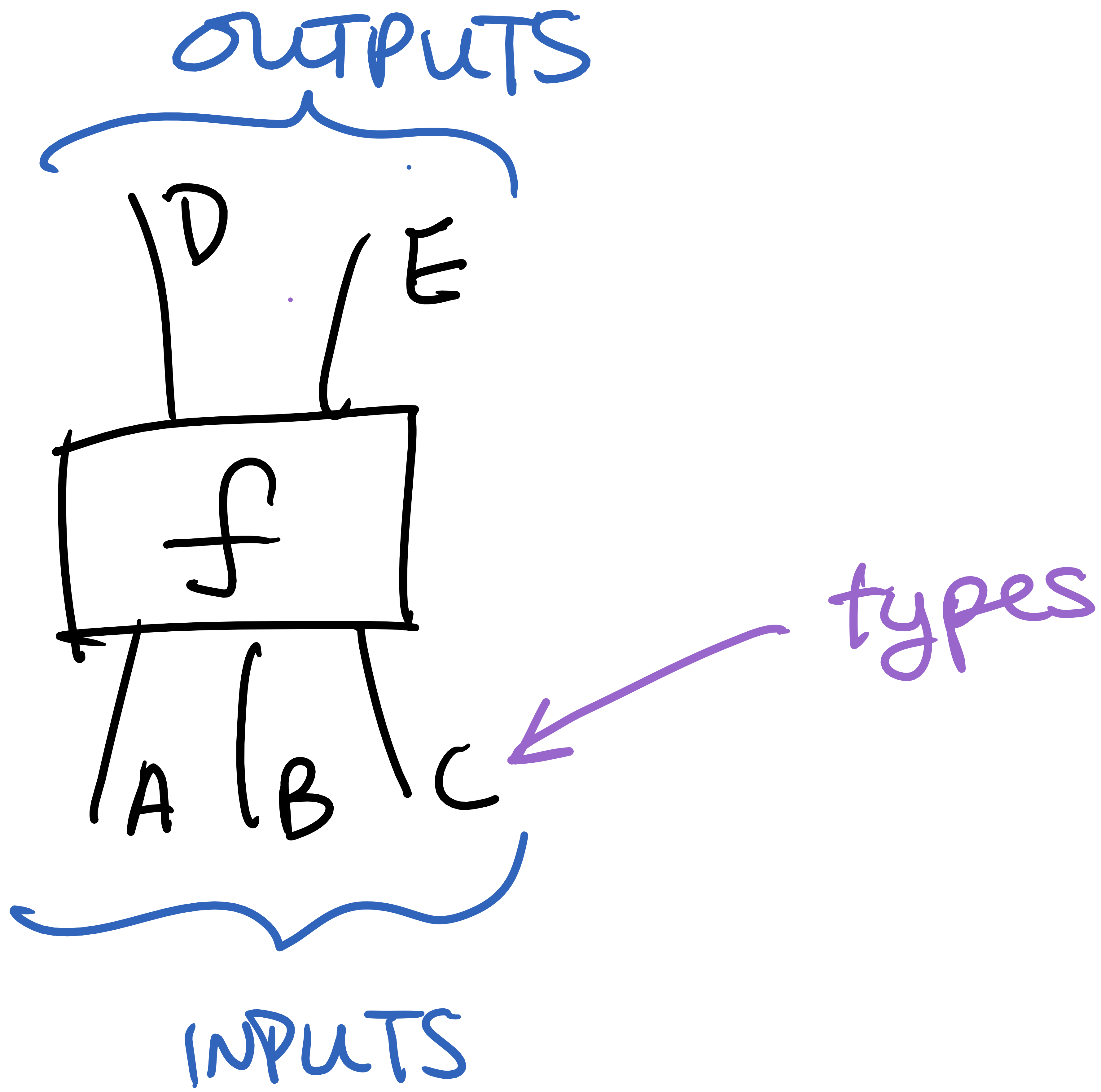
Picturing Quantum Processes I

Aleks Kissinger

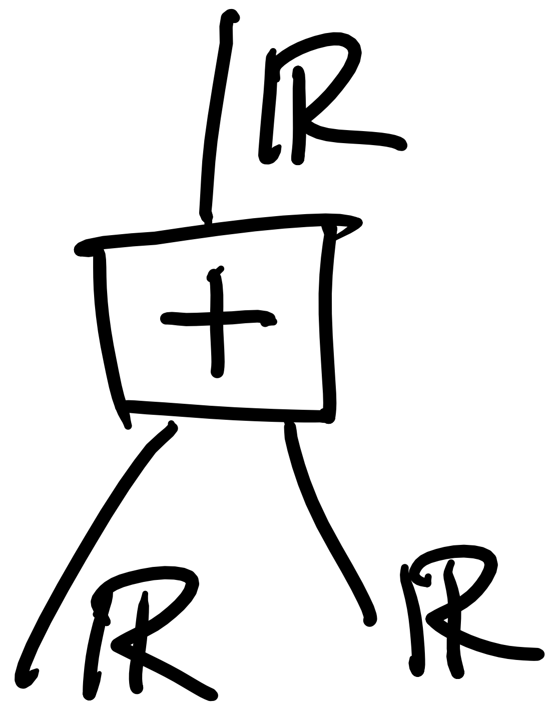
ISR 2019

Paris

PROCESSES



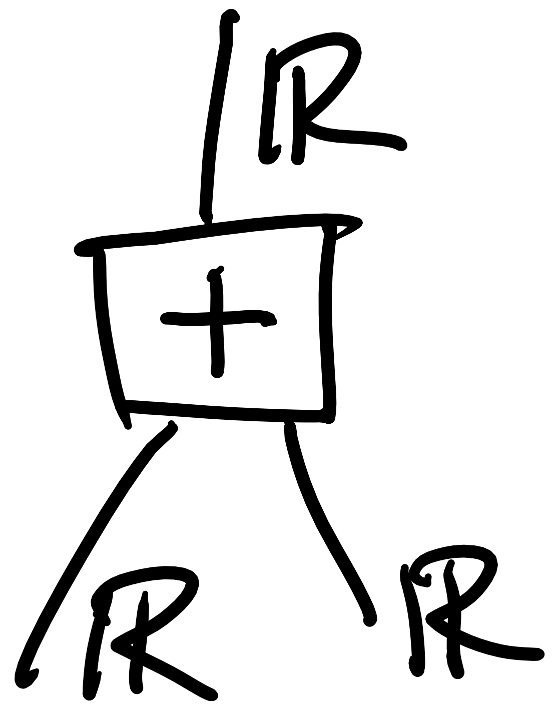
PROCESSES, e.g.

functions $\longrightarrow$ 

$$\text{“}+\text{”} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Processes, e.g.

functions →



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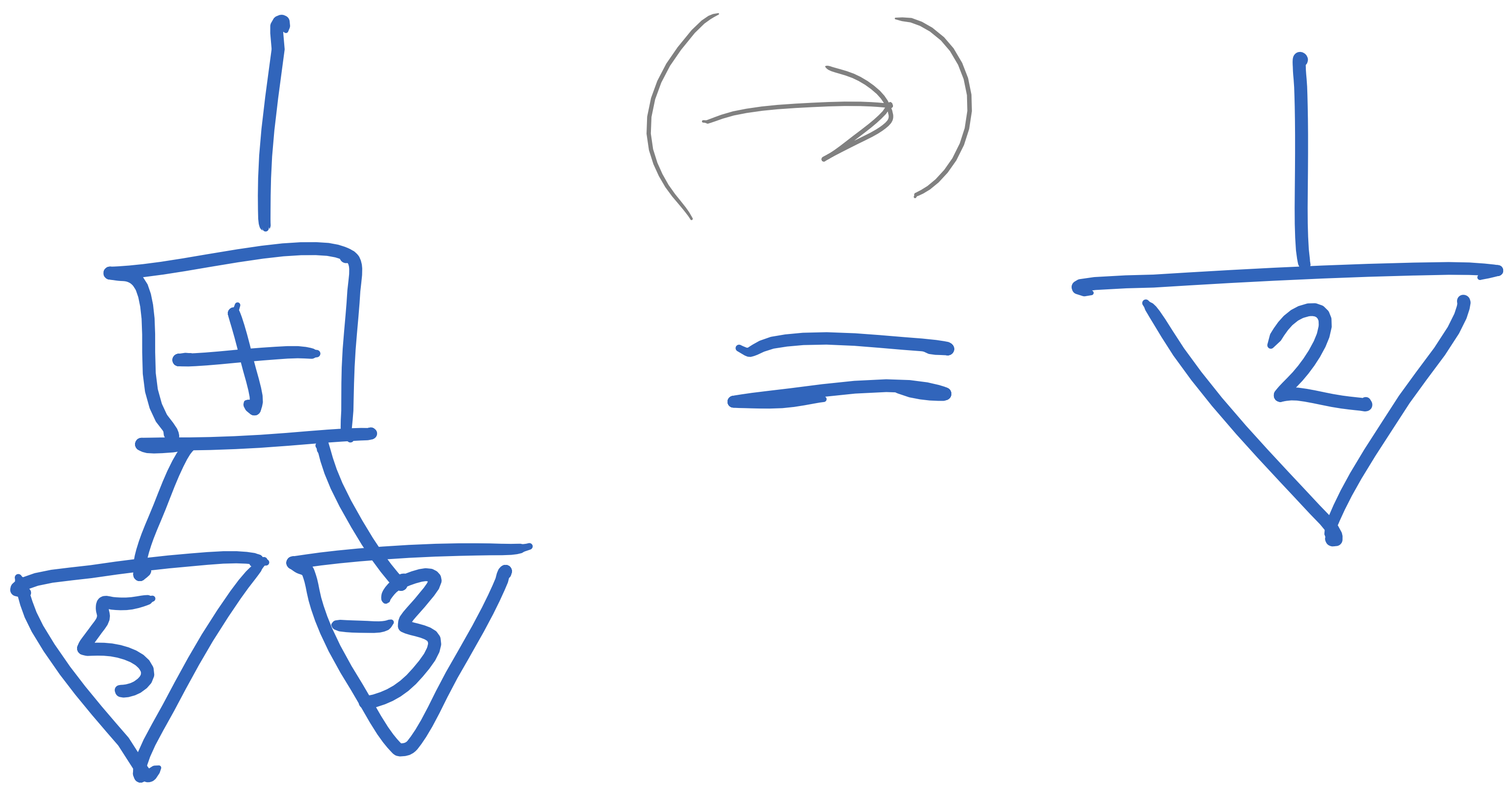
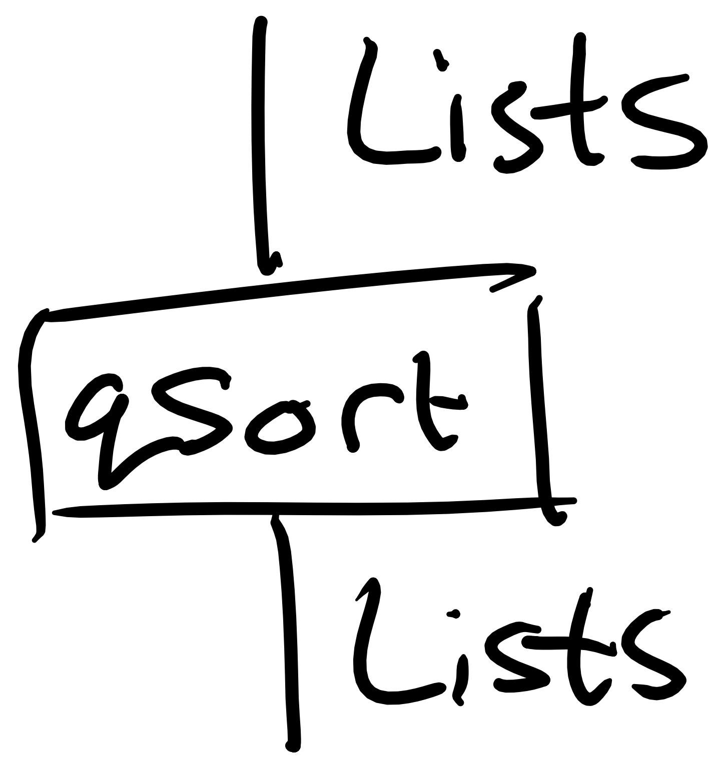
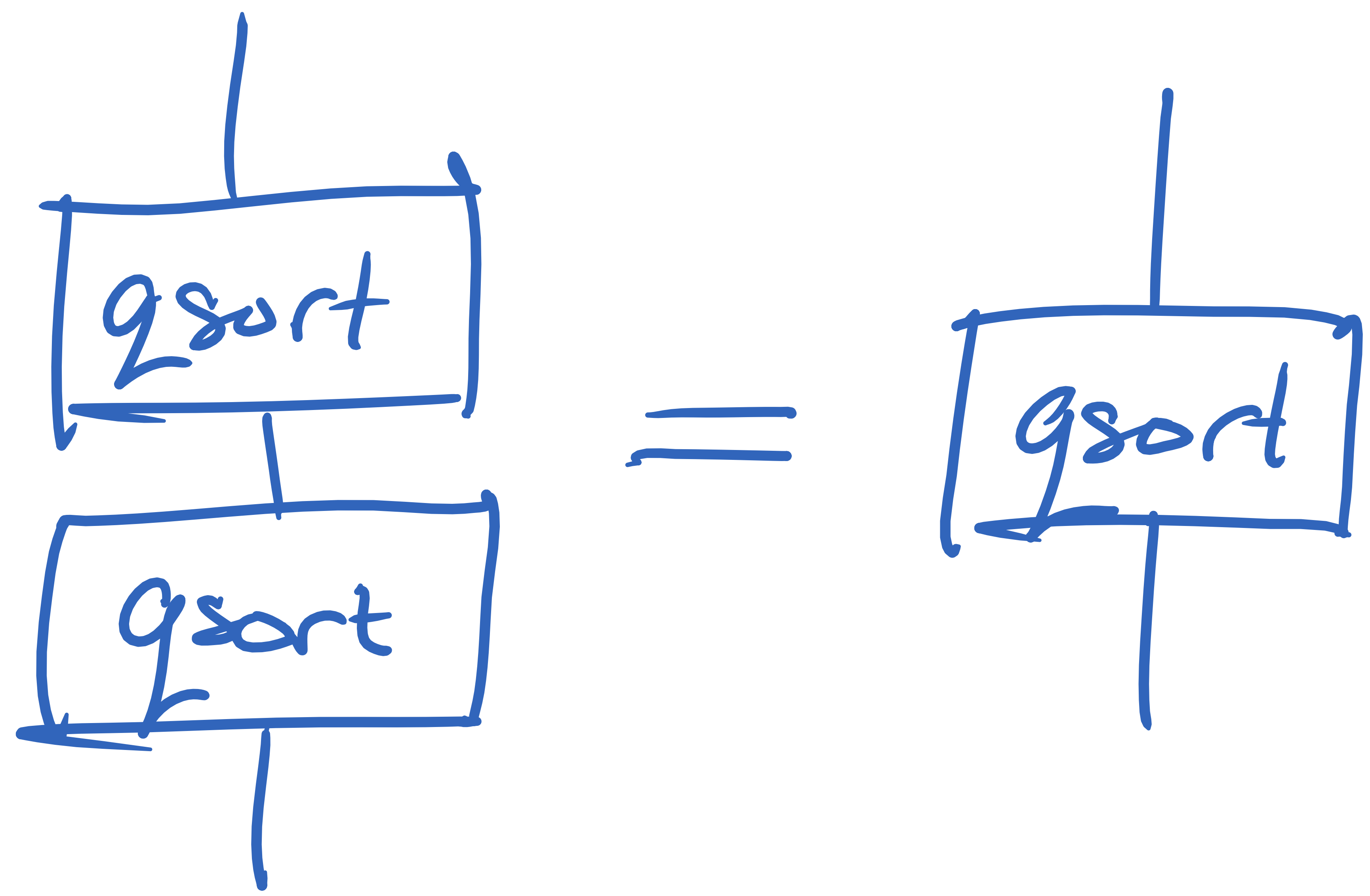


PROCESSES, e.g.

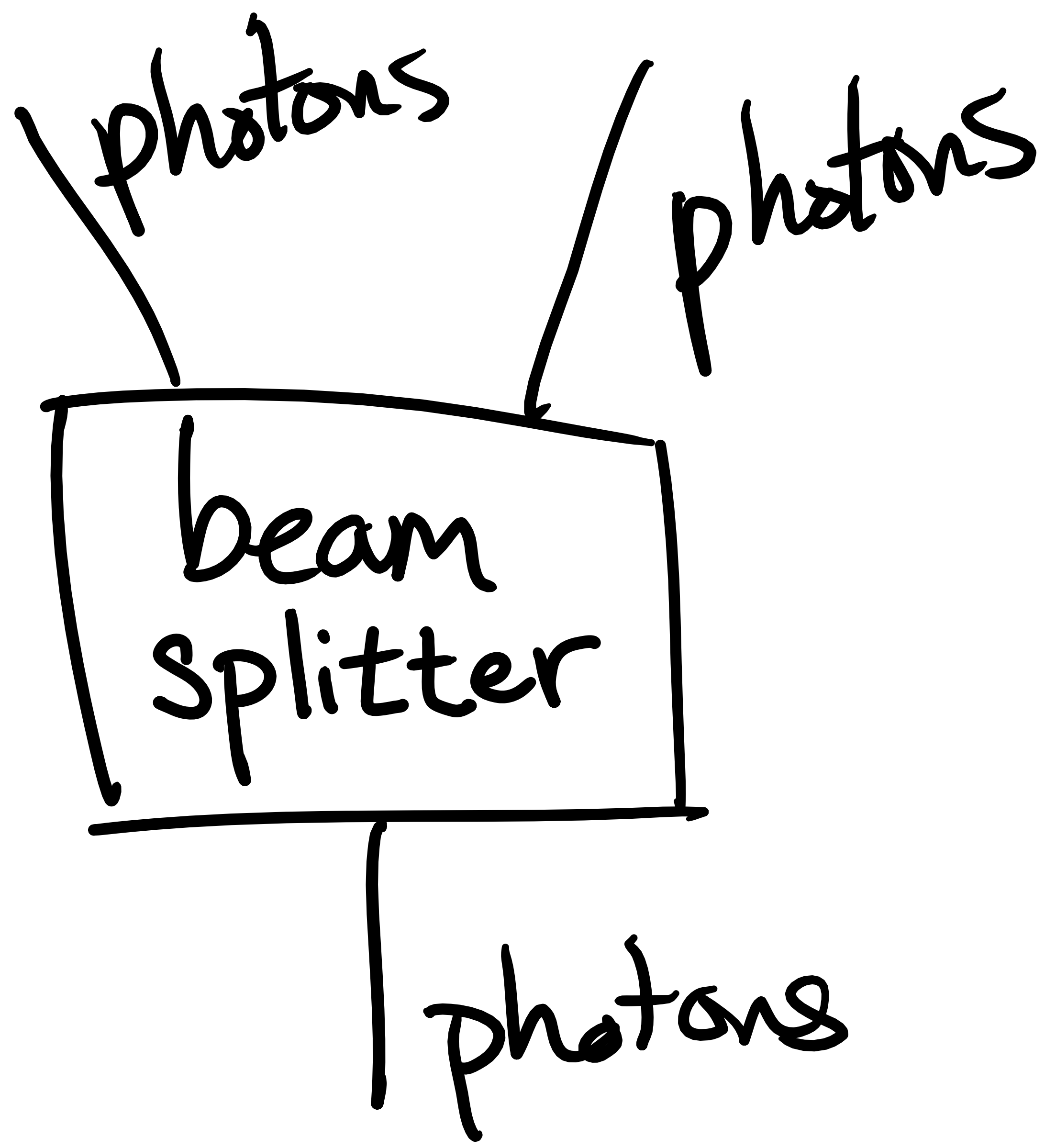
programs →





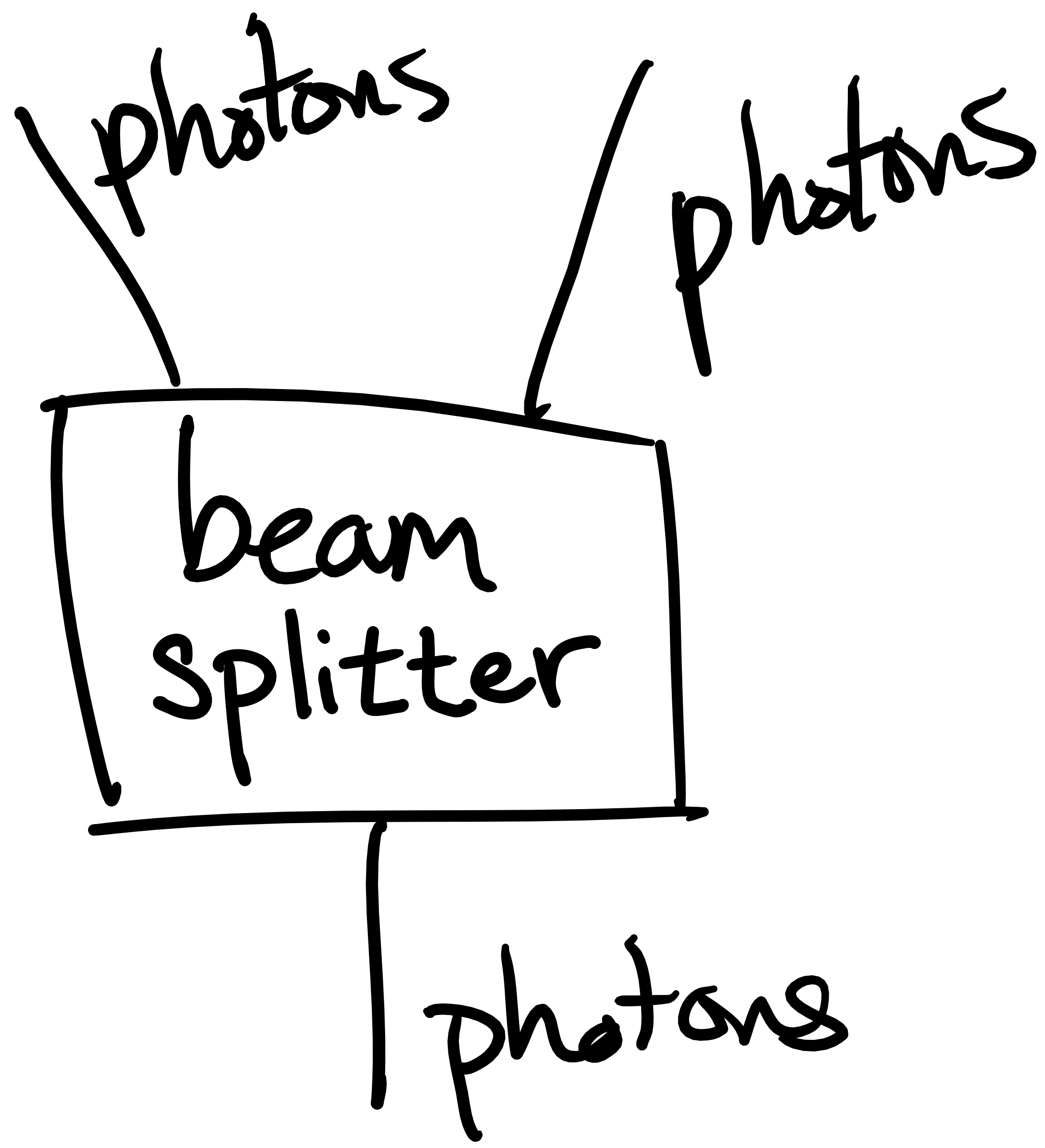
PROCESSES, e.g.

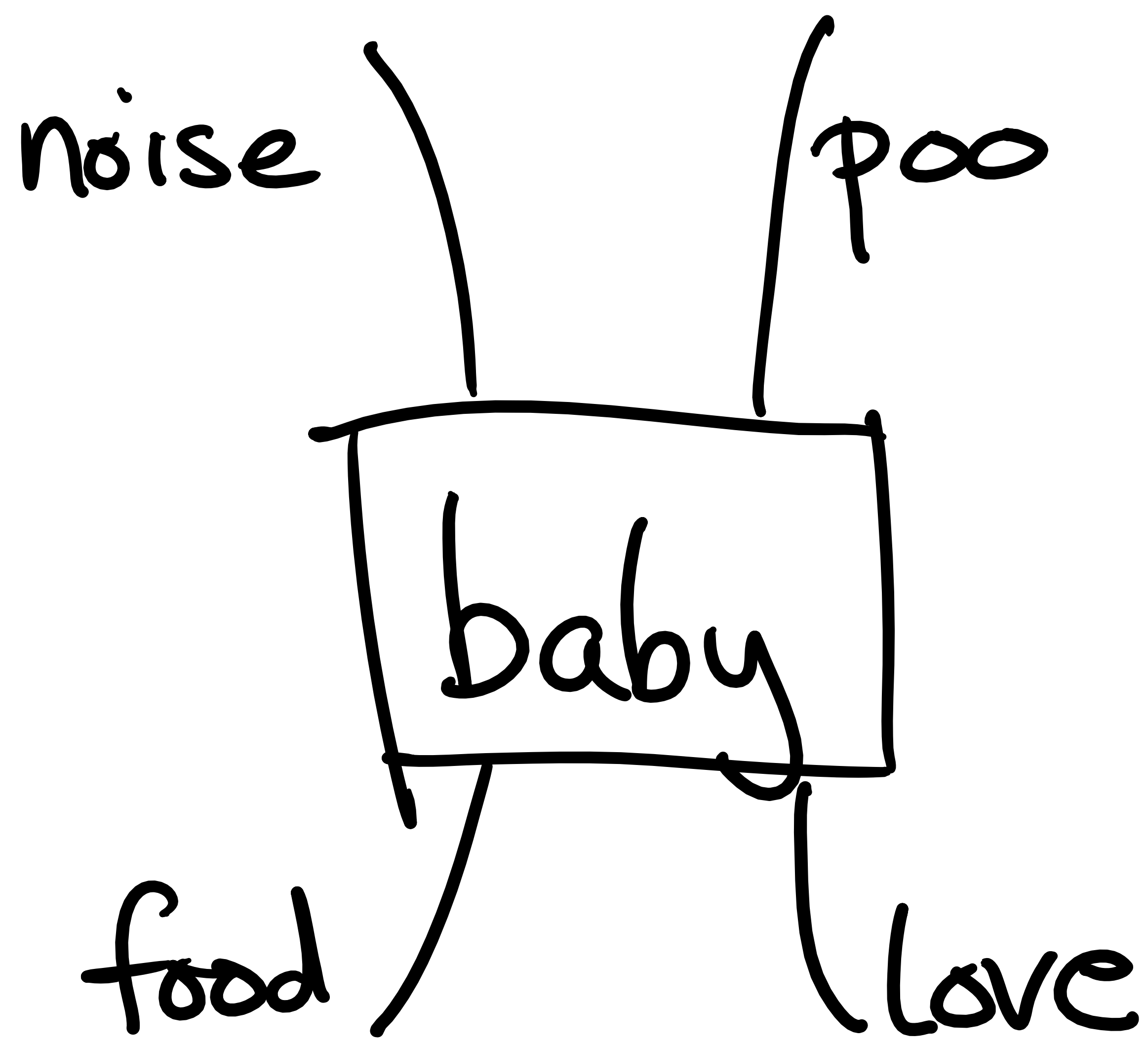
stuff ⟶



PROCESSES, e.g.

stuff ⟶





Process Theories

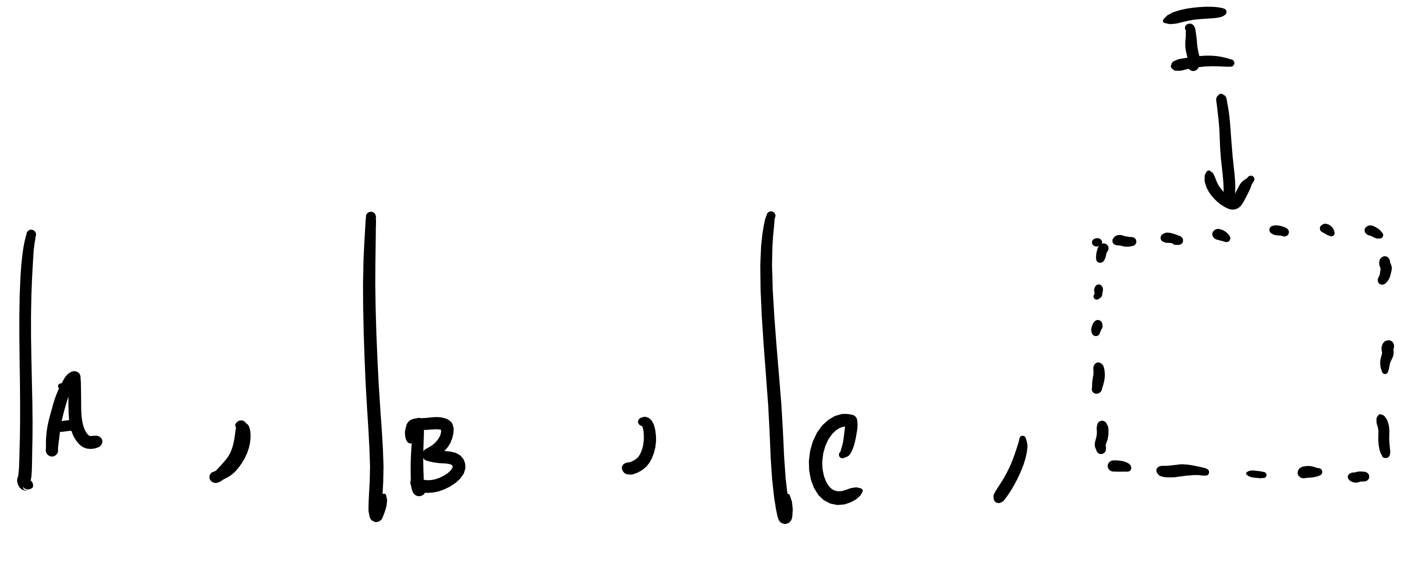
PROCESS THEORIES

:=

"a collection of processes that makes sense to plug together"

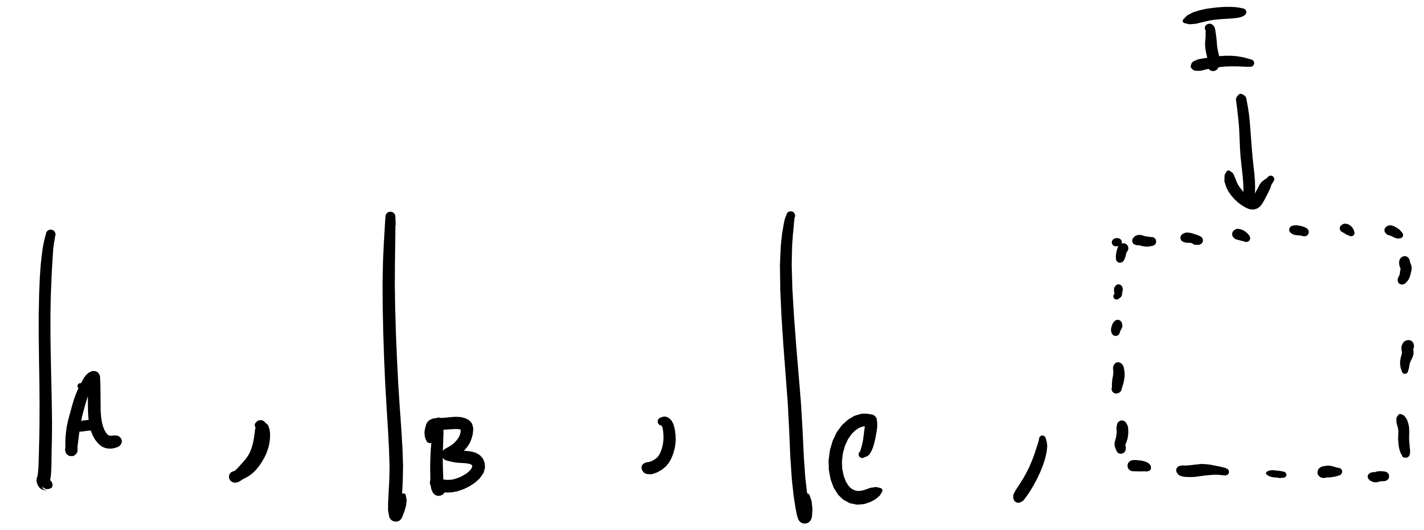
PROCESS THEORIES

- [redacted]
- [redacted]

1. a collection of types 

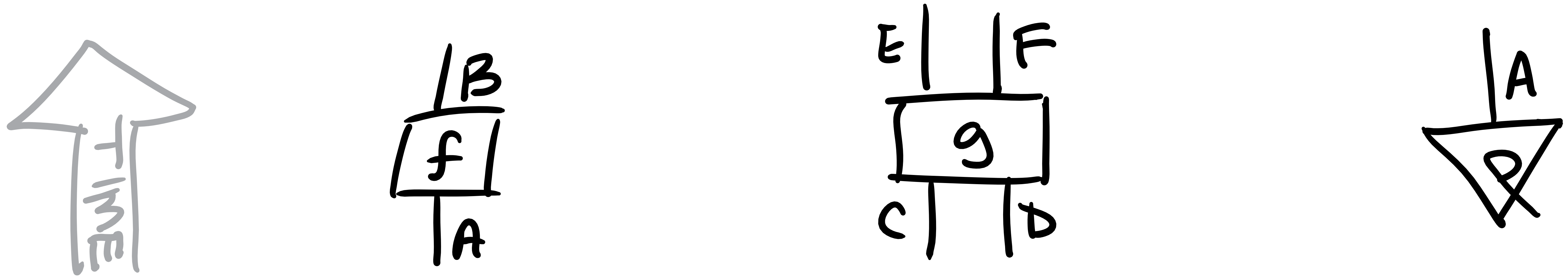
PROCESS THEORIES

1. a collection of types

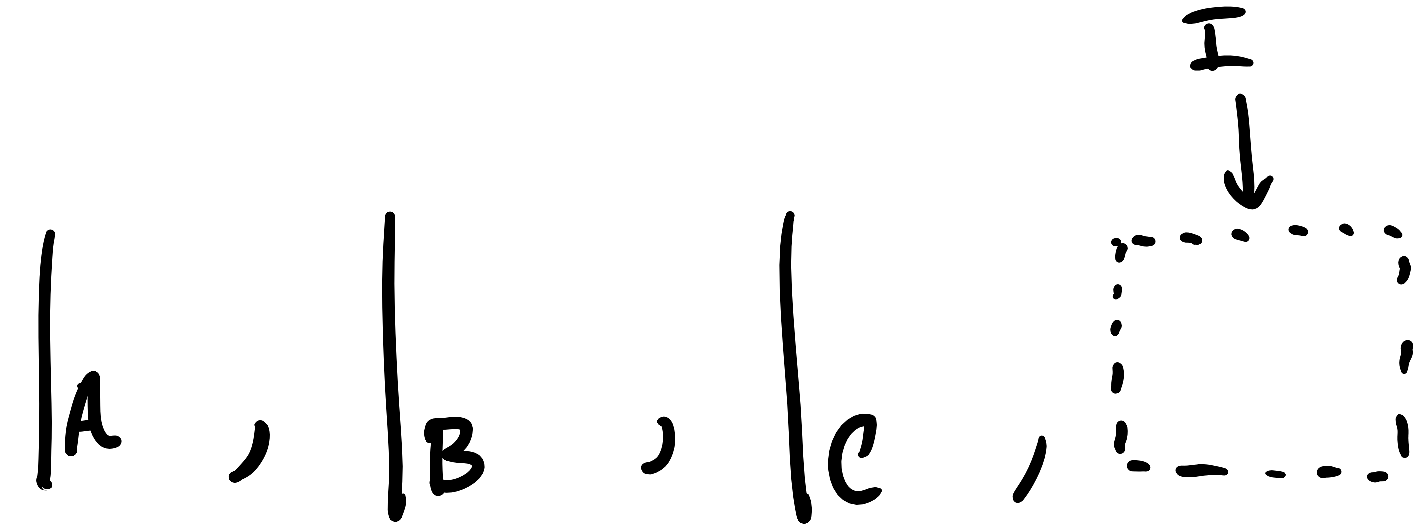


2. a collection of processes:

$f : A \to B$ $\quad$ $g : C \otimes D \to E \otimes F$ $\quad$ $p : I \to A$



PROCESS THEORIES

1. a collection of types 

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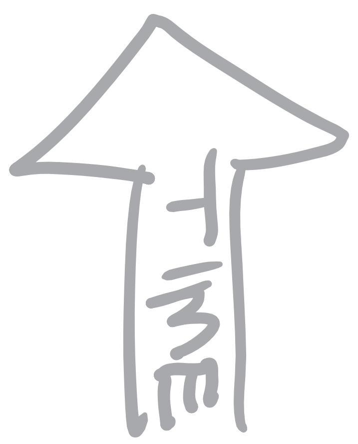
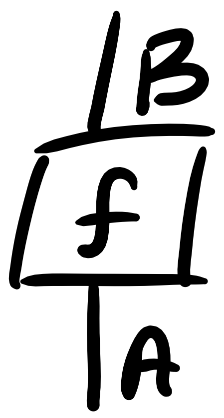
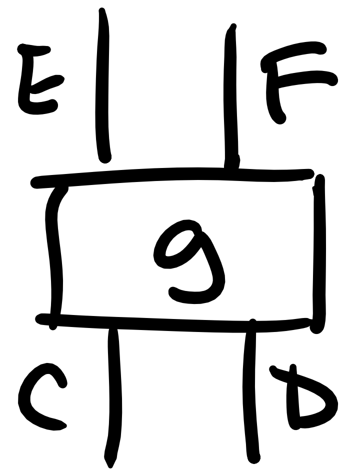
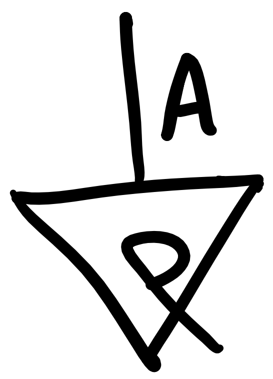




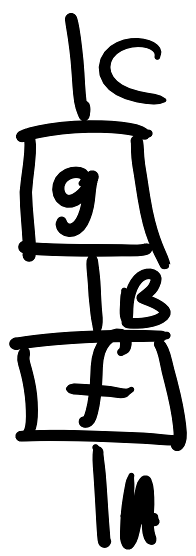


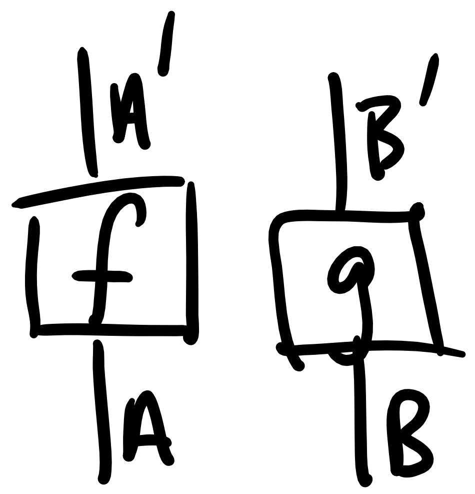


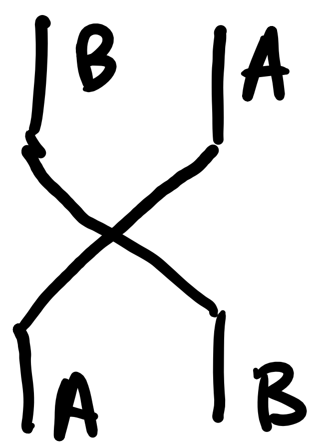
STATES

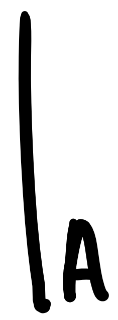
PROCESS THEORIES

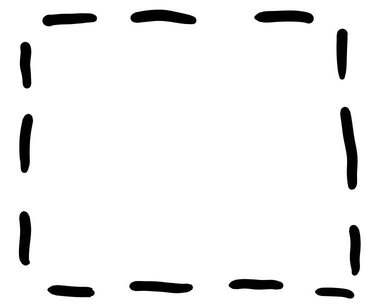
3. operations for building diagrams:

$g \circ f \Rightarrow$ 

$f \otimes g \Rightarrow$ 

$\sigma_{A,B} \Rightarrow$ 

$1_A \Rightarrow$ 

$1_I \Rightarrow$ 

PROCESS THEORIES

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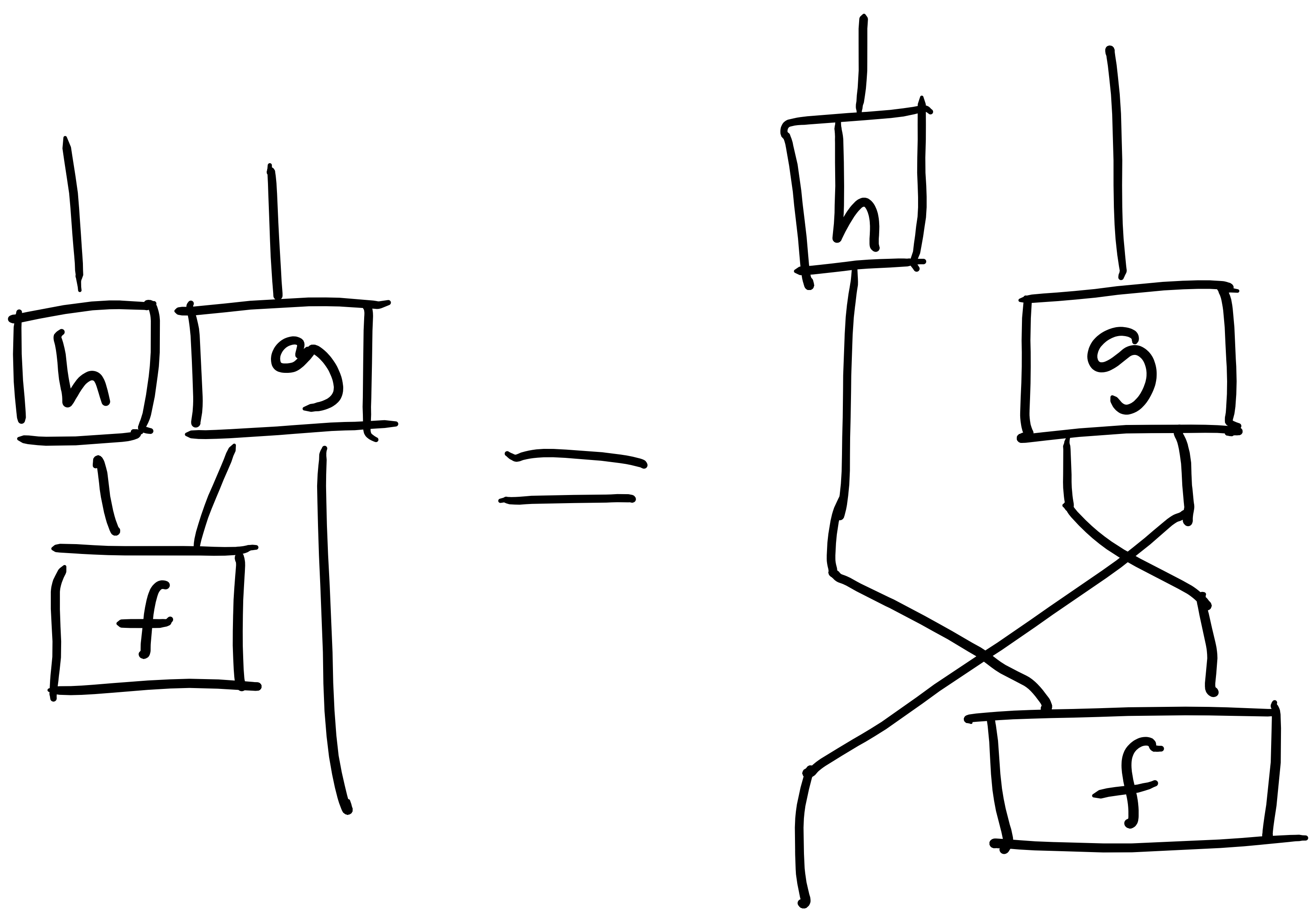
- [redacted]

4. Some rules (only connectivity matters)



Ex: <u>functions</u>

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types: Sets $A, B, C, \ldots$

Ex: functions

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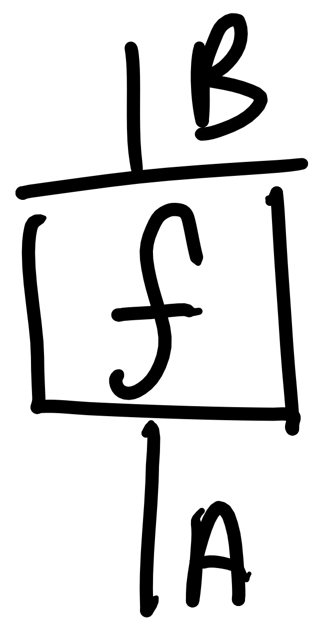
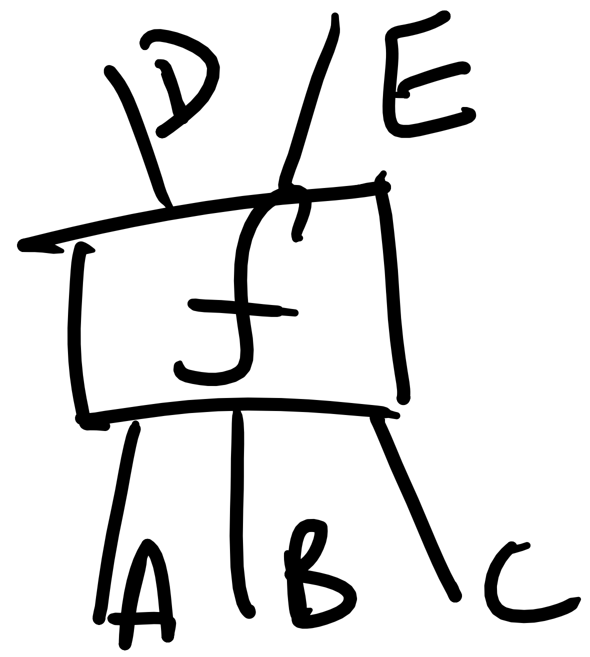
processes: functions  $:= f : A \to B$

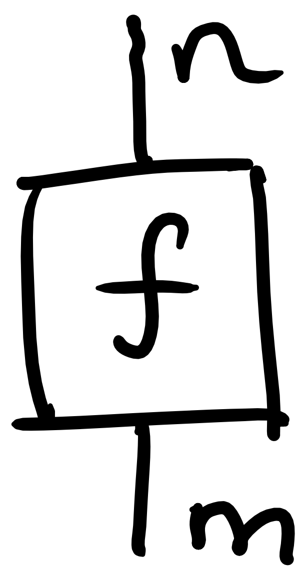
Ex: functions

types: Sets $A, B, C, \ldots$

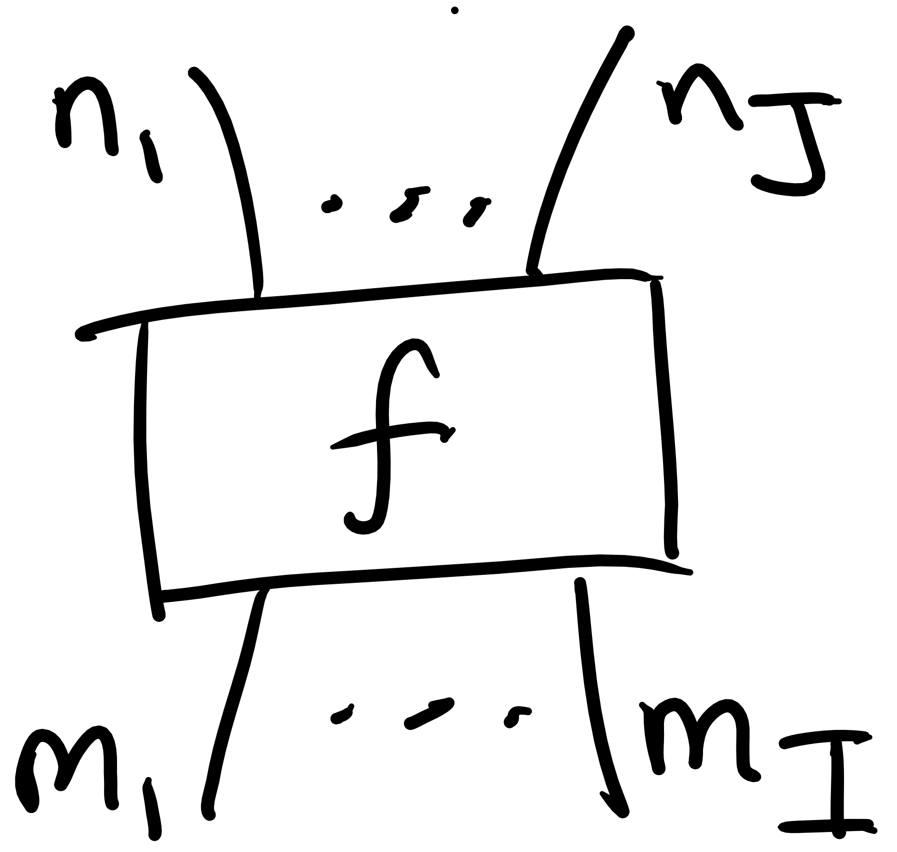
processes: functions  $:= f : A \times B \times C \to D \times E$

Ex: <u>matrices</u>

types: $n \in \mathbb{N}$ (dimension)

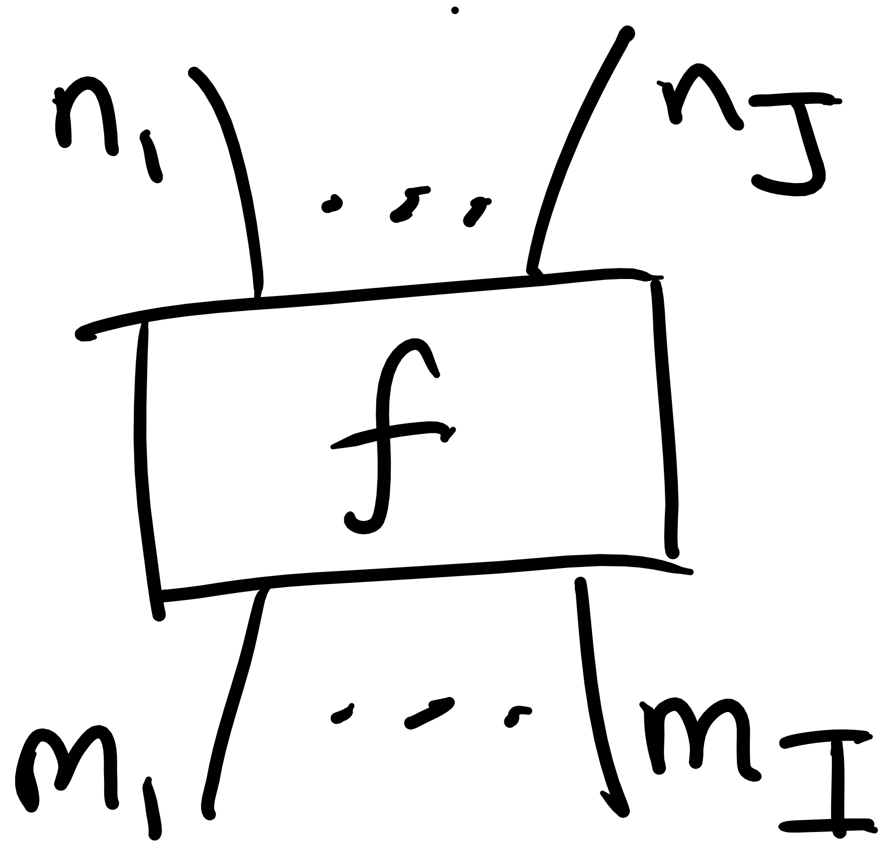
processes:  $n \times m$ matrices $\begin{pmatrix} f_{11} & \cdots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nm} \end{pmatrix}$

Ex: matrices



$(\prod n_j) \times (\prod m_i)$ matrices

Ex: matrices



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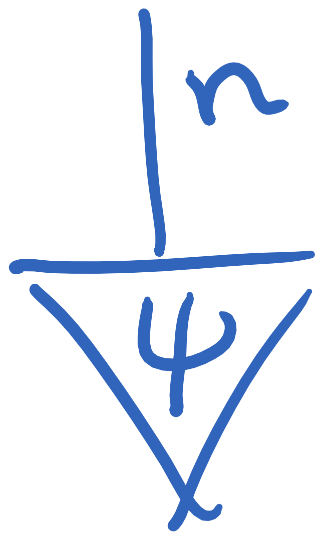
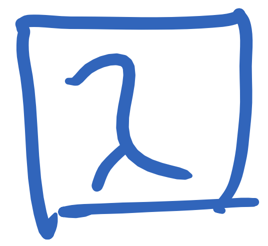


$n \times 1$

column vector

$1 \times m$

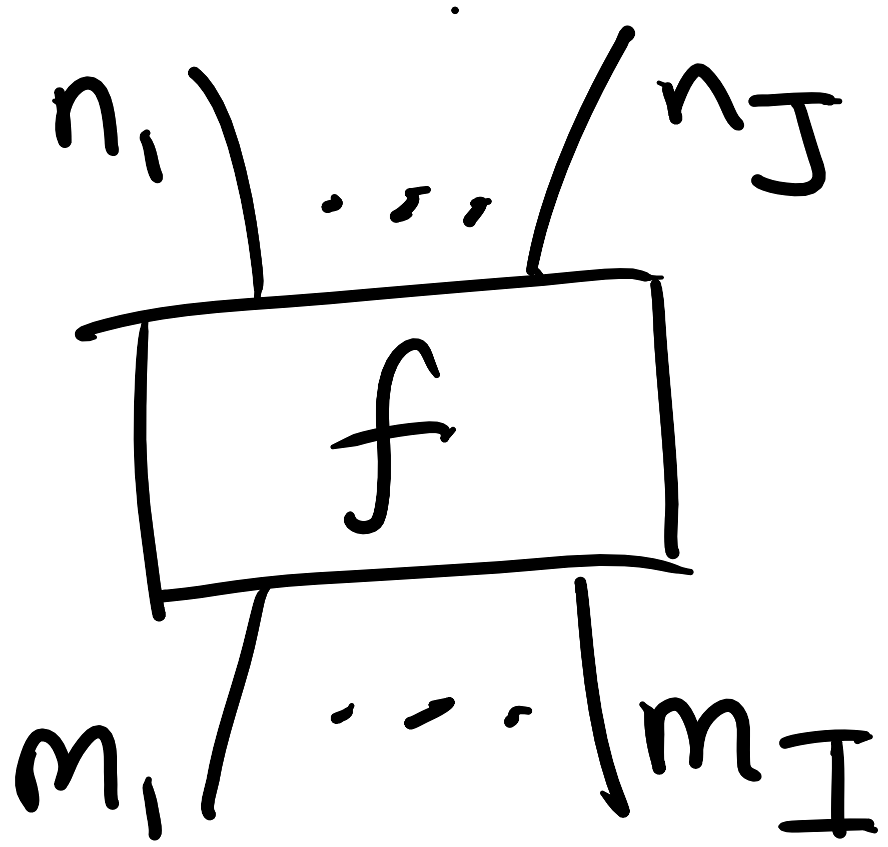
row vector



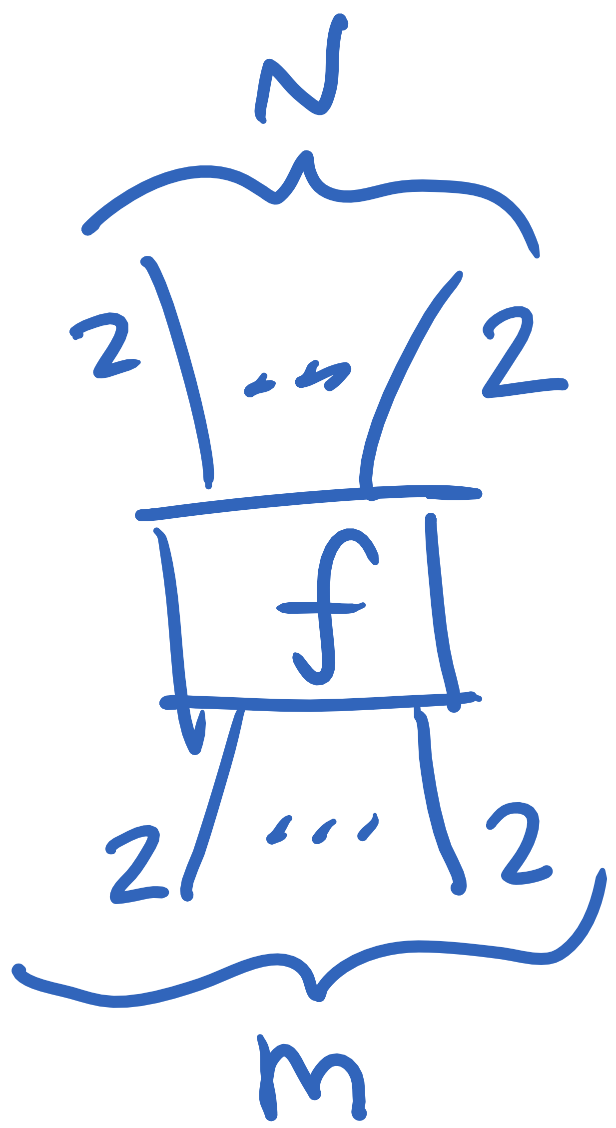
1×1

scalar

Ex: matrices

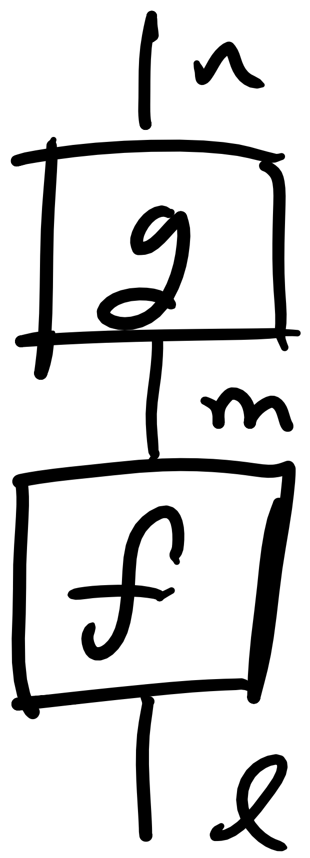


$\left(\prod n_j\right) \times \left(\prod m_i\right)$ matrices



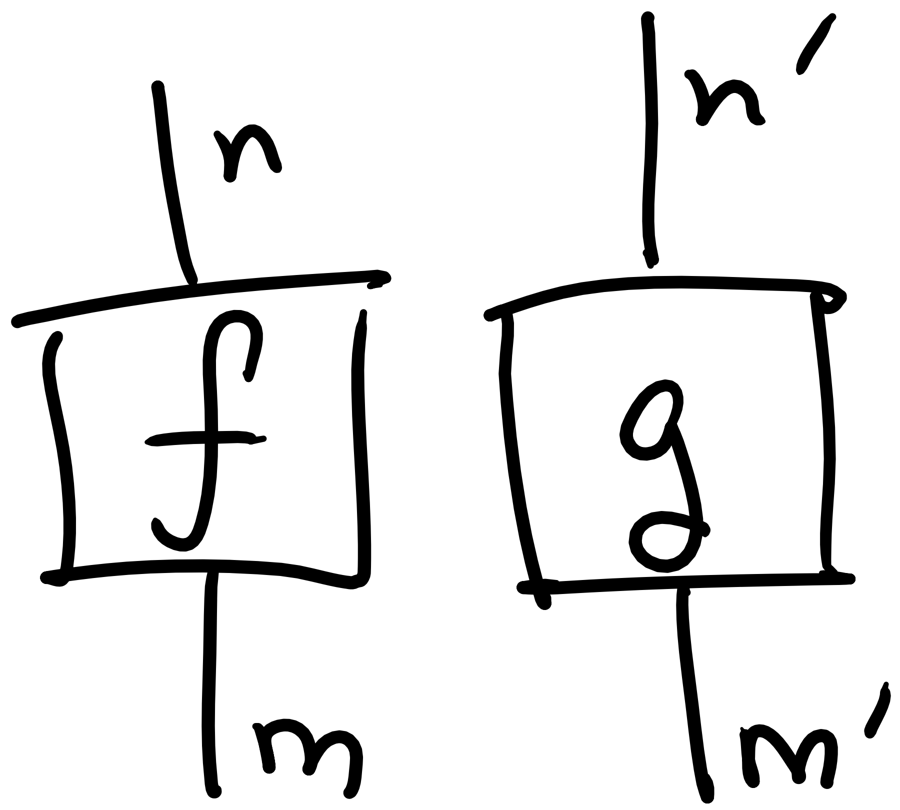
$2^N \times 2^M$

Composition



$$:= \underbrace{\overbrace{g}^{n\times m} \cdot \overbrace{f}^{m\times \ell}}_{n\times \ell}$$

Composition



$$:= \underbrace{\overbrace{f}^{n \times m} \otimes \overbrace{g}^{n' \times m'}}_{nn' \times mm'}$$

COMPOSITION

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

2×2 $\qquad$ 2×1

$$f \otimes g =$$

Composition

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

2×2 $\qquad$ 2×1

$$f \otimes g = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

4×2

Composition

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

2×2 $\qquad$ 2×1

$$f \otimes g = \left(\begin{array}{c|c} & \\ \hline & \end{array} \right)$$

4×2

Composition

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

2×2 $\qquad$ 2×1

$$f \otimes g = \left(\begin{array}{c|c} f_{11}\,g & f_{12}\,g \\ \hline f_{21}\,g & f_{22}\,g \end{array}\right)$$

4×2

Composition

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$2 \times 2 \qquad\qquad 2 \times 1$$

$$f \otimes g = \left(\begin{array}{c|c} \begin{matrix} f_{11} g_1 \\ f_{11} g_2 \end{matrix} & f_{12} g \\ \hline f_{21} g & f_{22} g \end{array}\right)$$

$$4 \times 2$$

Composition

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2×2 $\qquad$ 2×1

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4×2

Composition

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

2×2 $\qquad$ 2×1

$$f \otimes g = \left(\begin{array}{c|c} f_{11} g_1 & f_{12} g_1 \\ f_{11} g_2 & f_{12} g_2 \\ \hline f_{21} g_1 & f_{22} g_1 \\ f_{21} g_2 & f_{22} g_2 \end{array}\right)$$

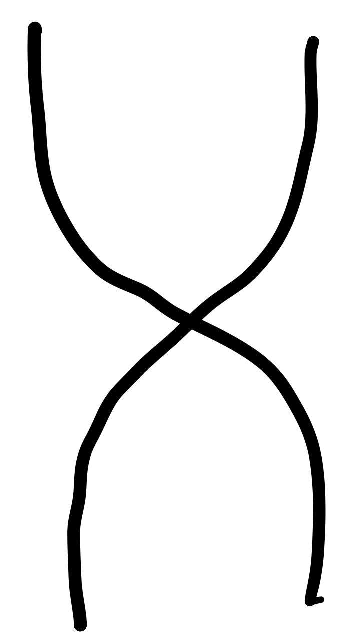
4×2

SWAP

$$v \otimes w = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix} \overset{?}{\rightsquigarrow} \begin{pmatrix} w_1 v_1 \\ w_1 v_2 \\ w_2 v_1 \\ w_2 v_2 \end{pmatrix} = w \otimes v$$

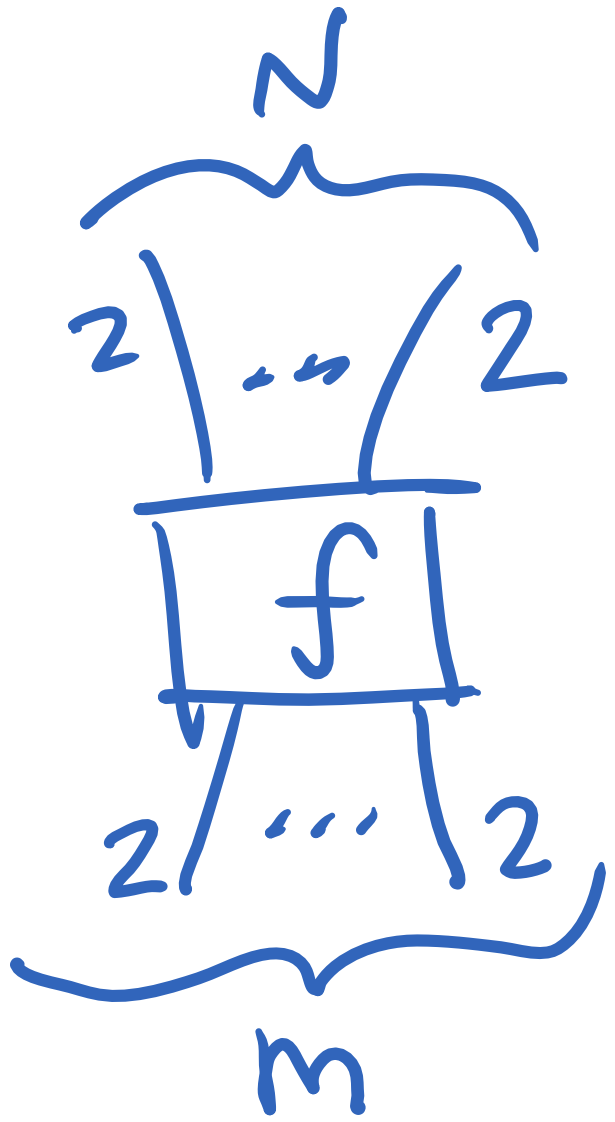
SWAP

$$v \otimes w = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} w_1 v_1 \\ w_1 v_2 \\ w_2 v_1 \\ w_2 v_2 \end{pmatrix} = w \otimes v$$



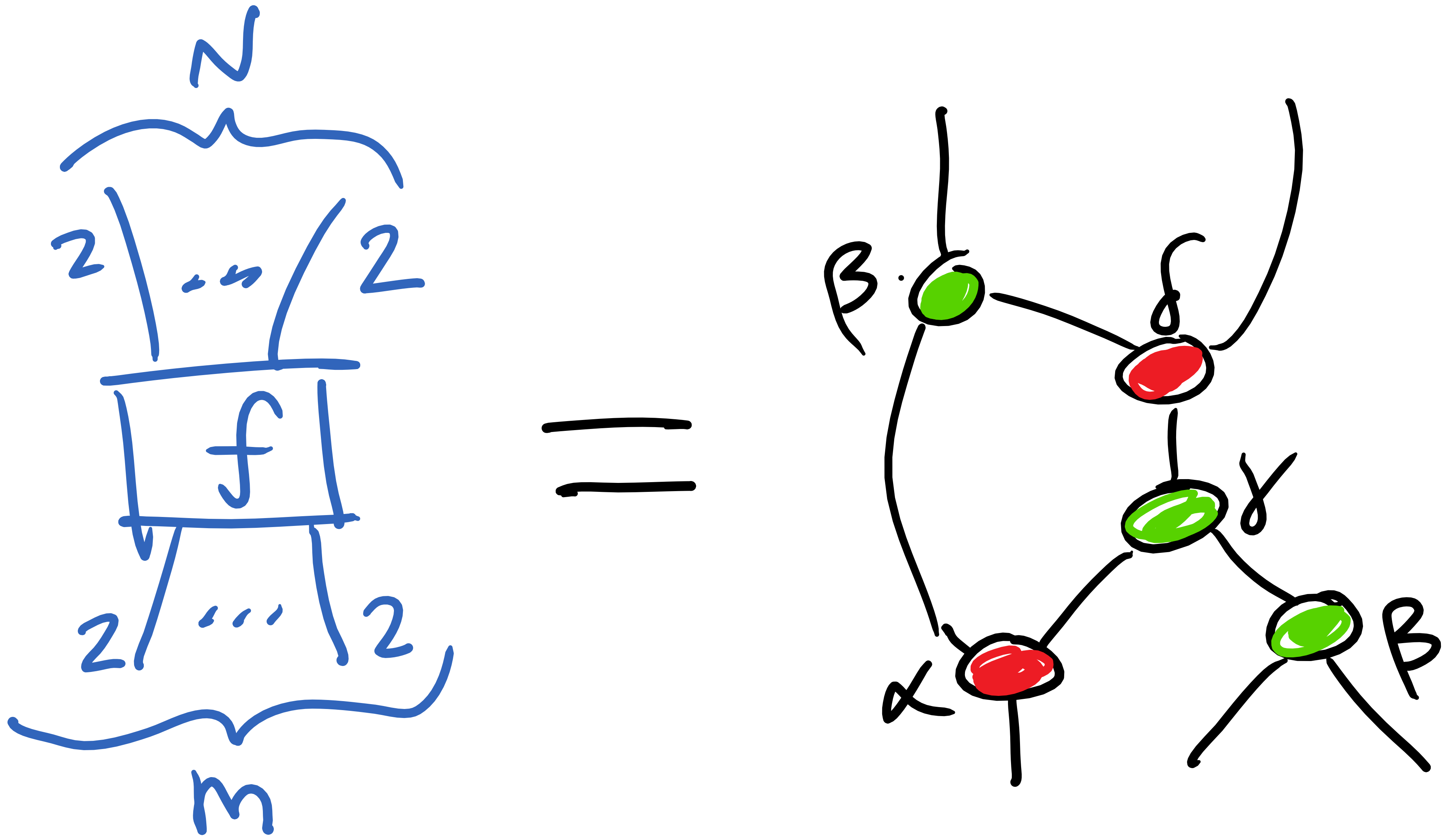
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

PROBLEM: Matrices relevant for QT are very large:



$2^N \times 2^M$

Problem: Matrices relevant for QT are very large:



Solution: Decompose into smaller pieces and use diagram rewrite rules.

Aside: Complex numbers

$$a + ib$$

Aside: Complex numbers

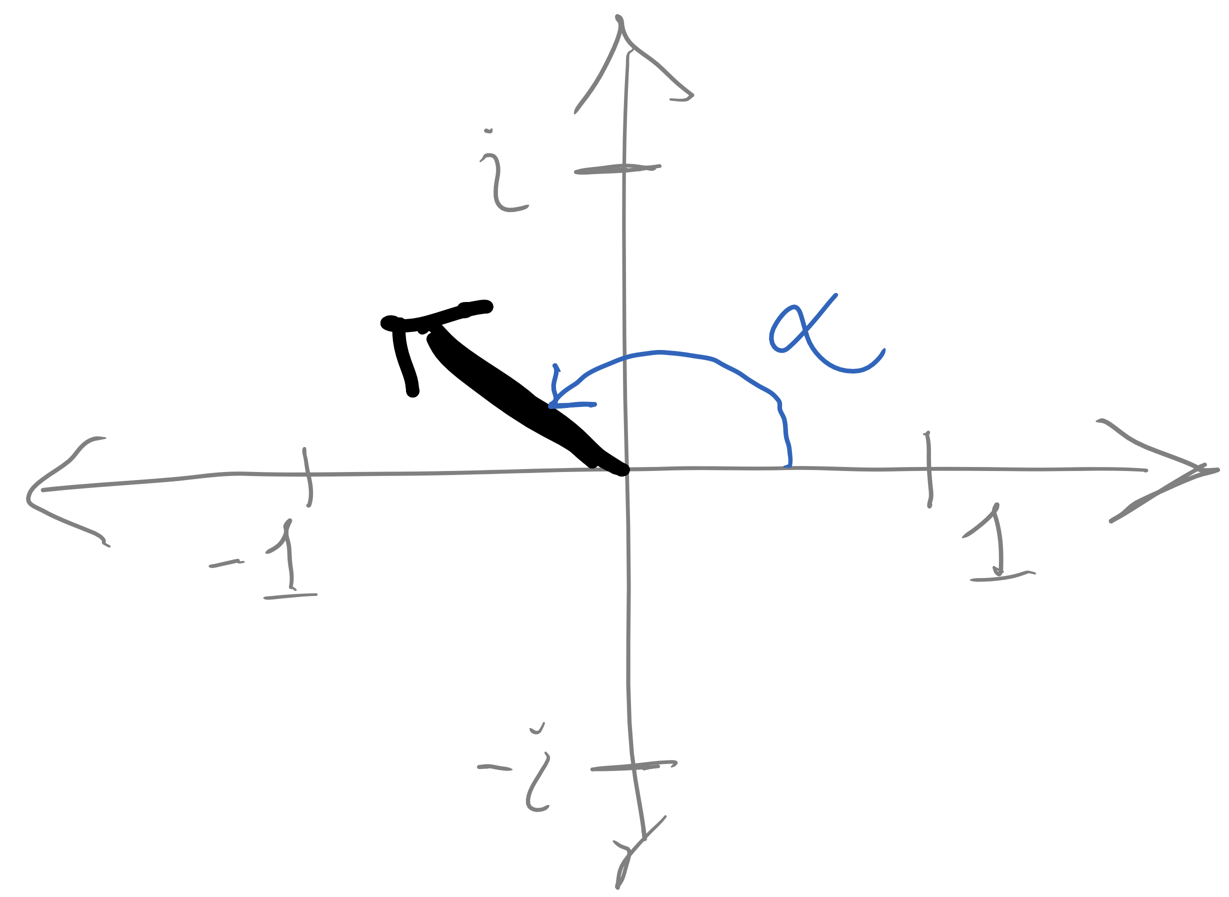
$$a + ib \rightsquigarrow re^{i\alpha}$$

Aside: Complex numbers

$$a + ib \rightsquigarrow \underbrace{r}_{\text{magnetude}}\,\underbrace{e^{i\alpha}}_{\text{phase}}$$

Aside: Complex numbers

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$



Aside: Complex numbers

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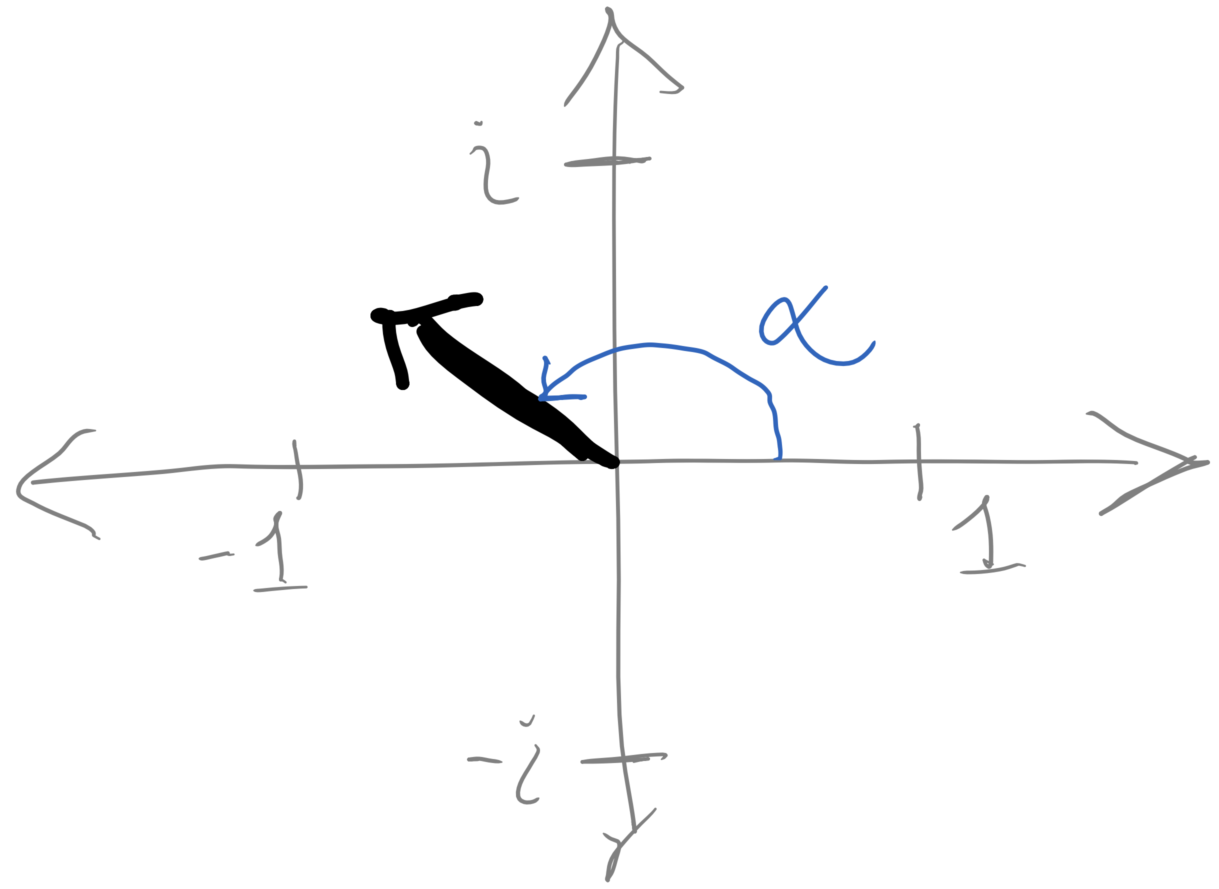
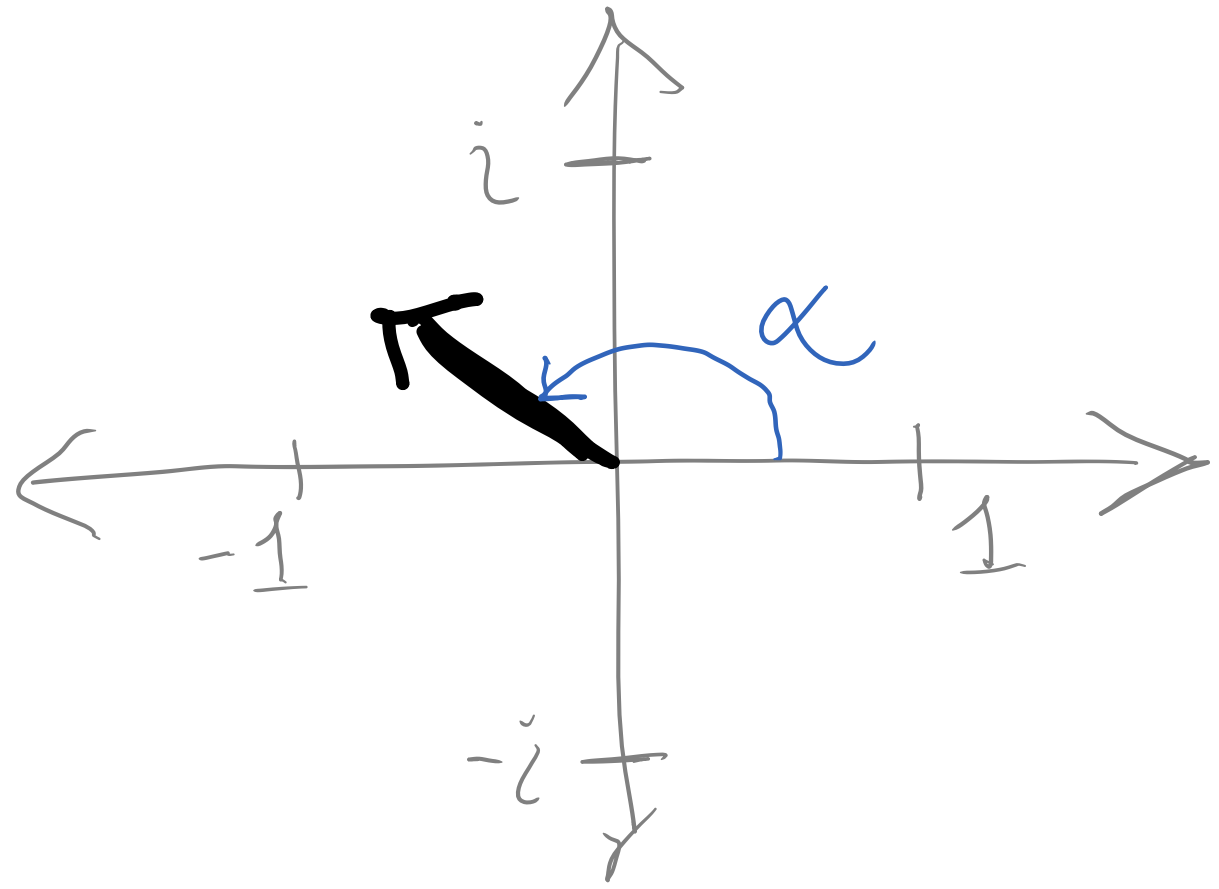


$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$$

Aside: Complex numbers

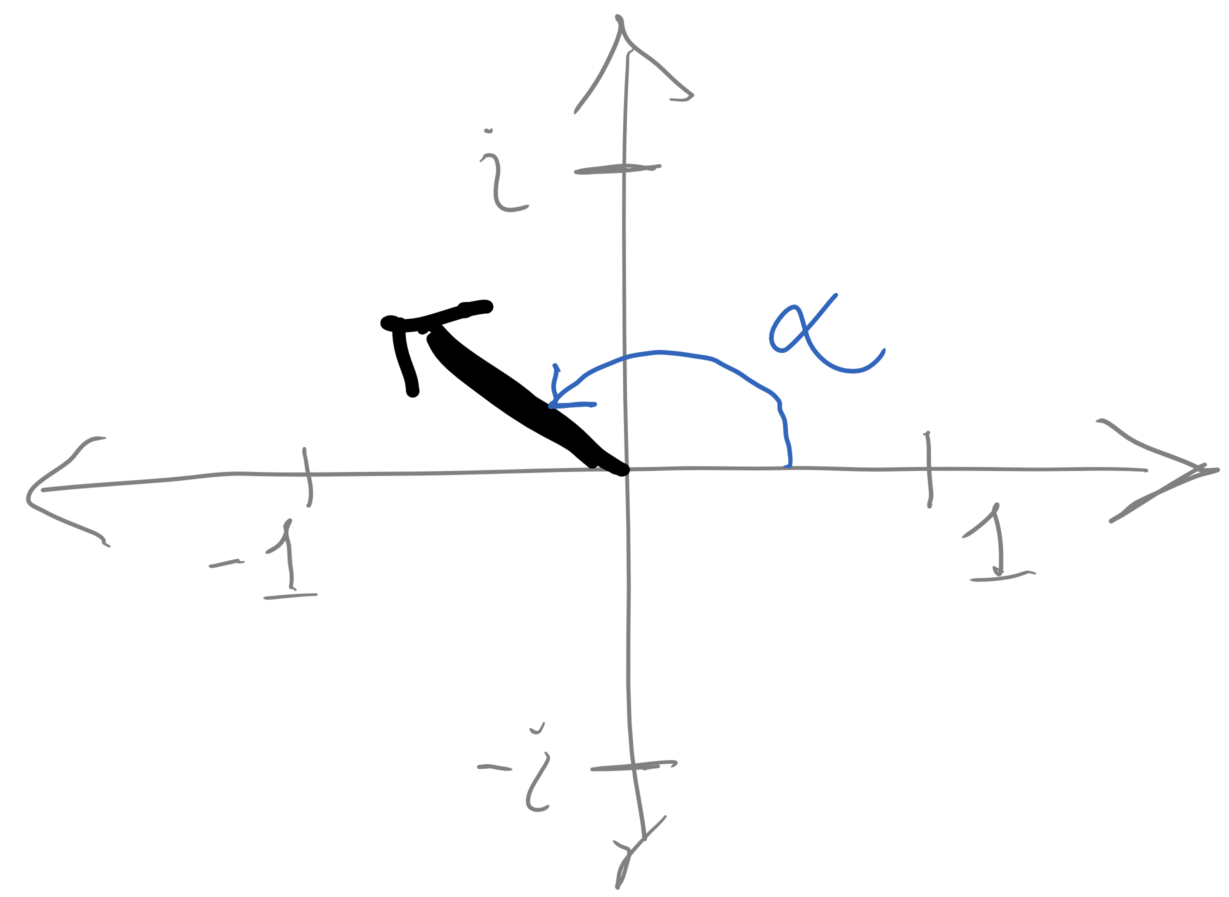
$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$



$$e^{i\alpha} + e^{i\alpha} = 2e^{i\alpha}$$

Aside: Complex numbers

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

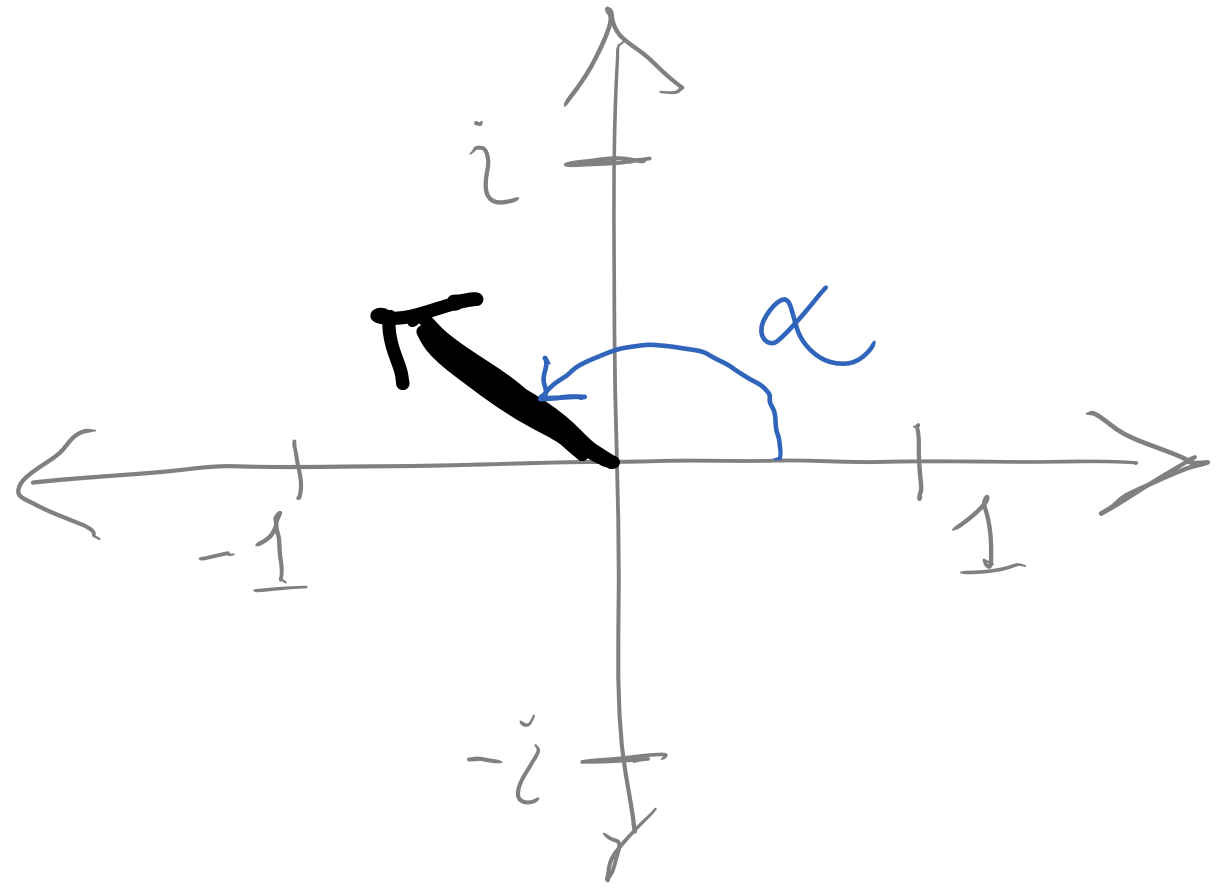


$$e^{i\alpha} + e^{i\alpha} = 2e^{i\alpha}$$

$$e^{i\alpha} + e^{i(\pi+\alpha)}$$

Aside: Complex numbers

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

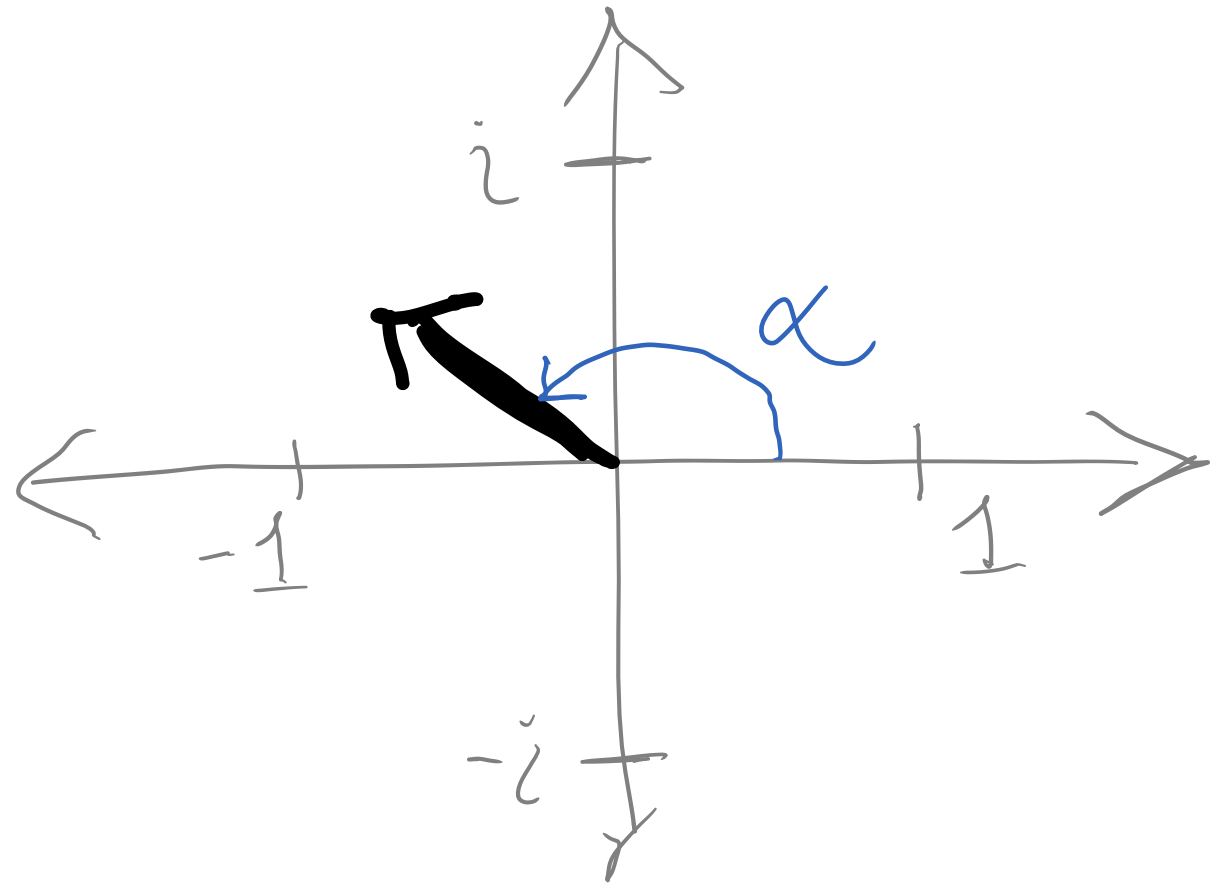


$$e^{i\alpha} + e^{i\alpha} = 2e^{i\alpha}$$

$$e^{i\alpha} + e^{i(\pi+\alpha)} = e^{i\alpha} + e^{i\pi}e^{i\alpha}$$

Aside: Complex numbers

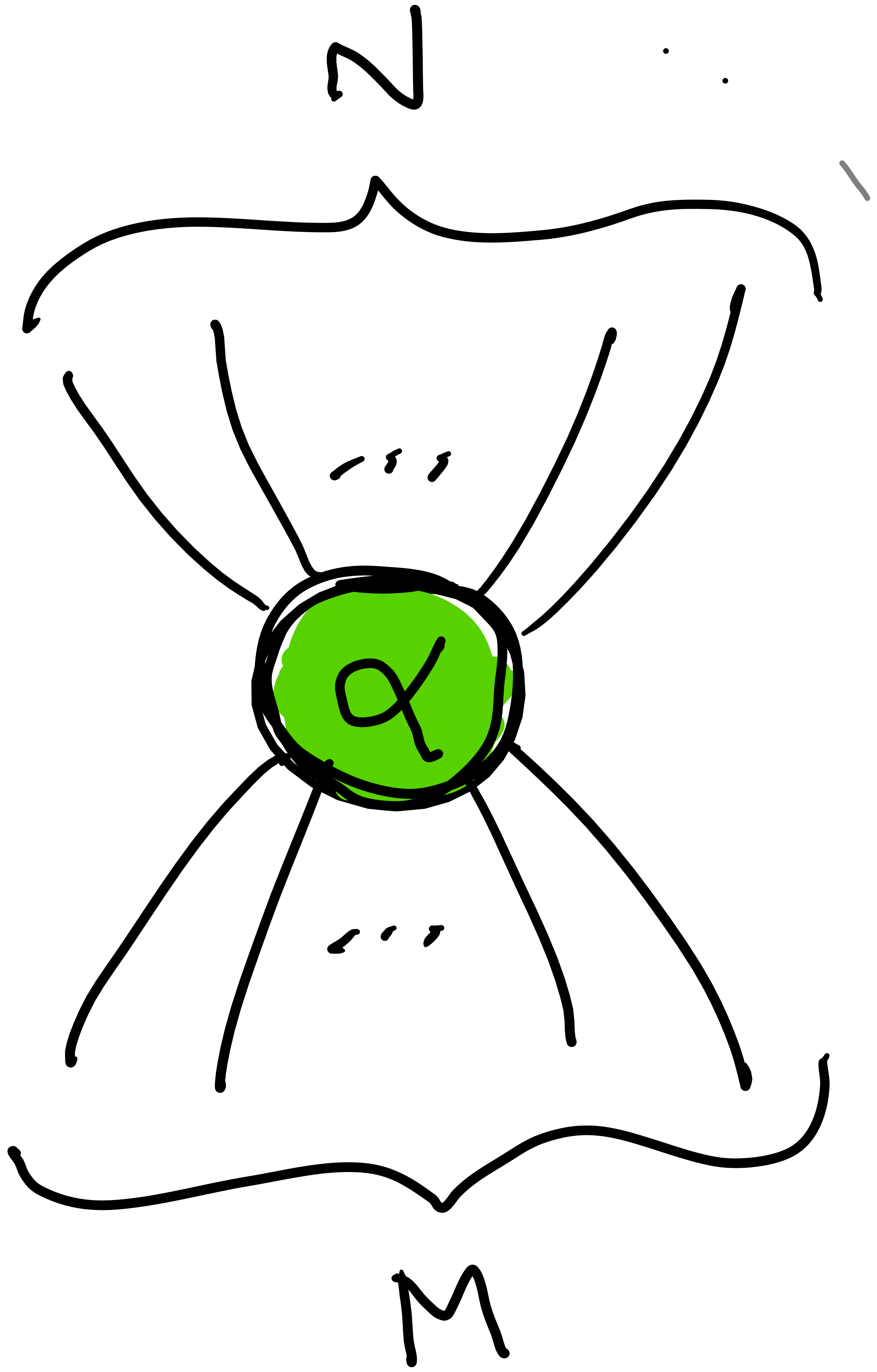
$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$



$$e^{i\alpha} + e^{i\alpha} = 2e^{i\alpha}$$

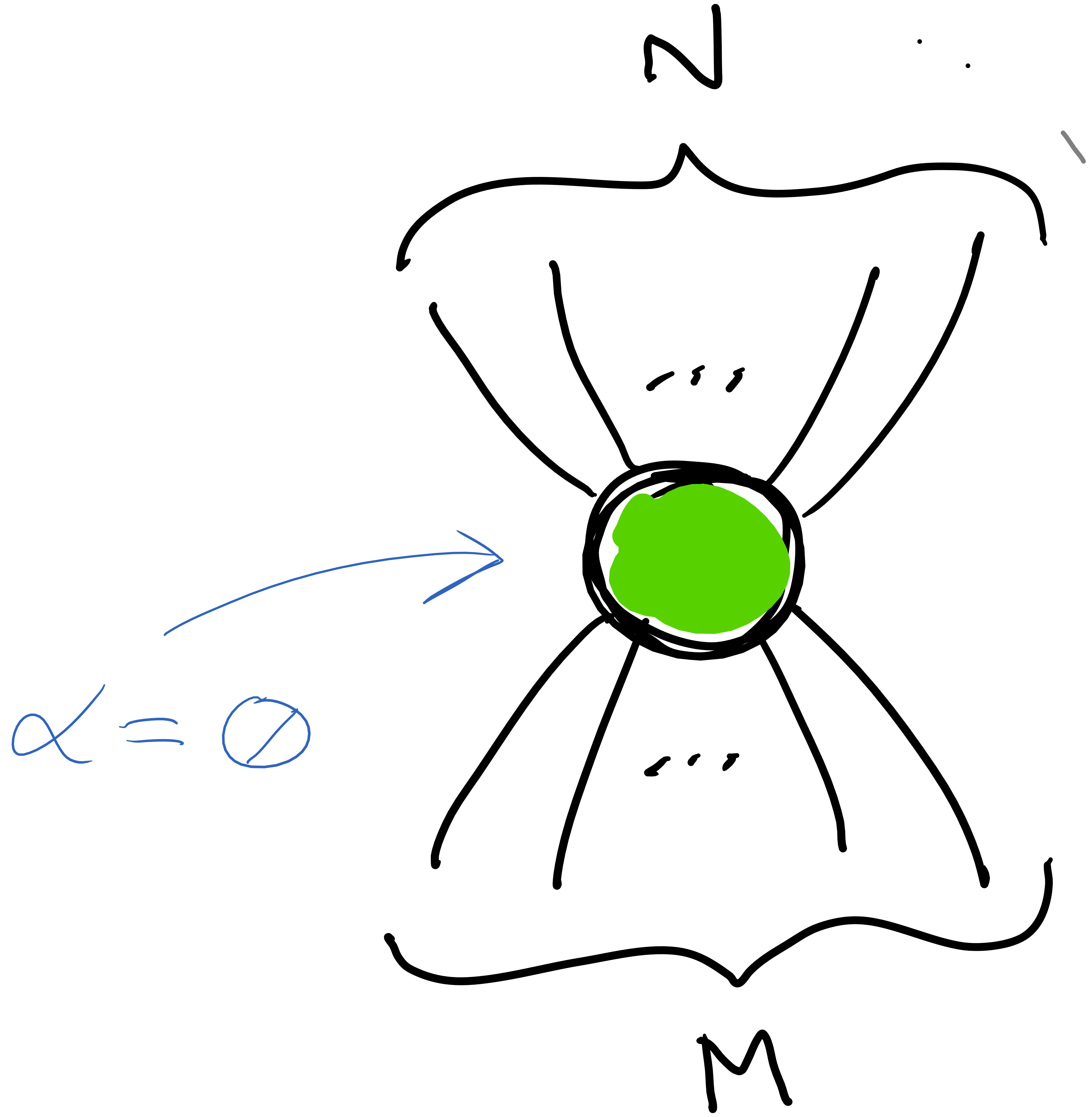
$$e^{i\alpha} + e^{i(\pi+\alpha)} = e^{i\alpha} + \cancel{e^{i\pi}}^{-1} e^{i\alpha} = \emptyset$$

SPIDERS



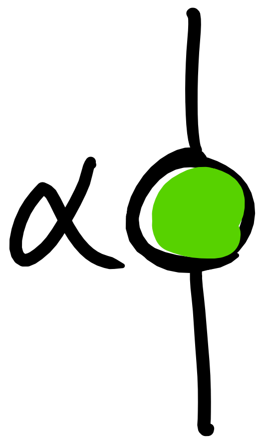
$$:= \begin{pmatrix} 1 & & \\ & \emptyset & \\ & & e^{i\alpha} \end{pmatrix}$$

SPIDERS



$$:= \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

SPIDERS



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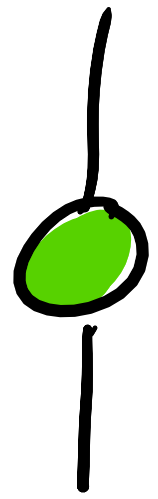

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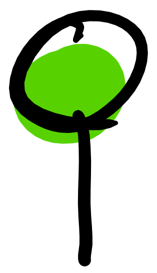

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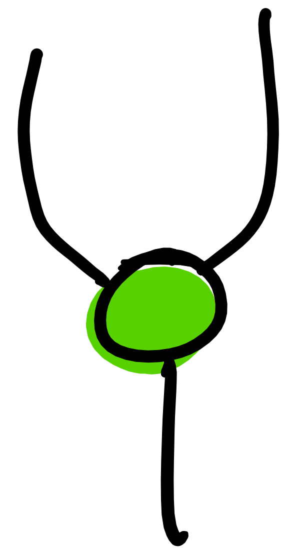

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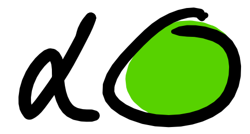

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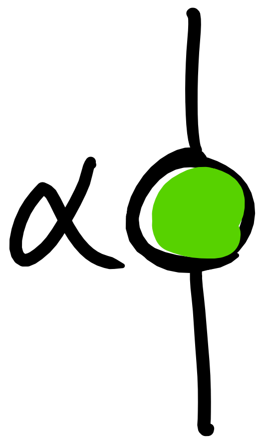
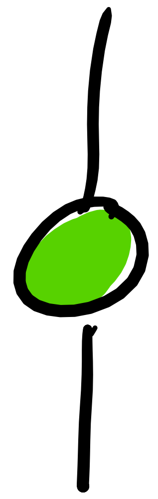
SPIDERS



$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



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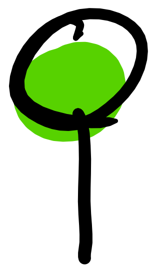

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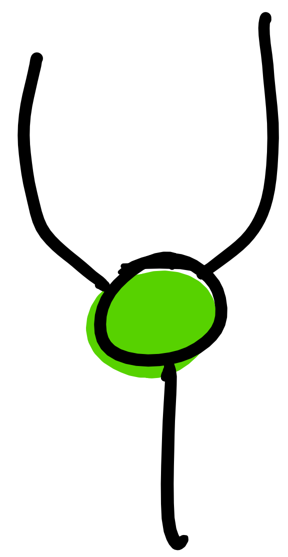

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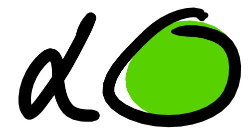

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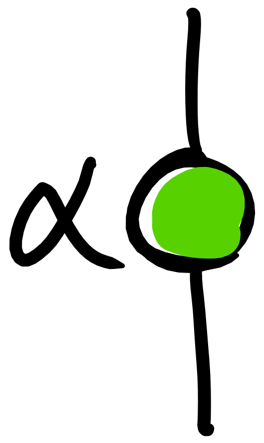
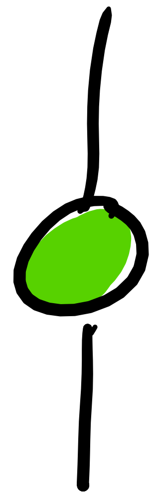
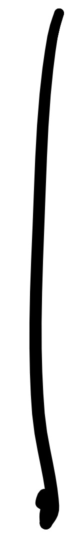
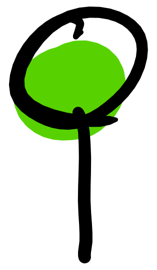


$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$





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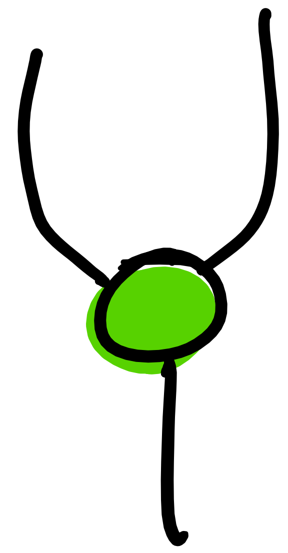

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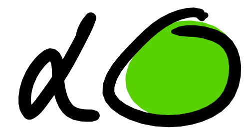

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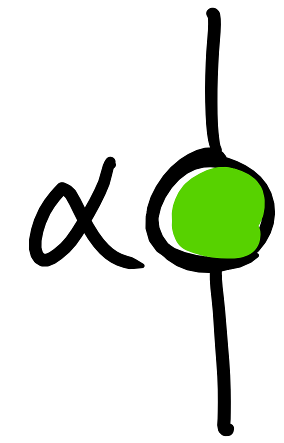
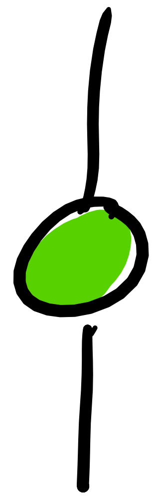
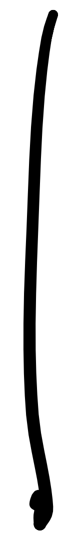
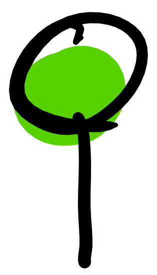
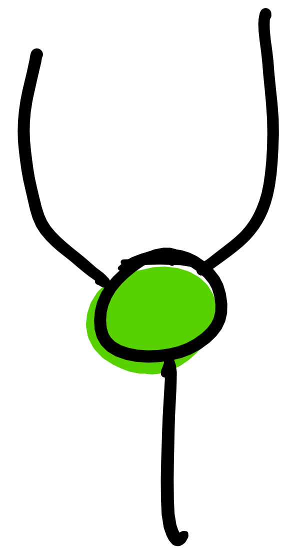
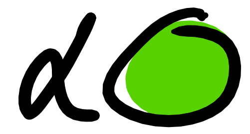
Spiders

 $= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$

 $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$ 

 $= \begin{pmatrix} 1 & 1 \end{pmatrix}$

 $=$

 $=$

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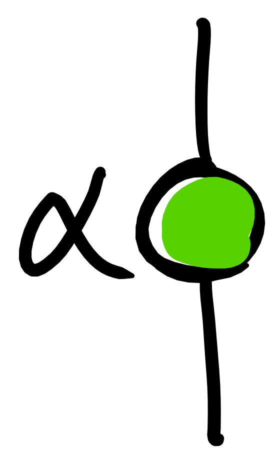
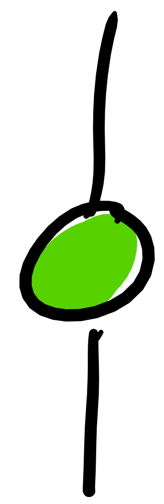
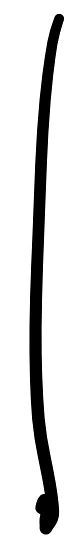
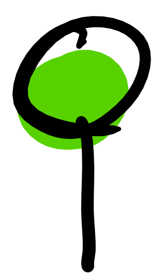
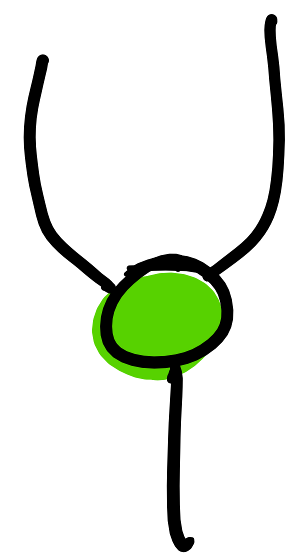

$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$


$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$



$$= \begin{pmatrix} 1 & 1 \end{pmatrix}$$


$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$


$$=$$

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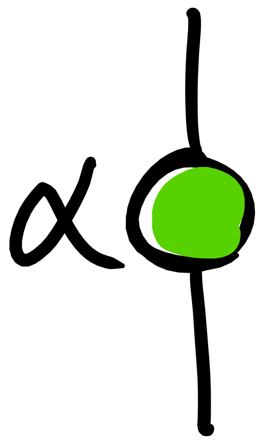
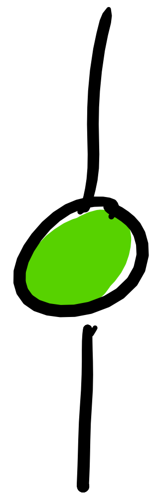
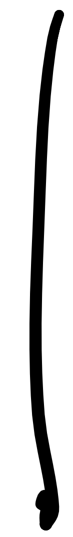
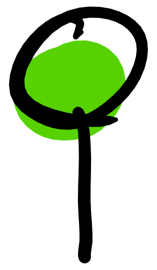


$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

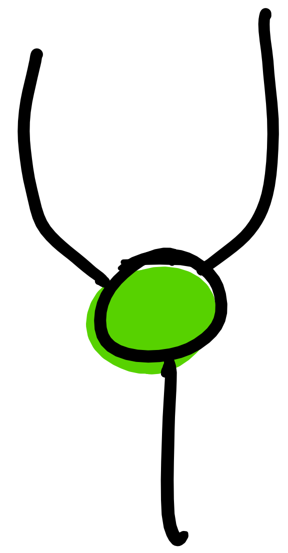


$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

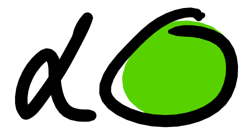




$$= \begin{pmatrix} 1 & 1 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$



$$= \begin{pmatrix} 1+e^{i\alpha} \end{pmatrix}$$

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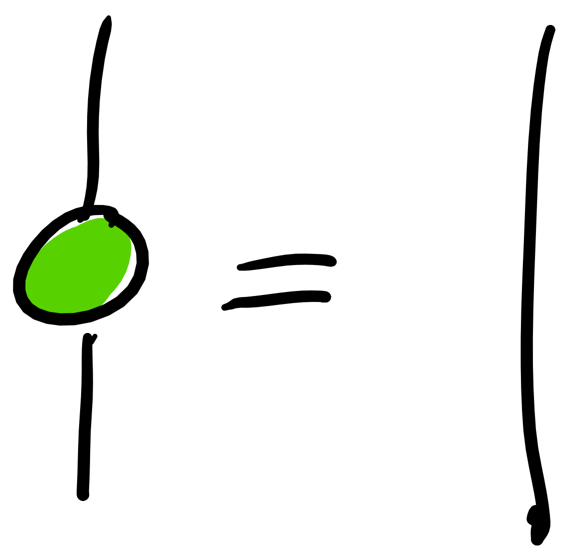

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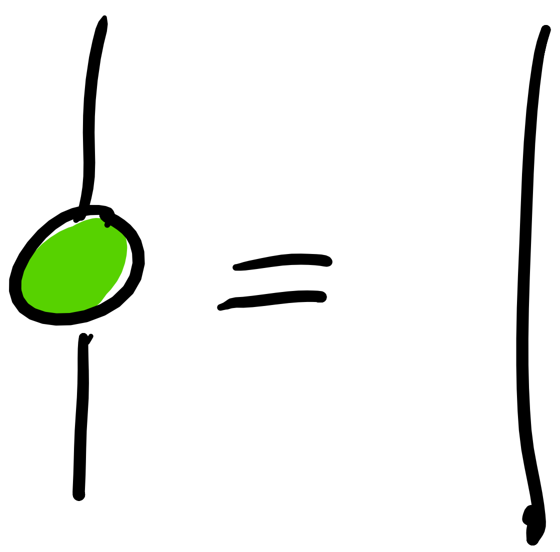

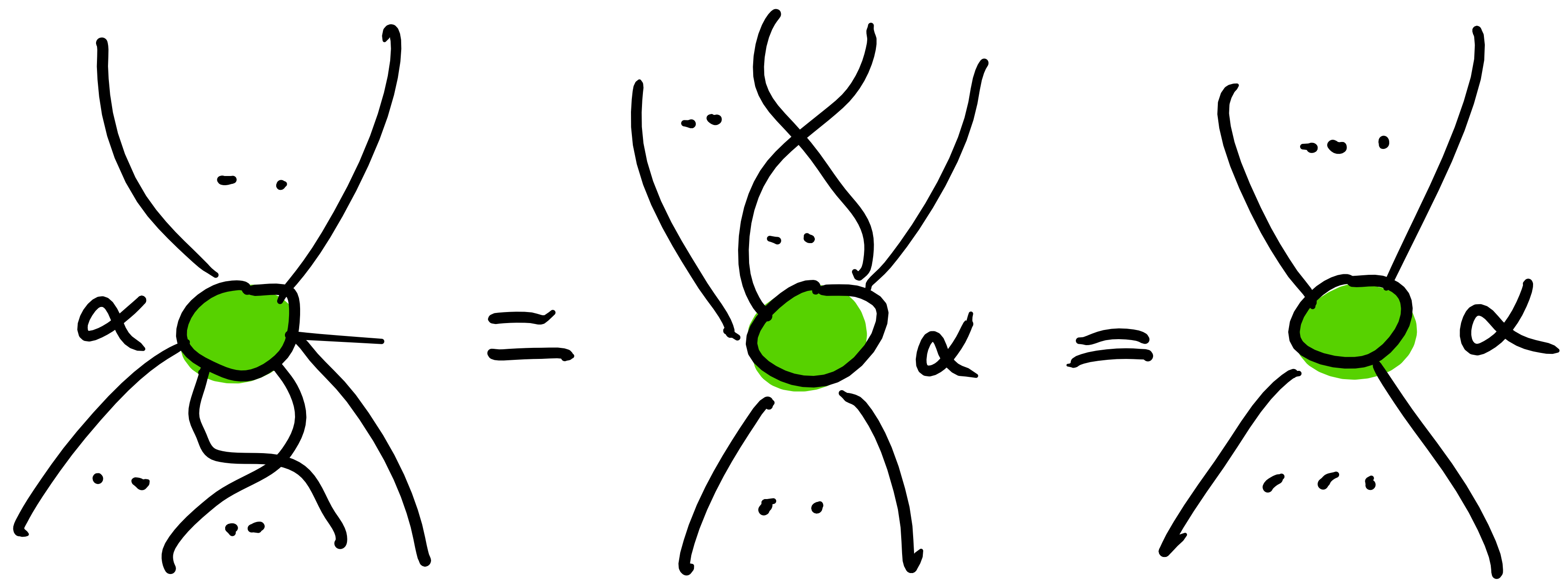

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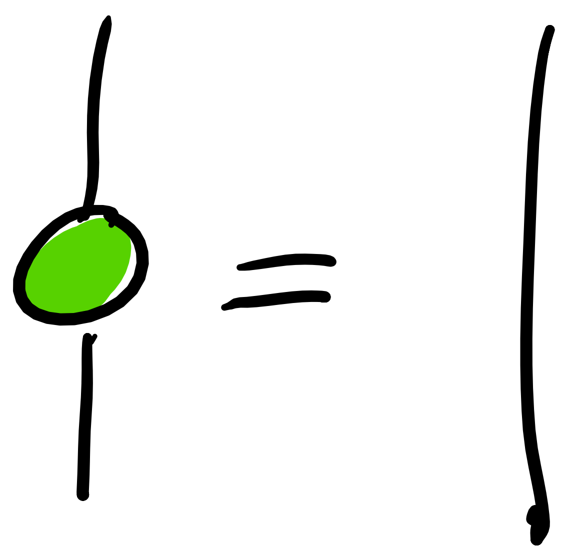
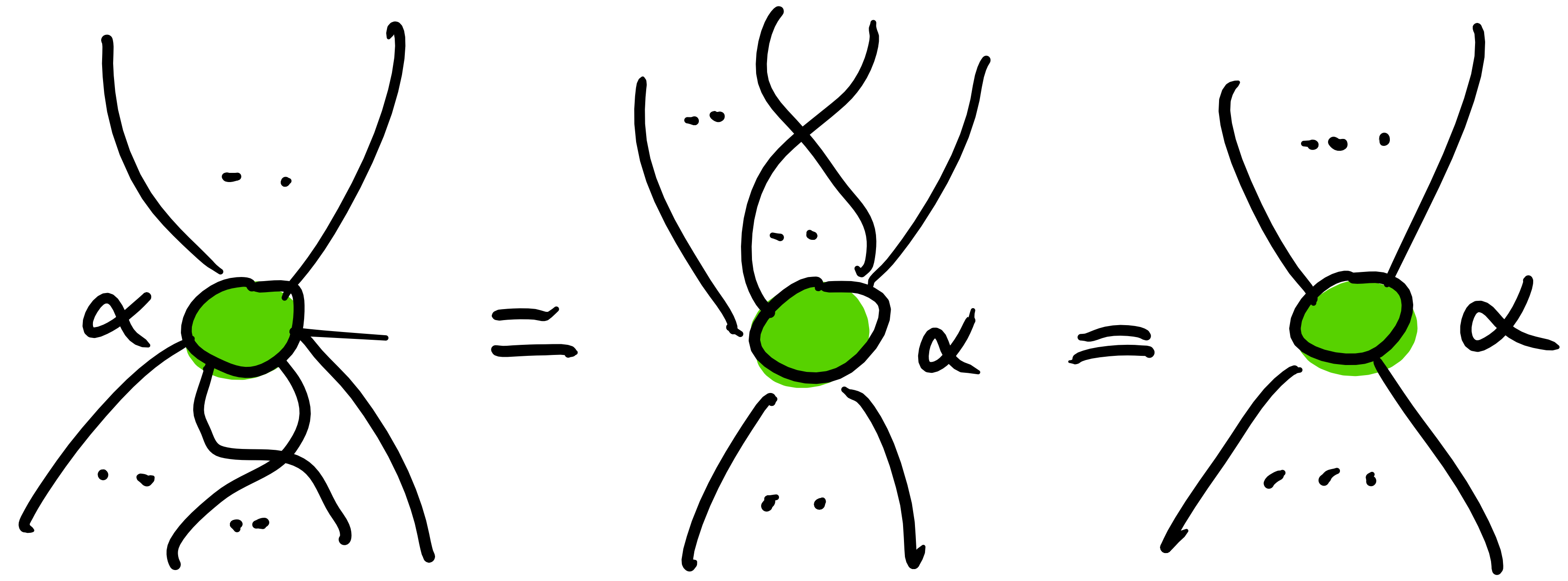
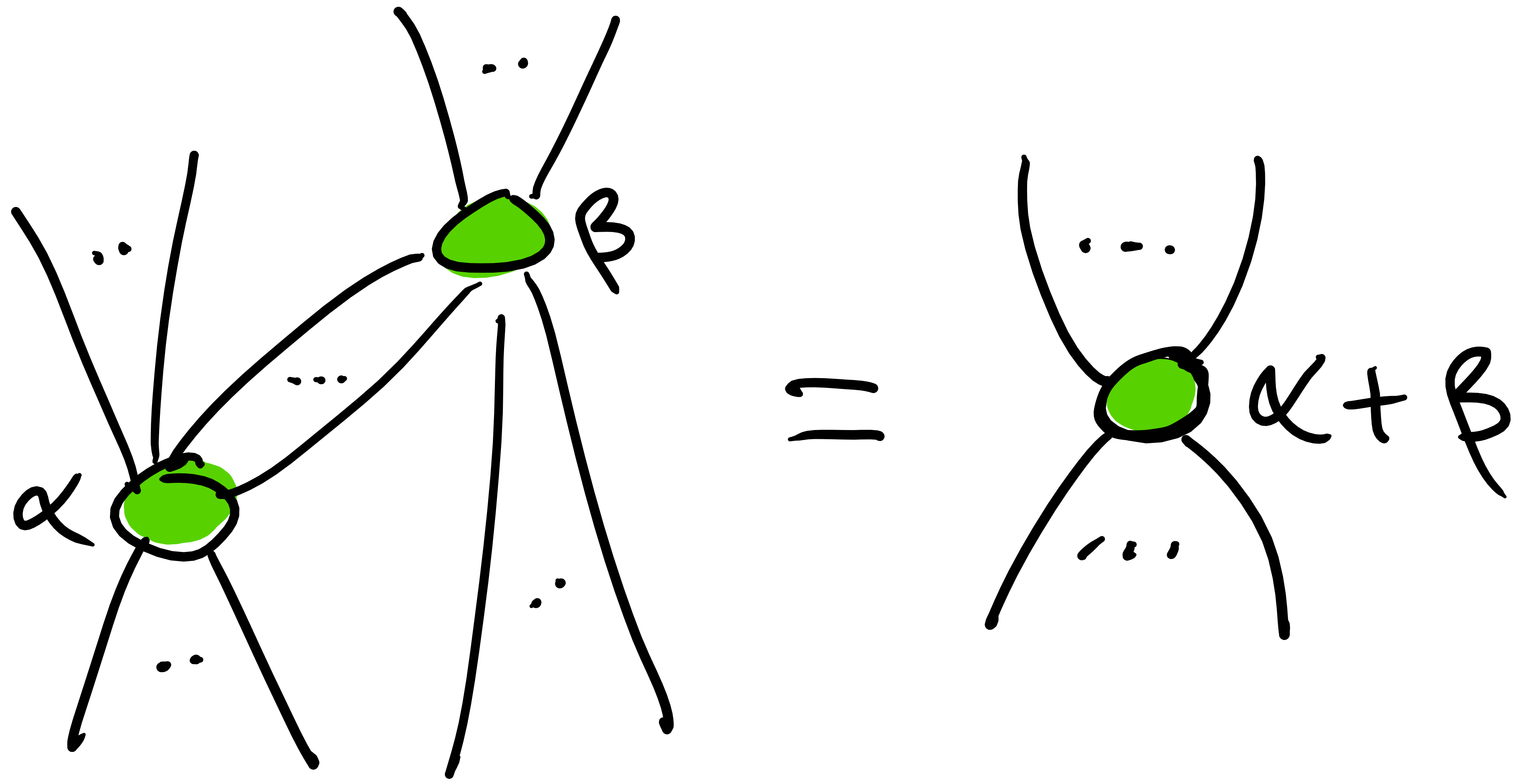
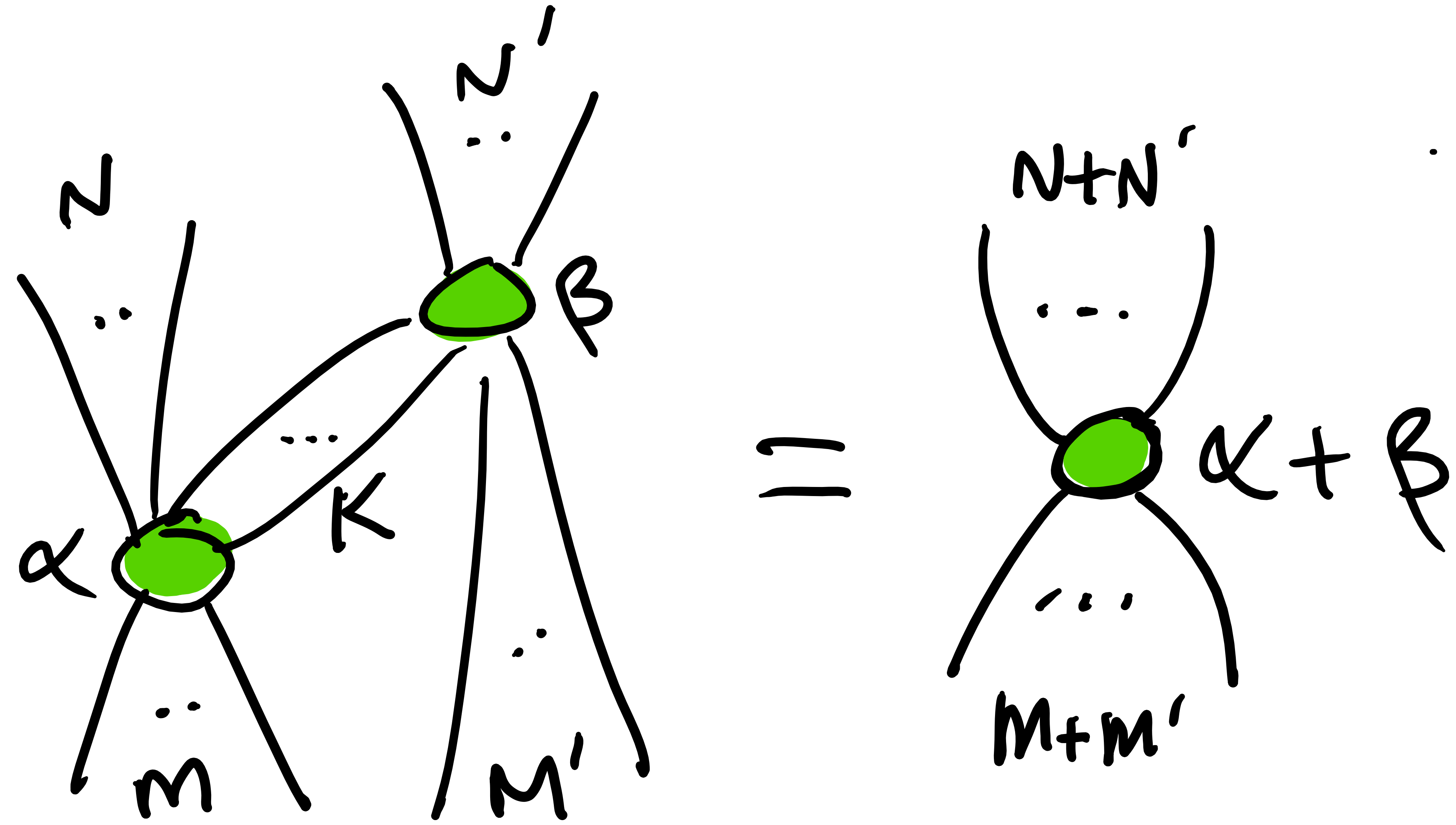
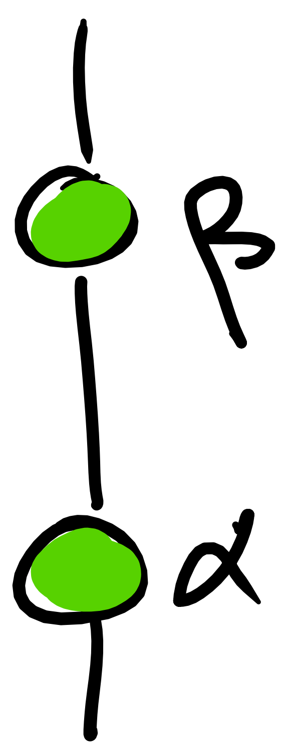


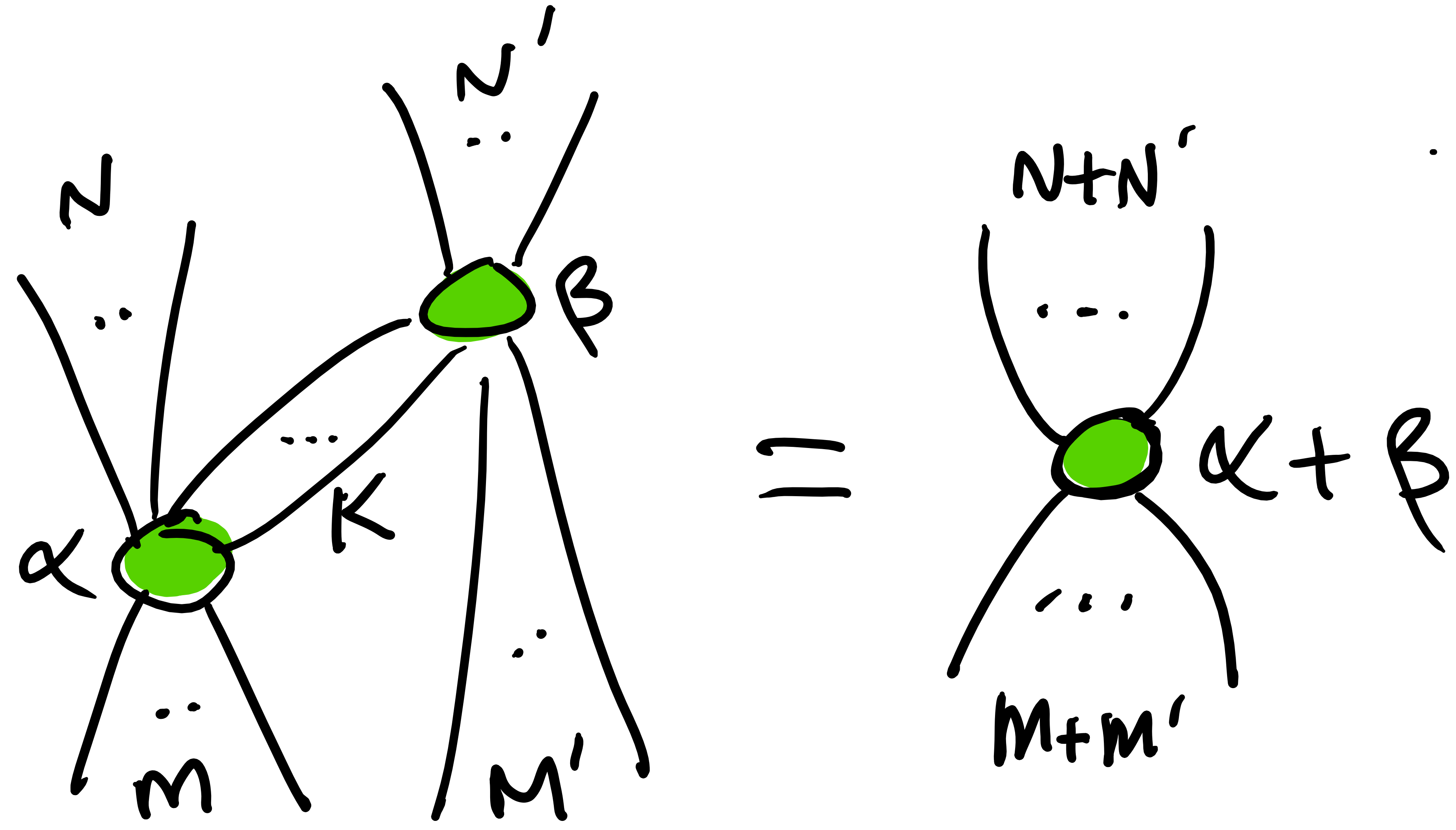




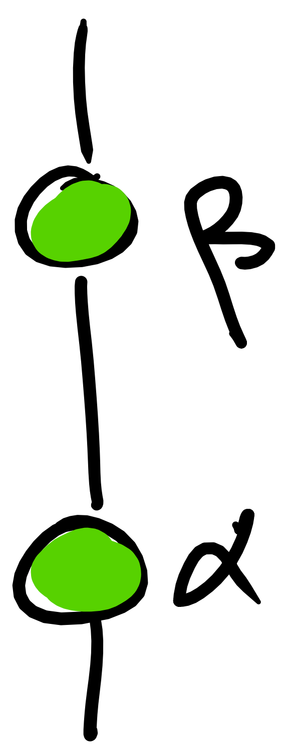


$(K \geq 1)$

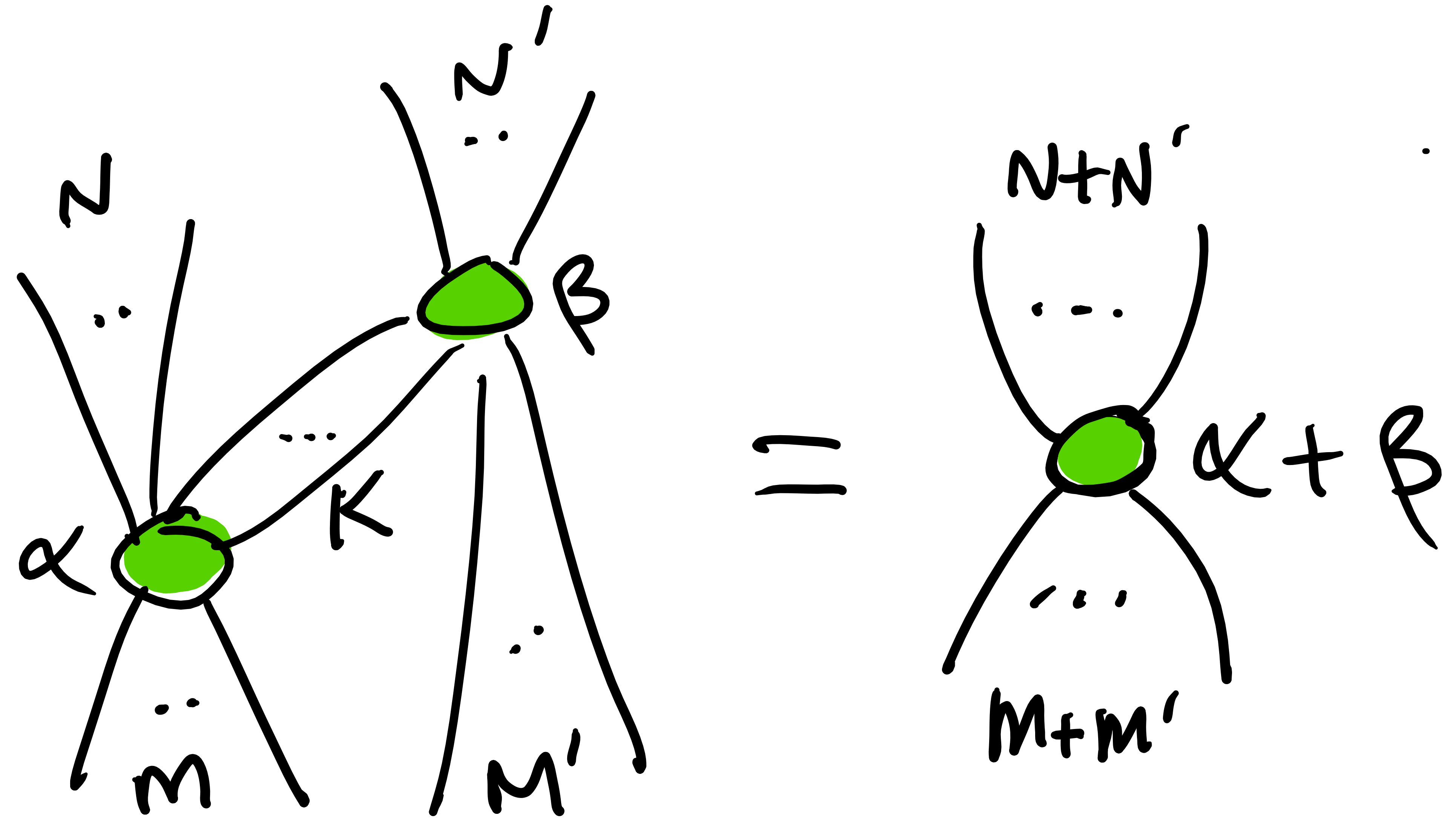




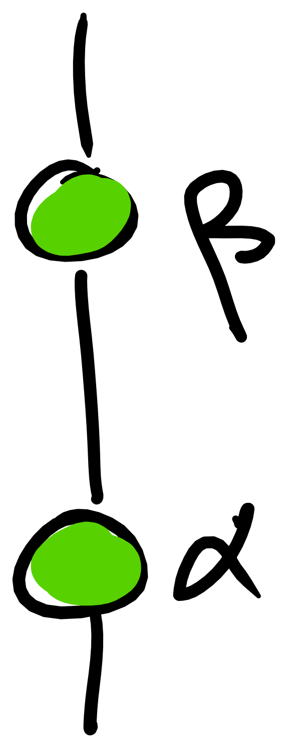
$(K \geq 1)$



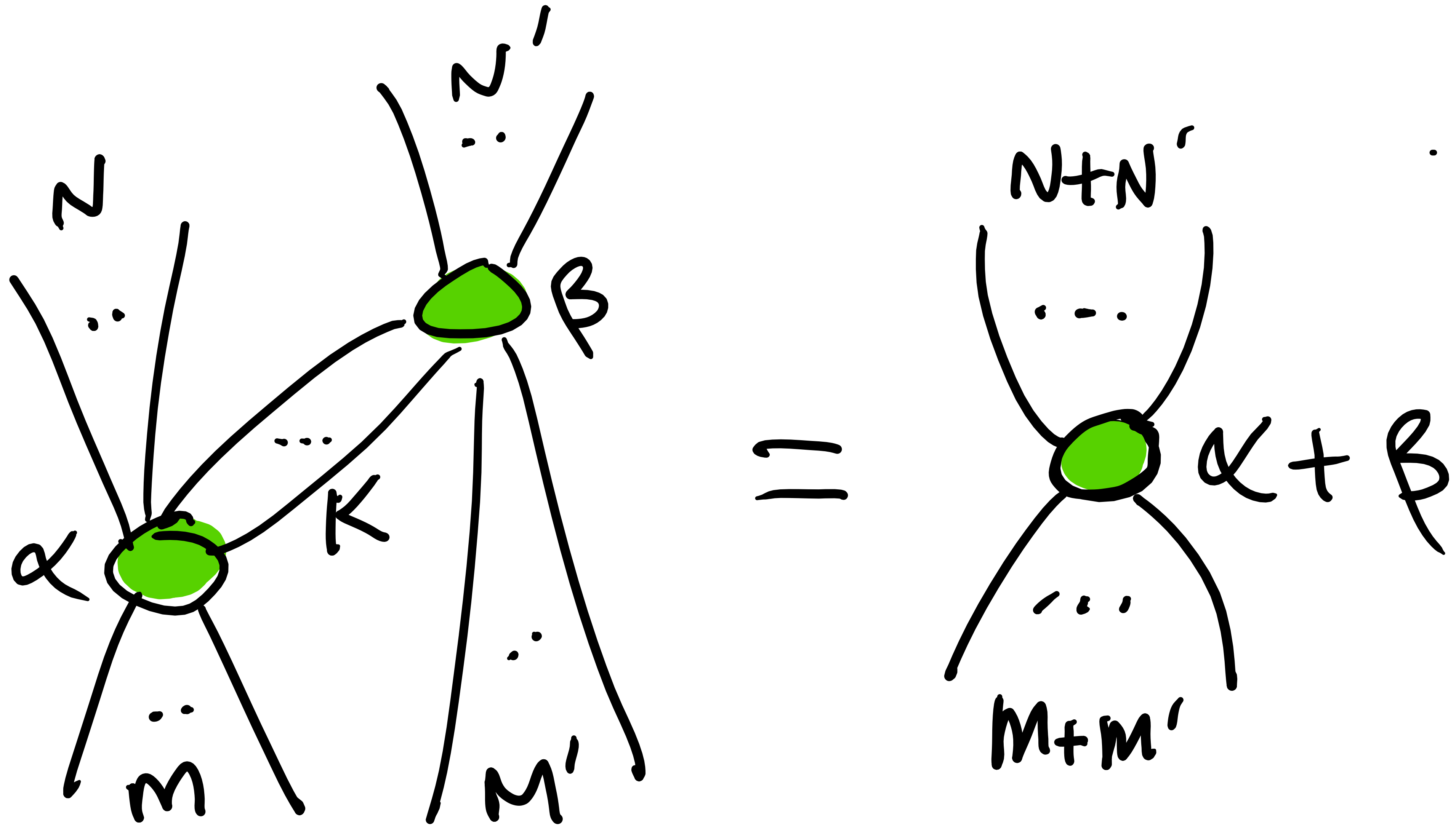
$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



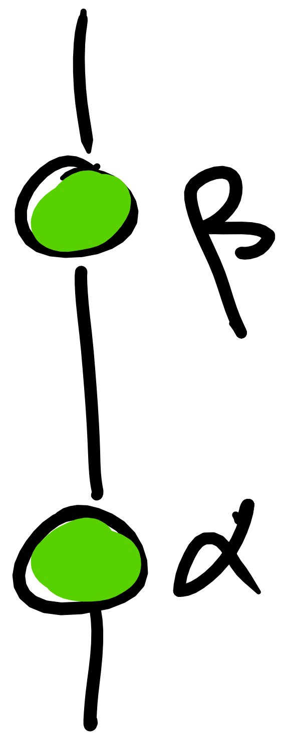
$(K \geq 1)$



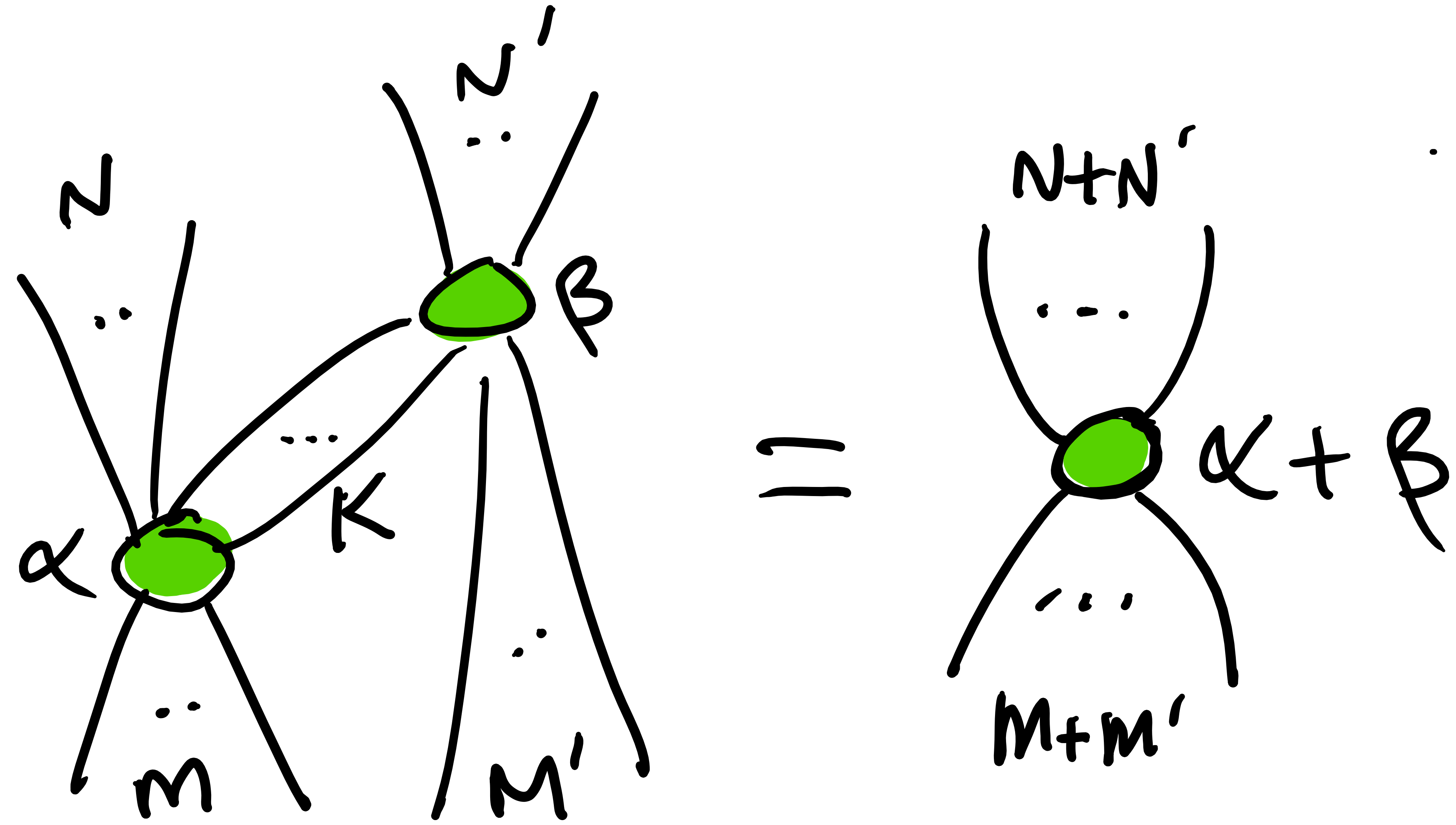
$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} e^{i\beta} \end{pmatrix}$$



$(K \geq 1)$



$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} e^{i\beta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\alpha+\beta)} \end{pmatrix}$$



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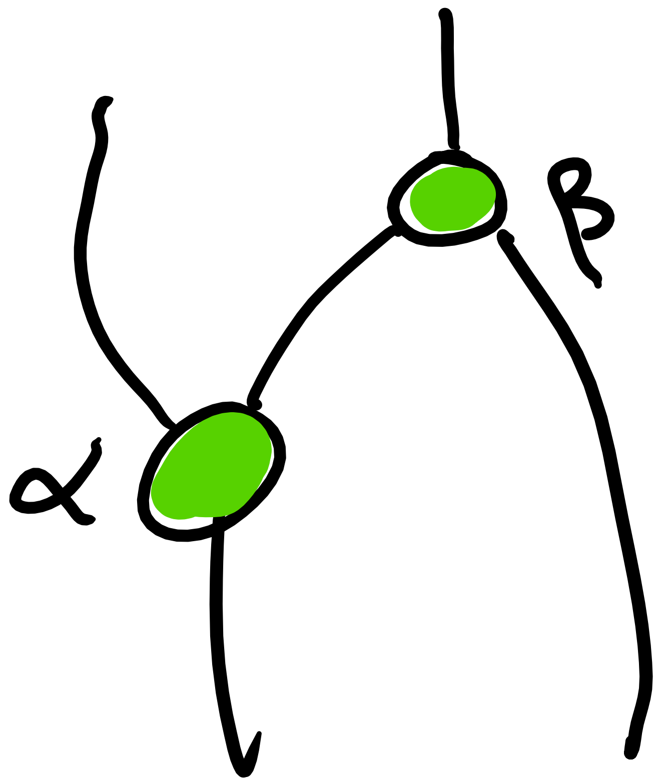

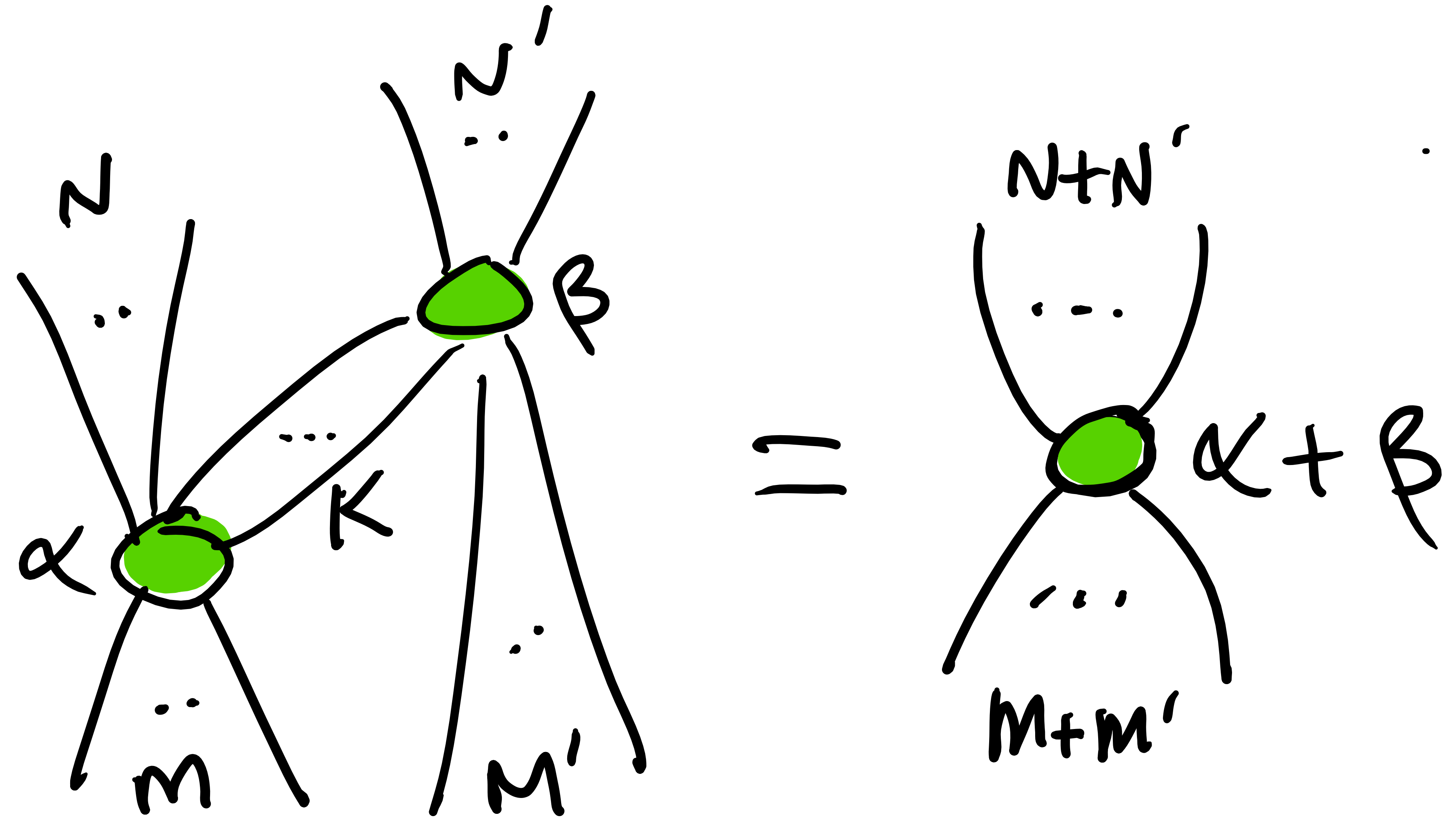

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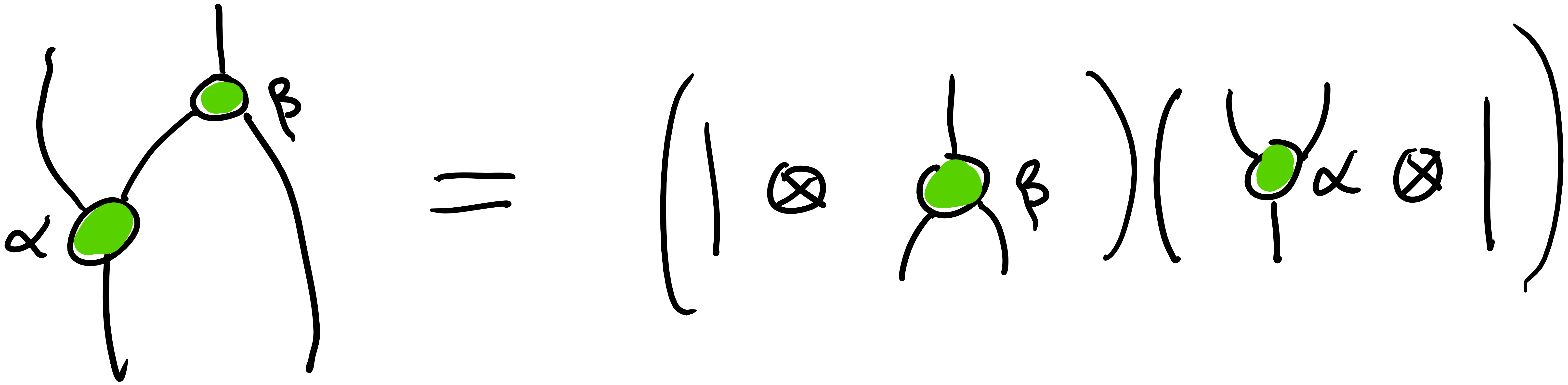

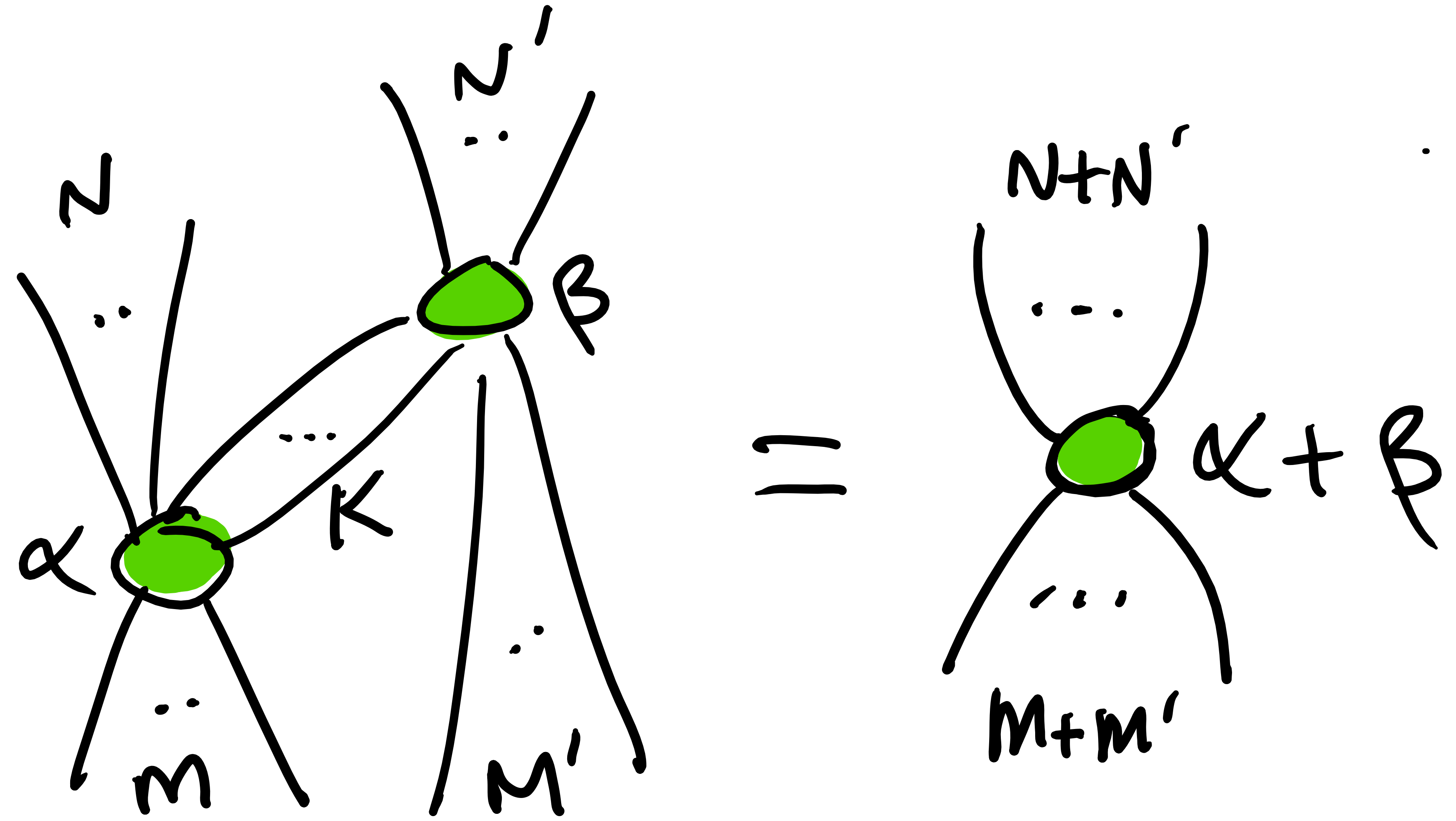

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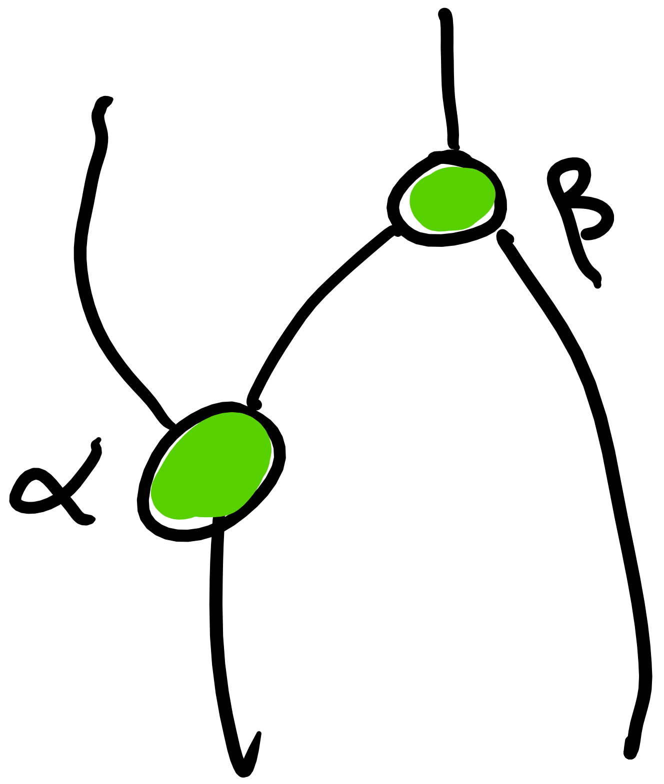
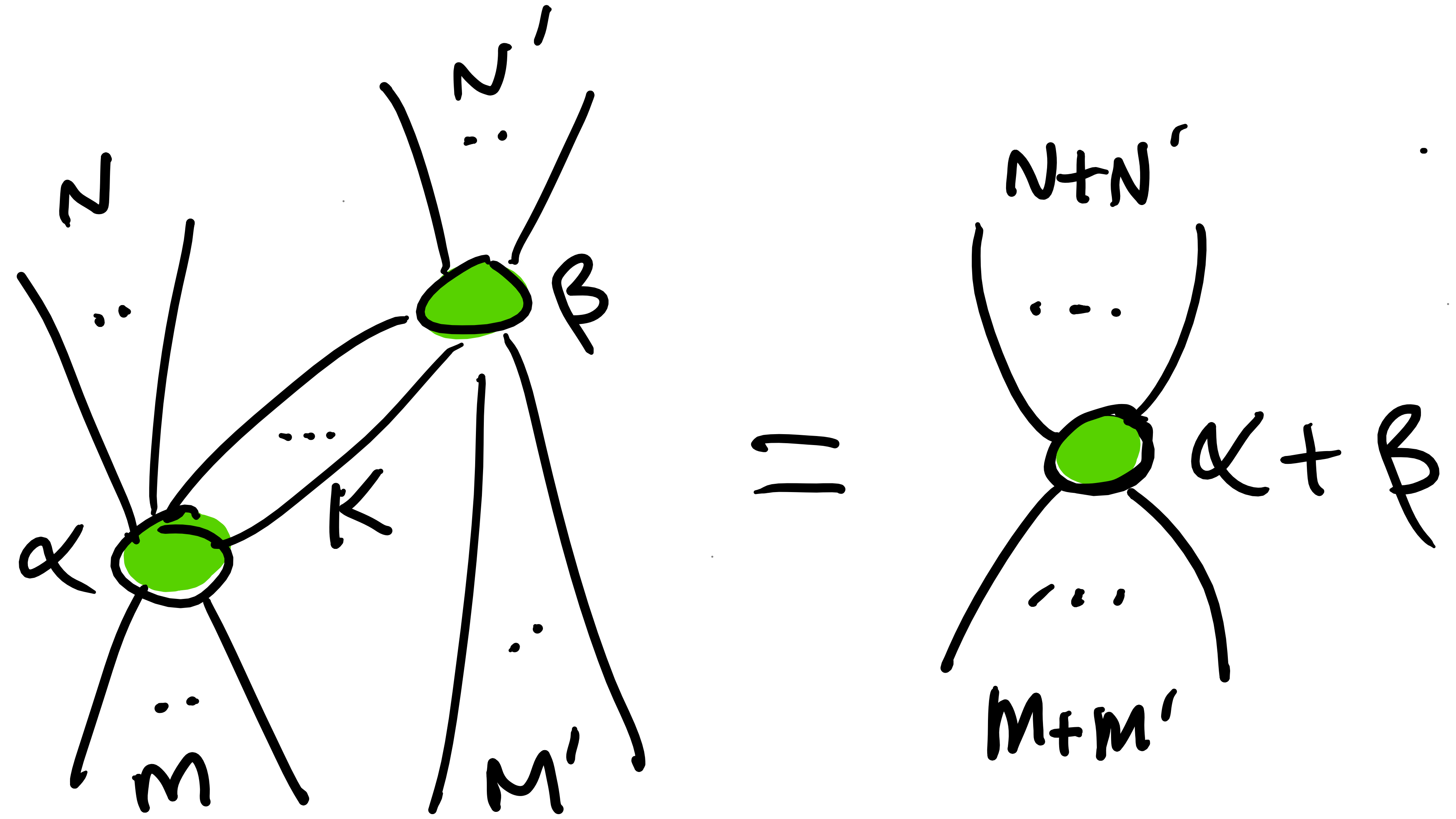
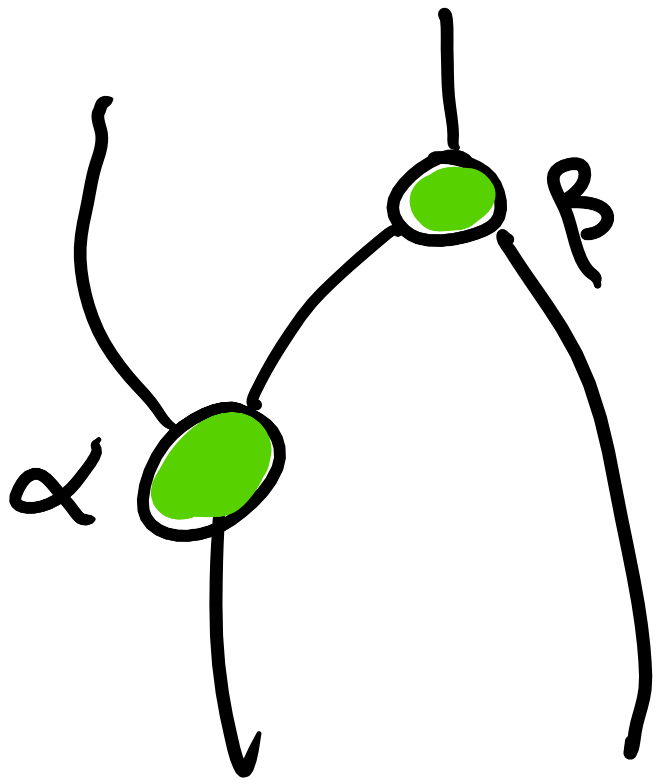


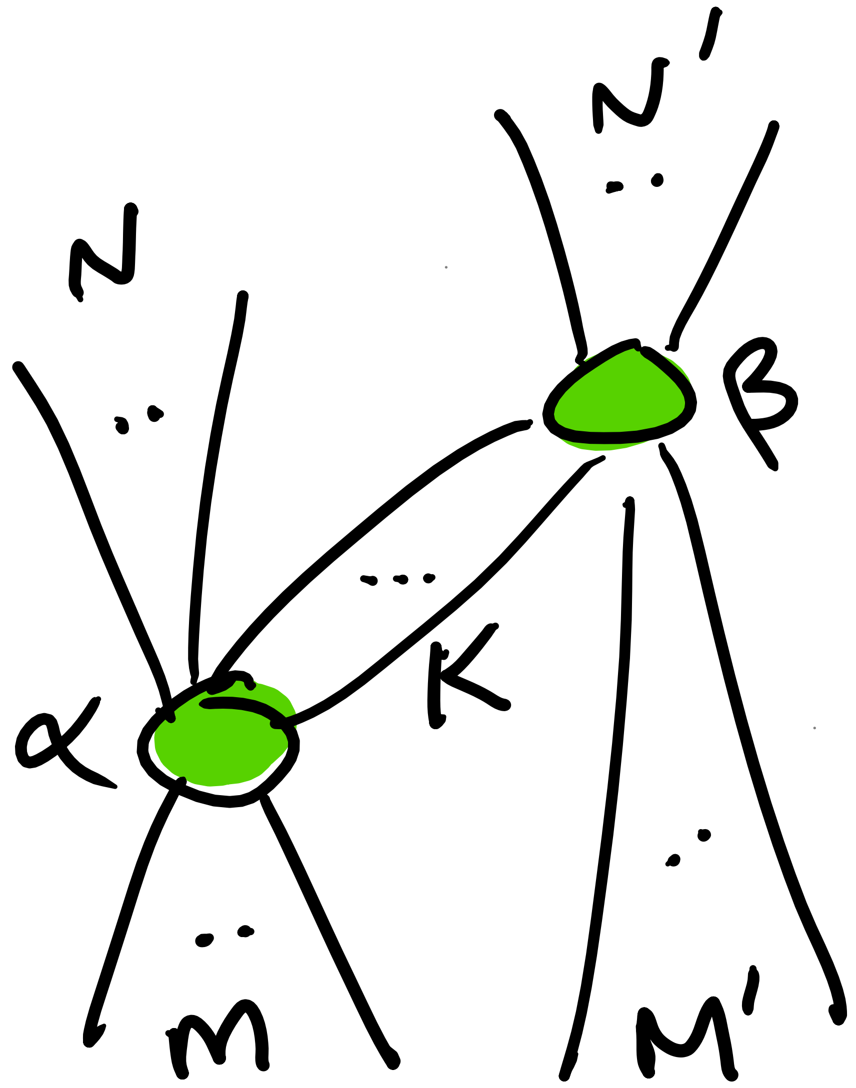
$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\beta} \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \right)$$



$(K \geq 1)$



$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\beta} \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \right)$$



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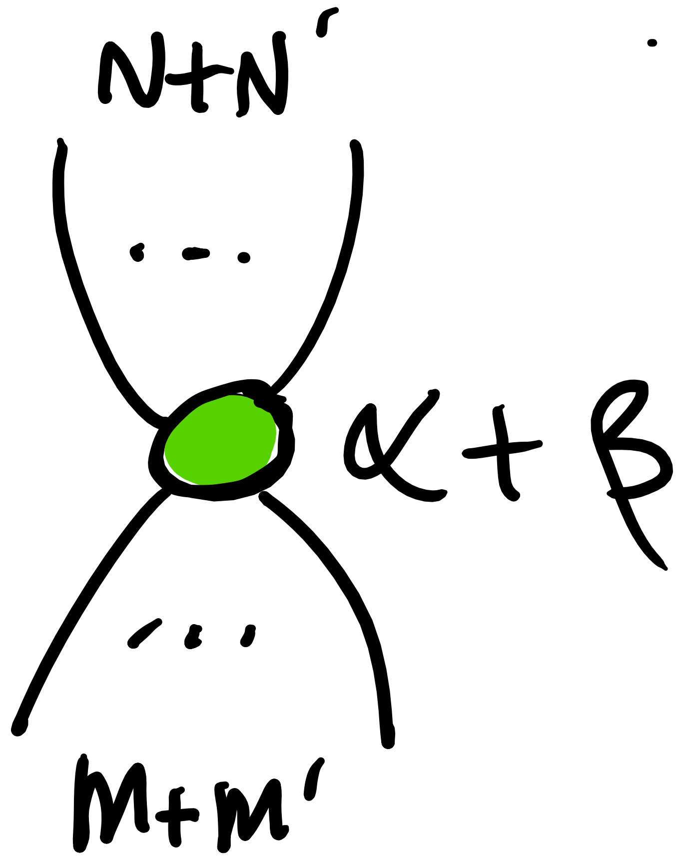
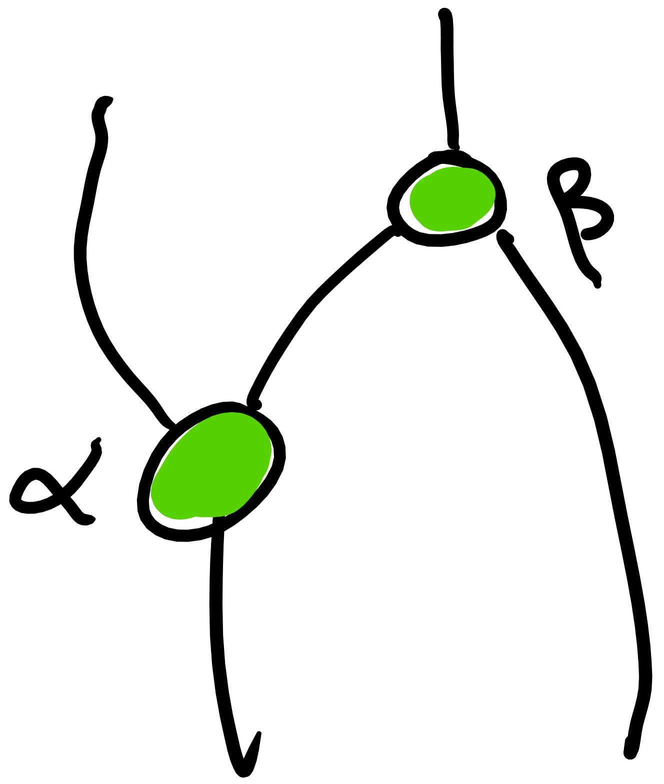
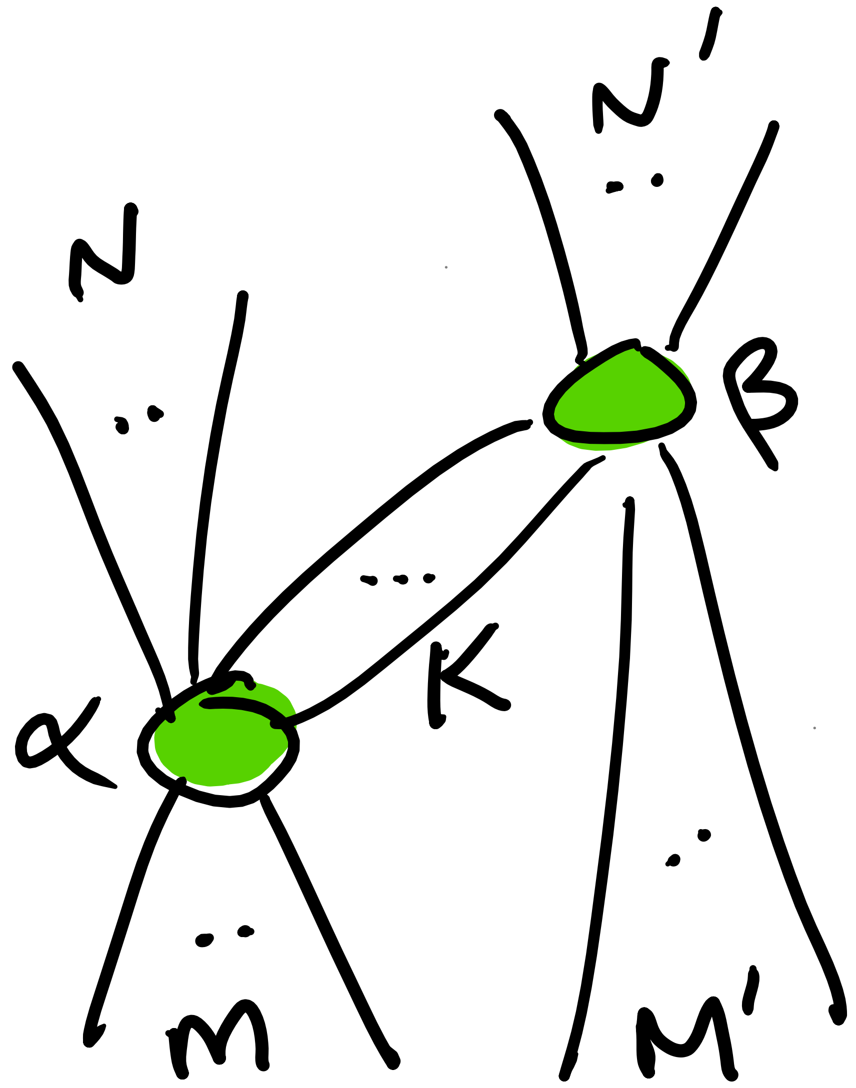


$(K \geq 1)$



$$= \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & e^{i\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\beta} \end{array}\right) \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & e^{i\alpha} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{array}\right)$$



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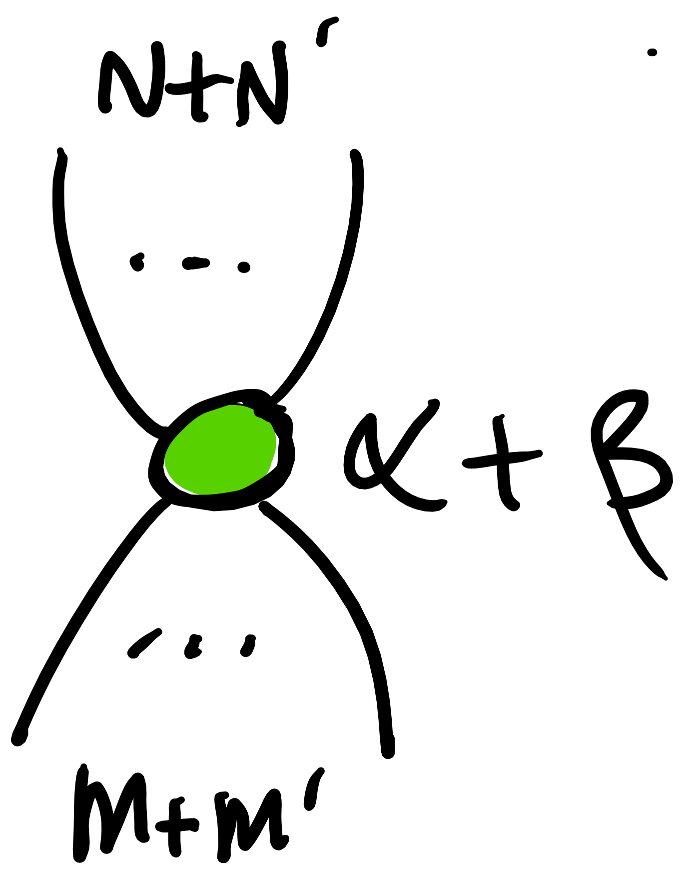

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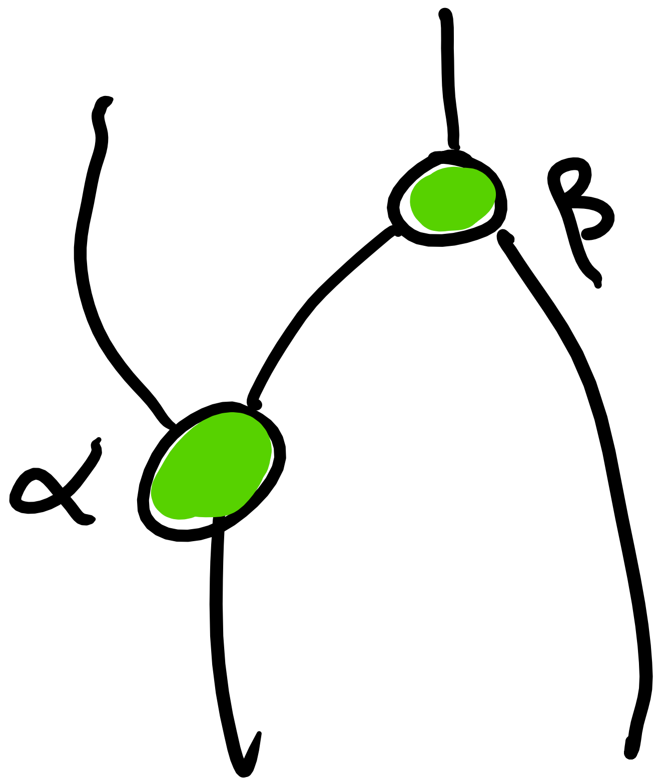
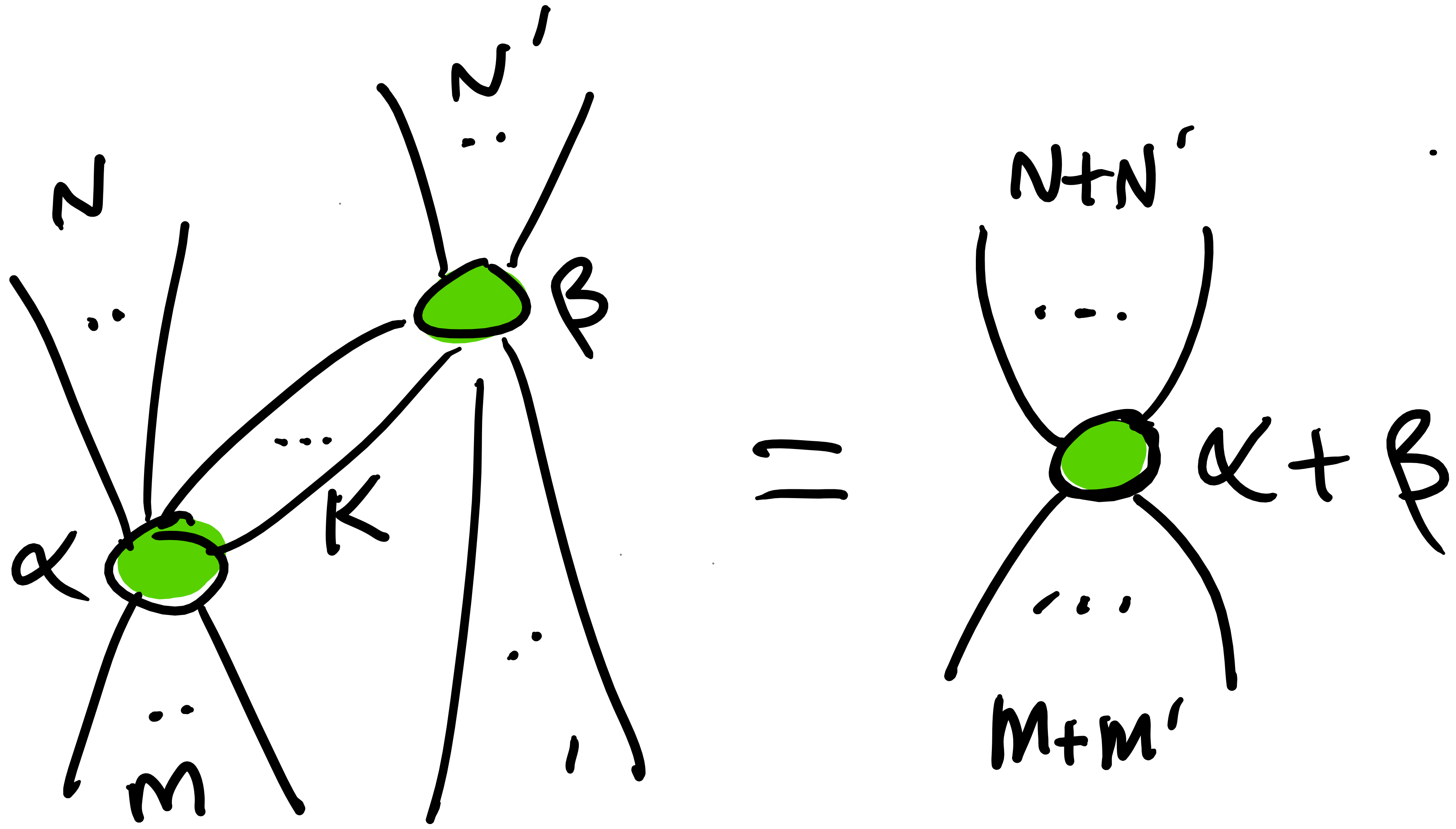
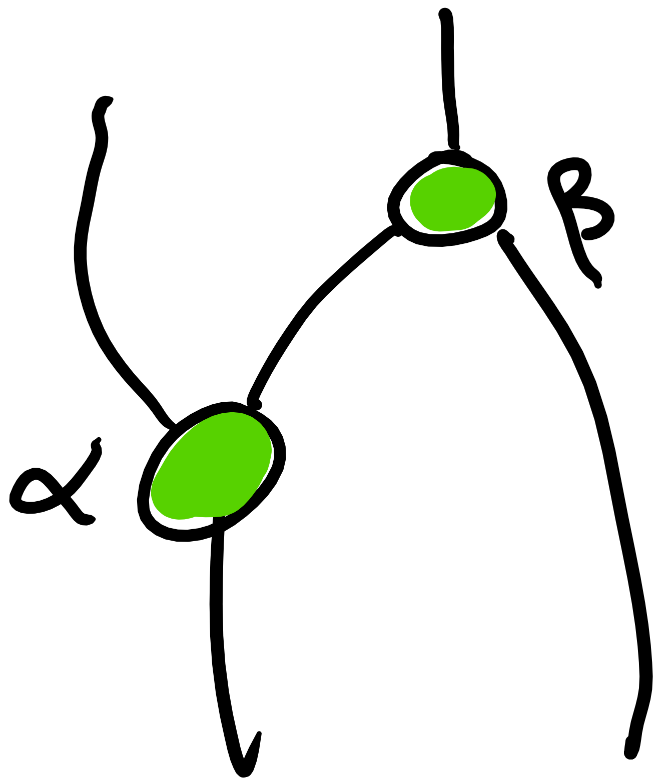


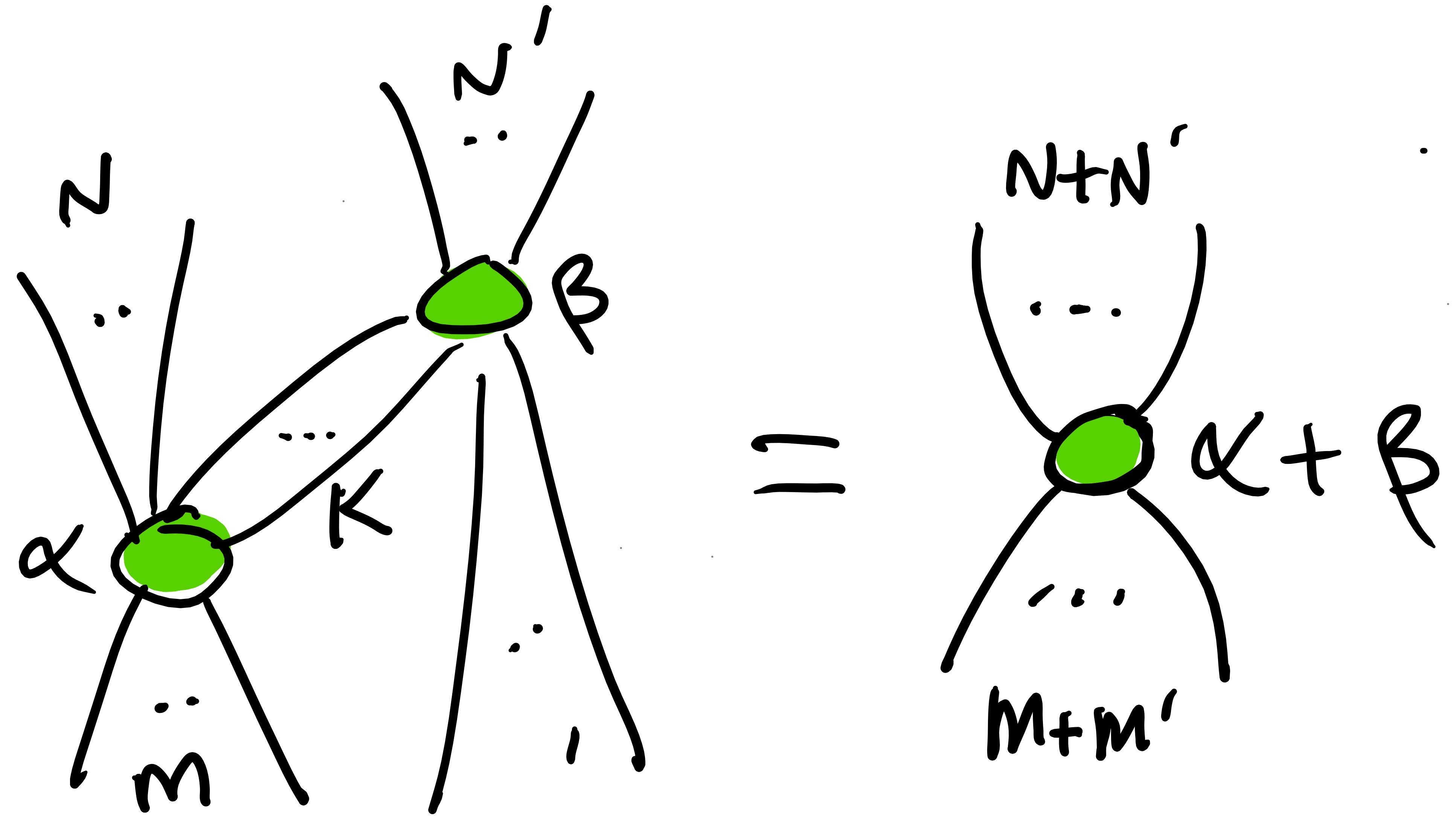
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha}e^{i\beta} \end{pmatrix}$$



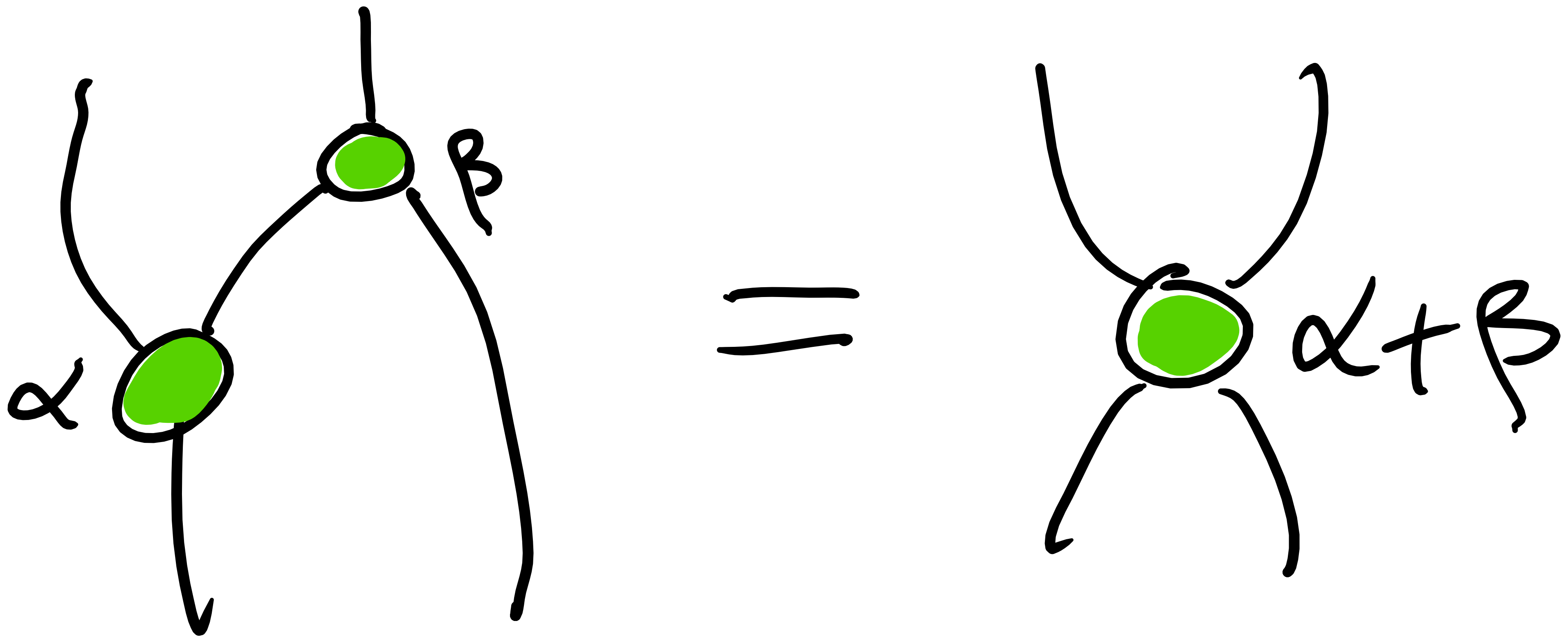
$(K \geq 1)$



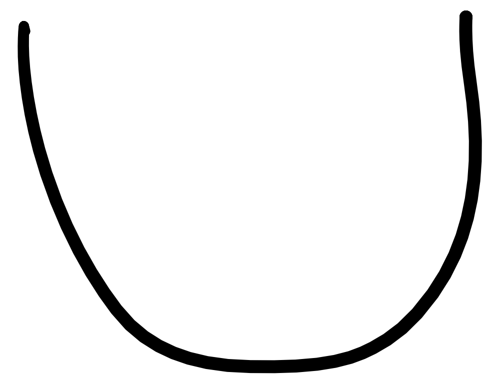
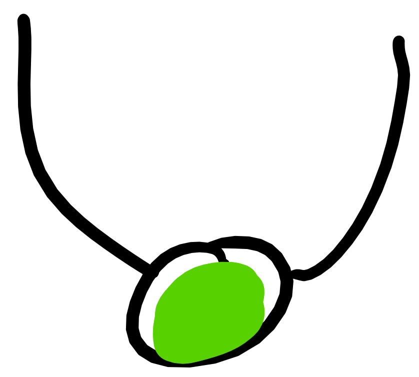
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i(\alpha+\beta)} \end{pmatrix}$$

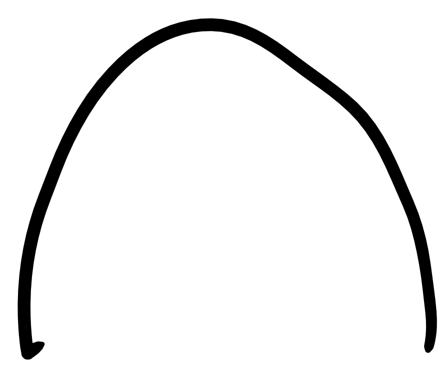
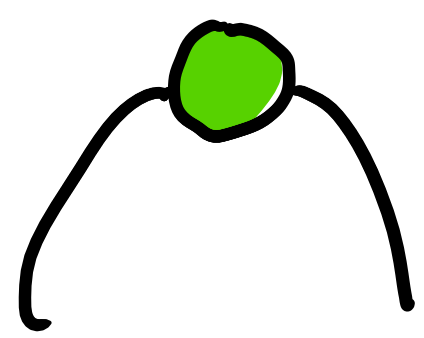


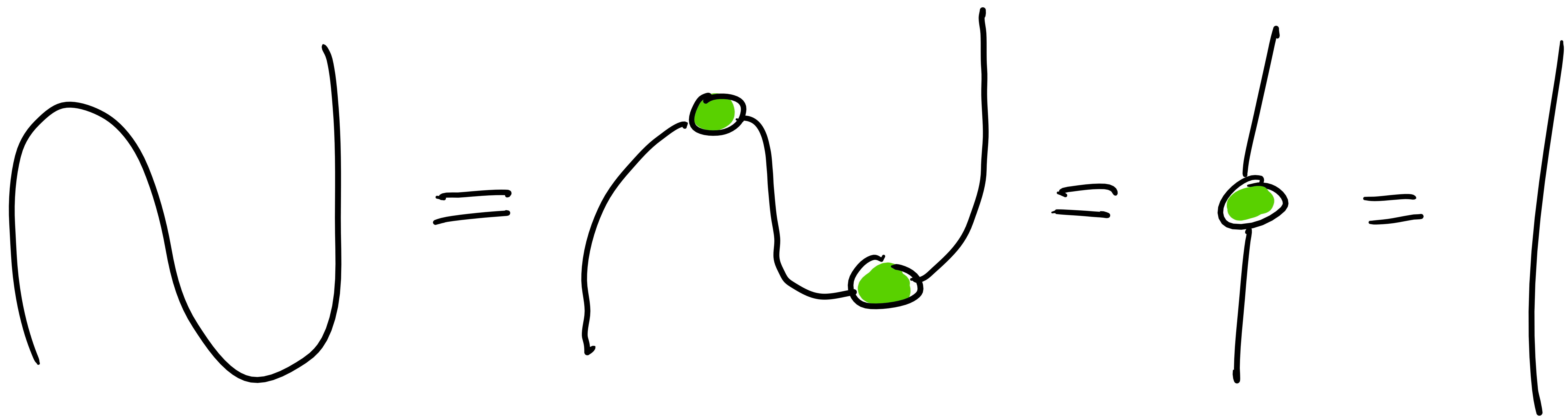
$(K \geq 1)$



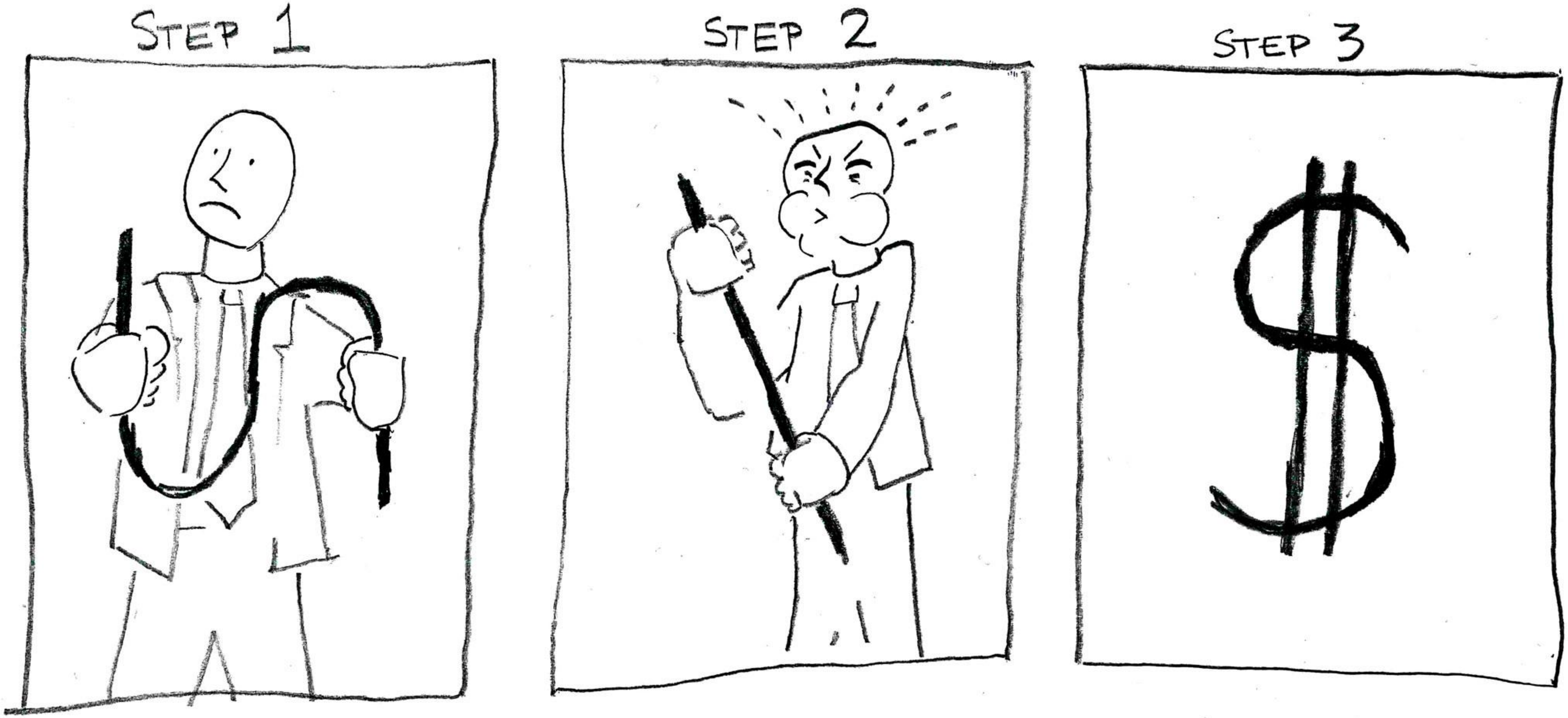
Cups & Caps

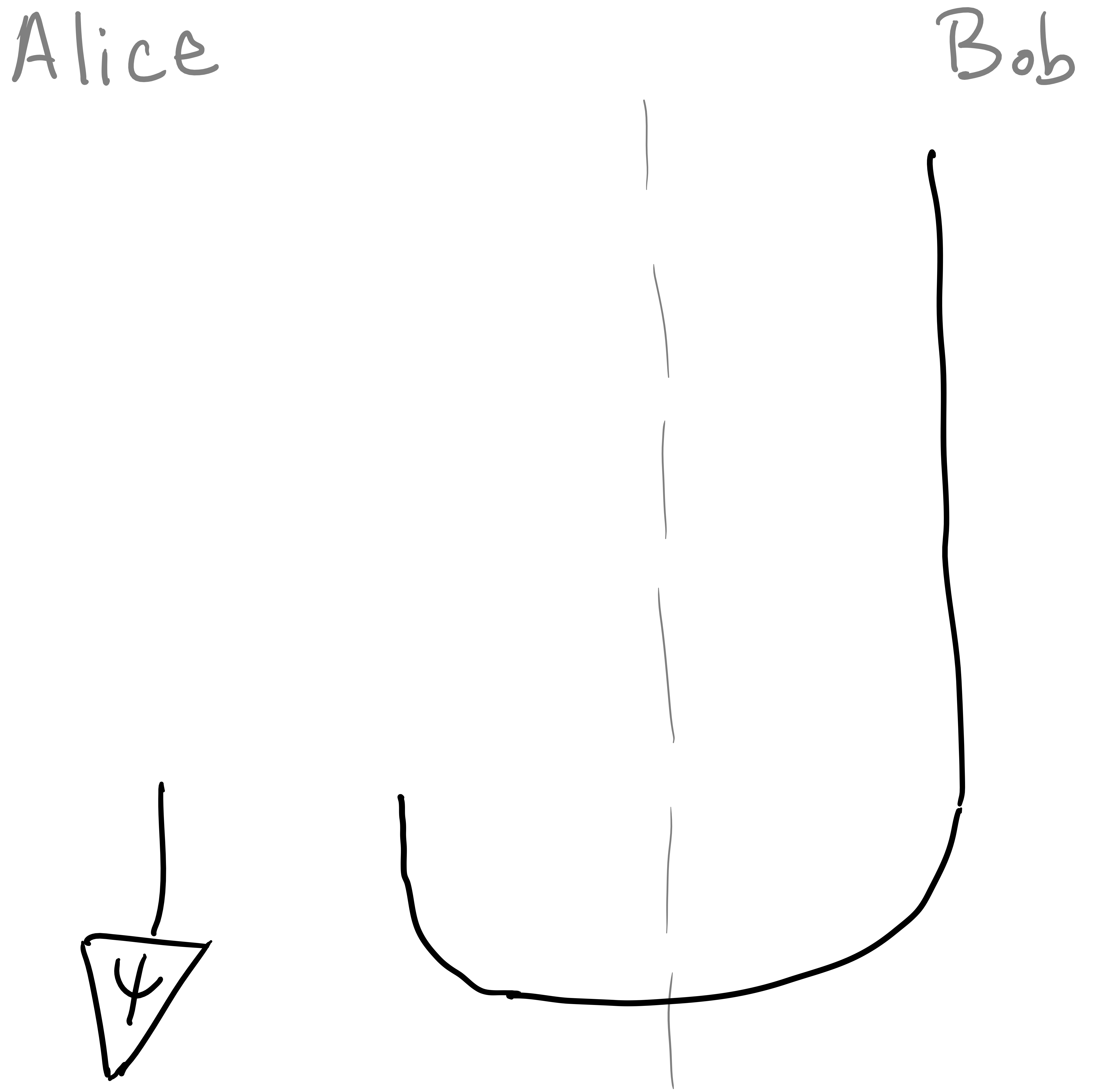
 $:=$ 

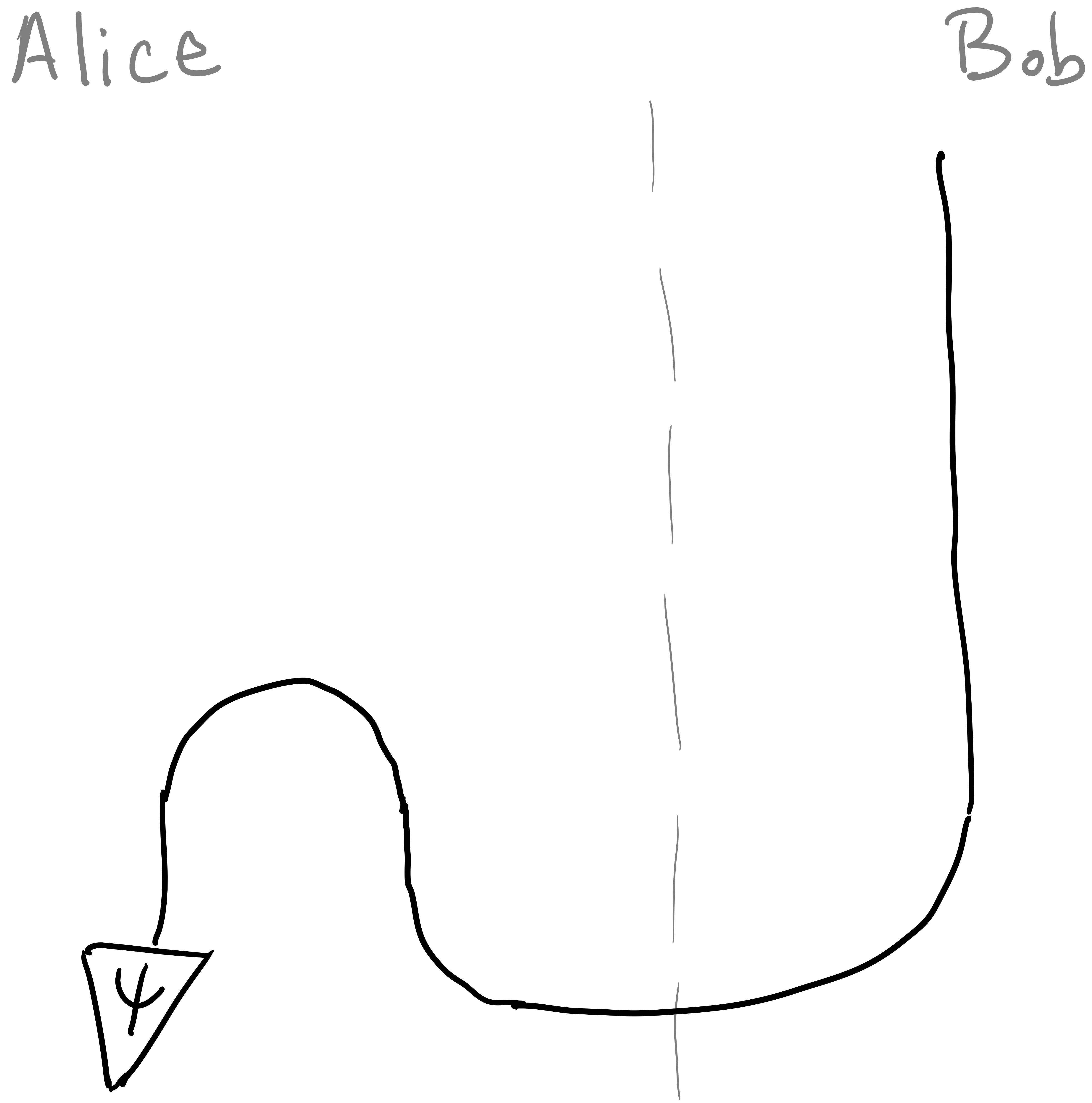
 $:=$ 

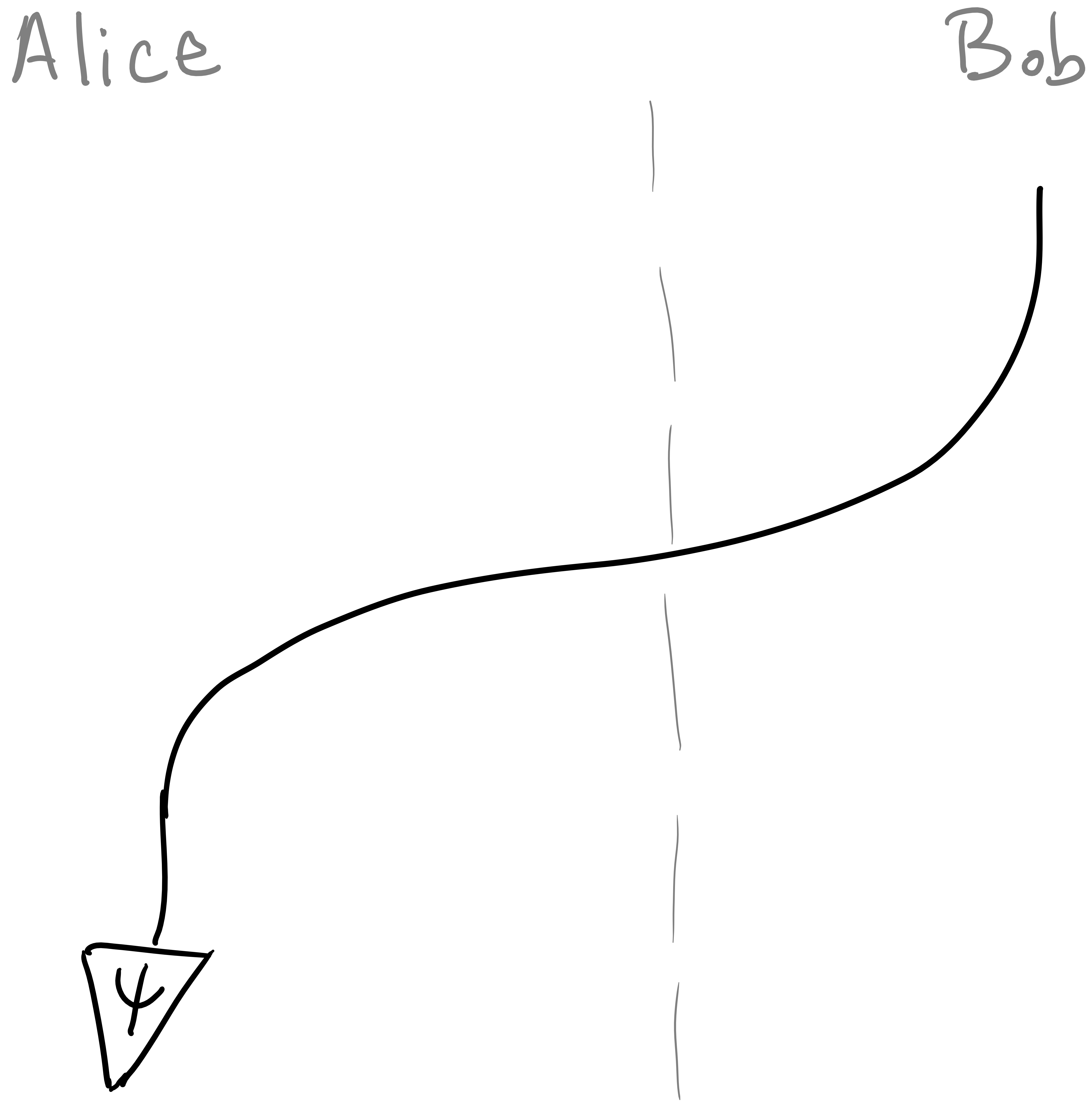


"THE FUNDAMENTALS OF CATEGORICAL QUANTUM MECHANICS"



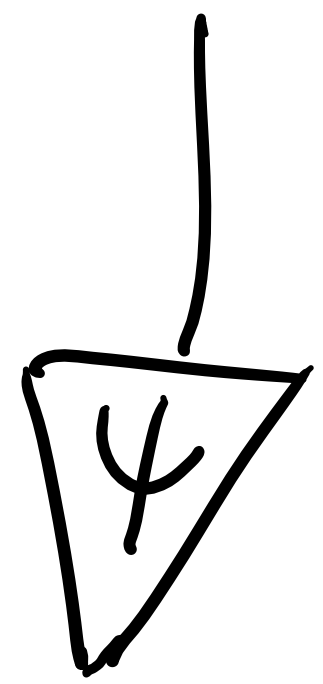

Alice
Bob
ψ


Alice
Bob
ψ

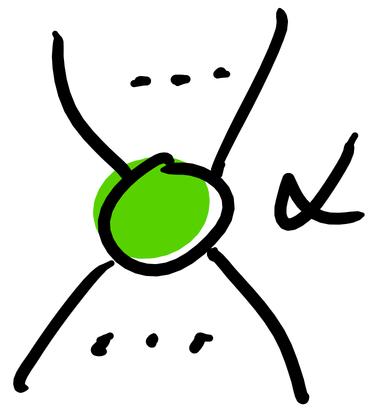

Alice
Bob
ψ

Alice

Bob



ZX-diagrams consist of:

I. Z spiders  $:= \begin{pmatrix} 1 & \\ & e^{i\alpha} \end{pmatrix}$

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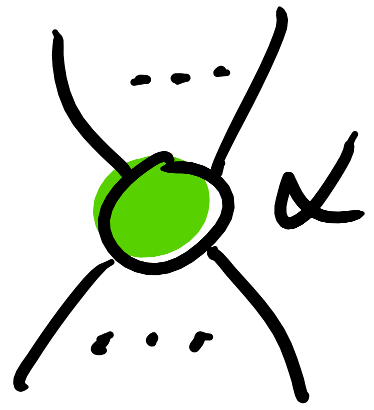
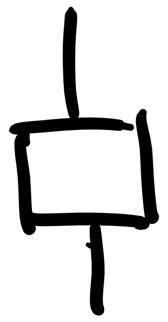
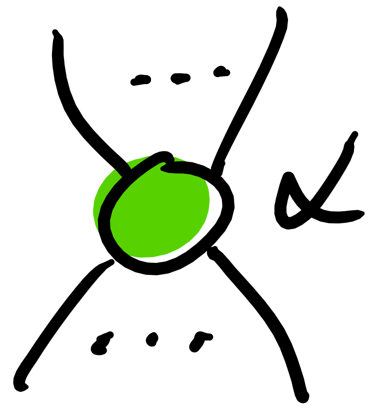
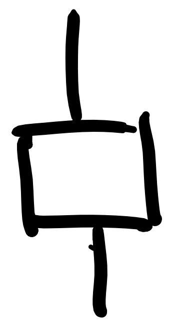
I. Z spiders  $:= \begin{pmatrix} 1 & \\ & e^{i\alpha} \end{pmatrix}$

II. colour-changer  $:= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

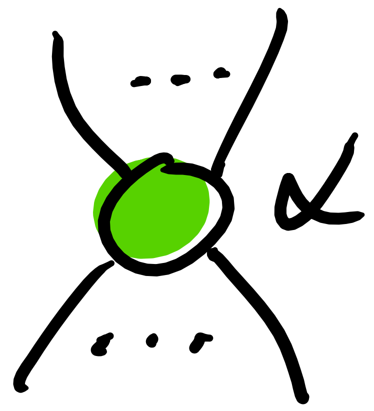
ZX-diagrams consist of:

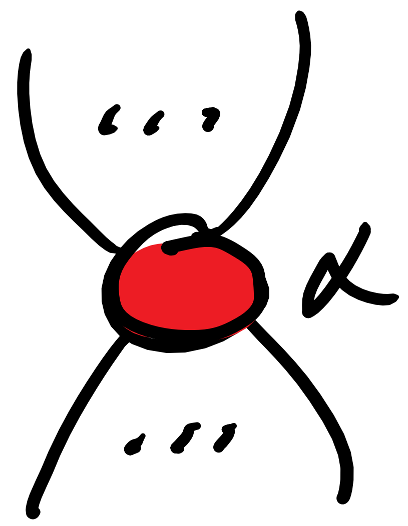
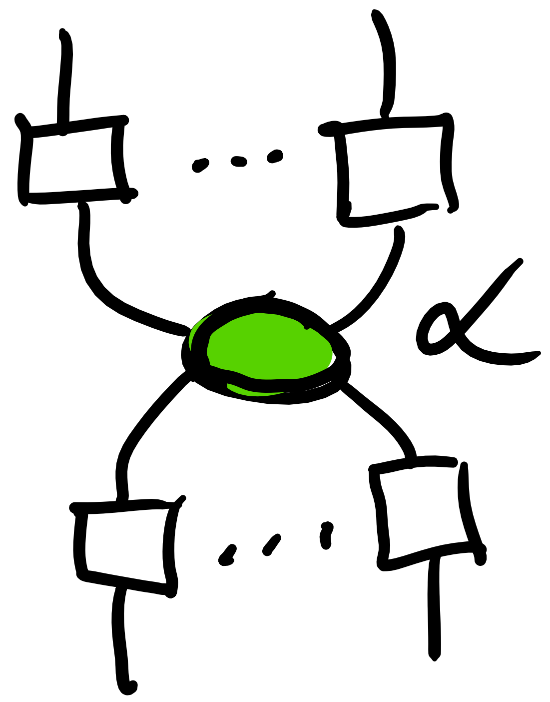
I. Z spiders  $:= \begin{pmatrix} 1 & \\ & e^{i\alpha} \end{pmatrix}$

II. colour-changer  $:= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

III. identity & swap.

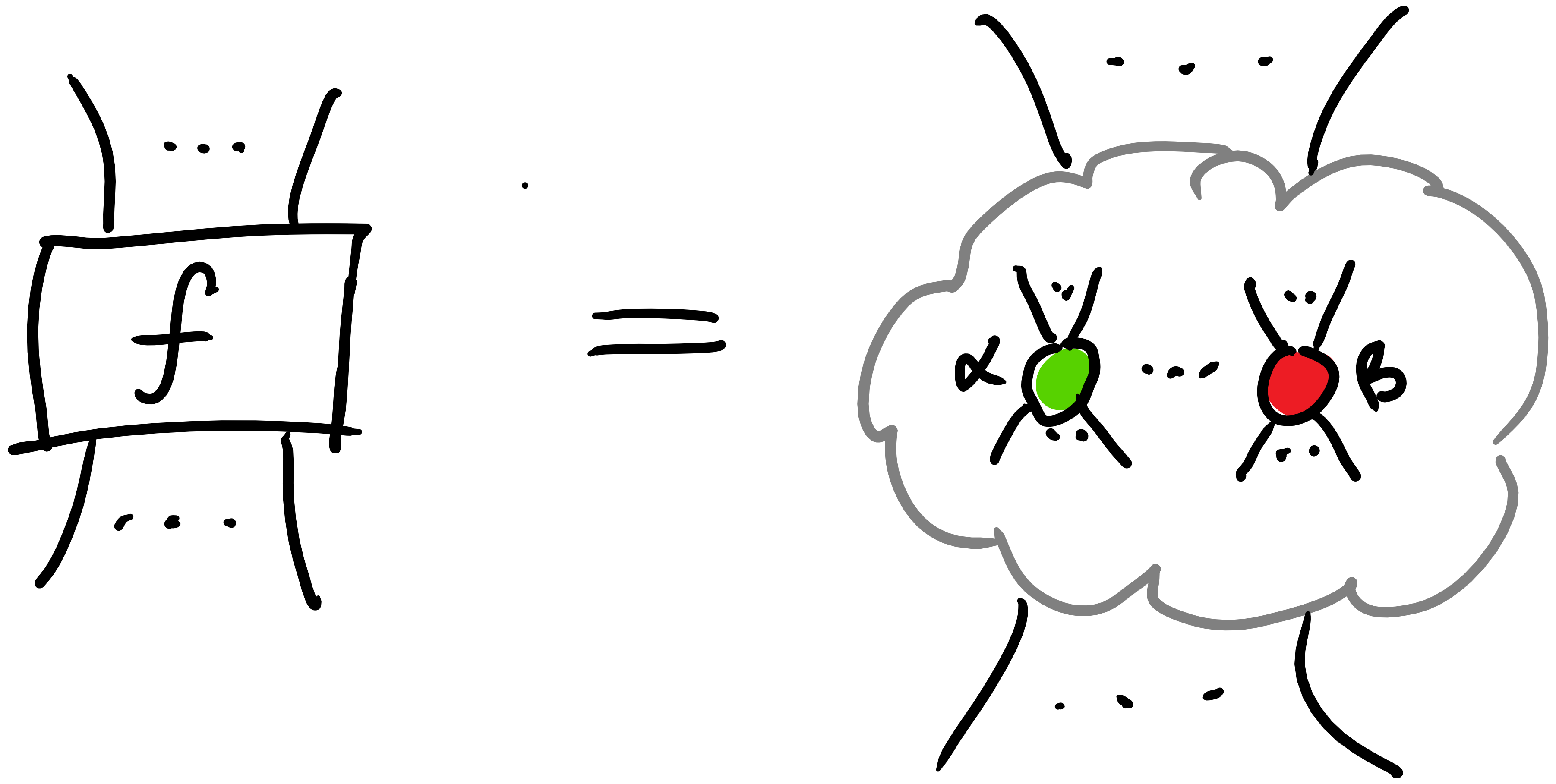
ZX-diagrams equivalently consist of:

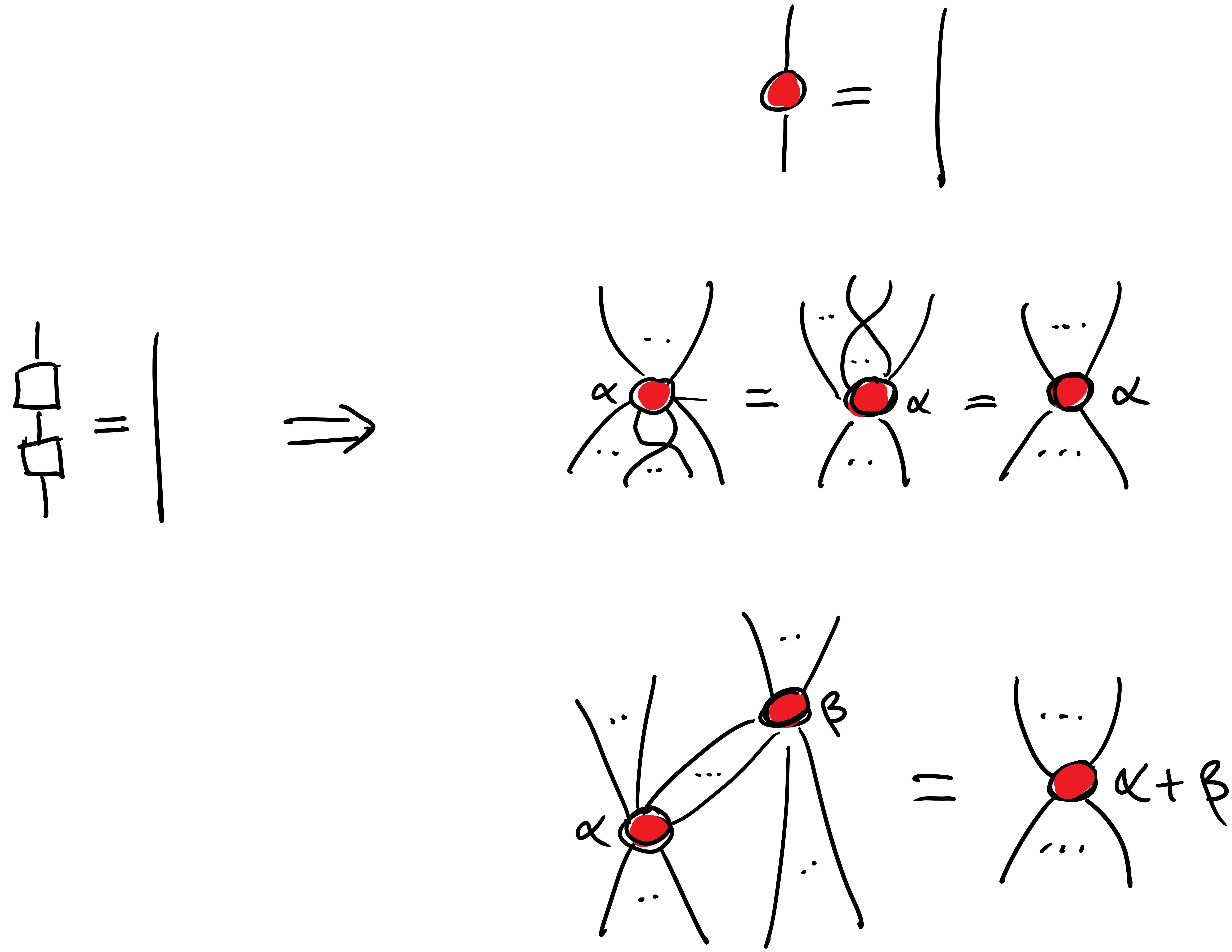
I. Z spiders  $:= \begin{pmatrix} 1 & \\ & e^{i\alpha} \end{pmatrix}$

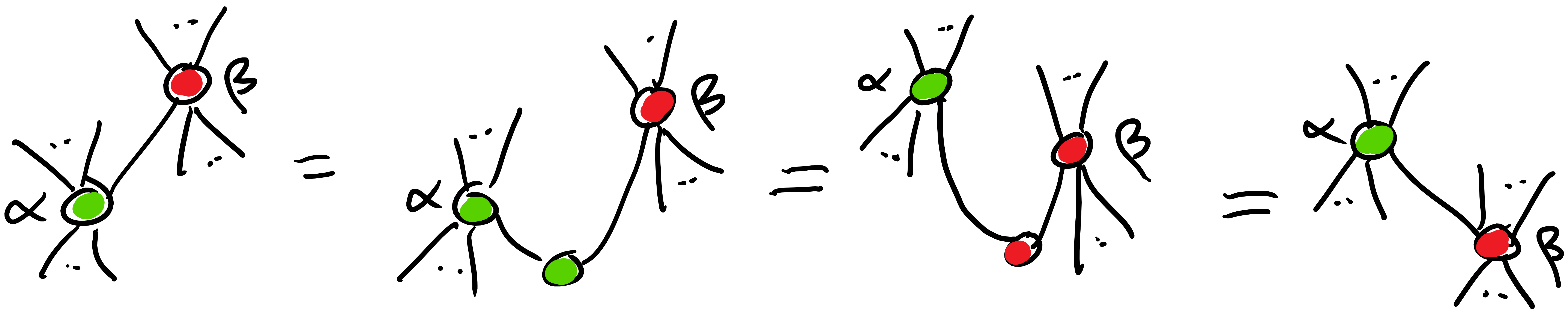
II. X spiders  $:=$ 

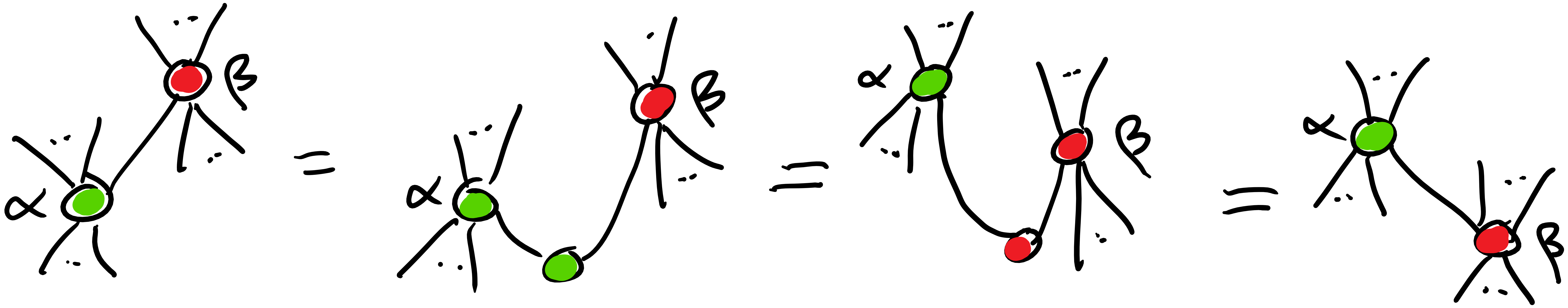
III. identity & swap.

THEOREM (Universality). Any $2^N \times 2^M$ matrix can be expressed as a ZX-diagram:




α
α
α
β
α
α+β

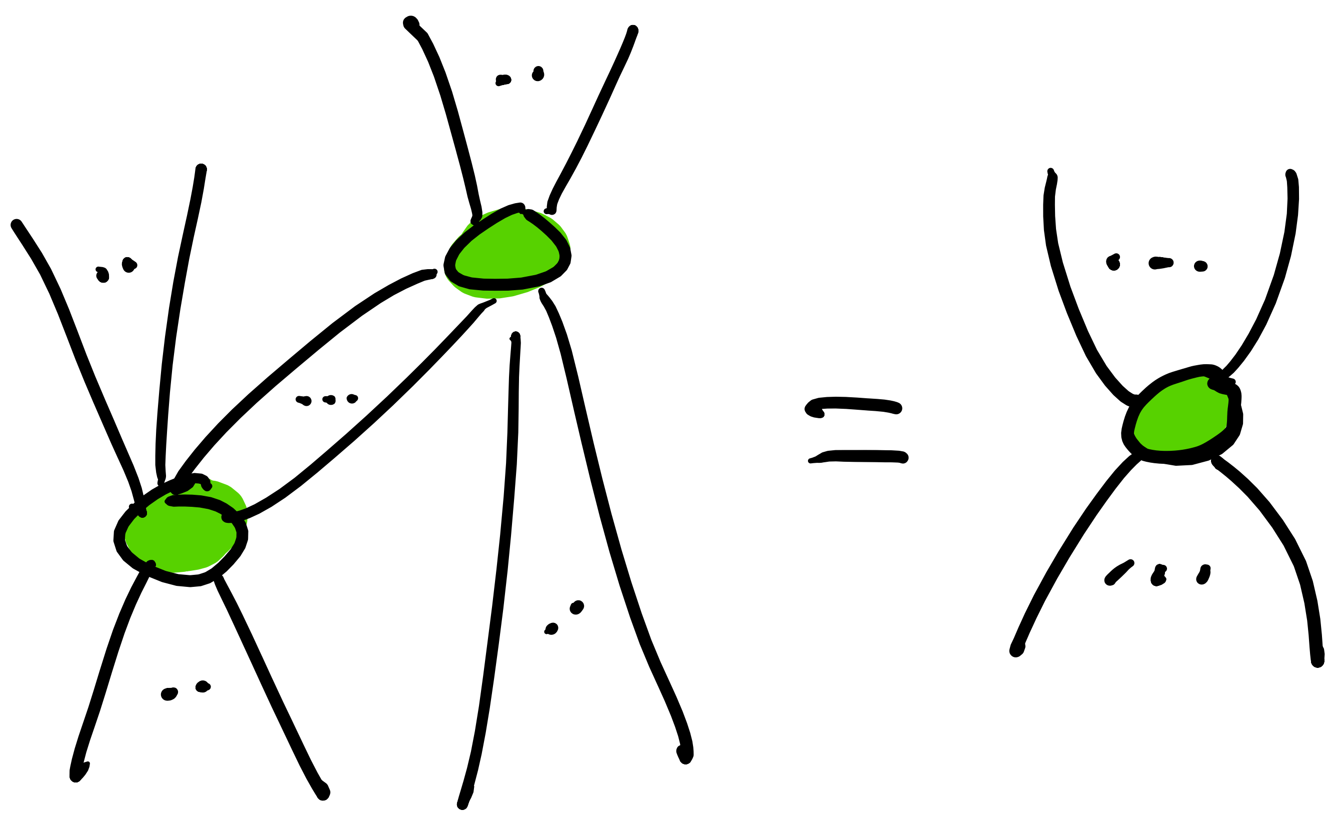

α
β

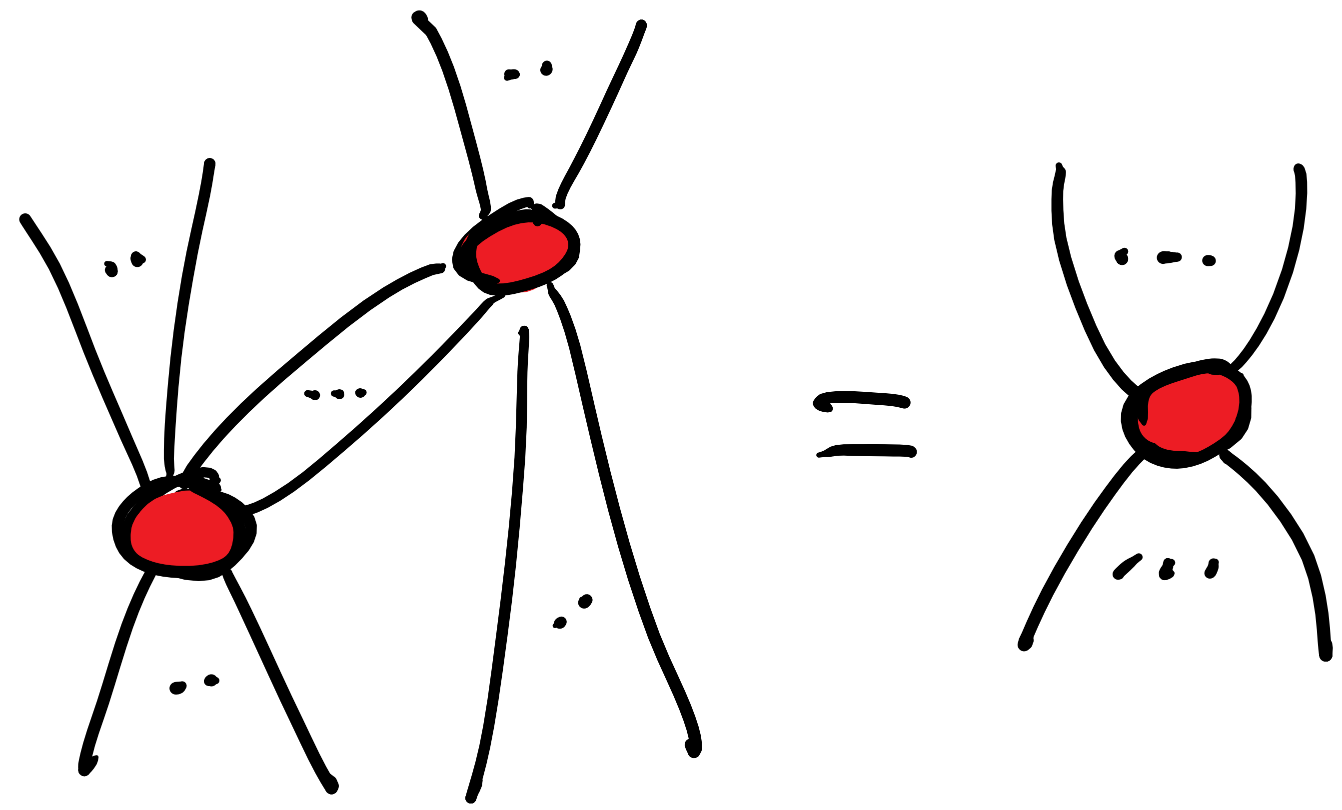


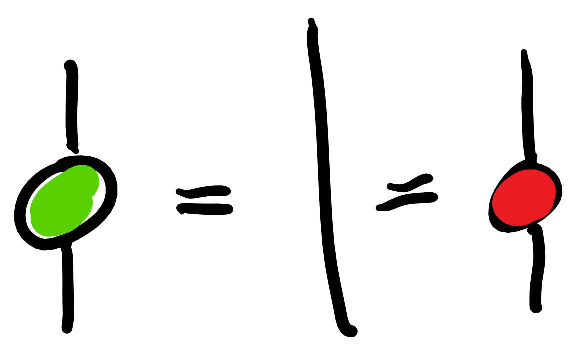
∴ ZX-diagrams are un-directed.

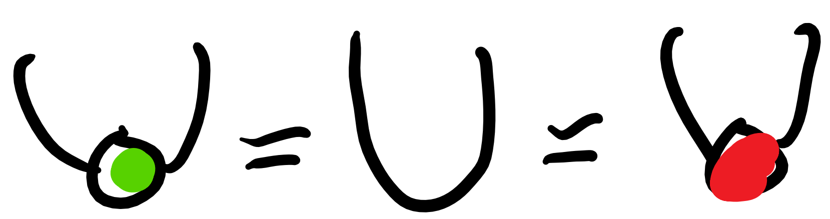
PHASE-FREE

ZX-calculus



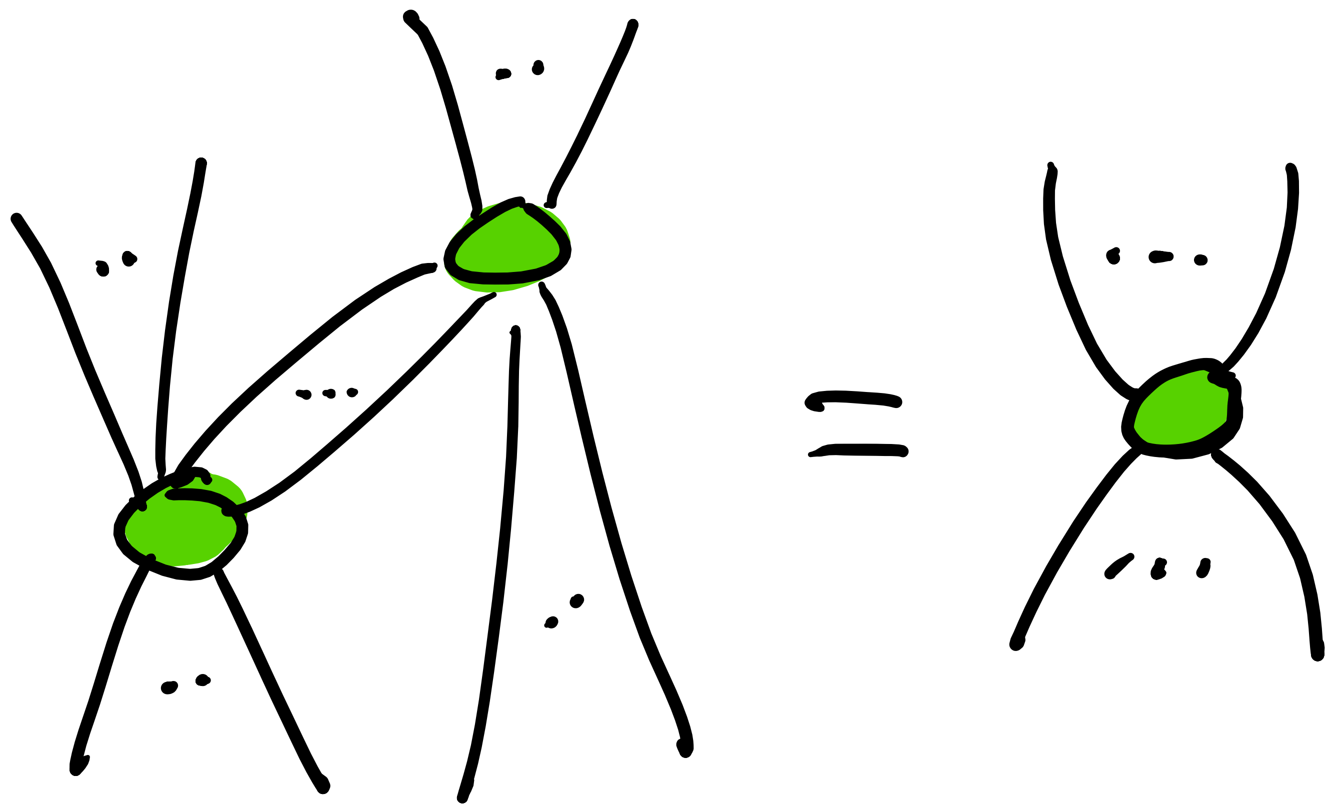


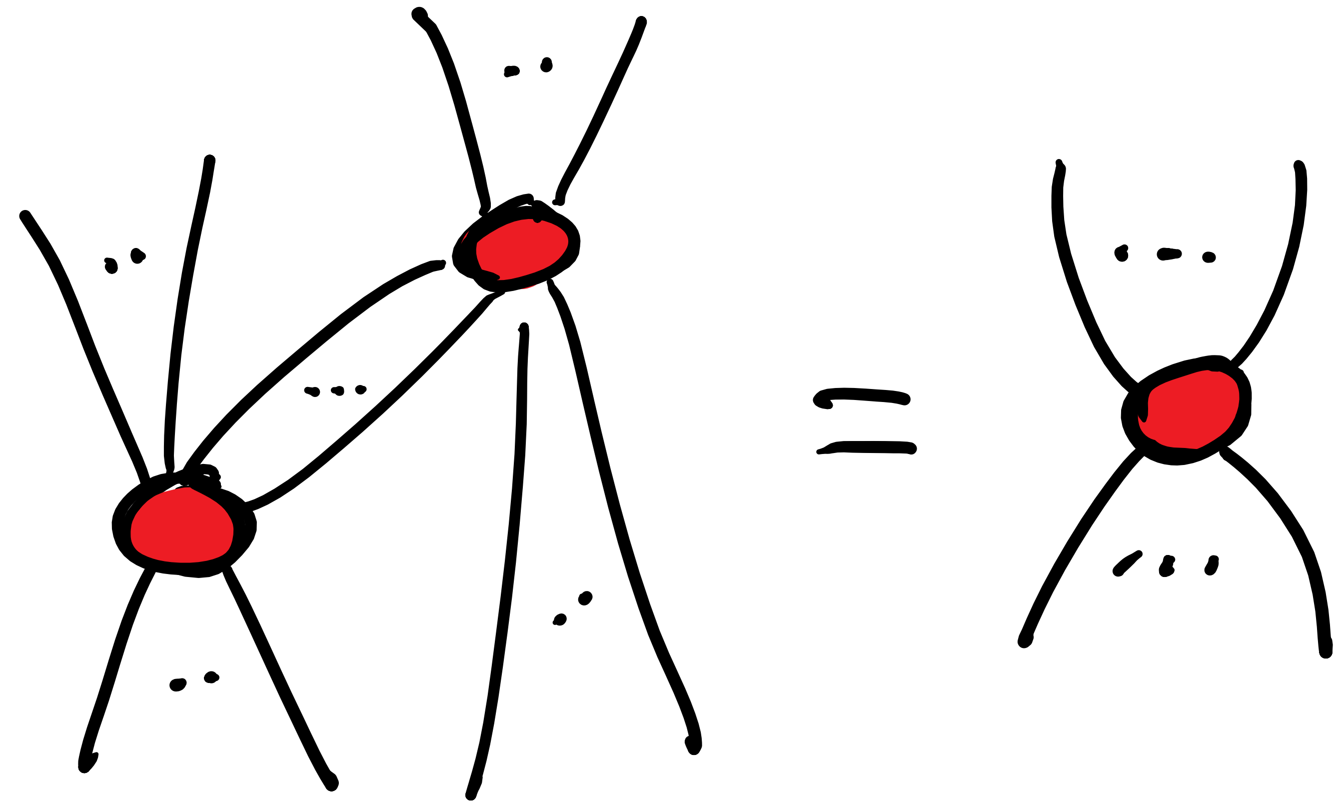


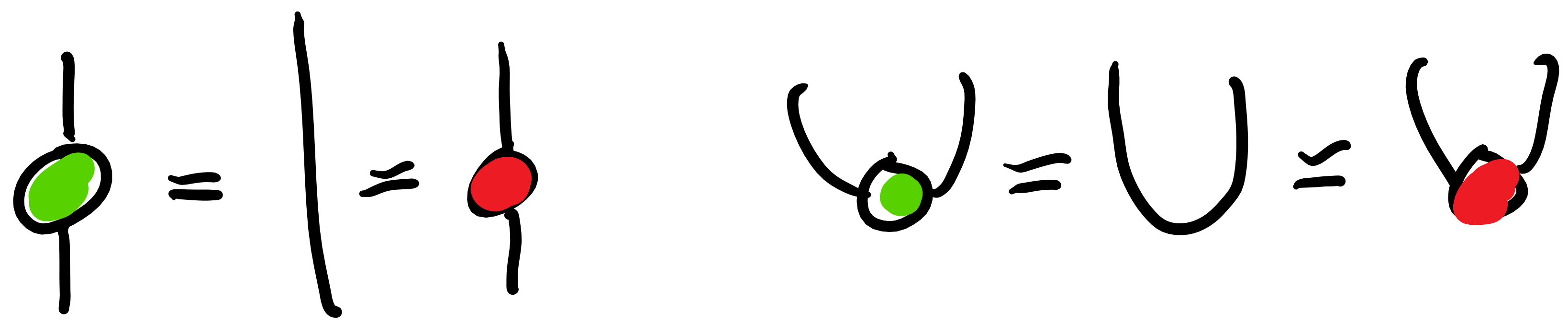


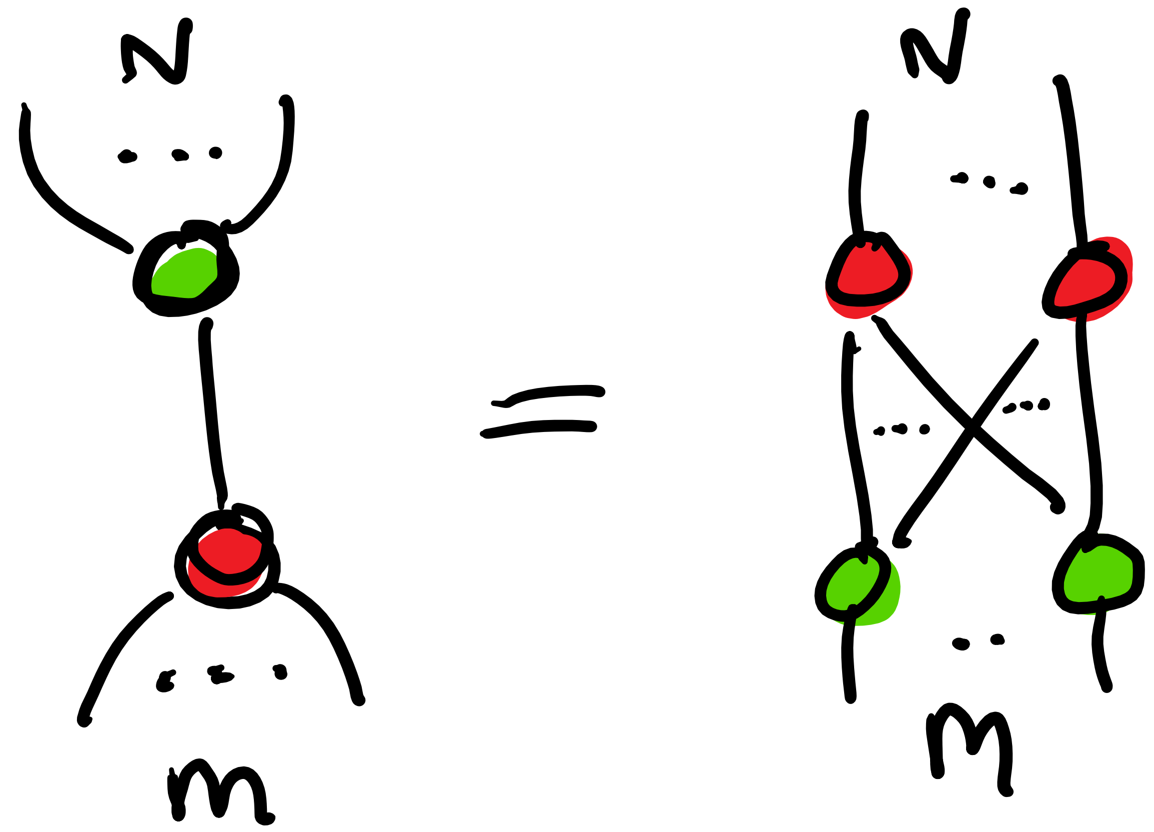
PHASE-FREE

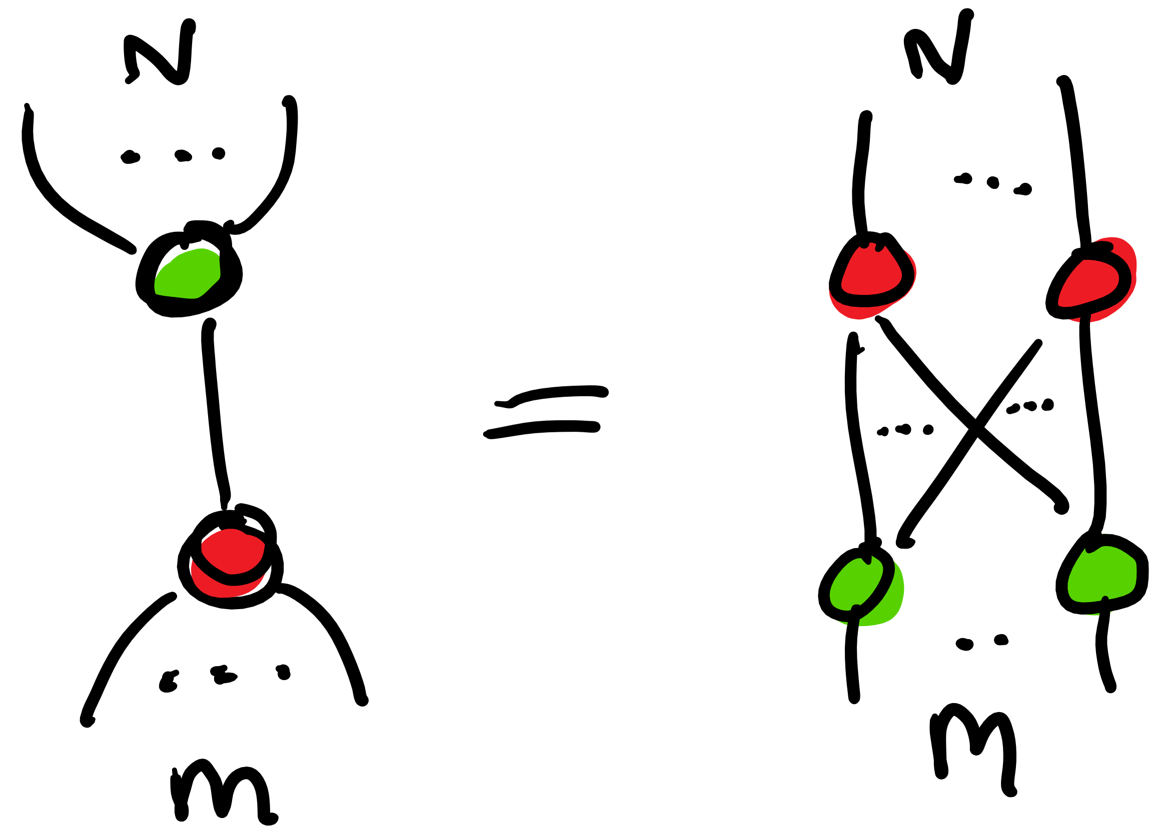
ZX-calculus



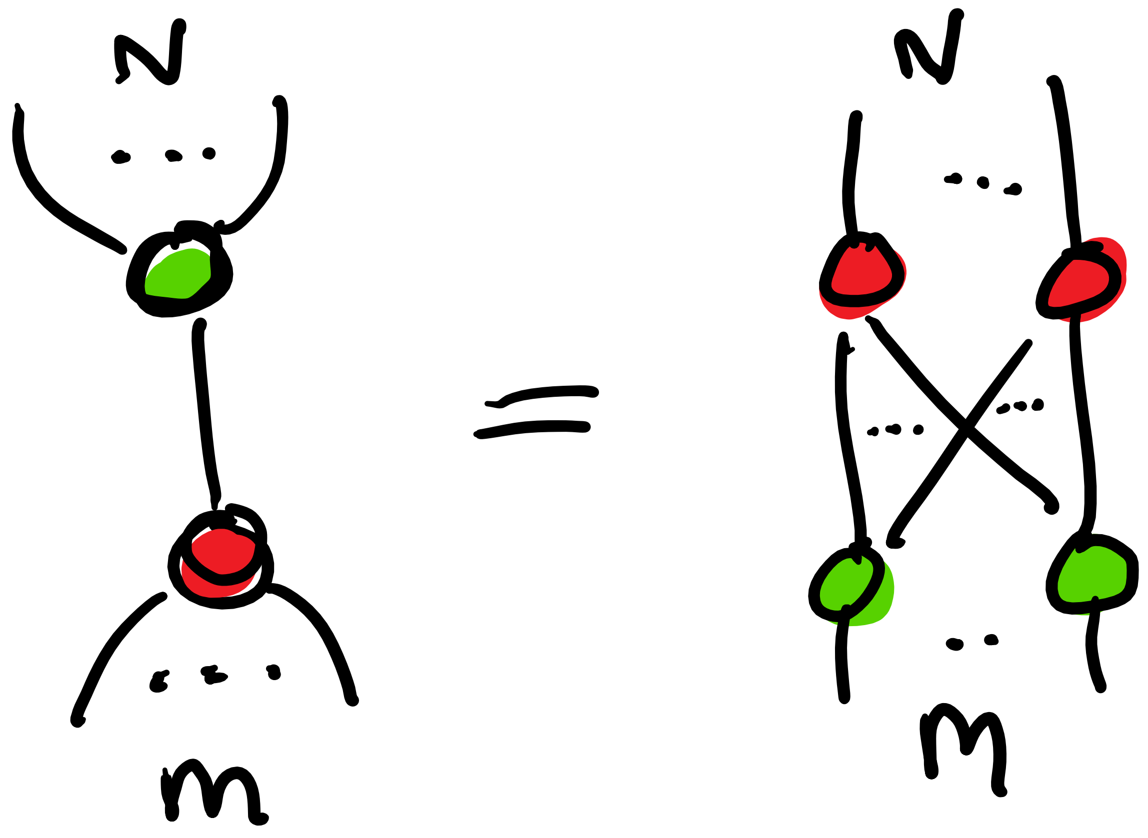






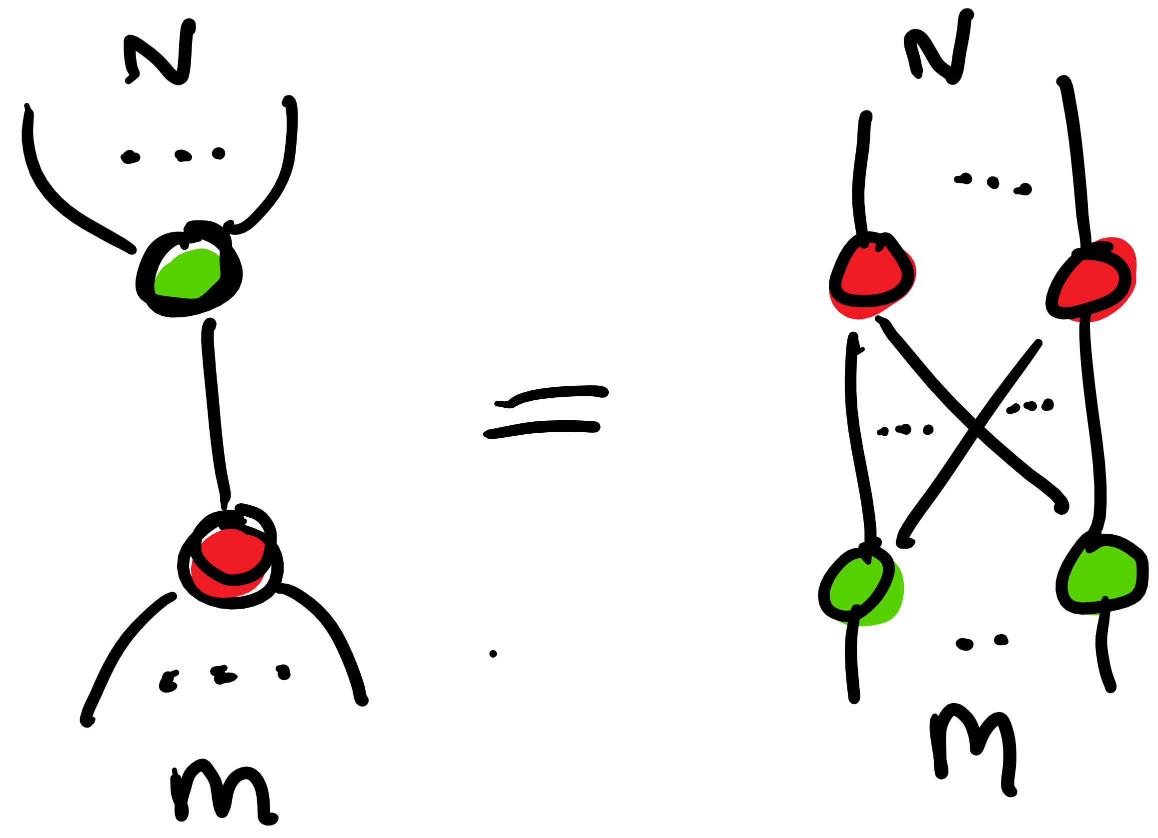


n
m
=
n
m



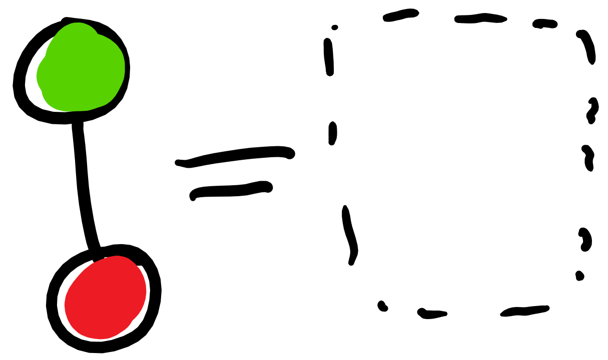
$$\Updownarrow$$

$m=N=0$ $\quad$ $M=0\ \ N=2$ $\quad$ $M=2\ \ N=0$ $\quad$ $M=N=2$

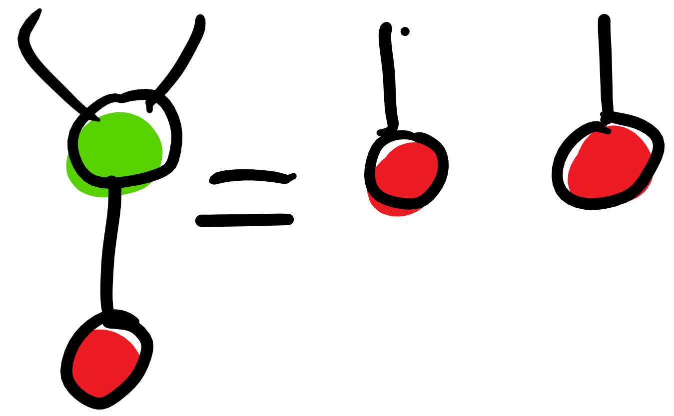


$$\Updownarrow$$

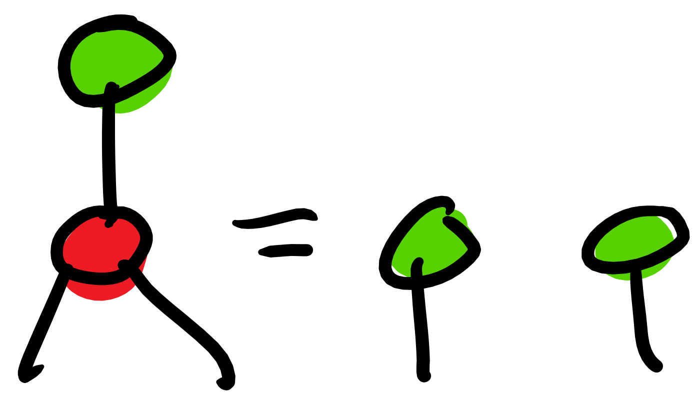
$m=N=0$



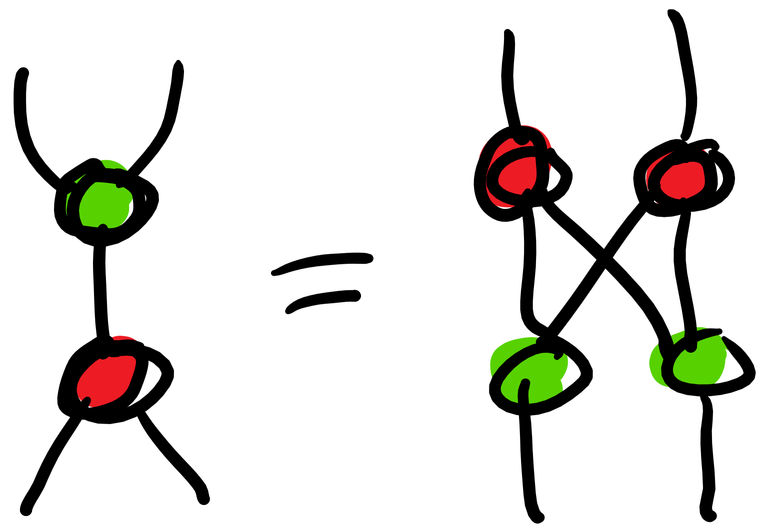
$M=0 \quad N=2$



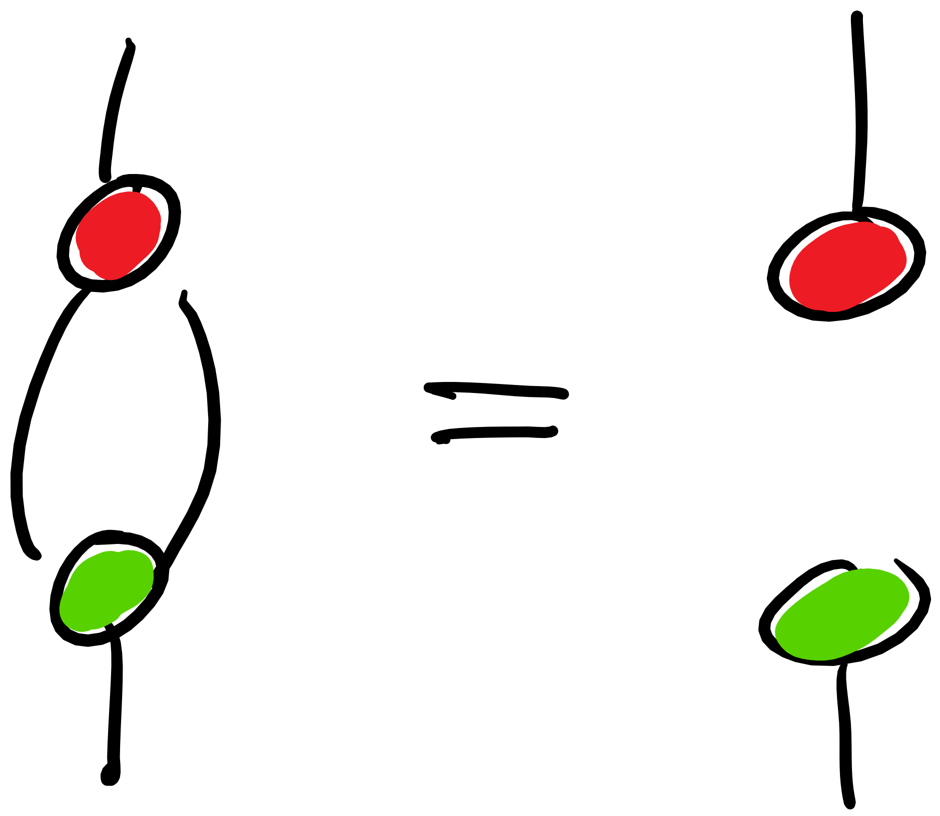
$M=2 \quad N=0$

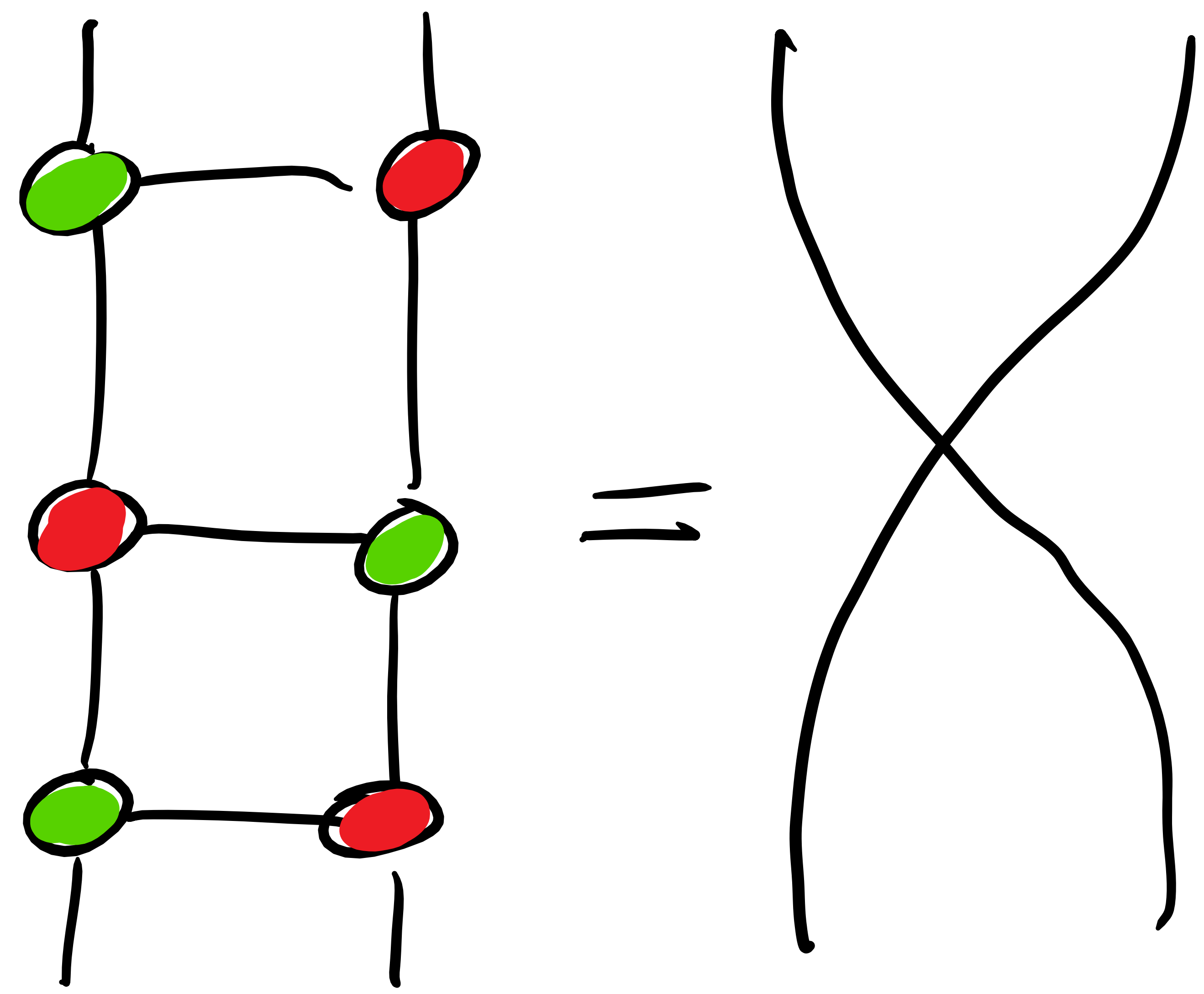


$M=N=2$



CONSEQUENCES





Consequences