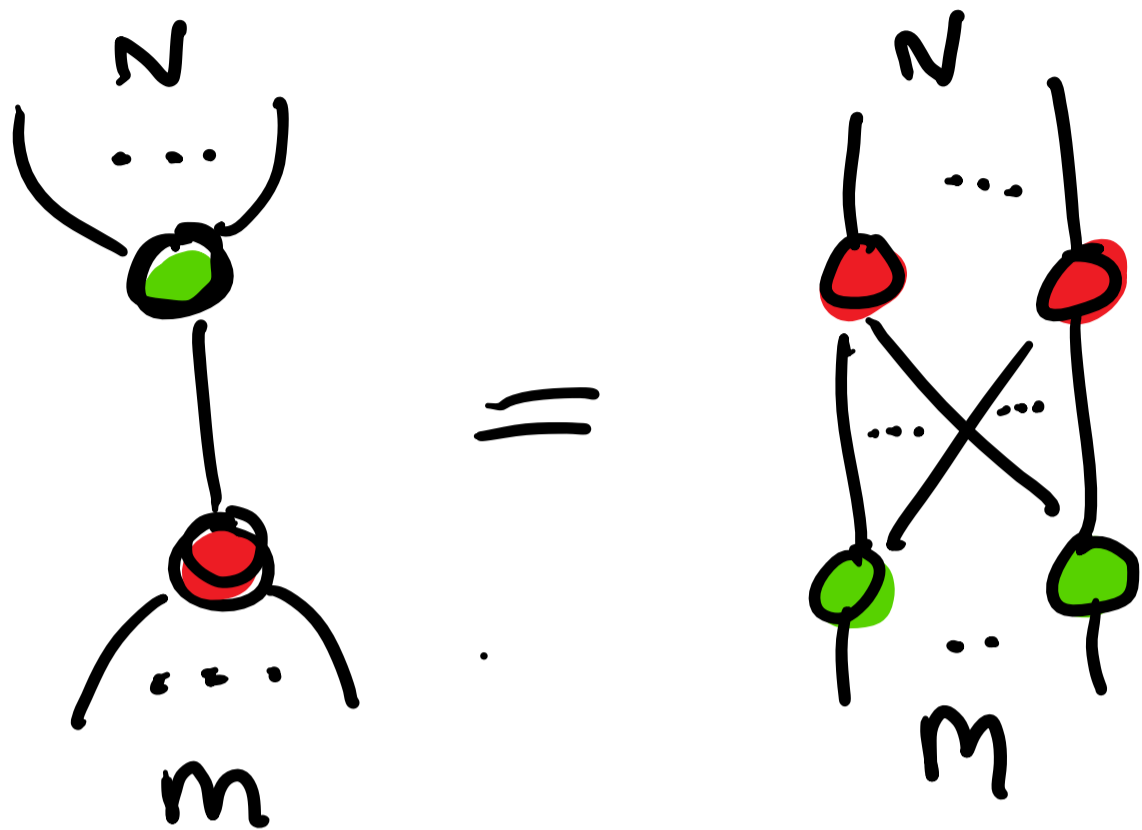


$m=n=0$

$m=0 \quad n=2$

$m=2 \quad n=0$

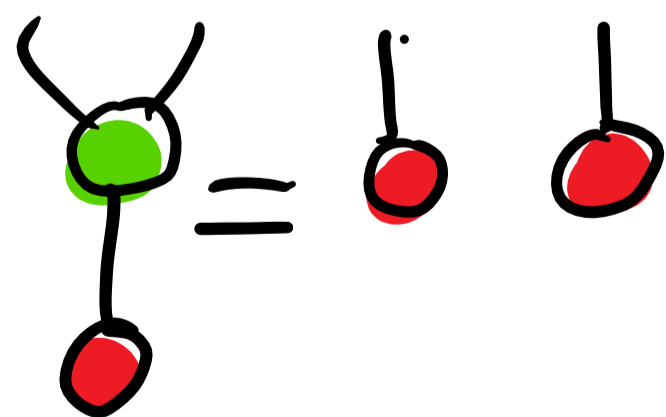
$m=n=2$



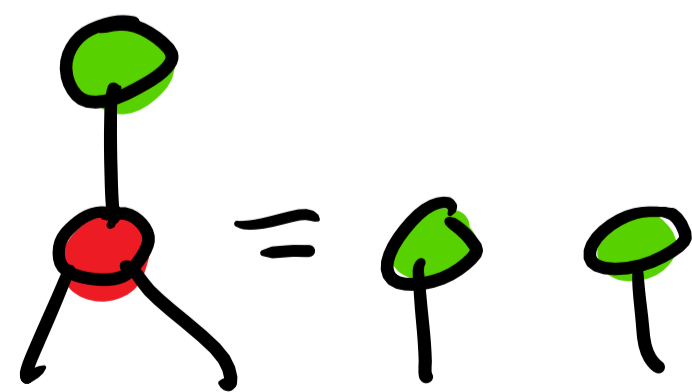
$M=N=0$



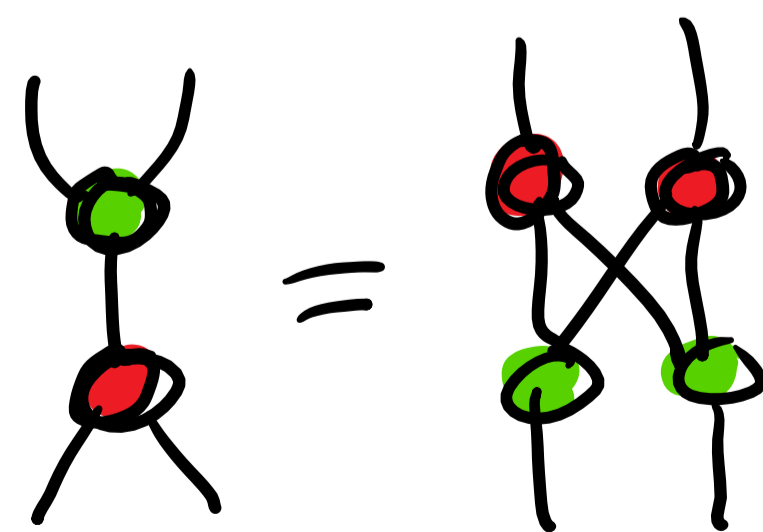
$M=0 \quad N=2$



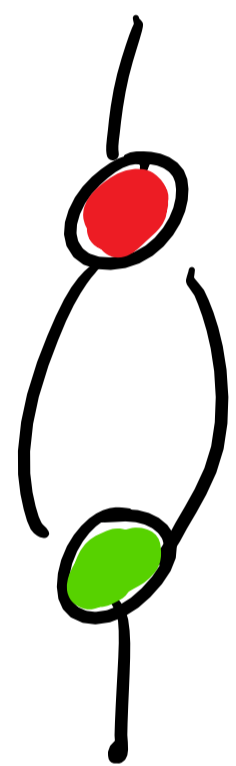
$M=2 \quad N=0$



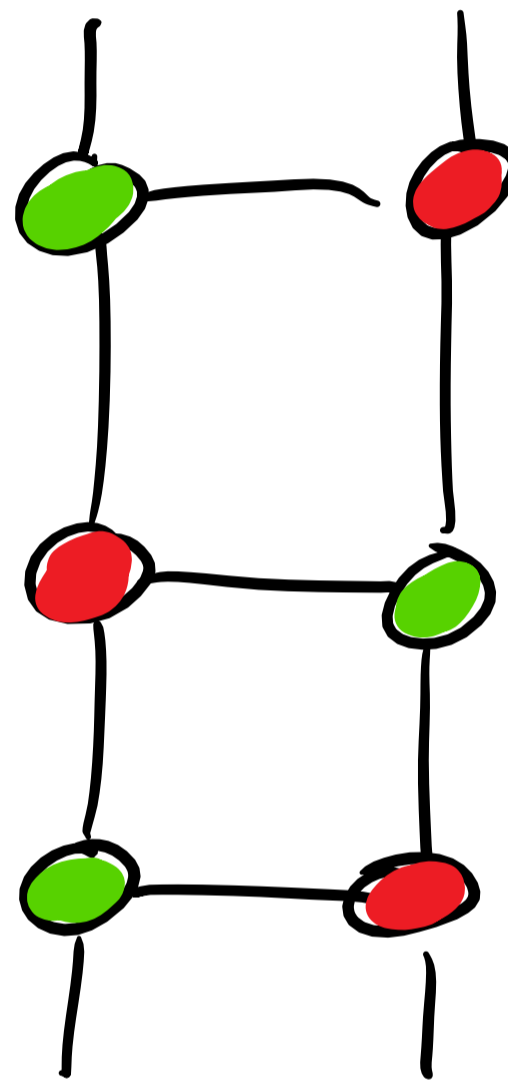
$M=N=2$



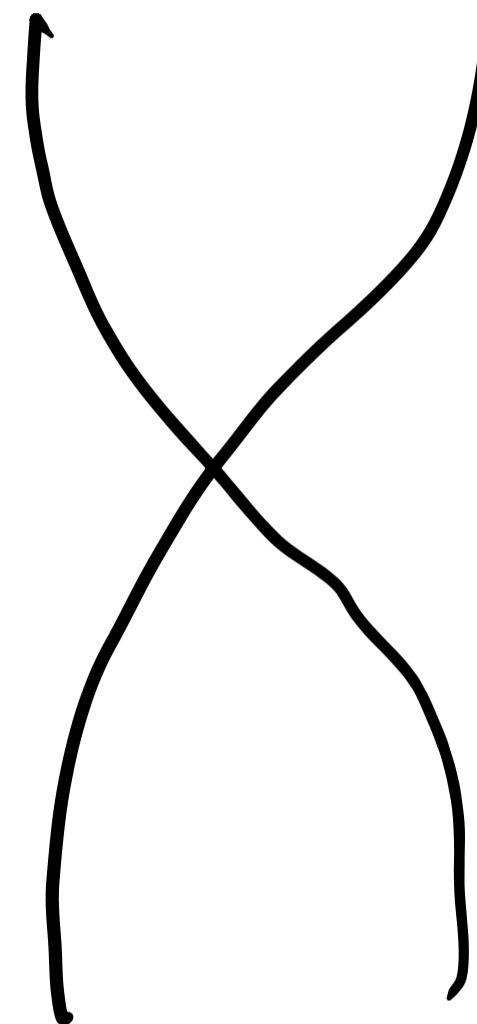
CONSEQUENCES



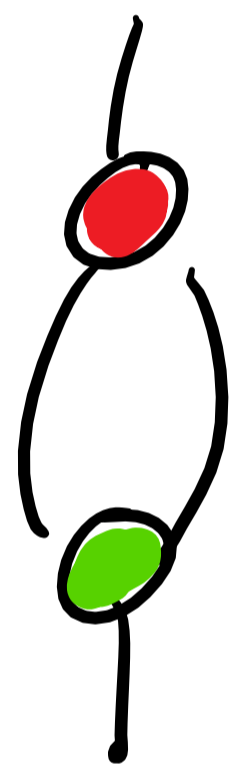
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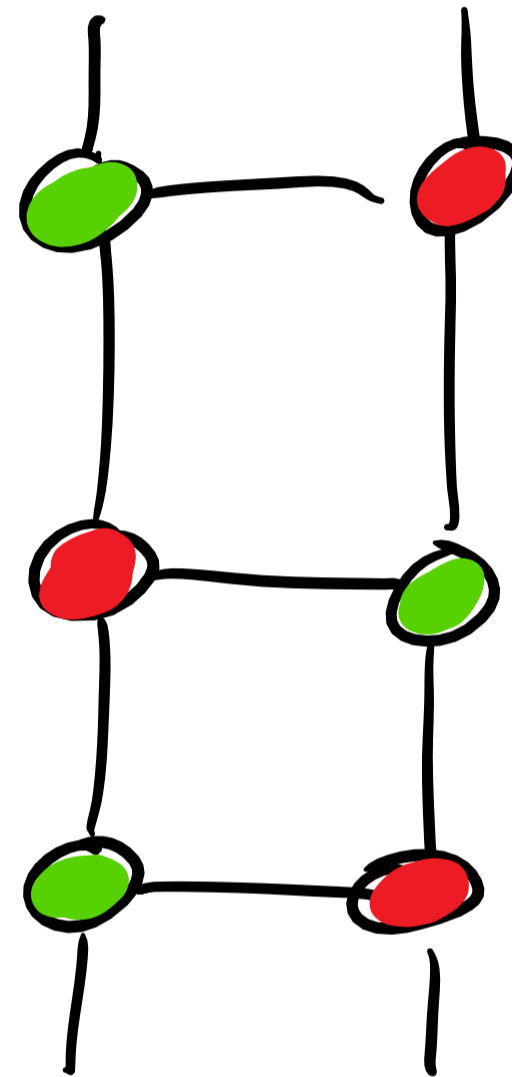
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CONSEQUENCES



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