

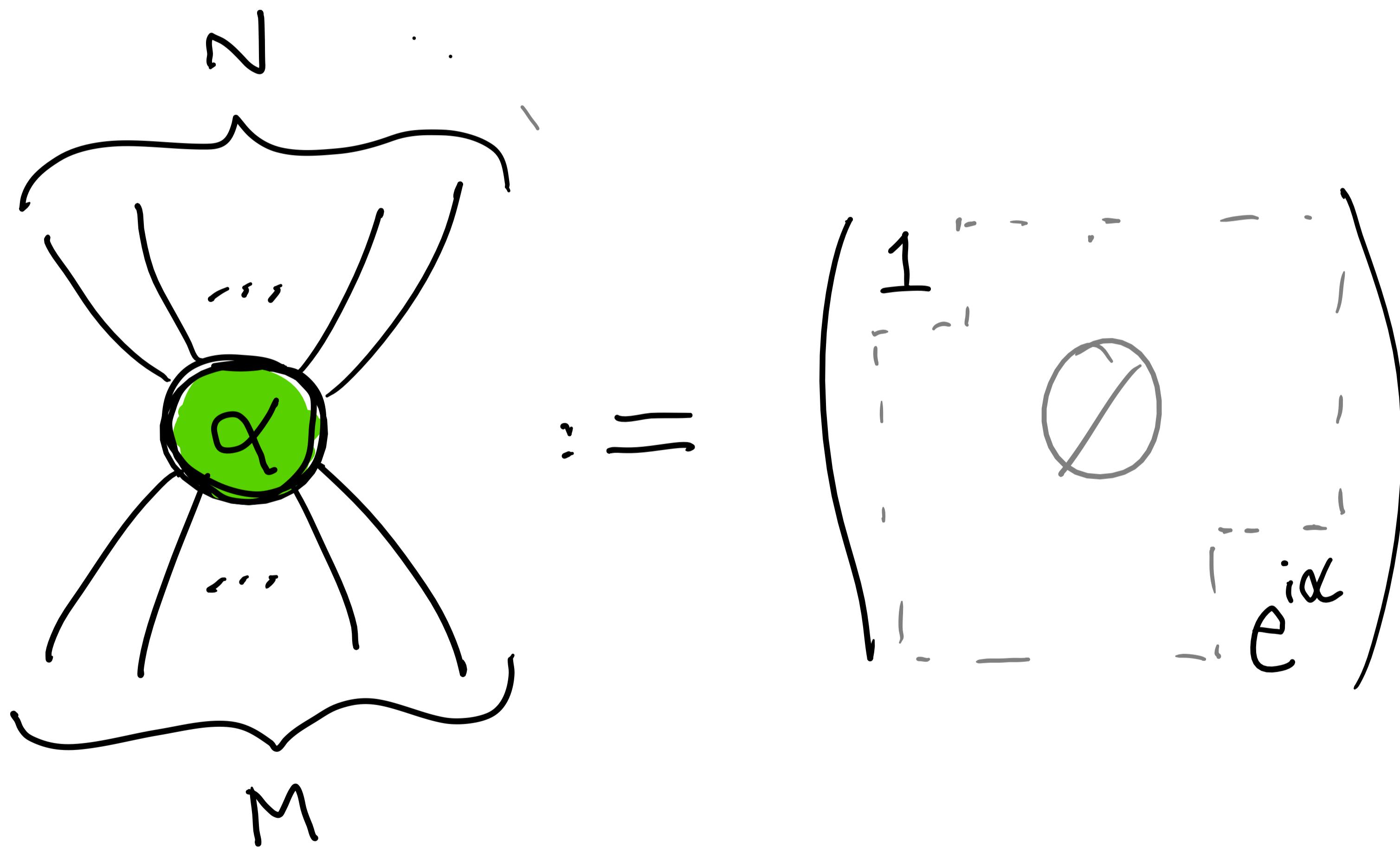
PICTURING QUANTUM PROCESSES II

ALEKS KISSINGER

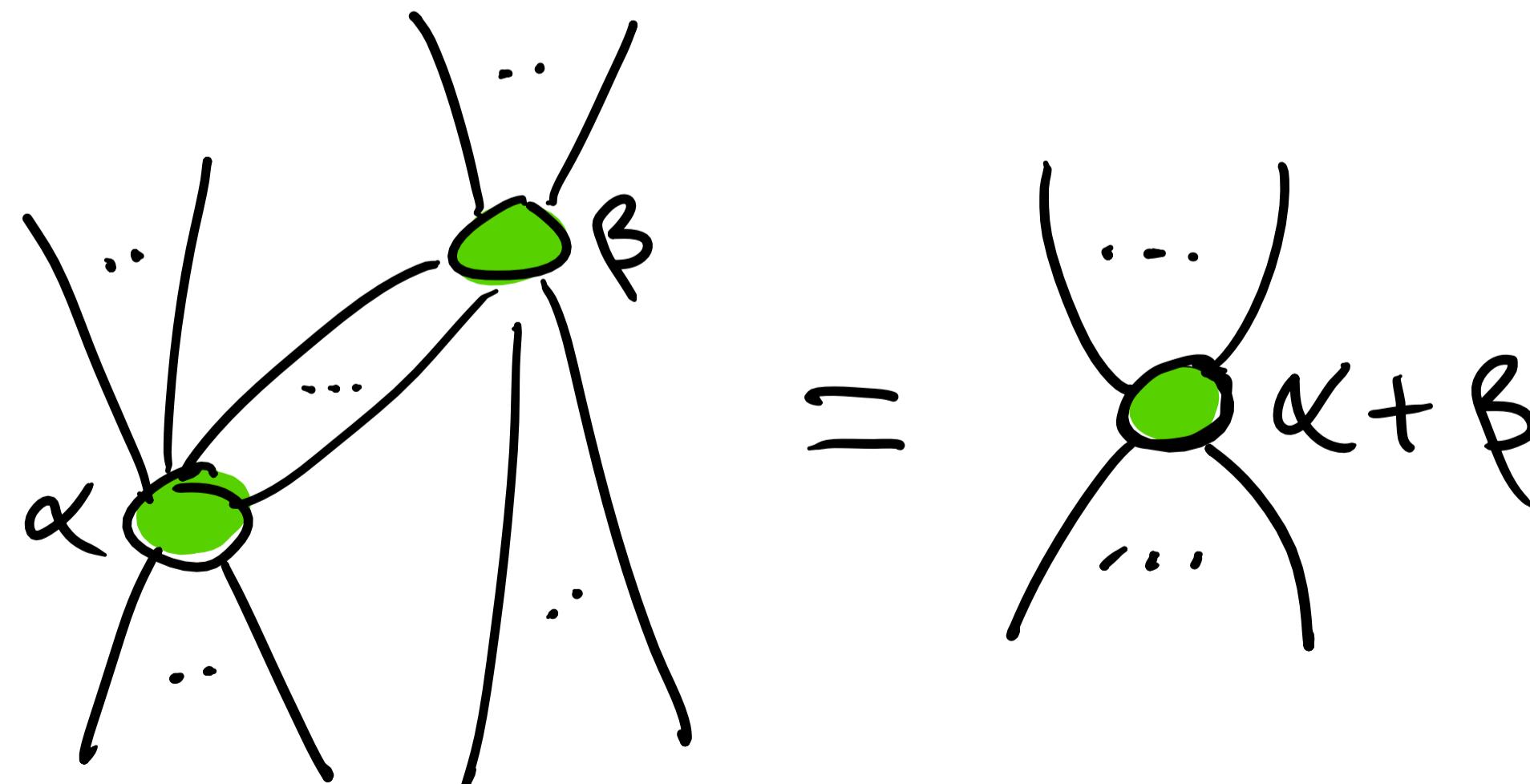
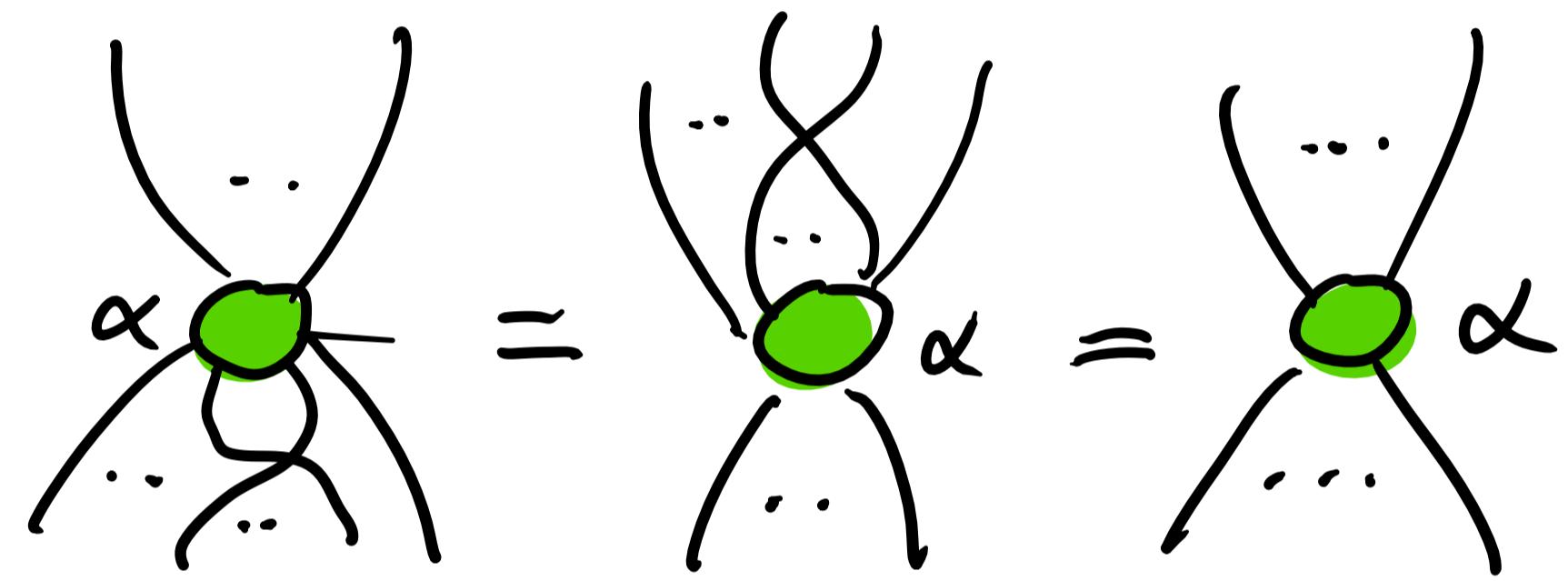
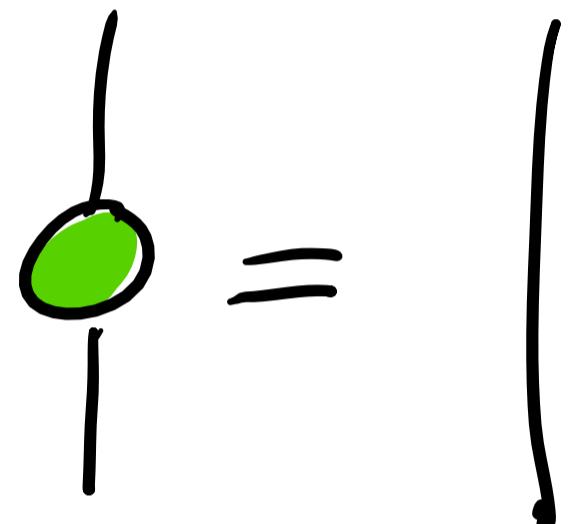
ISR 2019

Paris

SPIDERS

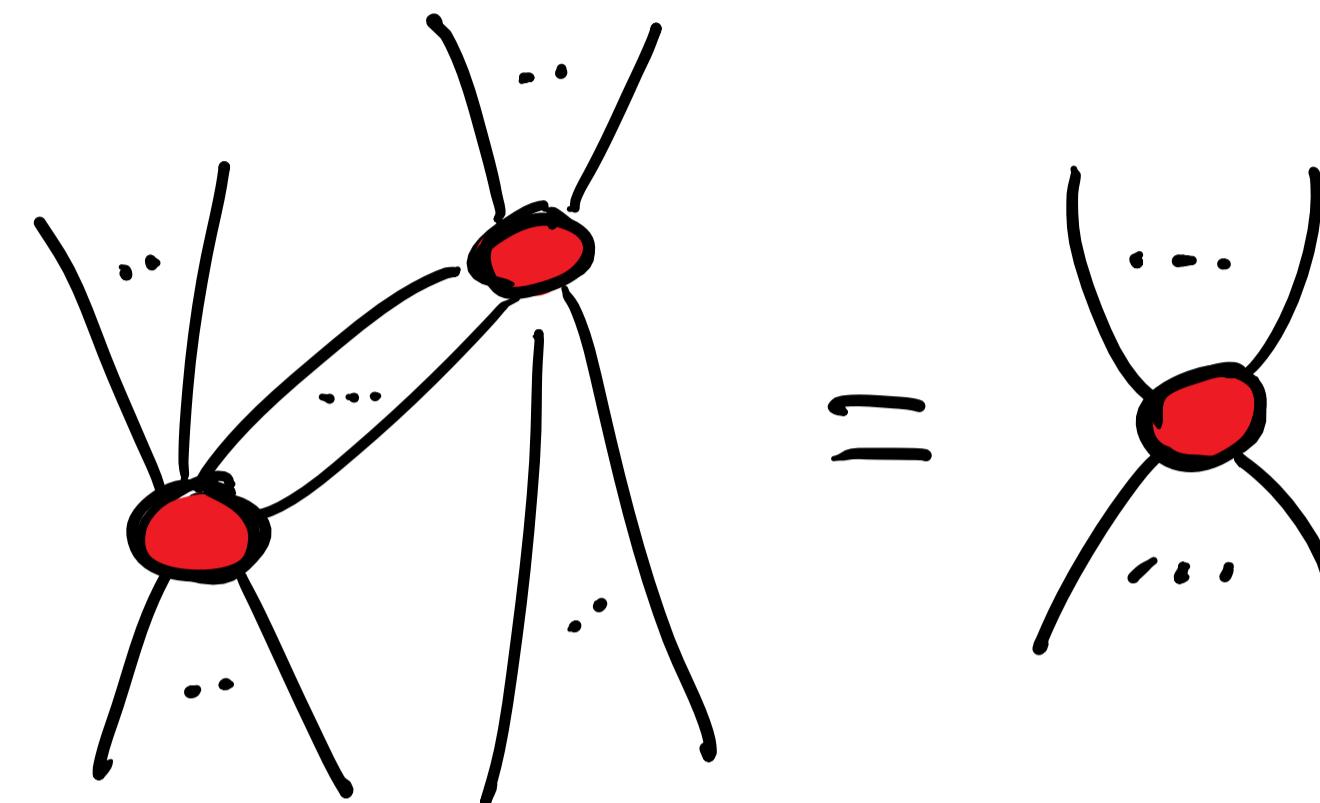
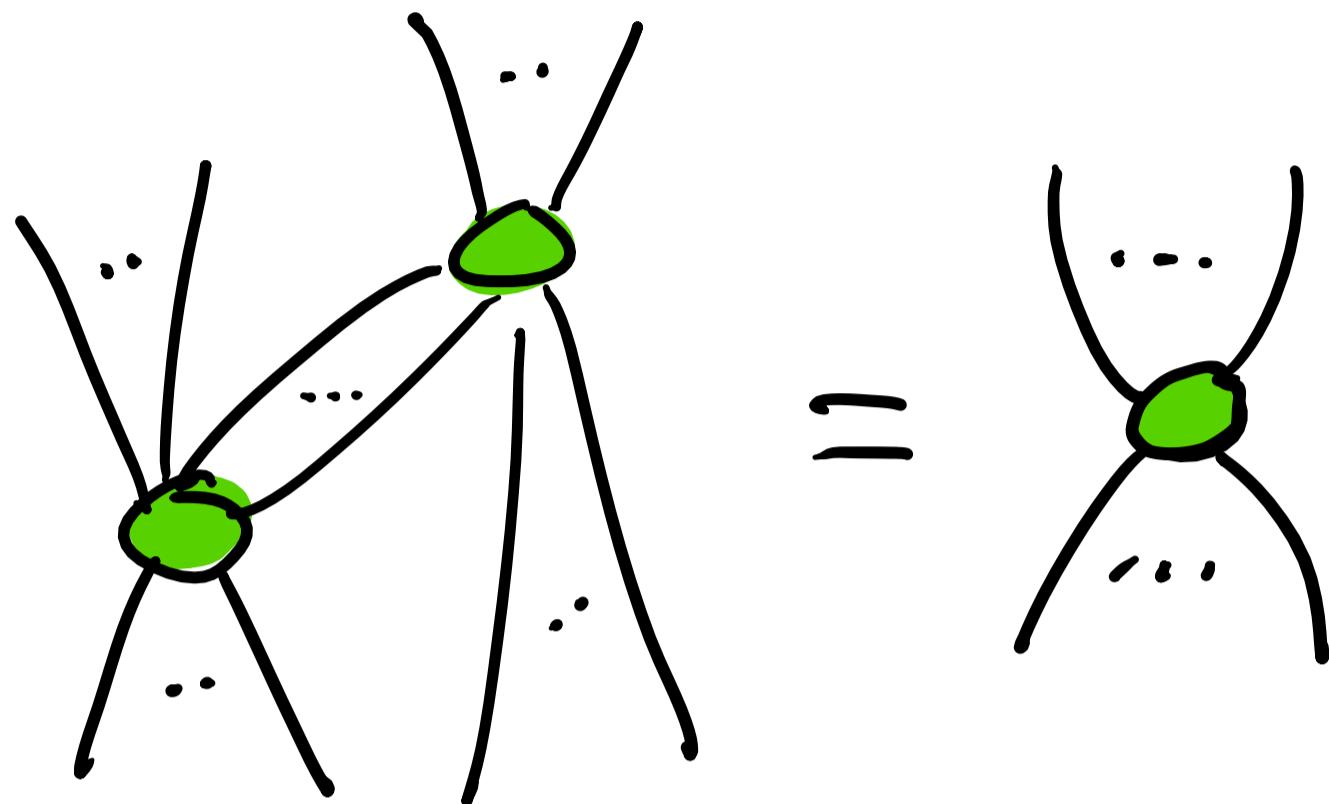


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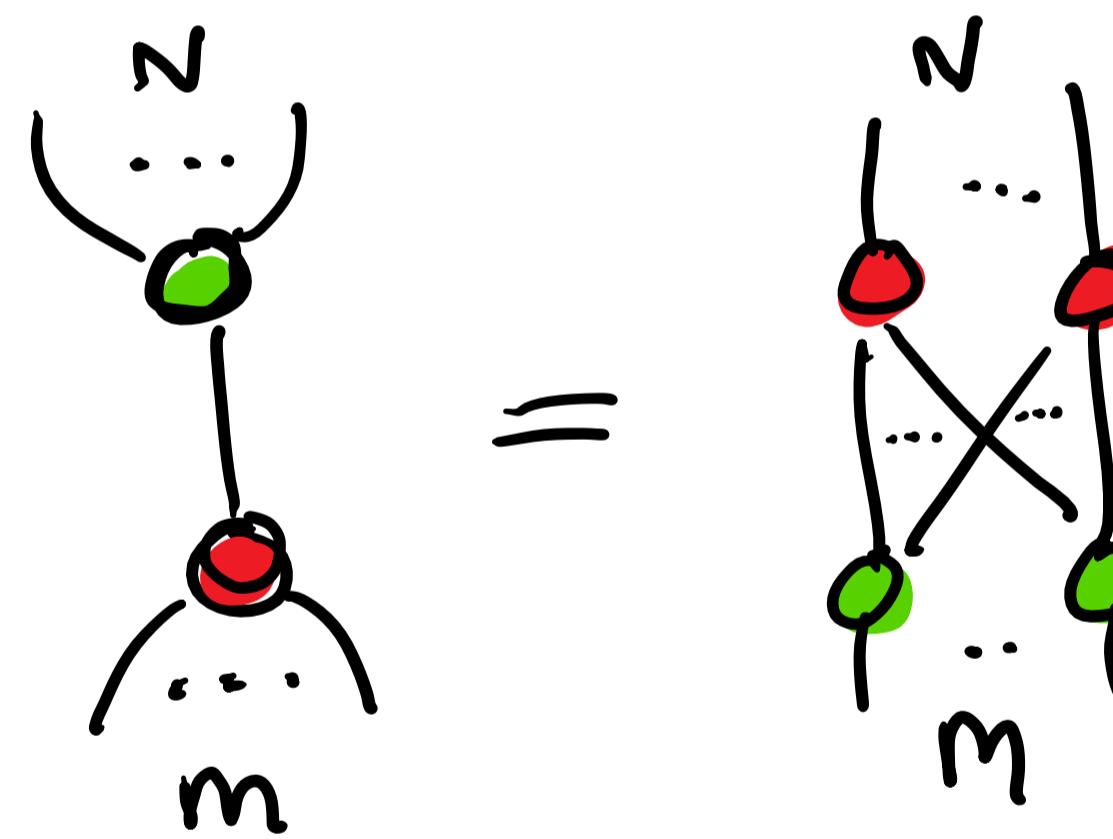


PHASE-
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ZX-calculus



$$\phi = \text{I} = \rho \quad \psi = U = \psi$$



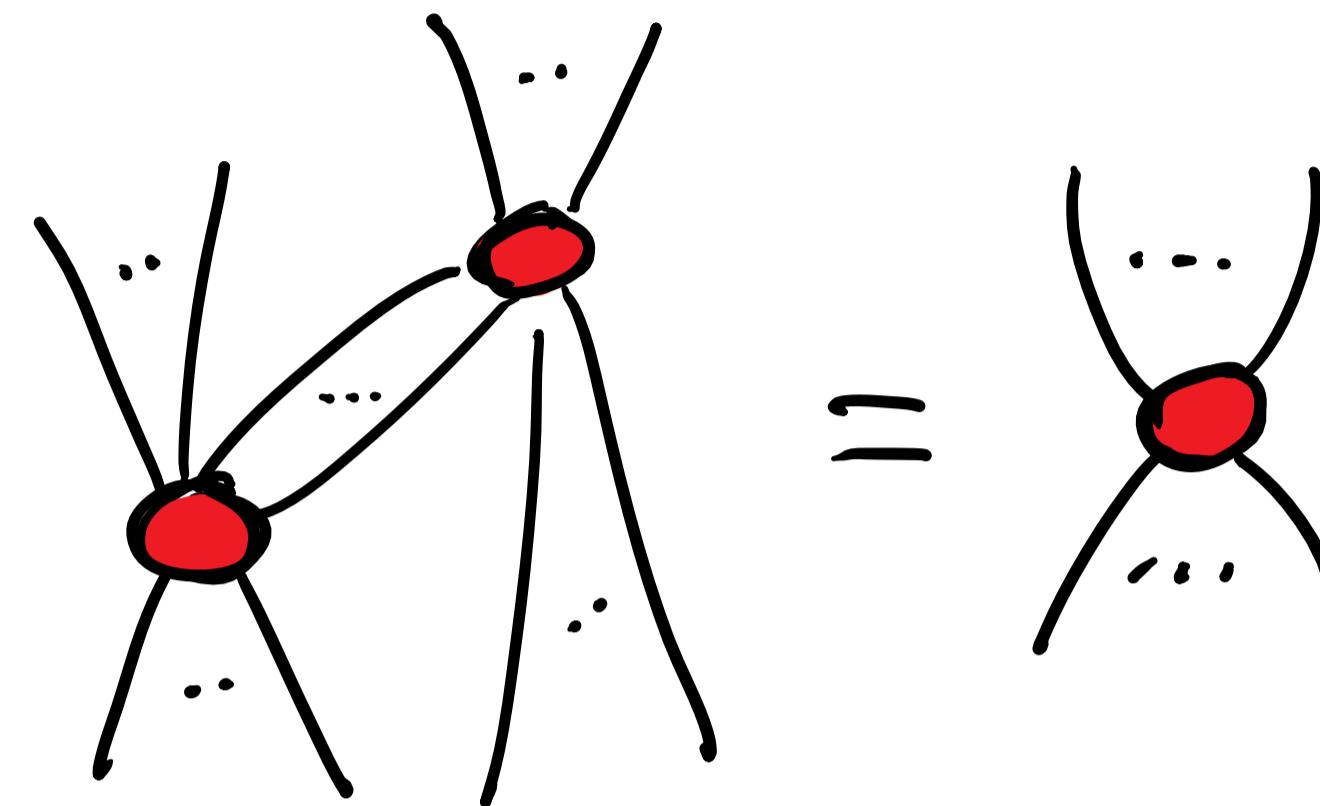
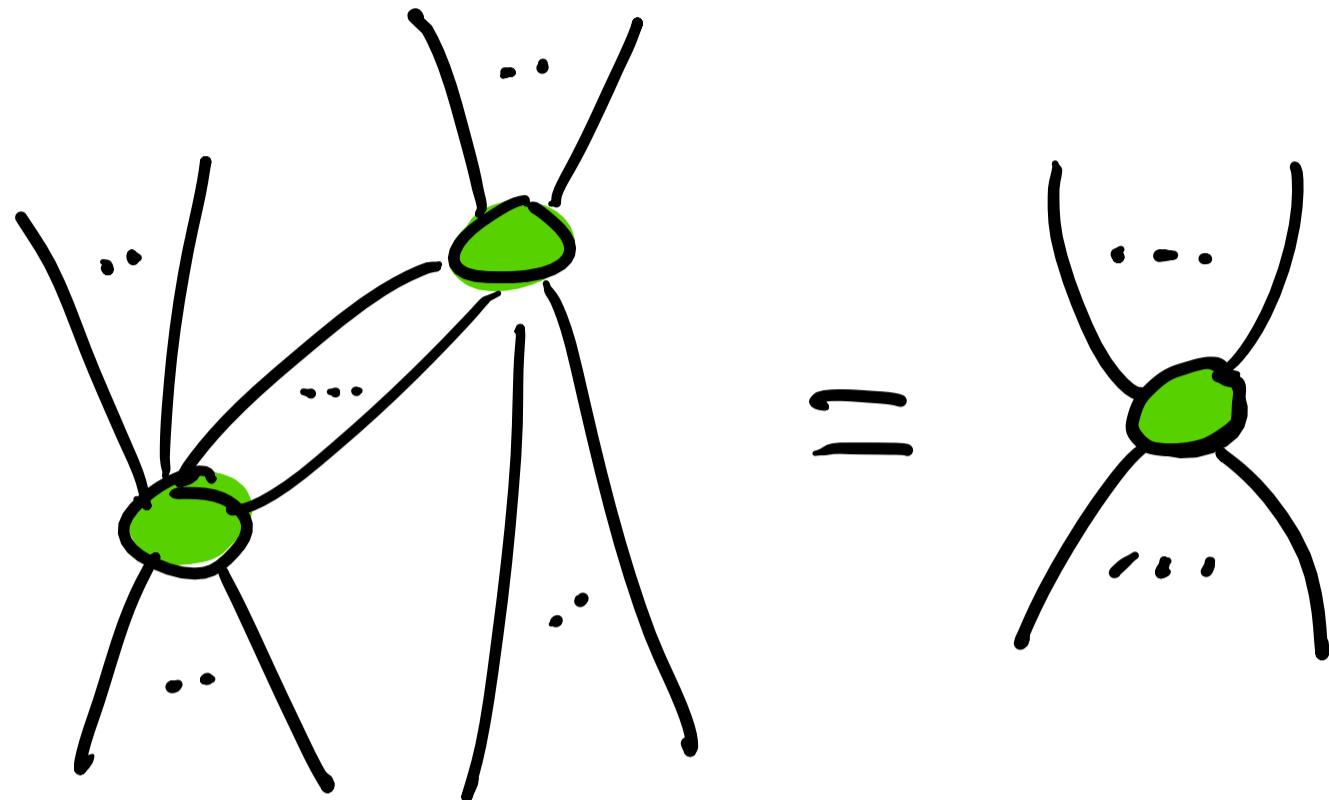
THEOREM The phase-free ZX-calculus ZX_0
is complete for phase-free ZX-diagrams.

$$[\mathcal{D}] = [\mathcal{D}'] \Rightarrow ZX_0 \vdash \mathcal{D} = \mathcal{D}'$$

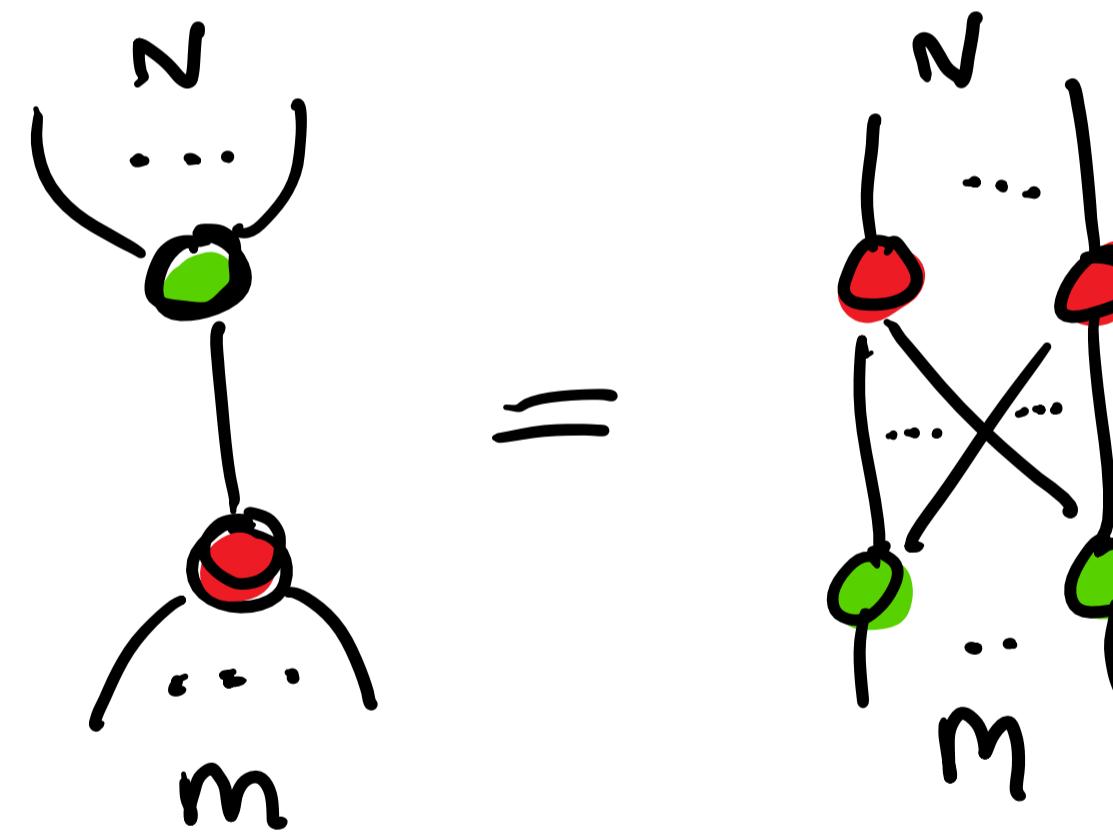
matrix of \mathcal{D}

PHASE-
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ZX-calculus

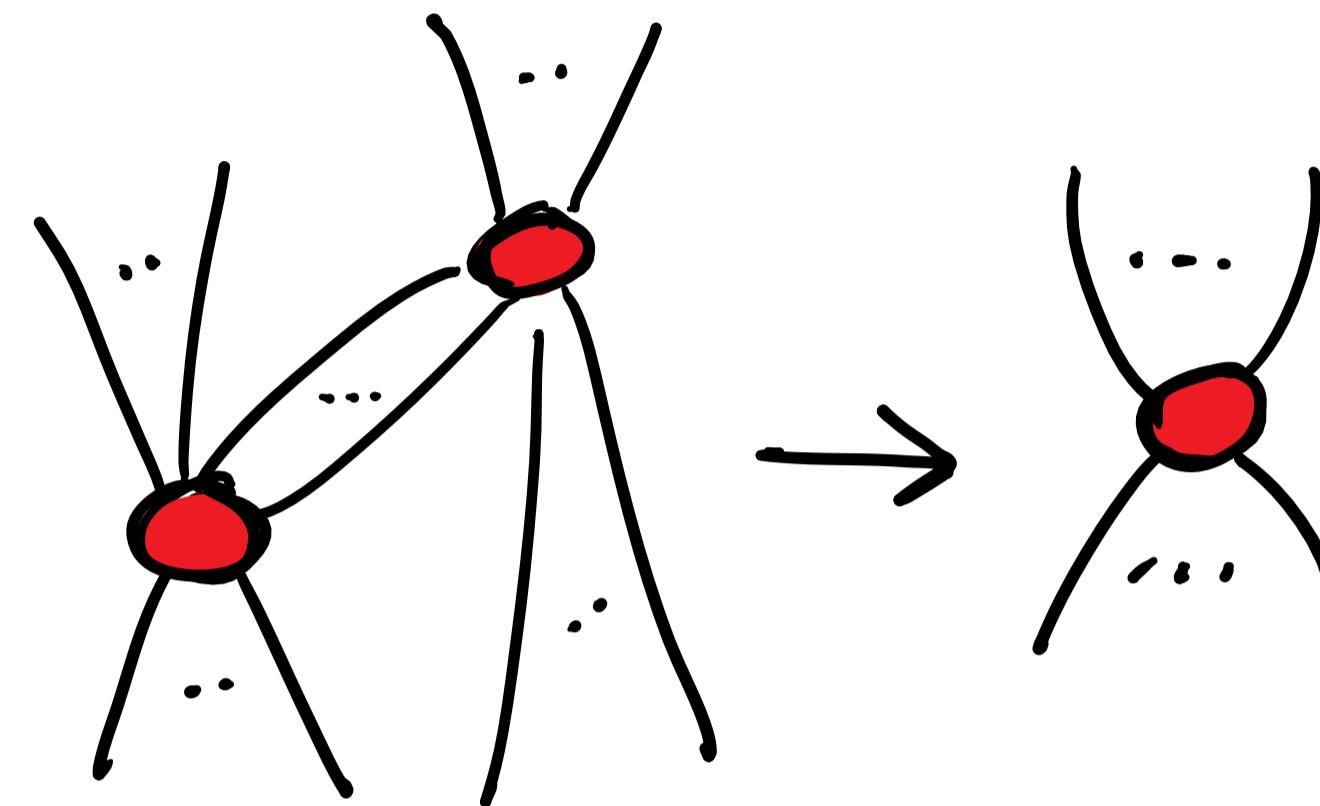
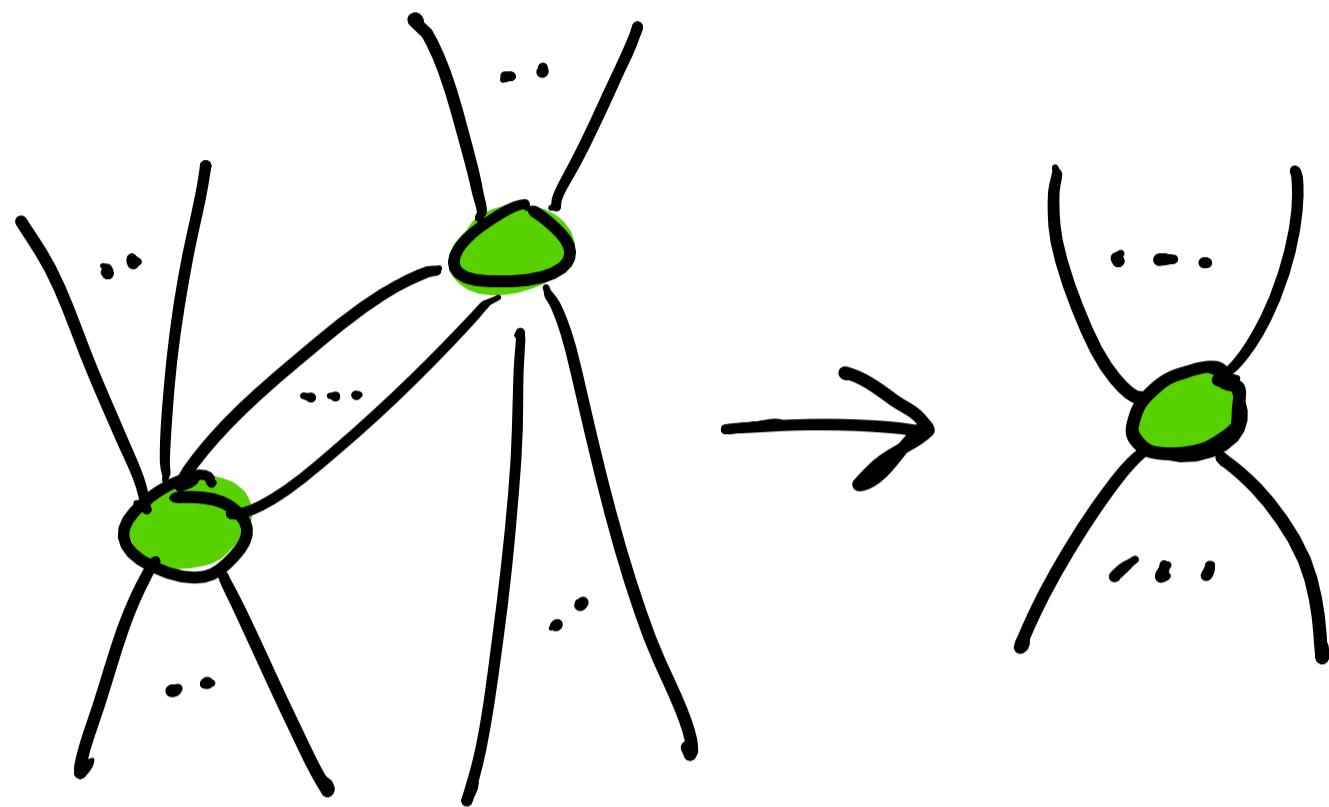


$$\phi = \text{I} = \rho \quad \psi = U = \psi$$



PHASE-
FREE

ZX-calculus

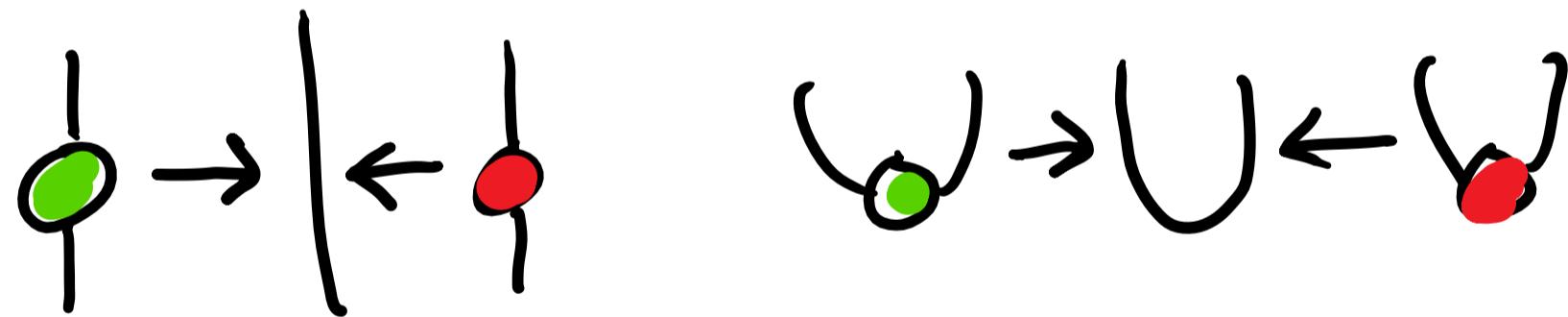
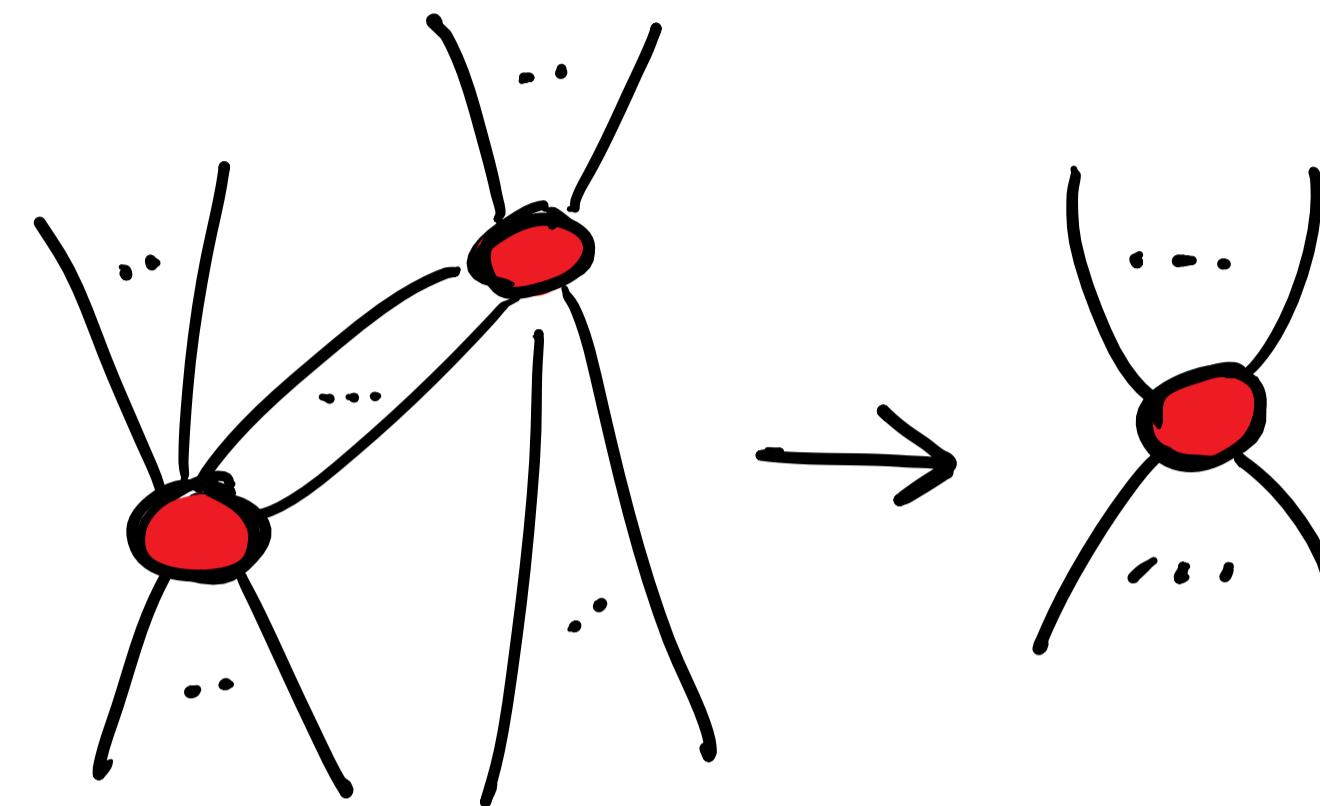
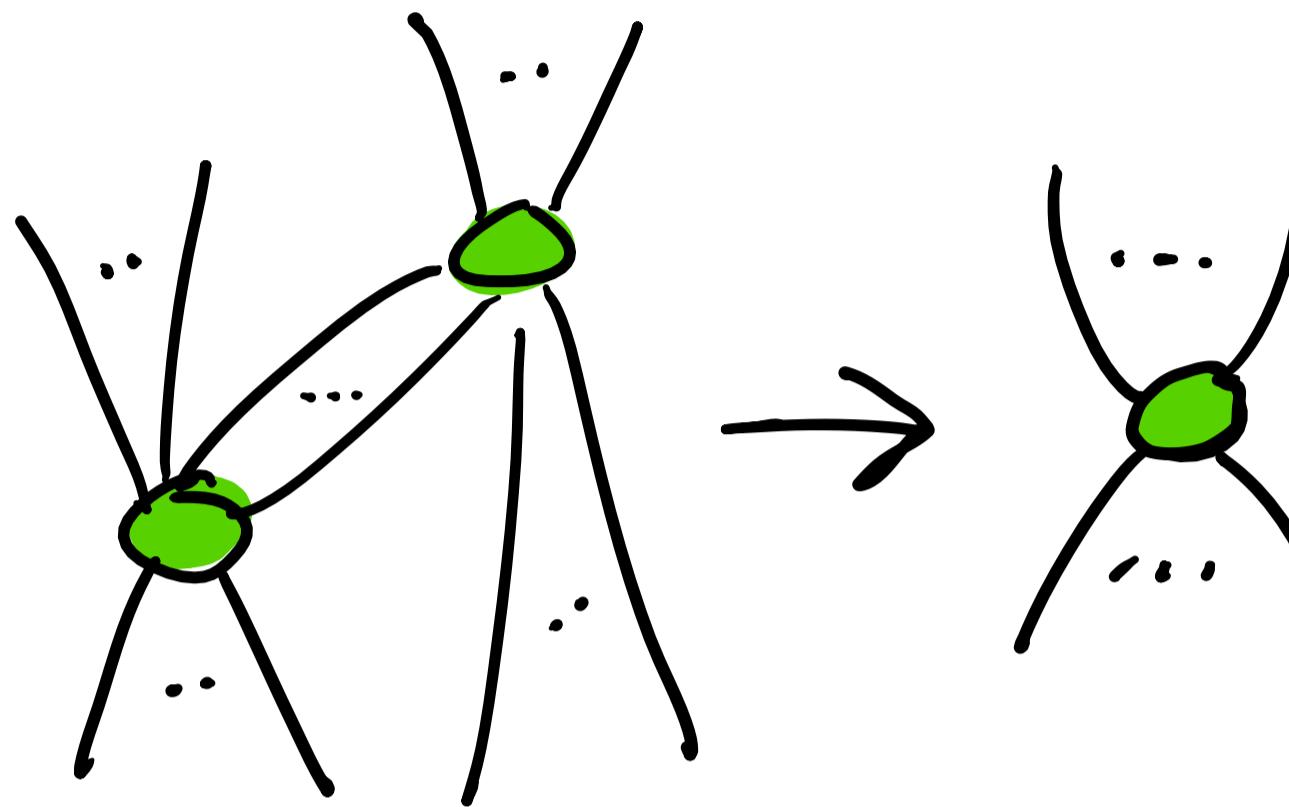


$$\phi = I = \phi \quad \psi = U = \psi$$

A ZX-calculus diagram showing the identity of a phase node. On the left, a green node labeled 'n' is connected to a red node labeled 'm'. An equals sign follows. On the right, the same two nodes are shown, but the green node is now red and the red node is now green, illustrating that the nodes are equivalent under phase exchange.

PHASE-
FREE

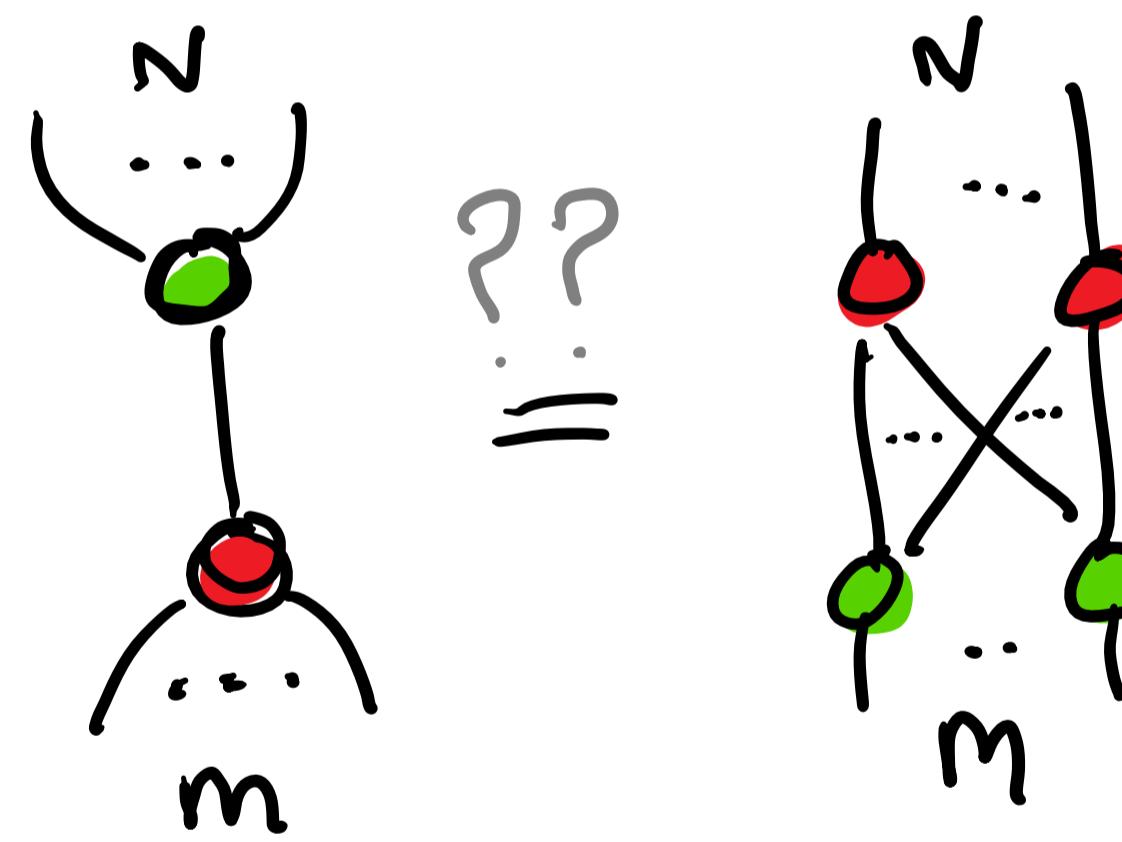
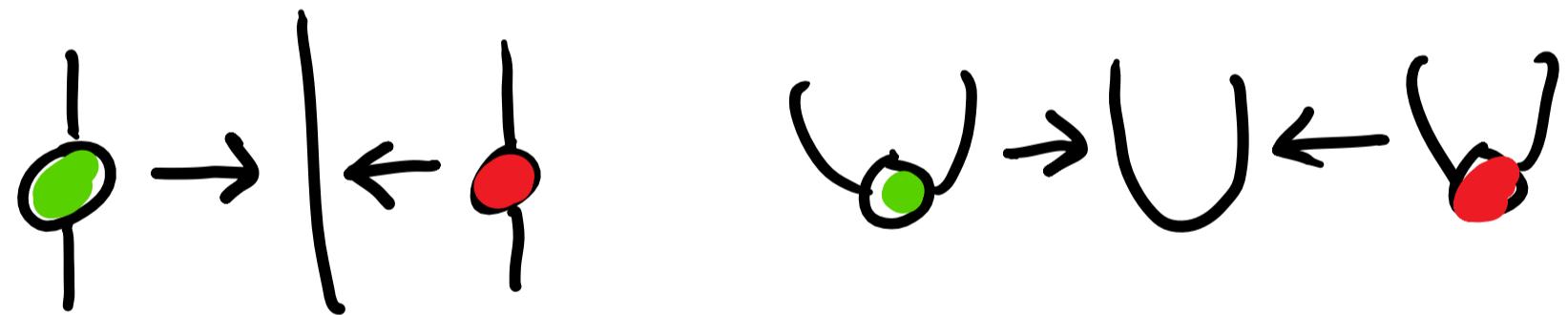
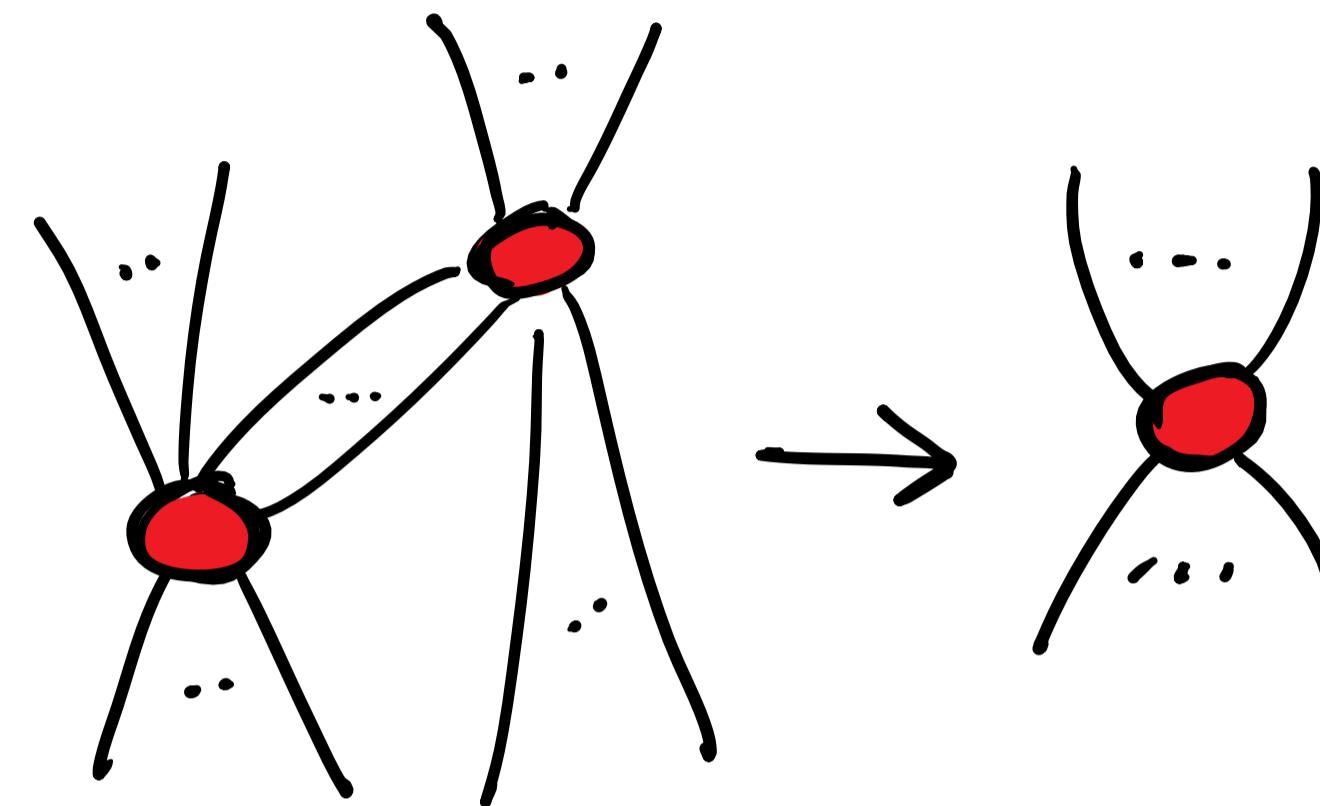
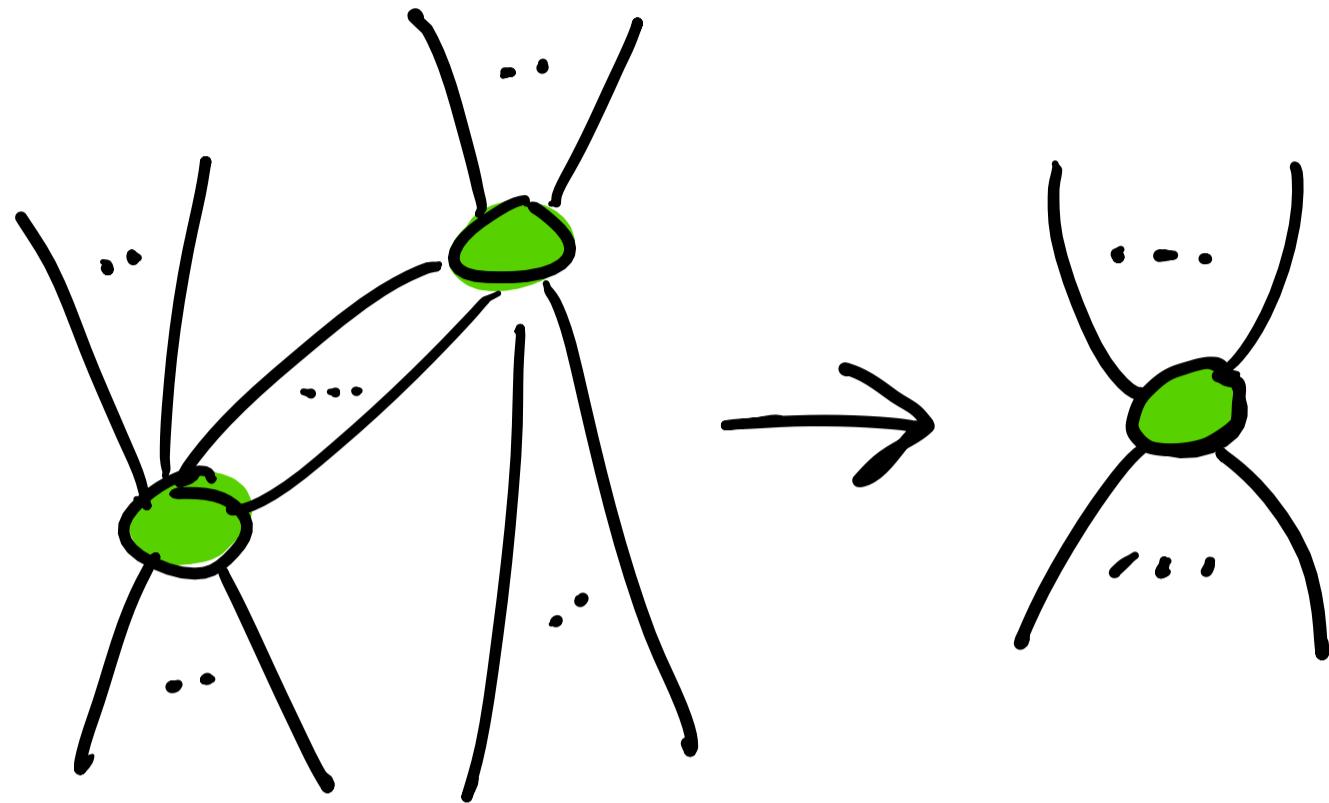
ZX-calculus

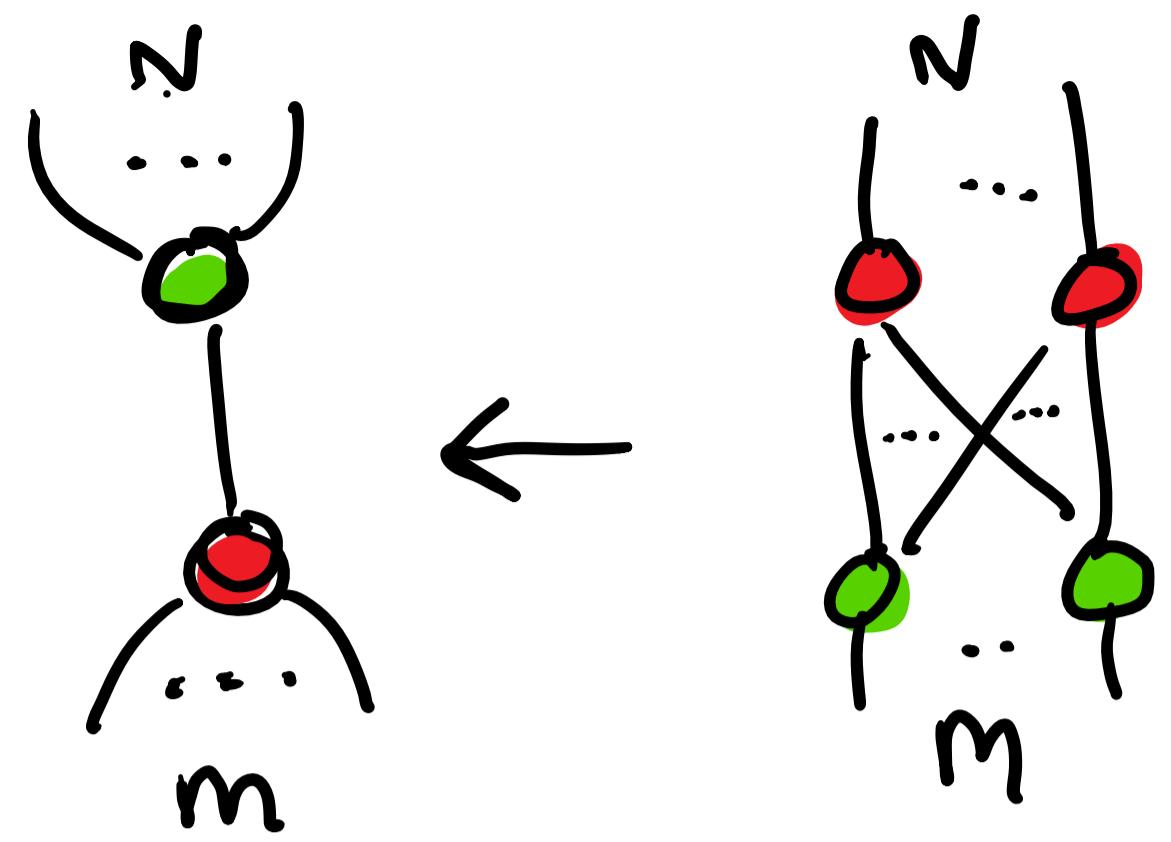


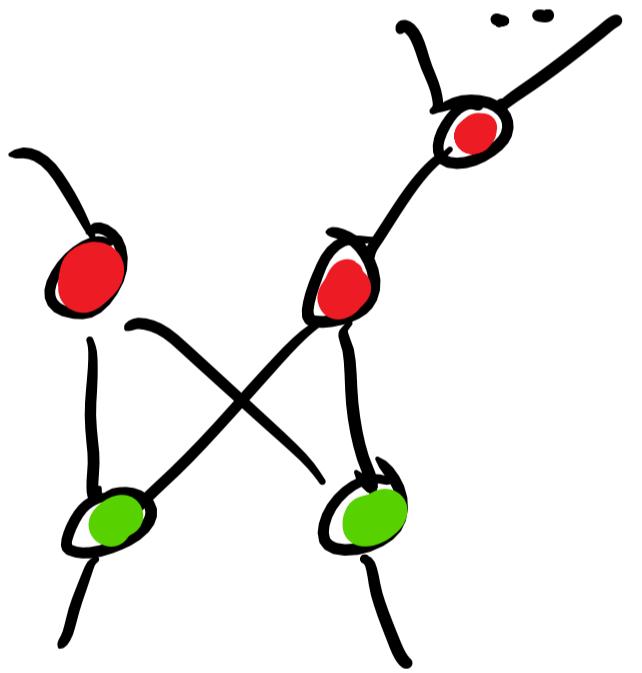
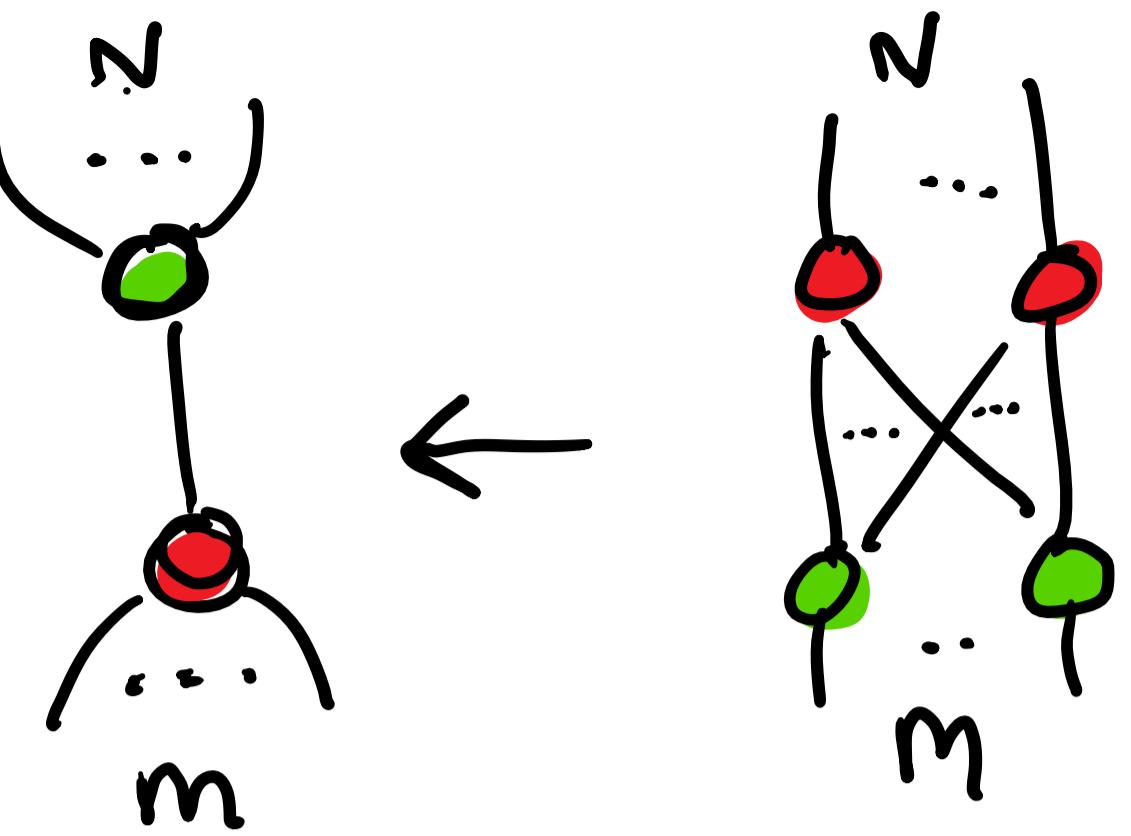
A ZX-diagram identity showing the equivalence of two configurations. On the left, a green node labeled 'n' is at the top, connected by a vertical wire to a red node labeled 'm'. On the right, a red node labeled 'n' is at the top, connected by a vertical wire to a green node labeled 'm'. An equals sign is between them.

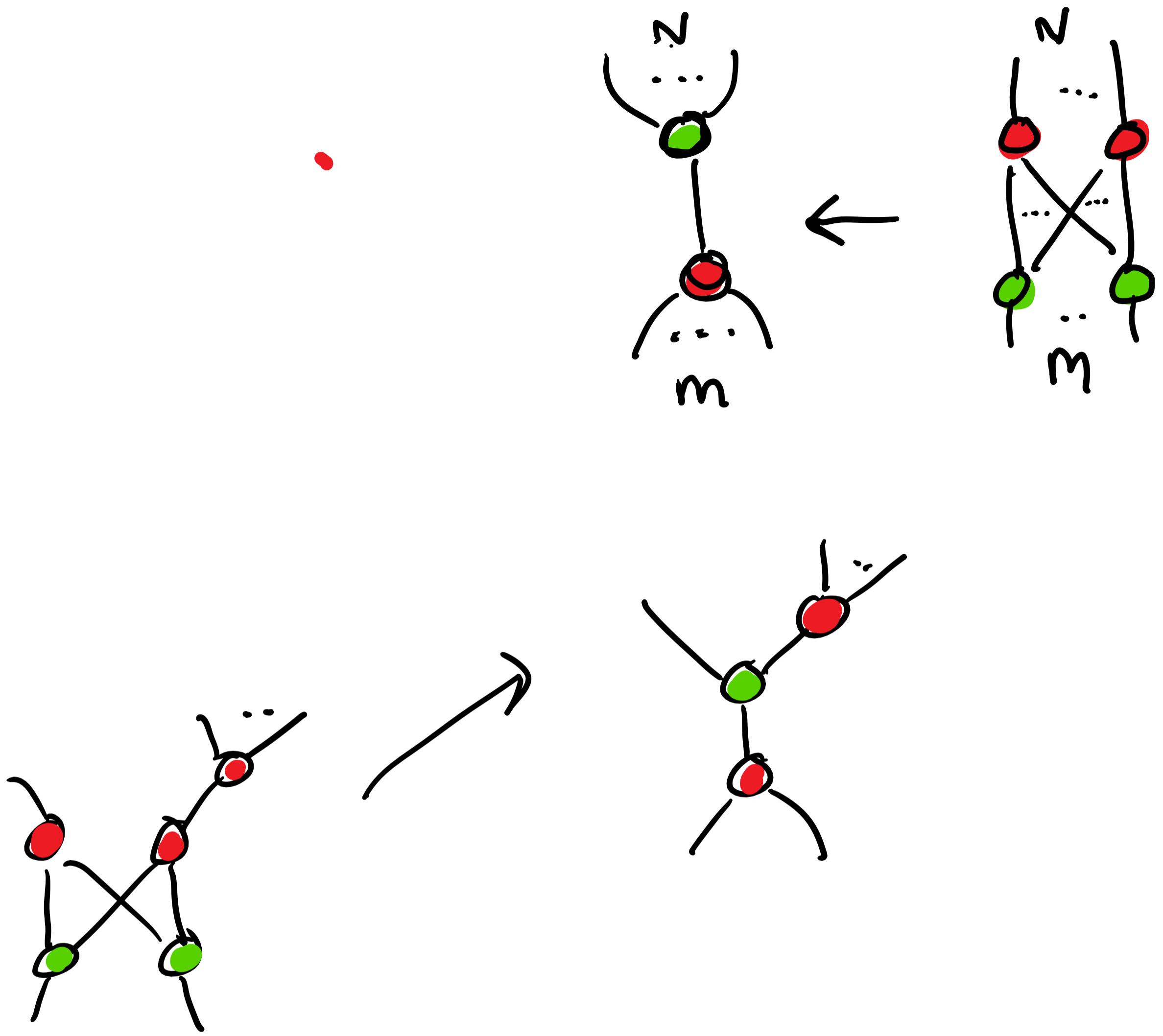
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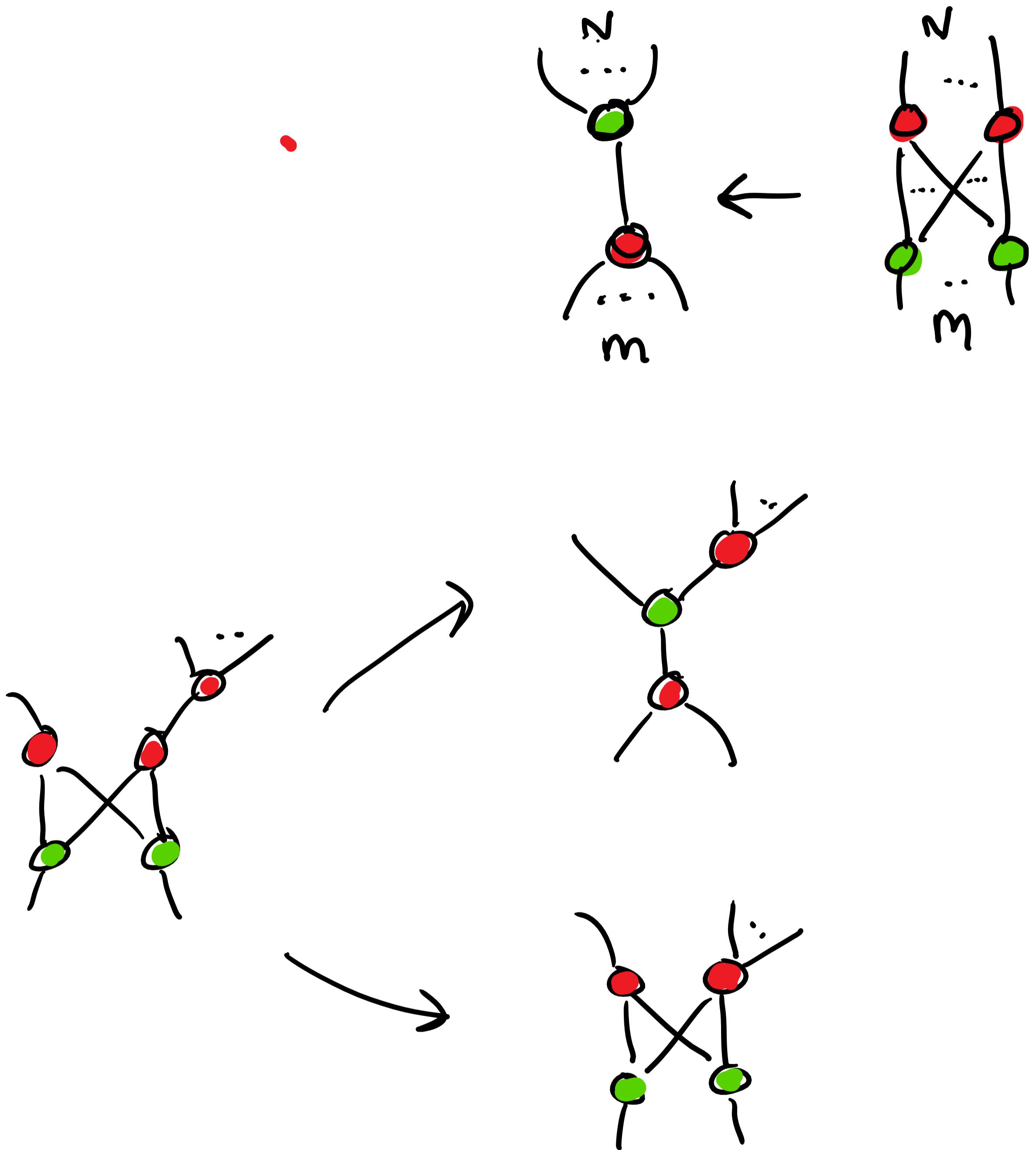
ZX-calculus

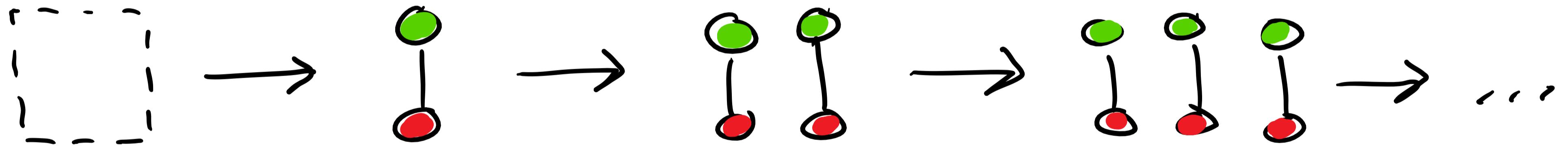
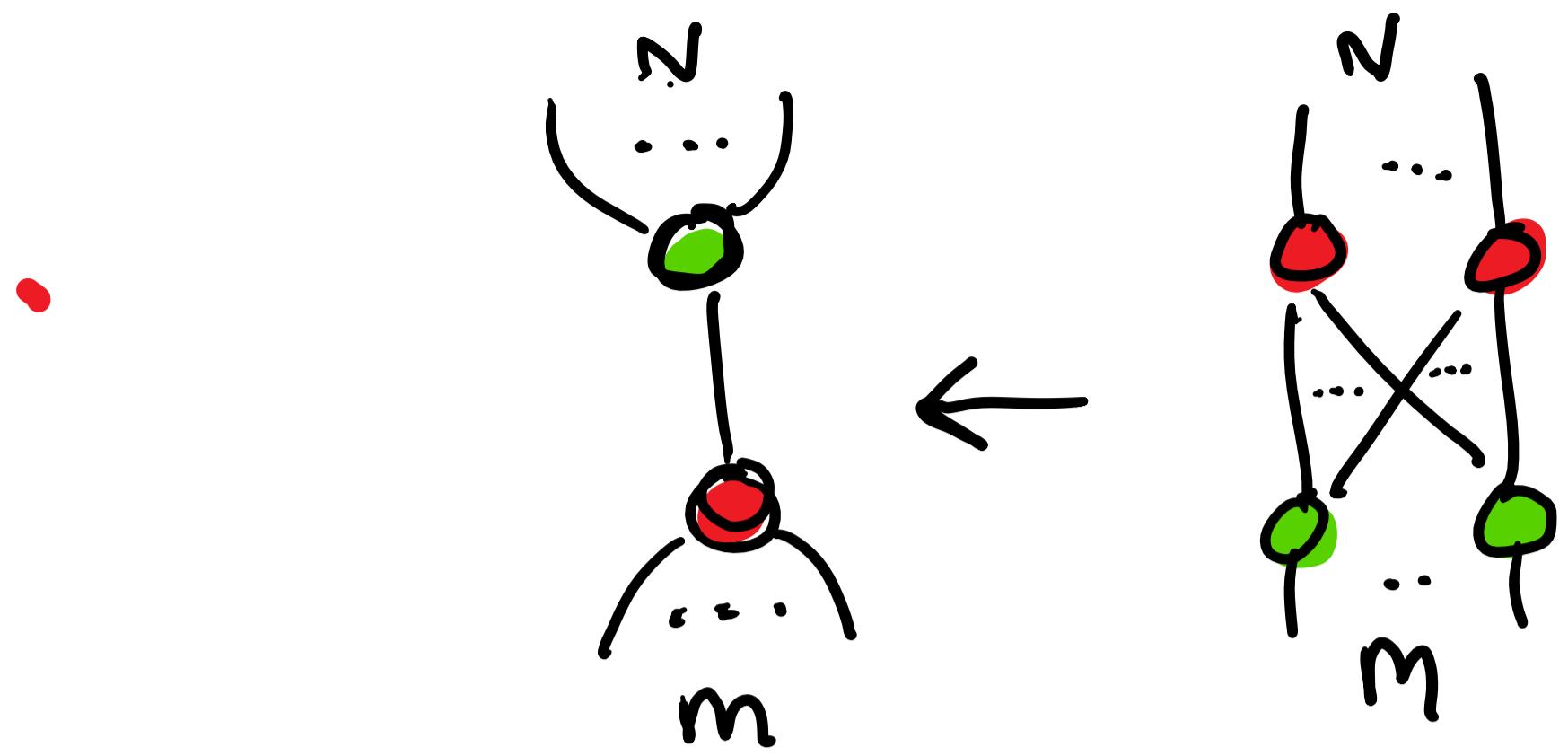




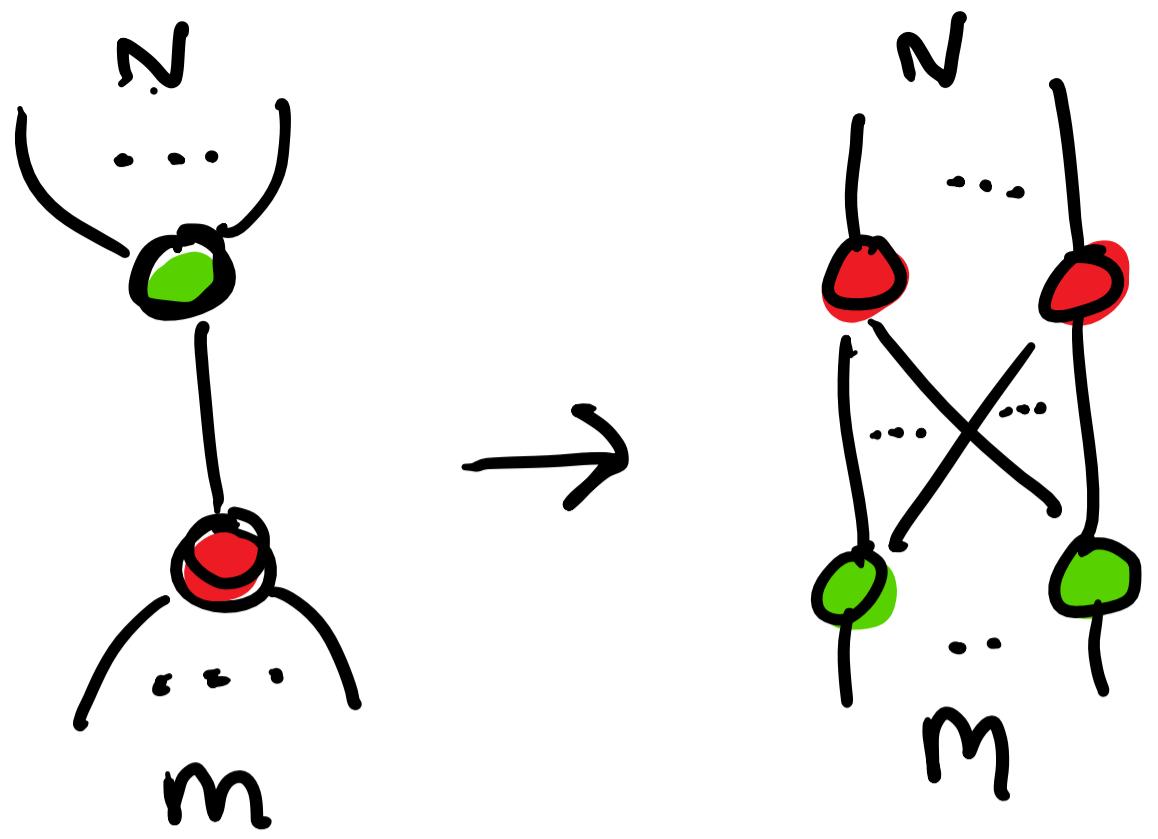


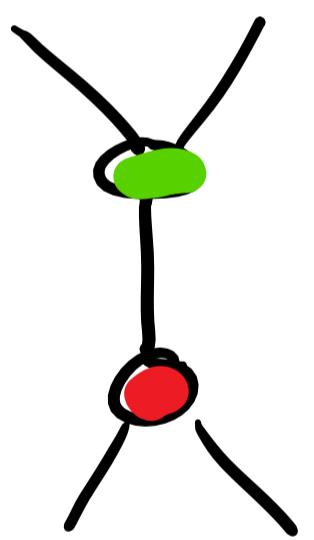
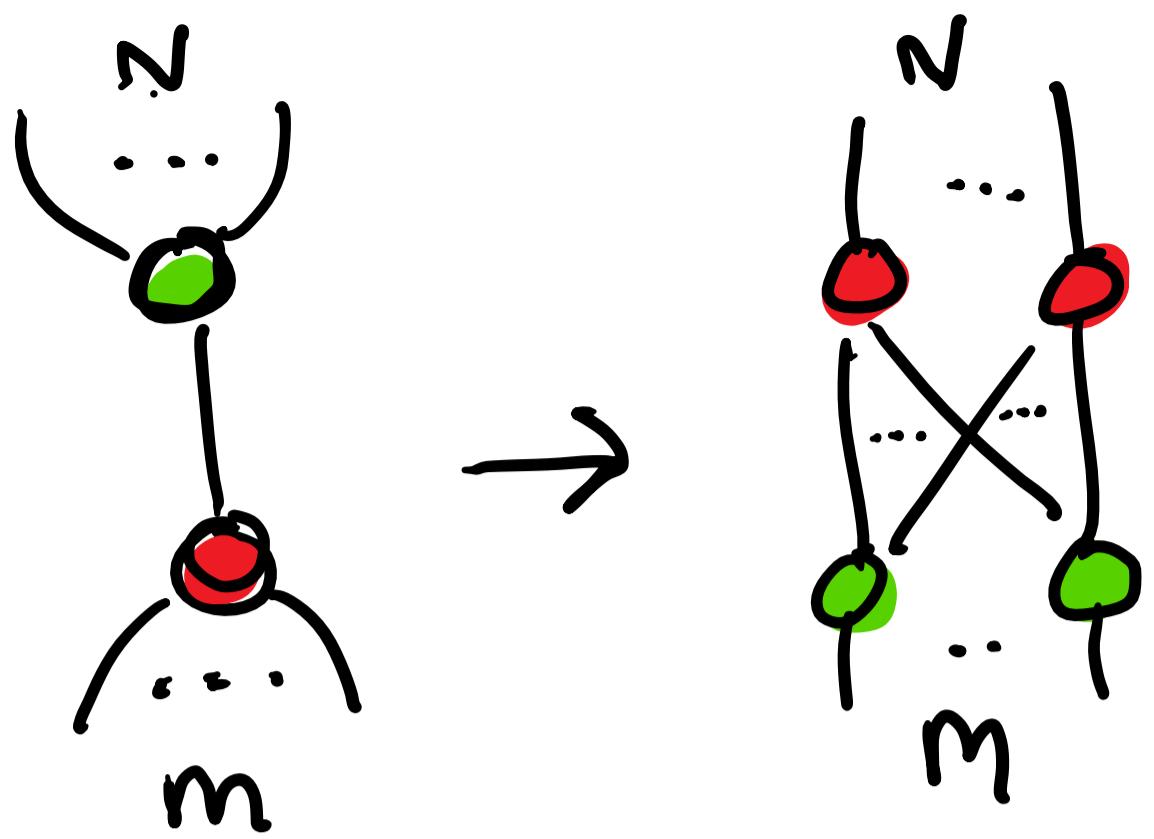


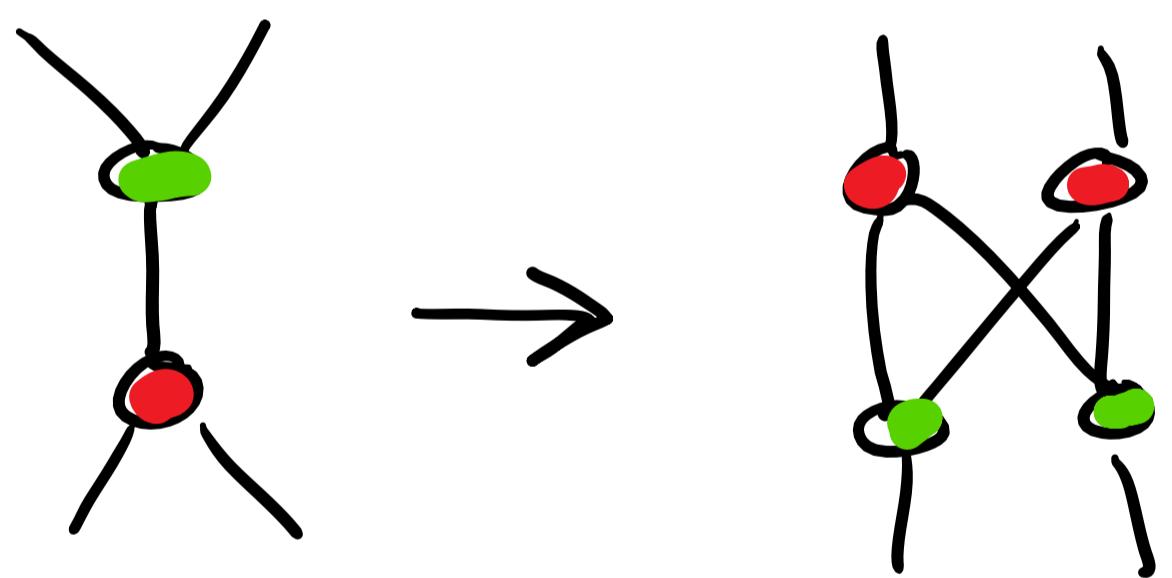
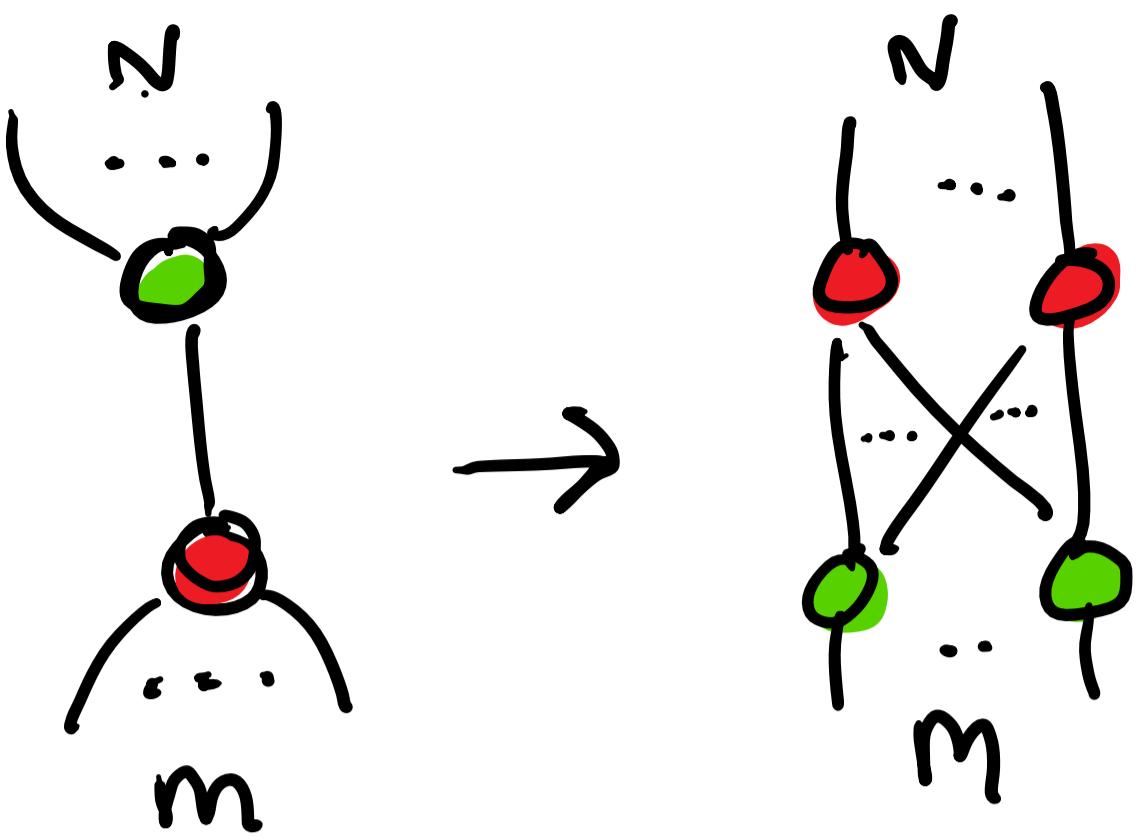


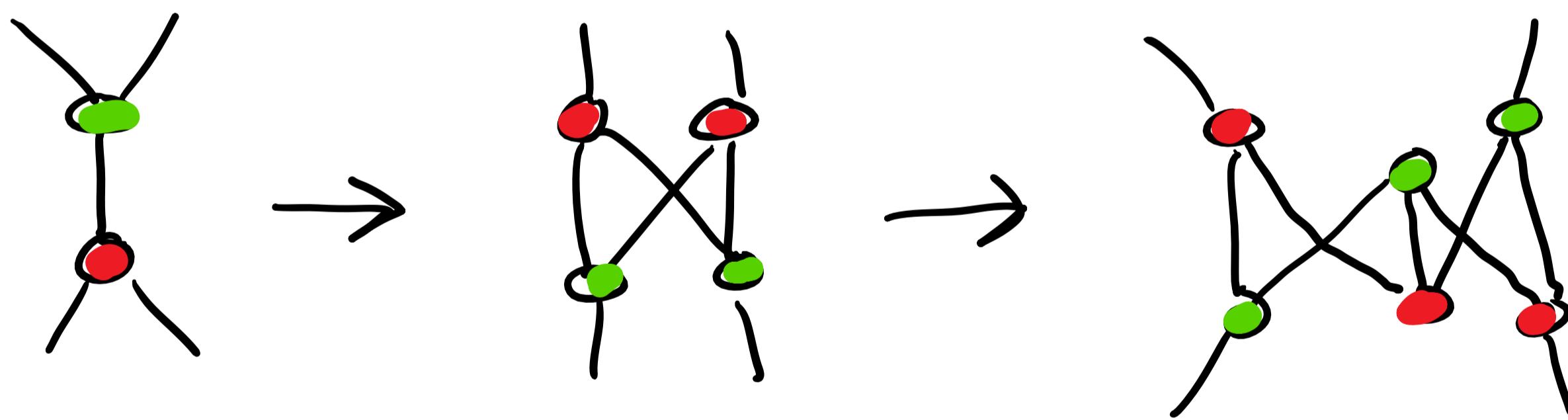
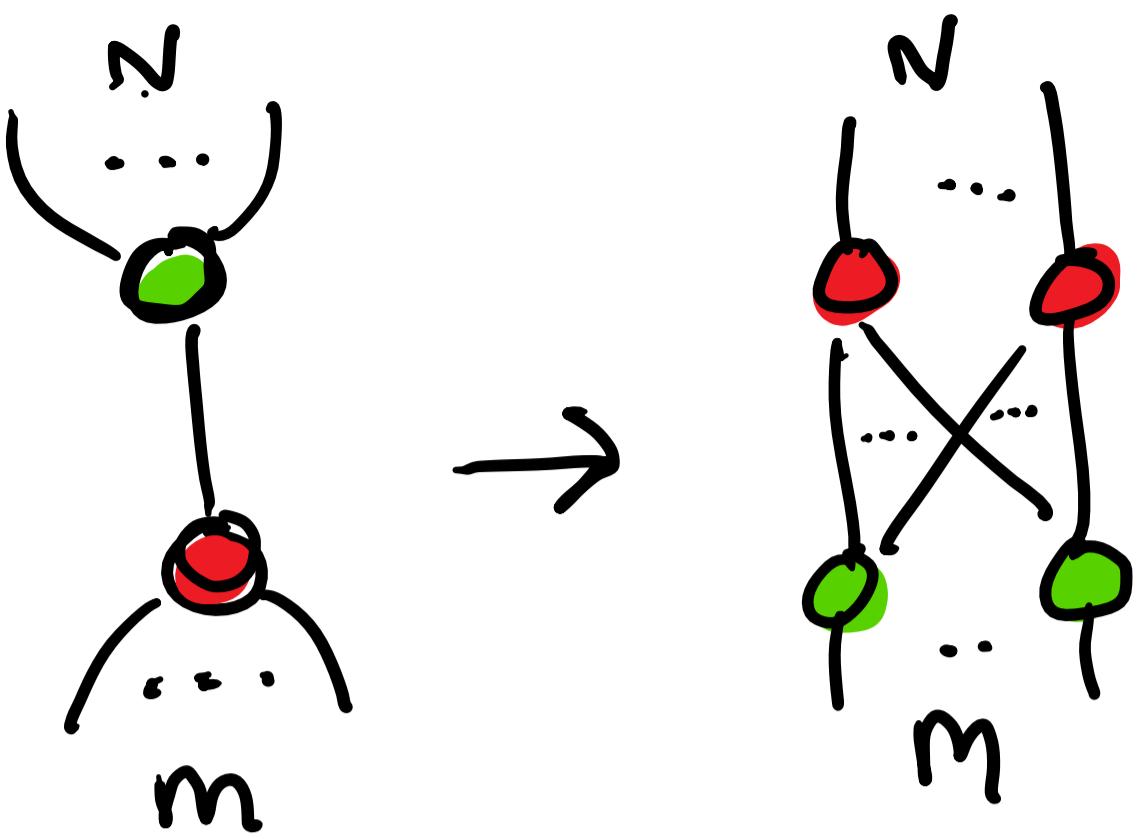


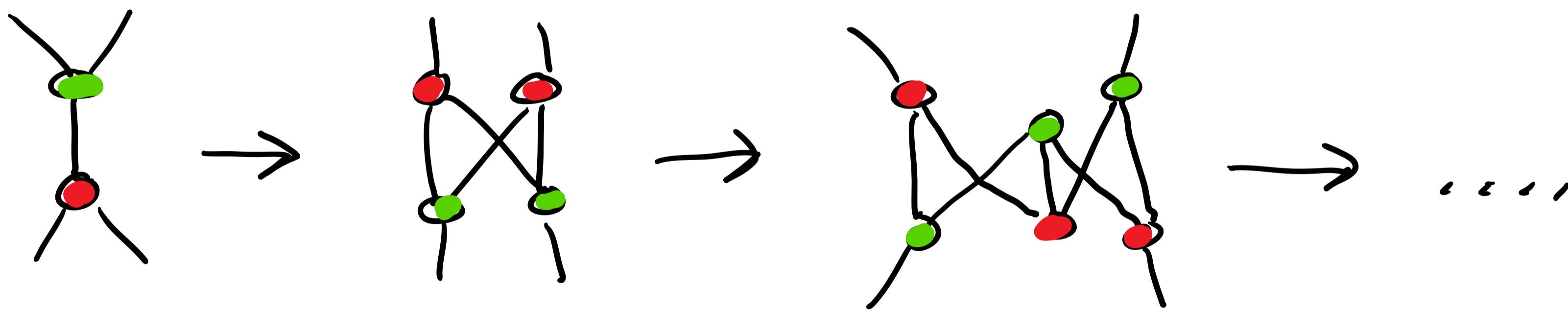
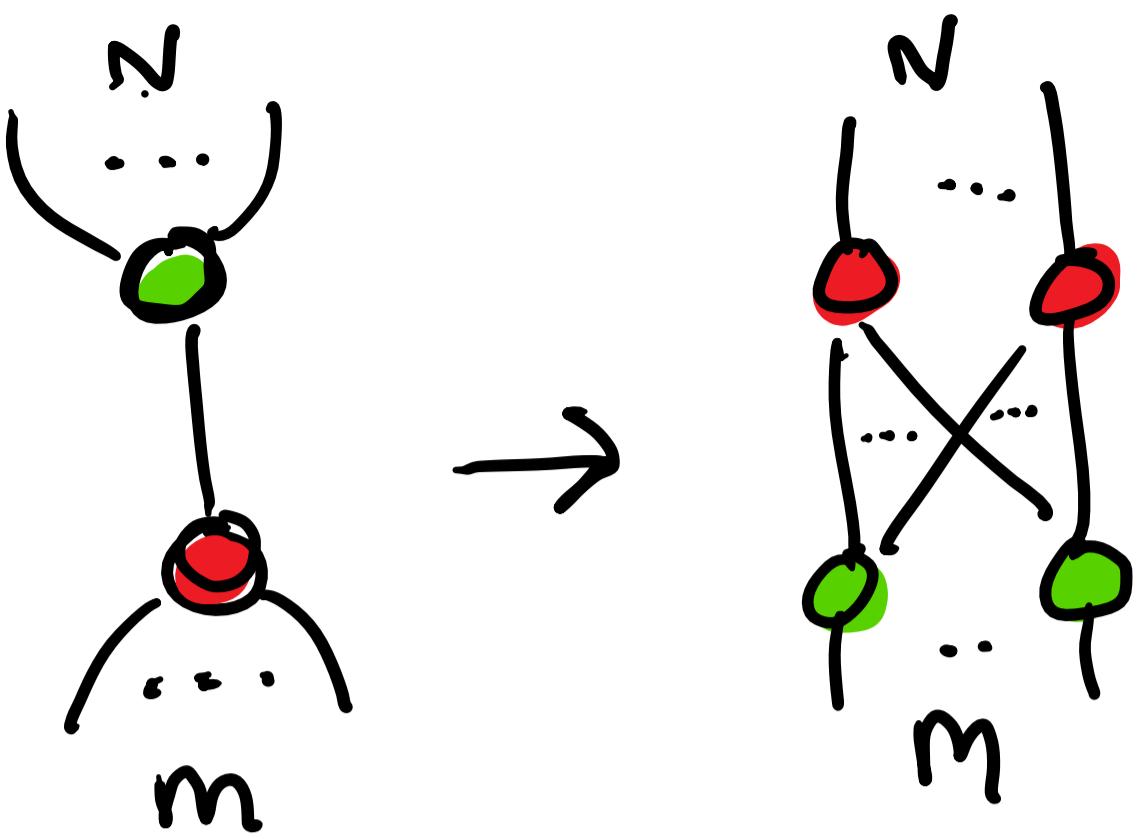
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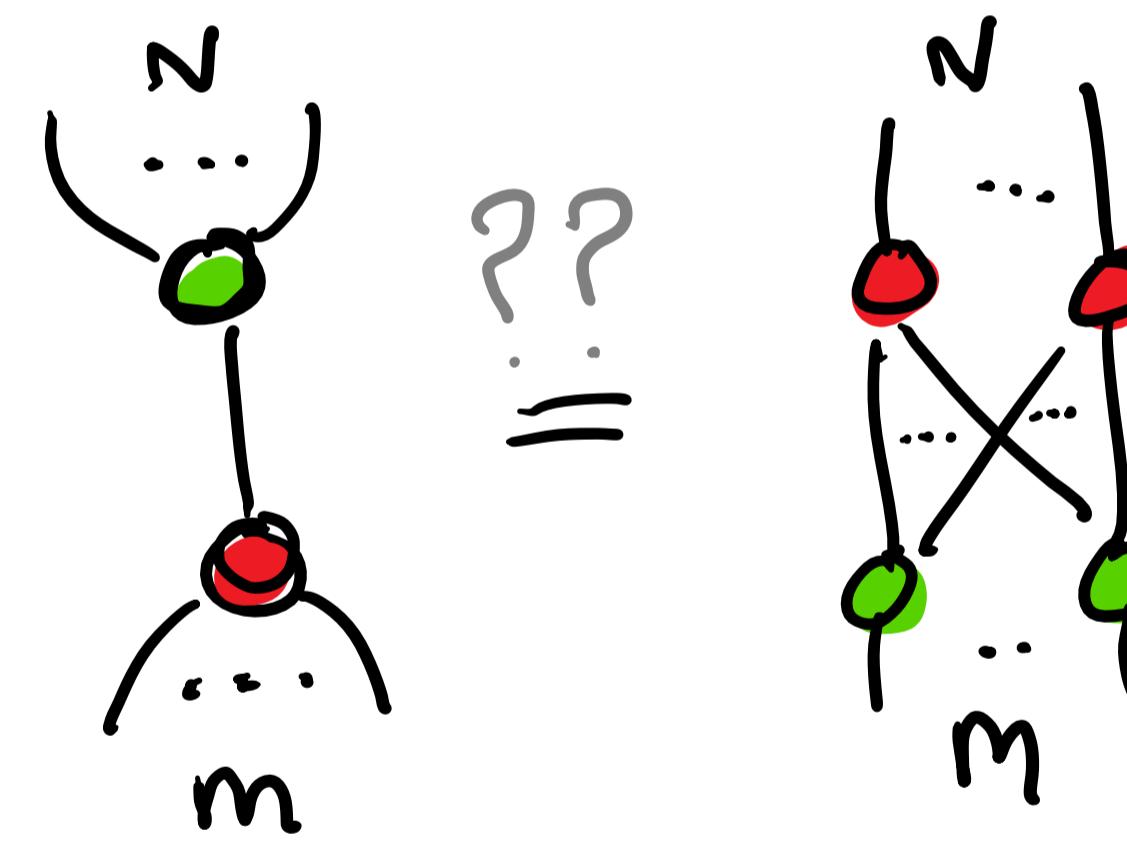
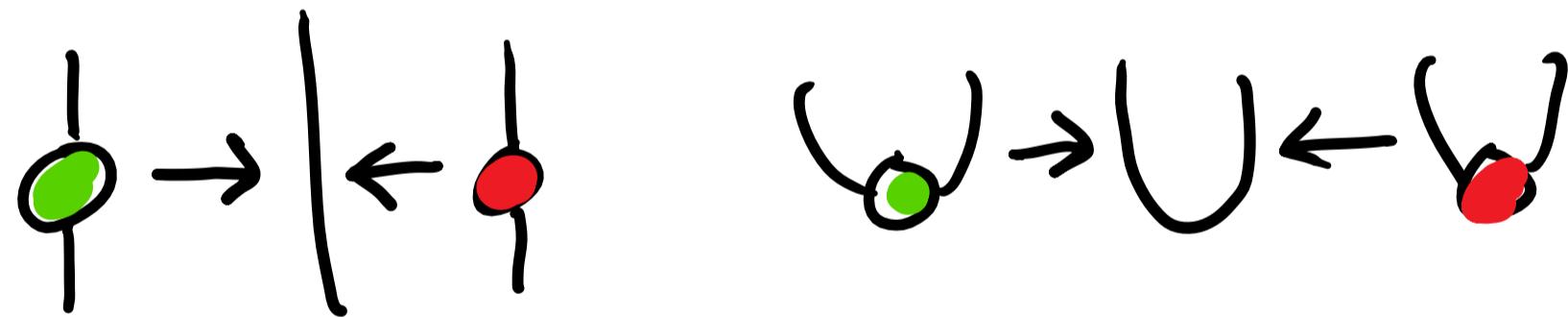
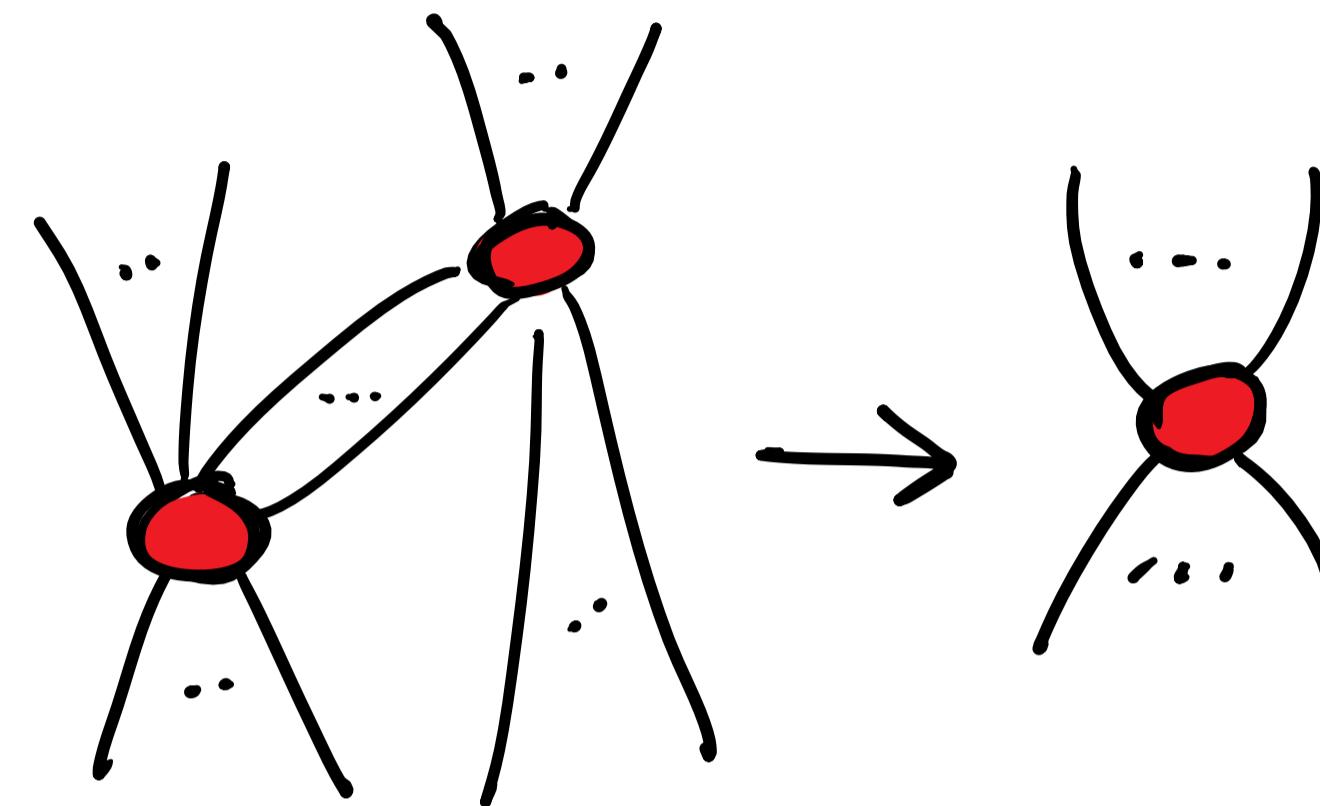
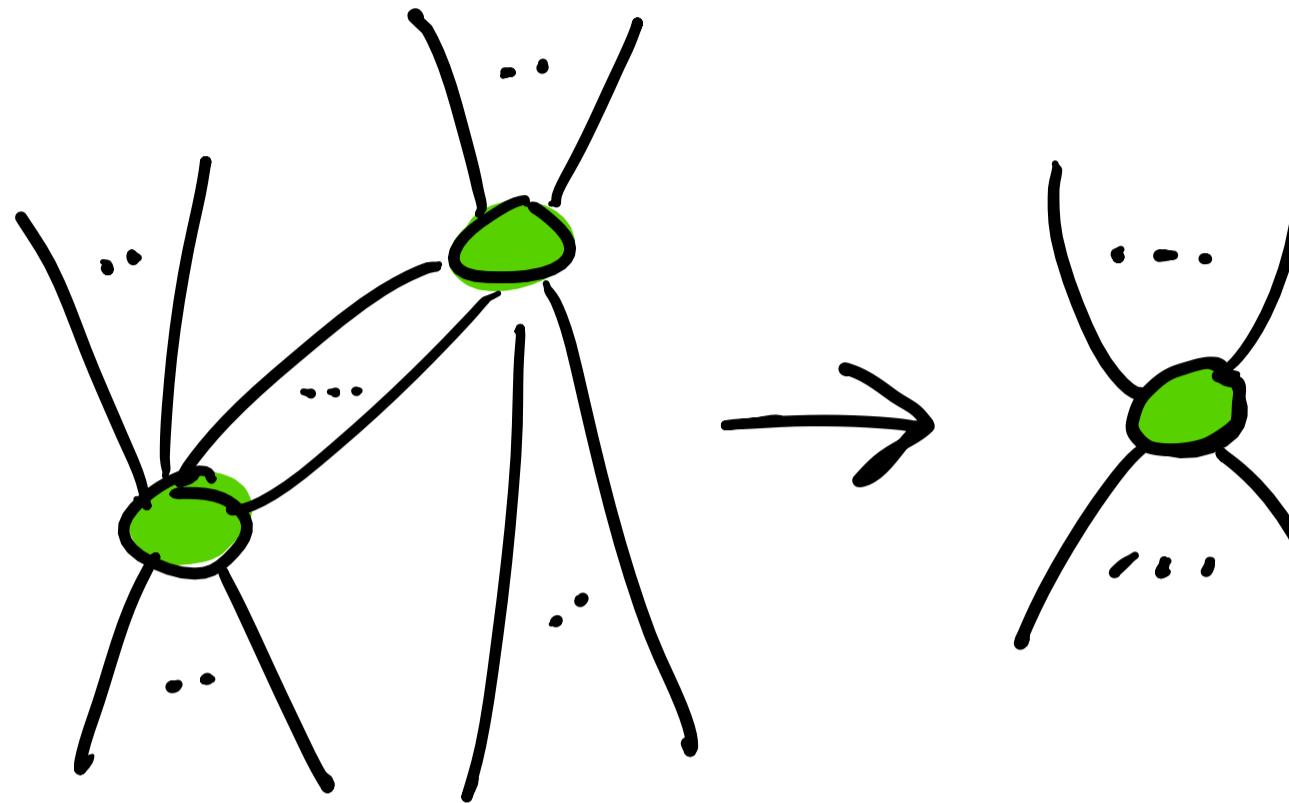






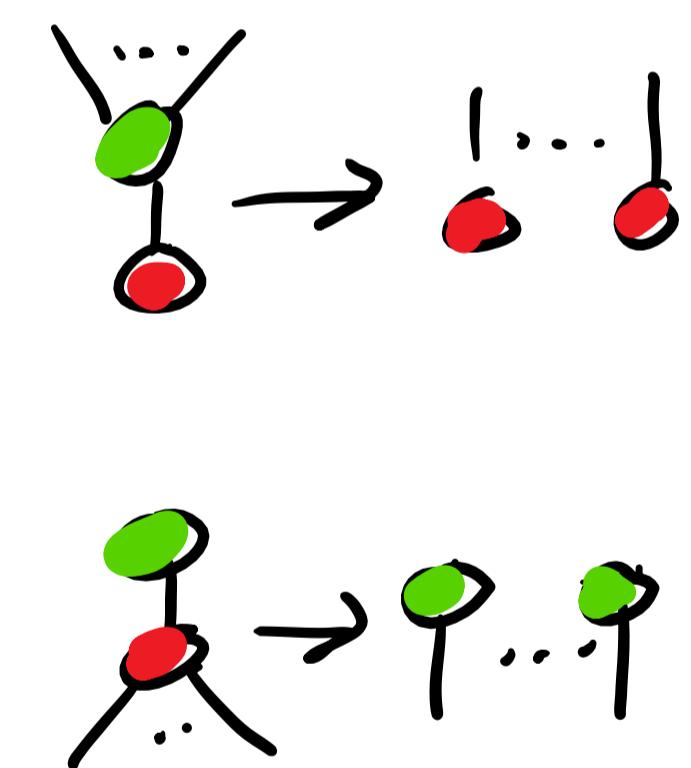
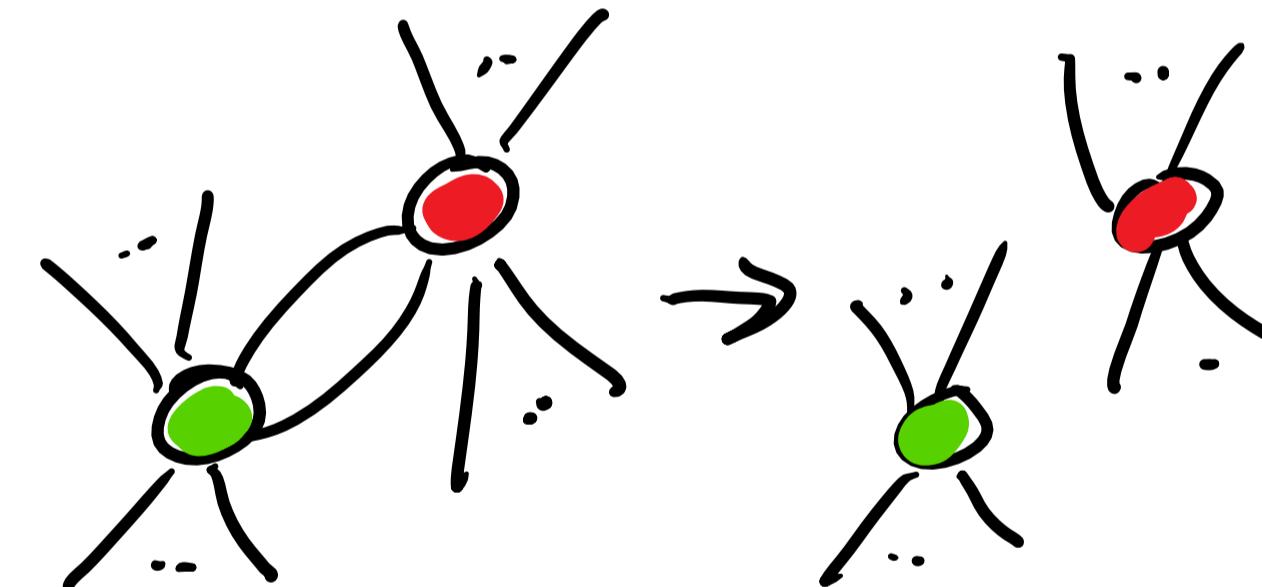
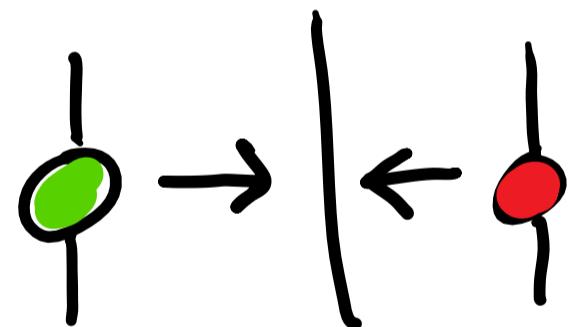
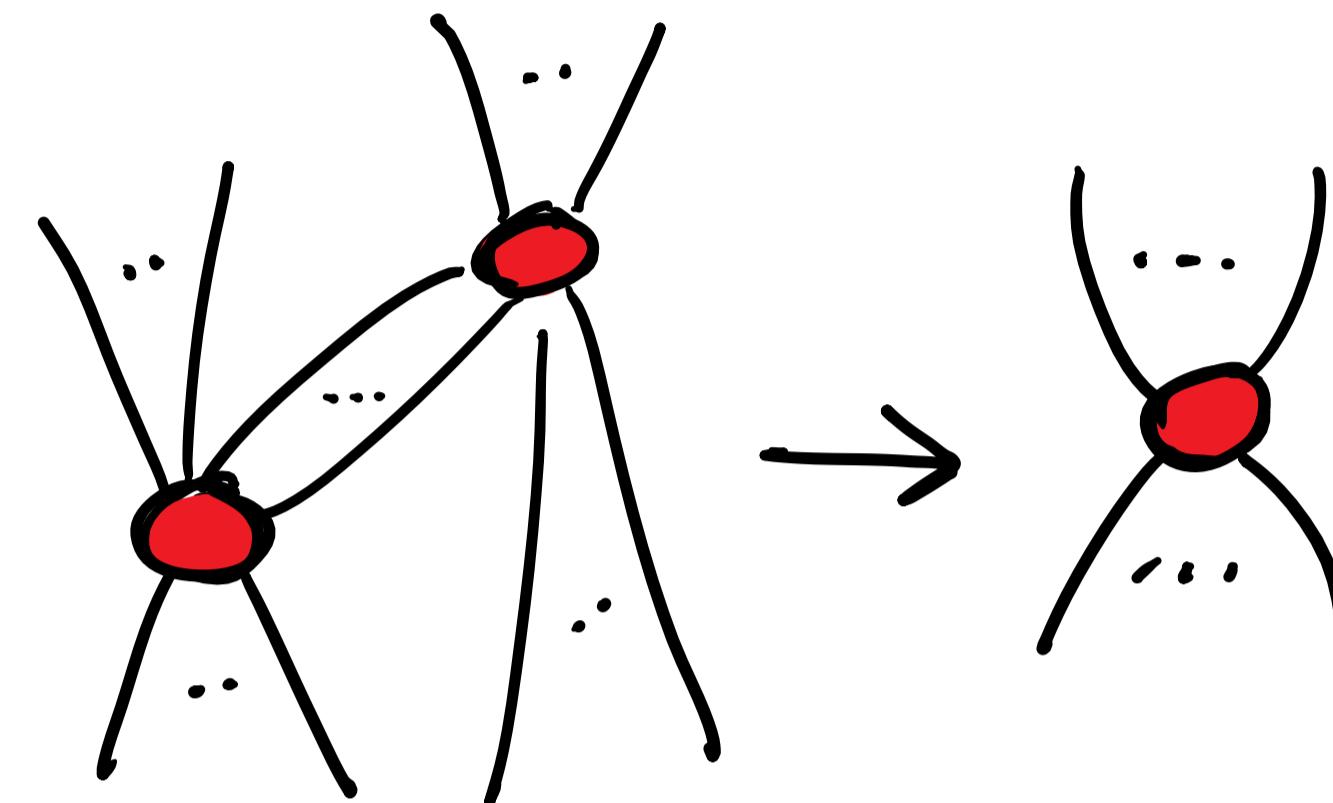
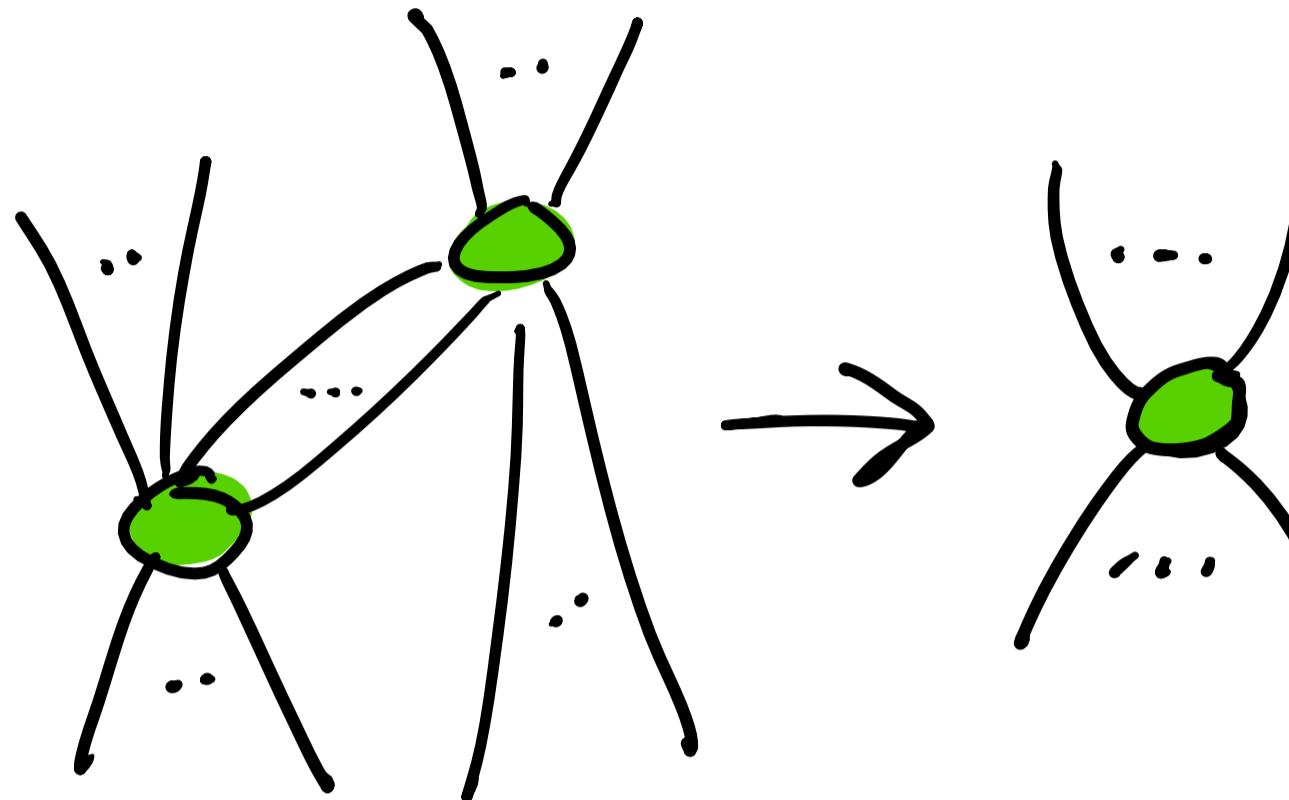
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ZX-calculus



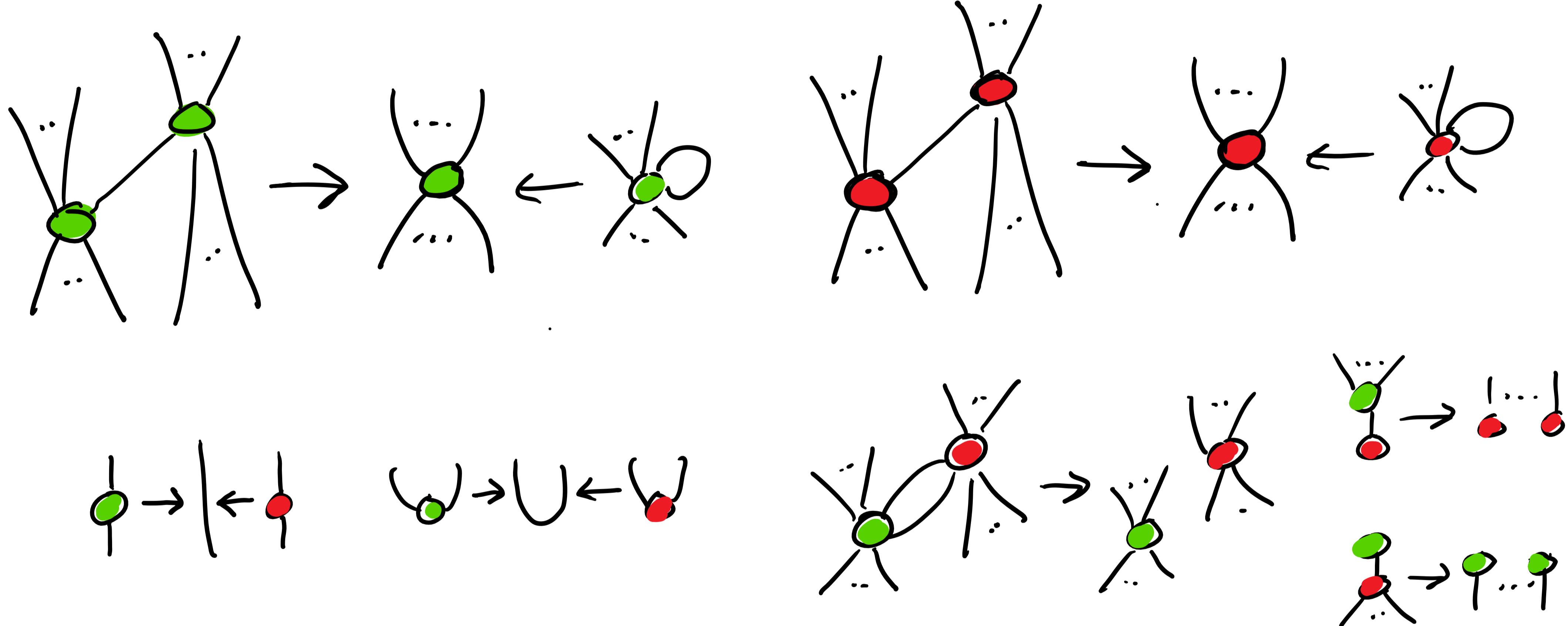
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ZX-calculus



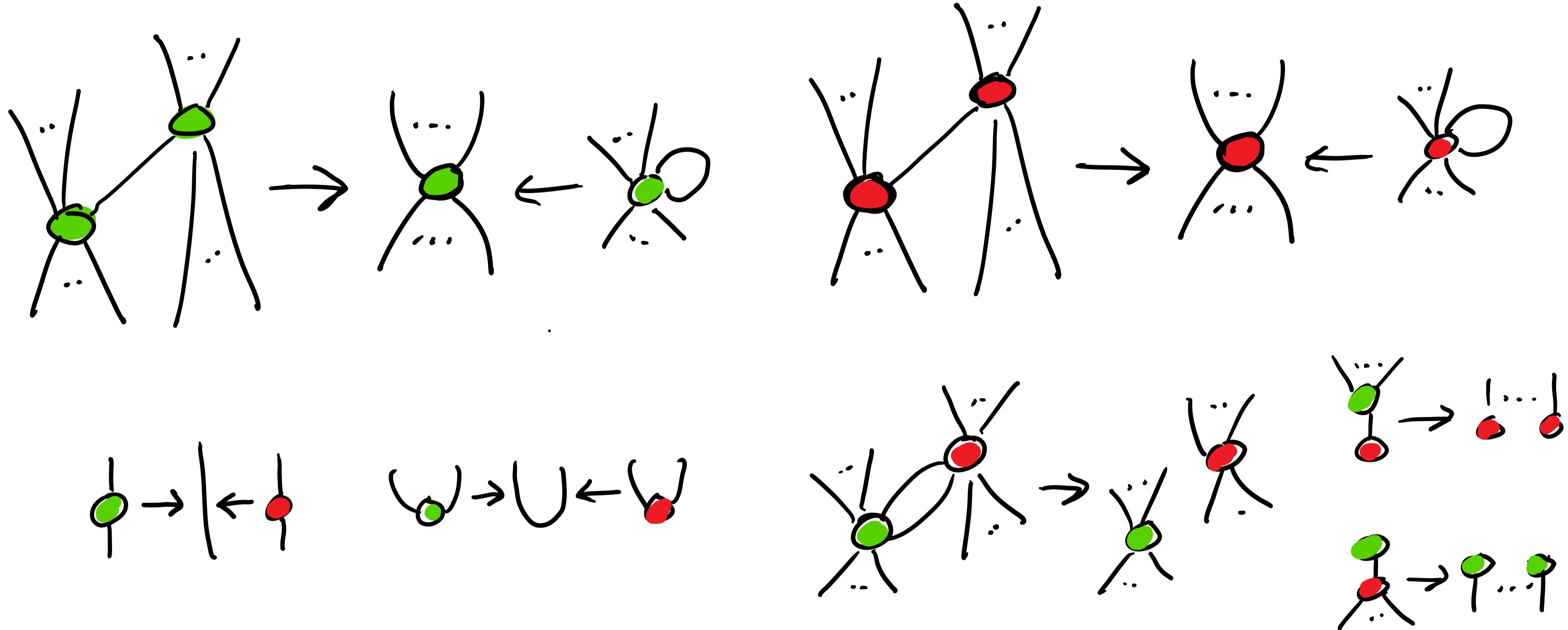
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ZX-calculus



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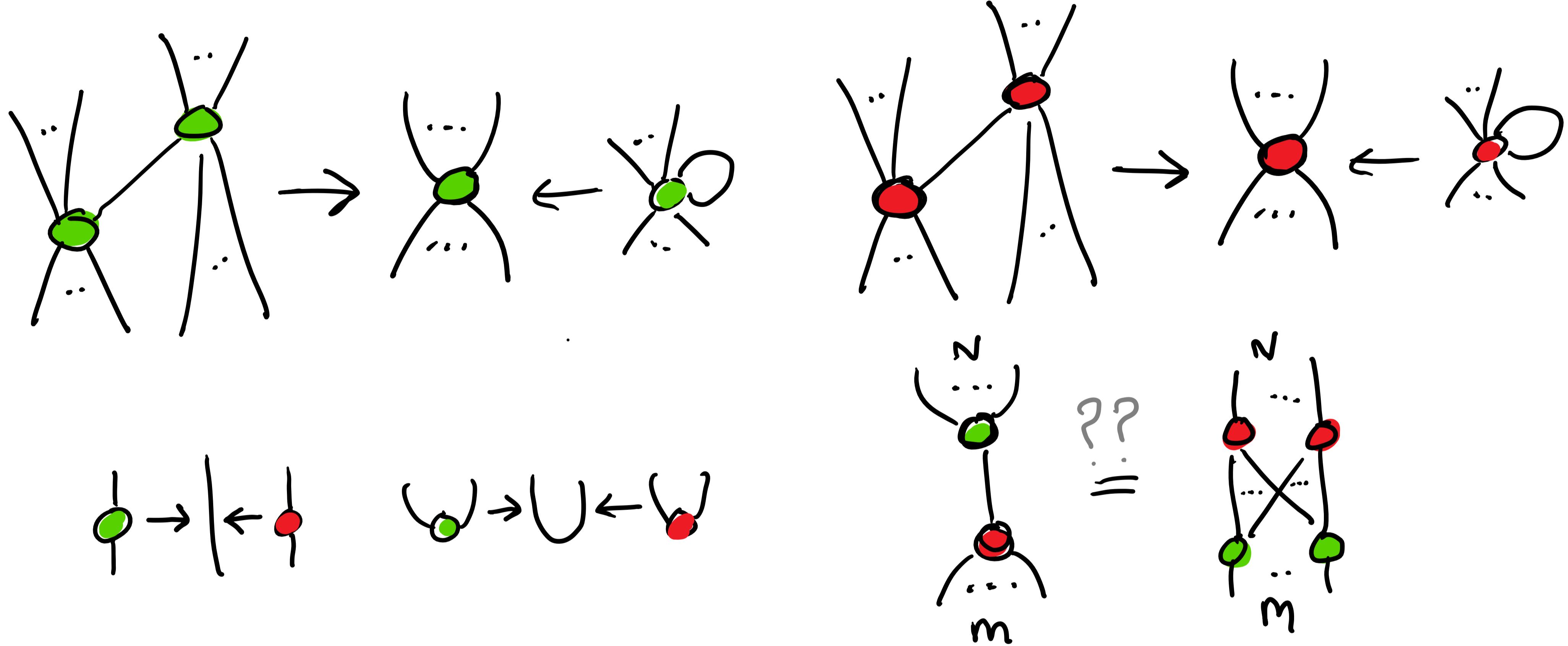
ZX-calculus



→ terminating and confluent.

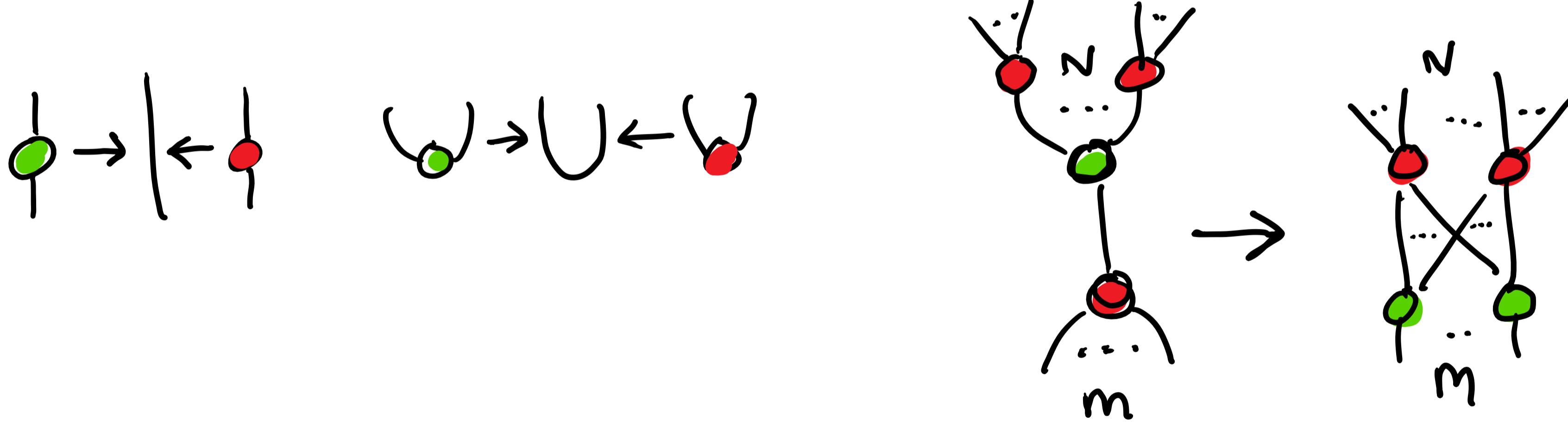
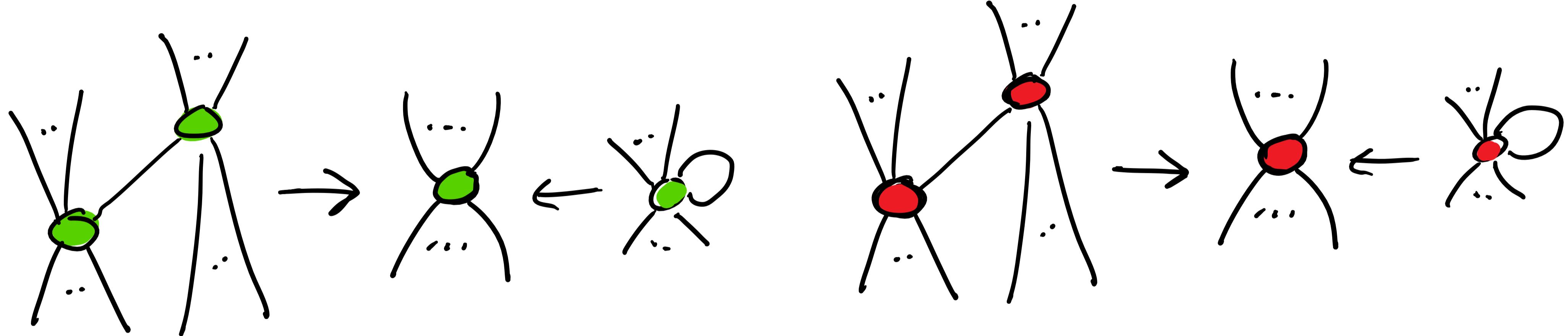
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ZX-calculus



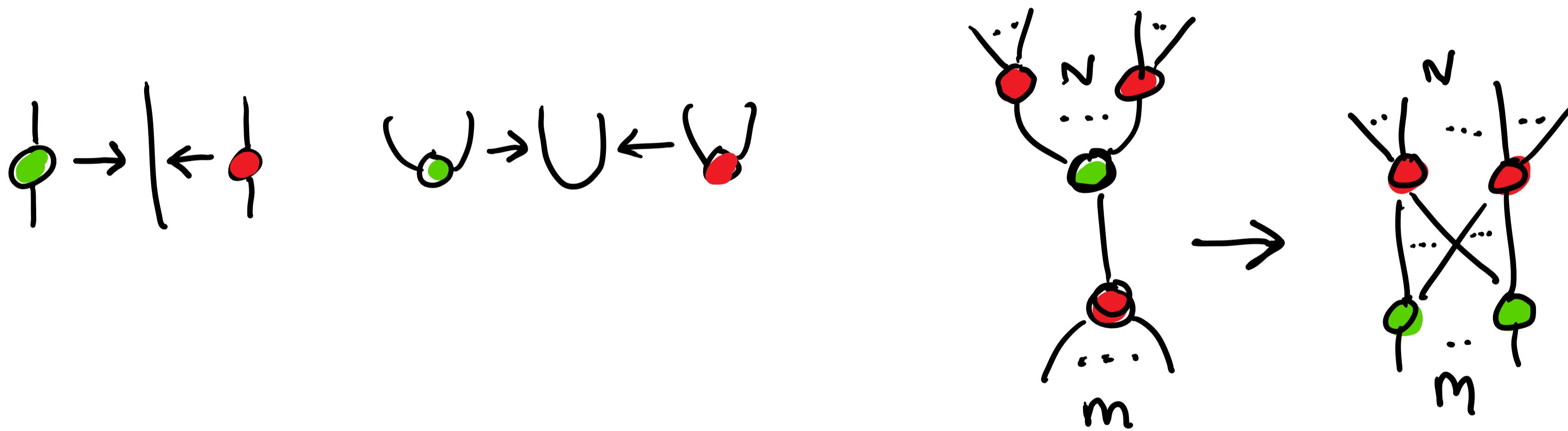
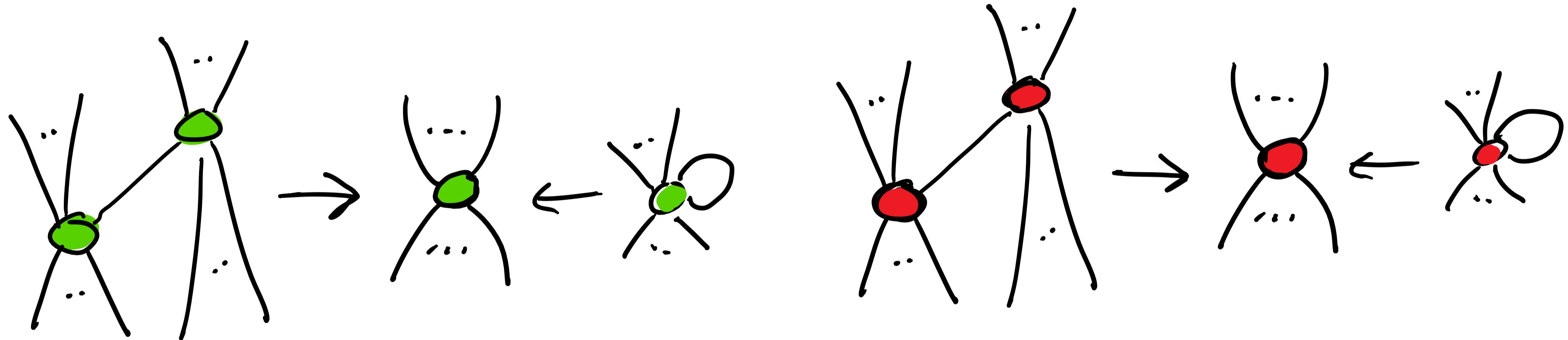
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ZX-calculus



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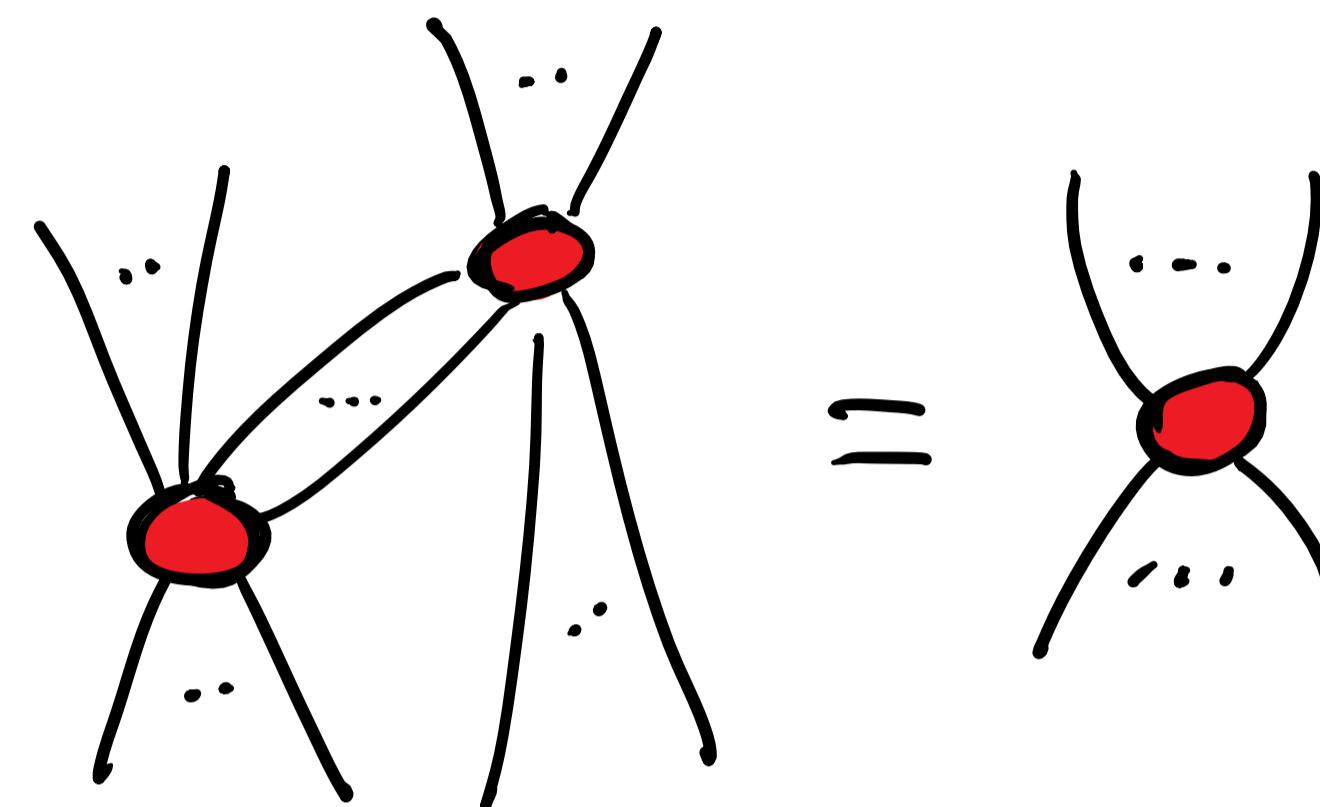
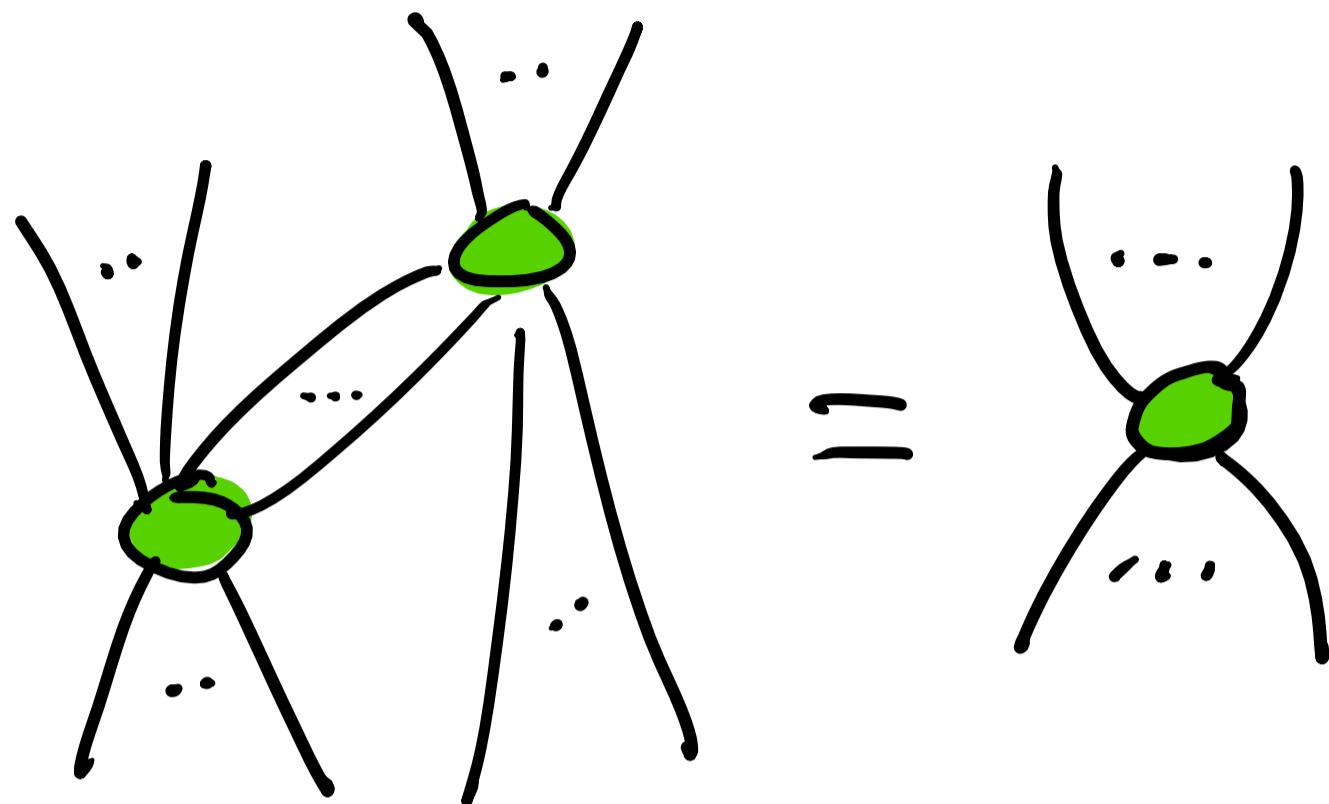
ZX-calculus



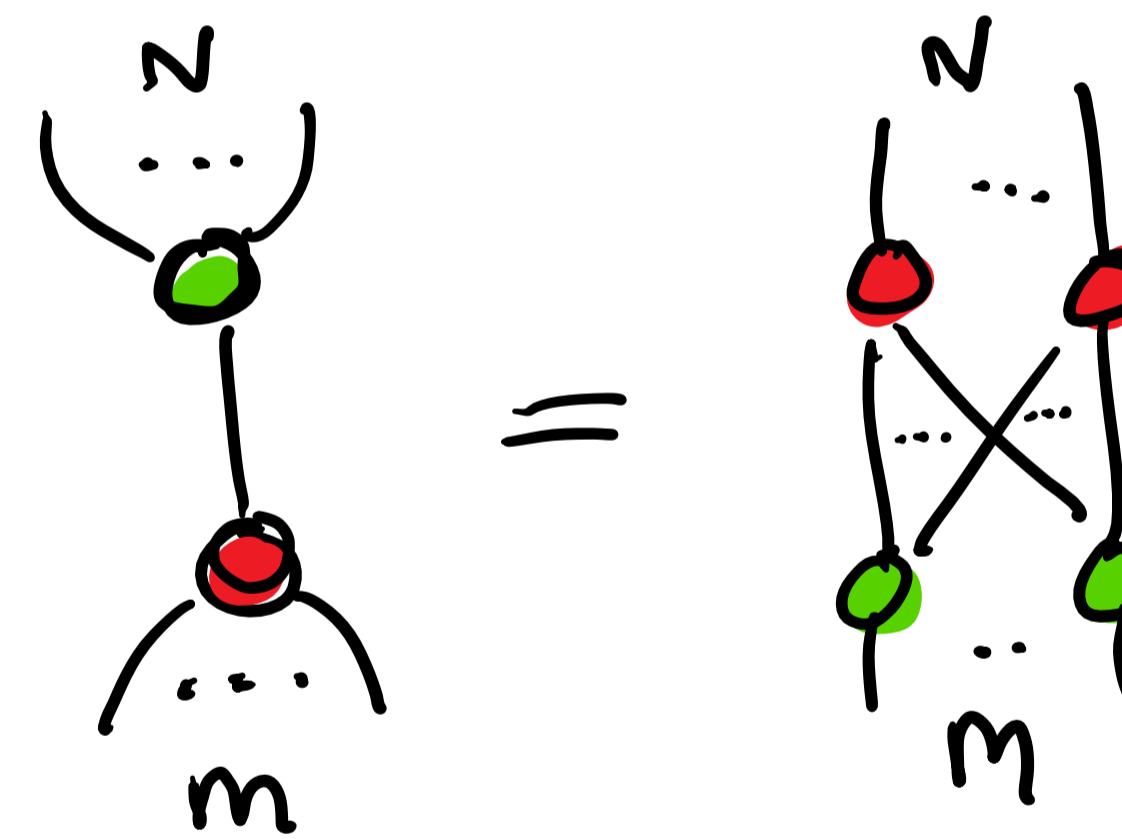
→ terminating with pseudo-NF's

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ZX-calculus



$$\phi = \text{I} = \rho \quad \psi = U = \psi$$



ZX - calculus

$$\begin{array}{c} \text{Diagram showing two green nodes labeled } \alpha \text{ and } \beta \text{ connected by a horizontal wire, with three wires exiting each node.} \\ = \\ \text{Diagram showing a single green node labeled } \alpha + \beta \text{ with three wires exiting it.} \end{array}$$

$$\begin{array}{c} \text{Diagram showing two red nodes labeled } \alpha \text{ and } \beta \text{ connected by a horizontal wire, with three wires exiting each node.} \\ = \\ \text{Diagram showing a single red node labeled } \alpha + \beta \text{ with three wires exiting it.} \end{array}$$

$$\begin{array}{c} \text{Diagram showing a green node labeled } \phi \text{ connected to a vertical wire, which then splits into two wires.} \\ = \\ \text{Diagram showing a vertical wire splitting into two wires, which then connect to two red nodes labeled } \phi. \end{array}$$

$$\begin{array}{c} \text{Diagram showing a green node labeled } \cup \text{ connected to a vertical wire, which then splits into two wires.} \\ = \\ \text{Diagram showing a vertical wire splitting into two wires, which then connect to two red nodes labeled } \cup. \end{array}$$

$$\begin{array}{c} \text{Diagram showing a green node labeled } \cap \text{ connected to a vertical wire, which then splits into two wires.} \\ = \\ \text{Diagram showing a vertical wire splitting into two wires, which then connect to two red nodes labeled } \cap. \end{array}$$

$$\begin{array}{c} \text{Diagram showing a green node labeled } N \text{ connected to a vertical wire, which then splits into two wires.} \\ = \\ \text{Diagram showing a vertical wire splitting into two wires, which then connect to two red nodes labeled } M. \end{array}$$

ZX -calculus

A diagram illustrating the addition of two Z -monomials. On the left, two green nodes labeled α and β are connected by a horizontal line. The node α has three outgoing lines, and the node β has three incoming lines. An equals sign follows this diagram. To the right is a single green node labeled $\alpha + \beta$, which has six outgoing lines, corresponding to the sum of the two original nodes.

A diagram illustrating the addition of two X -monomials. On the left, two red nodes labeled α and β are connected by a horizontal line. The node α has three outgoing lines, and the node β has three incoming lines. An equals sign follows this diagram. To the right is a single red node labeled $\alpha + \beta$, which has six outgoing lines, corresponding to the sum of the two original nodes.

Two diagrams illustrating basic operations. The first shows the identity operation: a green node labeled I is equal to a red node labeled I . The second shows the union operation: a green node labeled U is equal to a red node labeled U .

A diagram illustrating the multiplication of two Z -monomials. On the left, a green node labeled N is connected by a vertical line to a red node labeled m . An equals sign follows this diagram. To the right is a complex network of nodes: a red node labeled N is connected to a green node labeled m via several crossing lines, representing the product of the two monomials.

A diagram illustrating the multiplication of an X -monomial and a Z -monomial. On the left, a red node labeled α is connected to a green node labeled β via four small squares at the junction. An equals sign follows this diagram. To the right is a red node labeled α , representing the result of the multiplication.

ZX -calculus

$$\begin{array}{c} \text{Diagram showing } \alpha + \beta = \alpha + \beta \\ \text{Left: Two green nodes labeled } \alpha \text{ and } \beta \text{ connected by a wire.} \\ \text{Right: A single green node labeled } \alpha + \beta. \end{array}$$

$$\begin{array}{c} \text{Diagram showing } \alpha + \beta = \alpha + \beta \\ \text{Left: Two red nodes labeled } \alpha \text{ and } \beta \text{ connected by a wire.} \\ \text{Right: A single red node labeled } \alpha + \beta. \end{array}$$

$$\begin{array}{c} \text{Diagram showing } \phi = I = \phi \\ \text{Left: A green node labeled } \phi \text{ connected to a red node labeled } I. \\ \text{Right: A red node labeled } I \text{ connected to a green node labeled } \phi. \end{array}$$

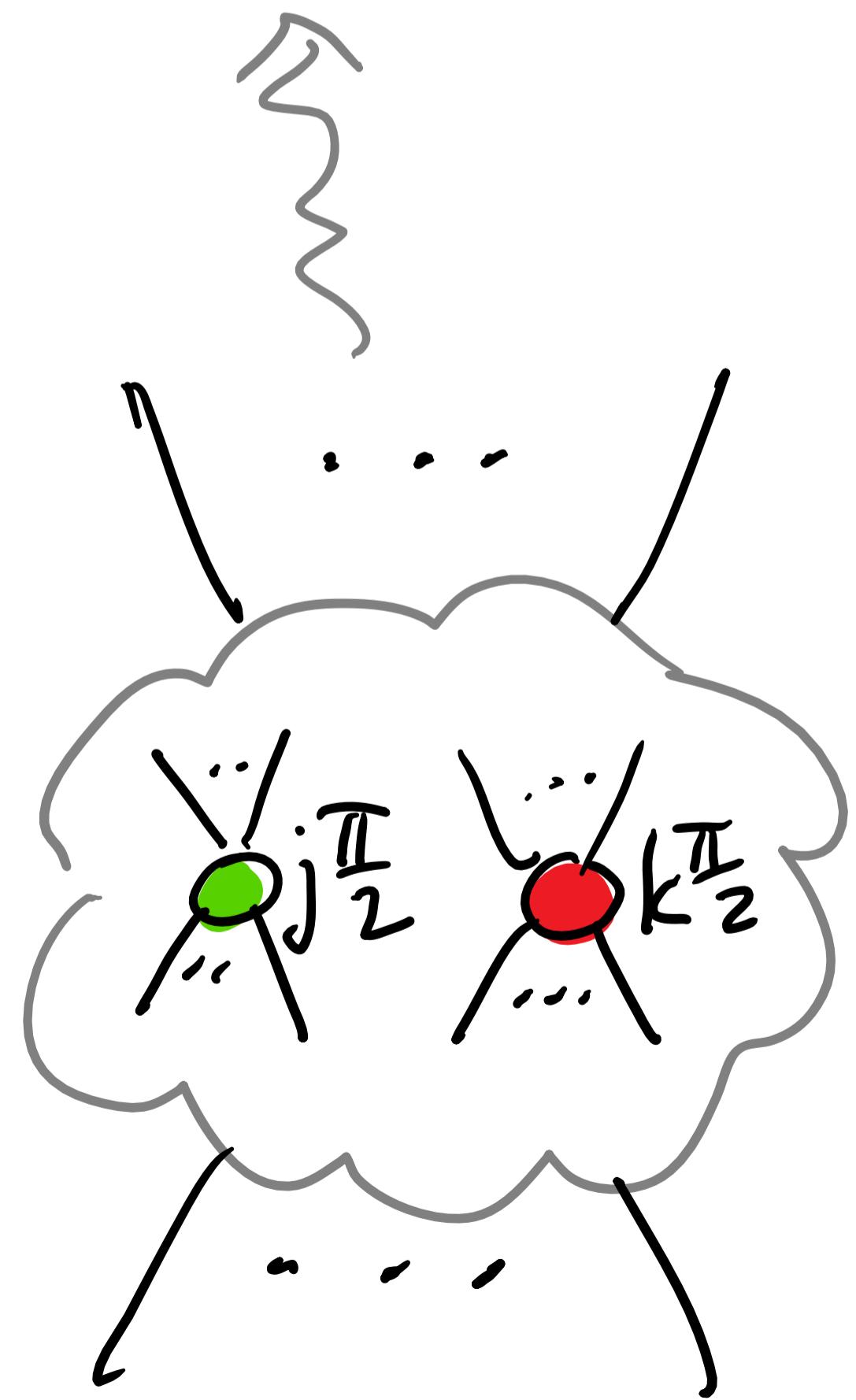
$$\begin{array}{c} \text{Diagram showing } \vee = \bigcup = \vee \\ \text{Left: A green node labeled } \vee \text{ connected to a red node labeled } \bigcup. \\ \text{Right: A red node labeled } \bigcup \text{ connected to a green node labeled } \vee. \\ \text{Below: A green node labeled } N \text{ connected to a red node labeled } m. \end{array}$$

$$\begin{array}{c} \text{Diagram showing } \alpha = \alpha \\ \text{Left: A green node labeled } \alpha \text{ with four wires entering it from the left.} \\ \text{Right: A red node labeled } \alpha \text{ with four wires exiting it to the right.} \end{array}$$

$$\begin{array}{c} \text{Diagram showing } \square := \text{ a sequence of three nodes:} \\ \text{Top: Green node labeled } \overline{\gamma}_2 \\ \text{Middle: Red node labeled } \overline{\gamma}_2 \\ \text{Bottom: Green node labeled } \overline{\gamma}_2 \end{array}$$

THEOREM The ZX-calculus is complete
for Clifford ZX-diagrams.

THEOREM The ZX-calculus is complete
for Clifford ZX-diagrams.



THEOREM Equality of Clifford ZX-diagrams
is decidable in poly time.

⇒ Quantum computations involving $\pi/2$
phases can be simulated efficiently
on a classical computer.

* Gottesman-Knill theorem

ZX -calculus

$$\begin{array}{c} \text{Diagram showing two green nodes labeled } \alpha \text{ and } \beta \text{ connected by a wire, with outputs } \dots \\ \text{Diagram showing a single green node labeled } \alpha + \beta \text{ with outputs } \dots \end{array} = \begin{array}{c} \dots \\ \text{Diagram showing a single green node labeled } \alpha + \beta \text{ with outputs } \dots \end{array}$$

$$\begin{array}{c} \text{Diagram showing two red nodes labeled } \alpha \text{ and } \beta \text{ connected by a wire, with outputs } \dots \\ \text{Diagram showing a single red node labeled } \alpha + \beta \text{ with outputs } \dots \end{array} = \begin{array}{c} \dots \\ \text{Diagram showing a single red node labeled } \alpha + \beta \text{ with outputs } \dots \end{array}$$

$$\begin{array}{c} \text{Diagram showing a green node } \phi \text{ and a red node } \text{I} \text{ connected by a wire, with outputs } \dots \\ \text{Diagram showing a green node } \text{I} \text{ and a red node } \text{I} \text{ connected by a wire, with outputs } \dots \\ \text{Diagram showing a green node } \text{U} \text{ and a red node } \text{U} \text{ connected by a wire, with outputs } \dots \\ \text{Diagram showing a green node } \text{U} \text{ and a red node } \text{U} \text{ connected by a wire, with outputs } \dots \end{array} = \begin{array}{c} \dots \\ \text{Diagram showing a green node labeled } m \text{ with outputs } \dots \\ \text{Diagram showing a red node labeled } m \text{ with outputs } \dots \end{array}$$

$$\begin{array}{c} \text{Diagram showing a green node labeled } \alpha \text{ with four square inputs, with outputs } \dots \\ \text{Diagram showing a red node labeled } \alpha \text{ with three square inputs, with outputs } \dots \end{array} = \begin{array}{c} \dots \\ \text{Diagram showing a red node labeled } \alpha \text{ with three square inputs, with outputs } \dots \end{array}$$

$$\begin{array}{c} \text{Diagram showing a square input node with outputs } \dots \\ \text{Diagram showing a red node labeled } \text{I} \text{ with three square inputs, with outputs } \dots \end{array} := \begin{array}{c} \dots \\ \text{Diagram showing a red node labeled } \text{I} \text{ with three square inputs, with outputs } \dots \\ \text{Diagram showing a green node labeled } \text{I} \text{ with three square inputs, with outputs } \dots \\ \text{Diagram showing a green node labeled } \text{I} \text{ with three square inputs, with outputs } \dots \end{array}$$

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ZX -calculus

Diagram illustrating the addition of two green nodes labeled α and β . The left side shows two separate green nodes with multiple outgoing wires. An equals sign follows, and the right side shows a single green node with all the outgoing wires from both the left and right nodes combined.

Diagram illustrating the addition of two red nodes labeled α and β . The left side shows two separate red nodes with multiple outgoing wires. An equals sign follows, and the right side shows a single red node with all the outgoing wires from both the left and right nodes combined.

Diagrams illustrating the identity node I and the swap node U . The first diagram shows a green node I with one outgoing wire, followed by an equals sign and another green node I with one outgoing wire. The second diagram shows a green node U with two outgoing wires, followed by an equals sign and a red node U with two outgoing wires.

Diagram illustrating the multiplication of two nodes labeled m and n . The left side shows a green node n connected to a red node m , with an equals sign. The right side shows a red node n connected to a green node m .

Diagram illustrating the effect of a node labeled α on four input wires. The left side shows a green node α with four incoming wires and four outgoing wires. An equals sign follows, and the right side shows a red node α with four outgoing wires.

Diagram illustrating the definition of a node β_2 . It shows a red node β_2 with three outgoing wires, followed by a colon and a definition. The definition consists of three green nodes labeled π_2 , $\bar{\pi}_2$, and π_2 stacked vertically.

Diagram illustrating the equality between two nodes labeled β and β' . The left side shows a green node β with three outgoing wires, followed by an equals sign and a red node β' with three outgoing wires. The right side shows a red node β' with three outgoing wires, followed by an equals sign and a green node β' with three outgoing wires.

THEOREM The universal ZX-calculus
is complete for all ZX-diagrams.

THEOREM The universal ZX-calculus
is complete for all ZX-diagrams.

* Wang o Ng 2017

* PRESNTATION Vilmart 2018

→ equality is decidable, but NOT efficient

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(unless $BQP \subseteq P$)



quantum
efficient

Quantum
Theory

qubit

Quantum
Theory

QUANTUM STATES

$$\left| \psi \right\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

normalised

QUANTUM STATES

$$|\psi\rangle = \frac{1}{\sqrt{2}} |\alpha\rangle_B |\beta\rangle_A$$

QUANTUM STATES

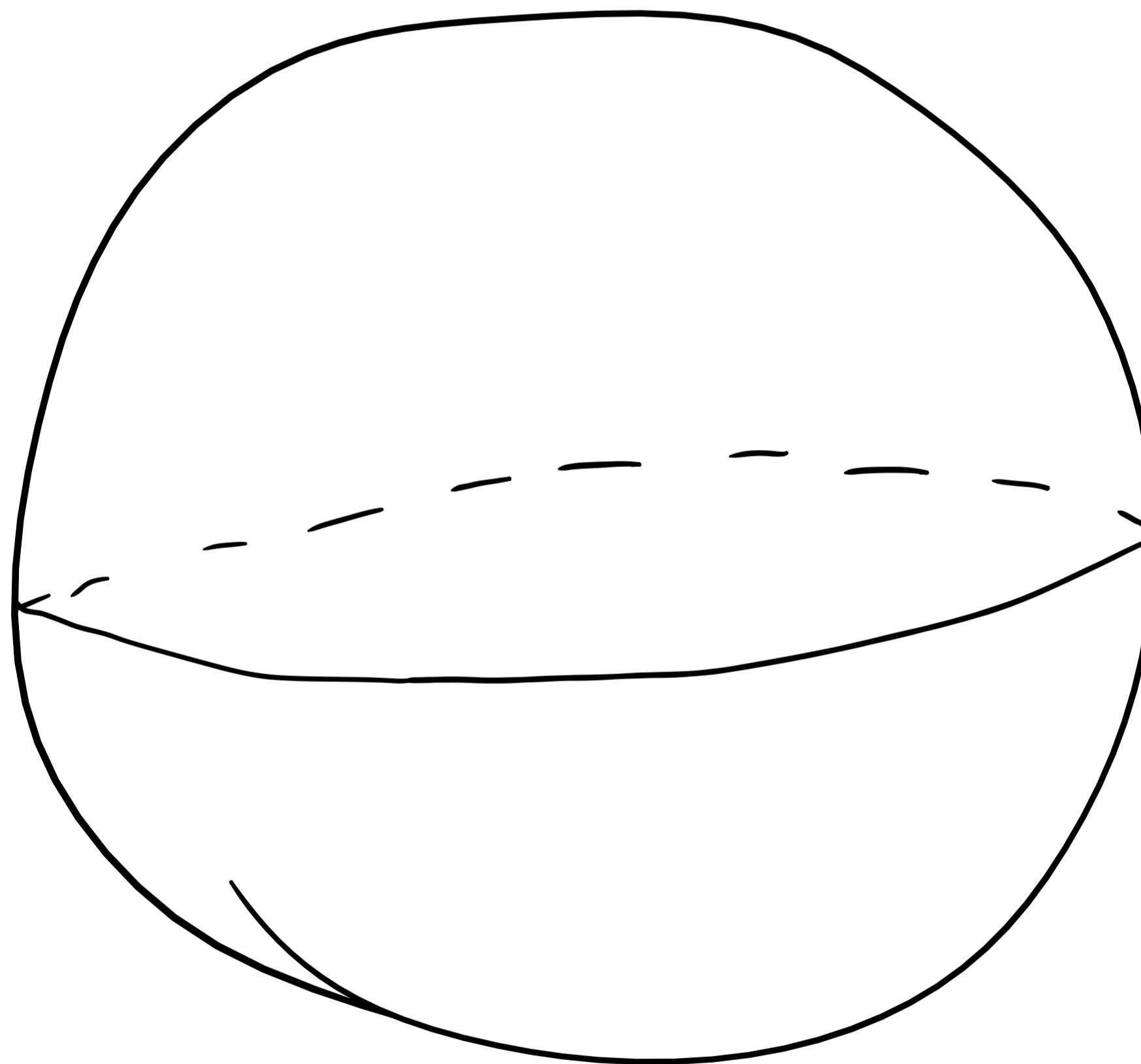
global phase

$$|\psi\rangle^2 = \frac{e^{i\gamma}}{\sqrt{2}} |\alpha\rangle \langle \beta| + \frac{e^{-i\gamma}}{\sqrt{2}} |\beta\rangle \langle \alpha|$$

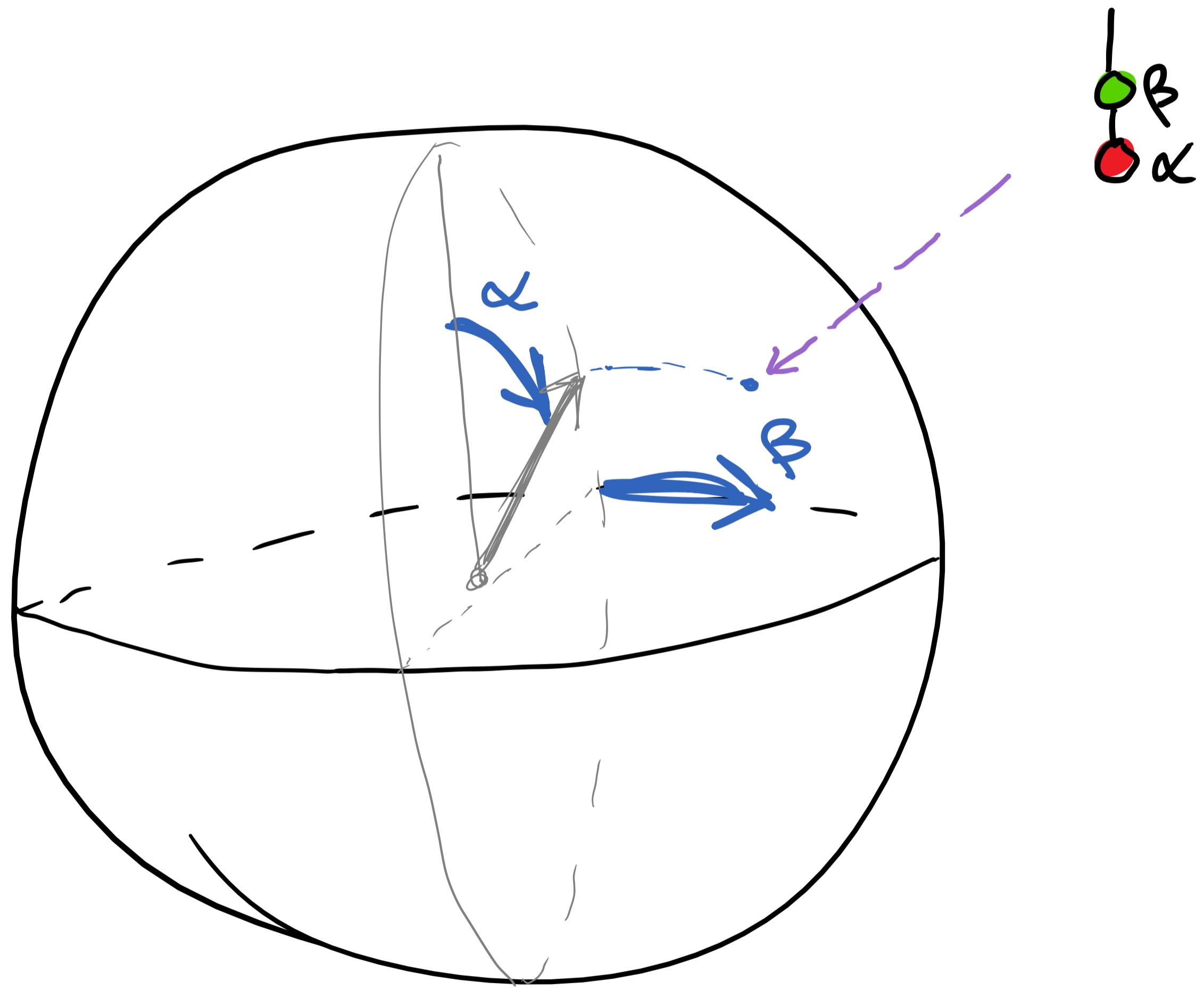
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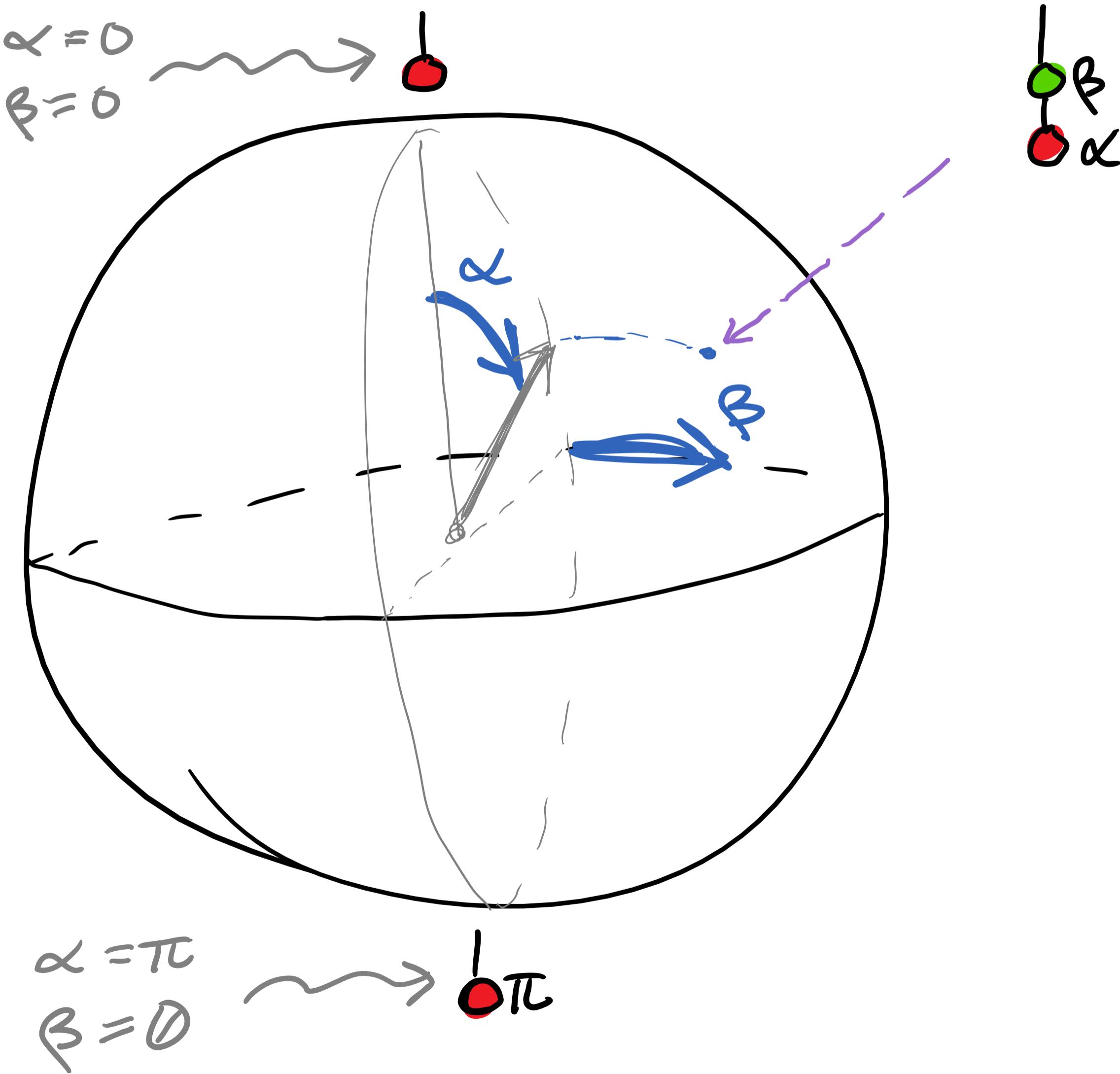
Bloch Sphere



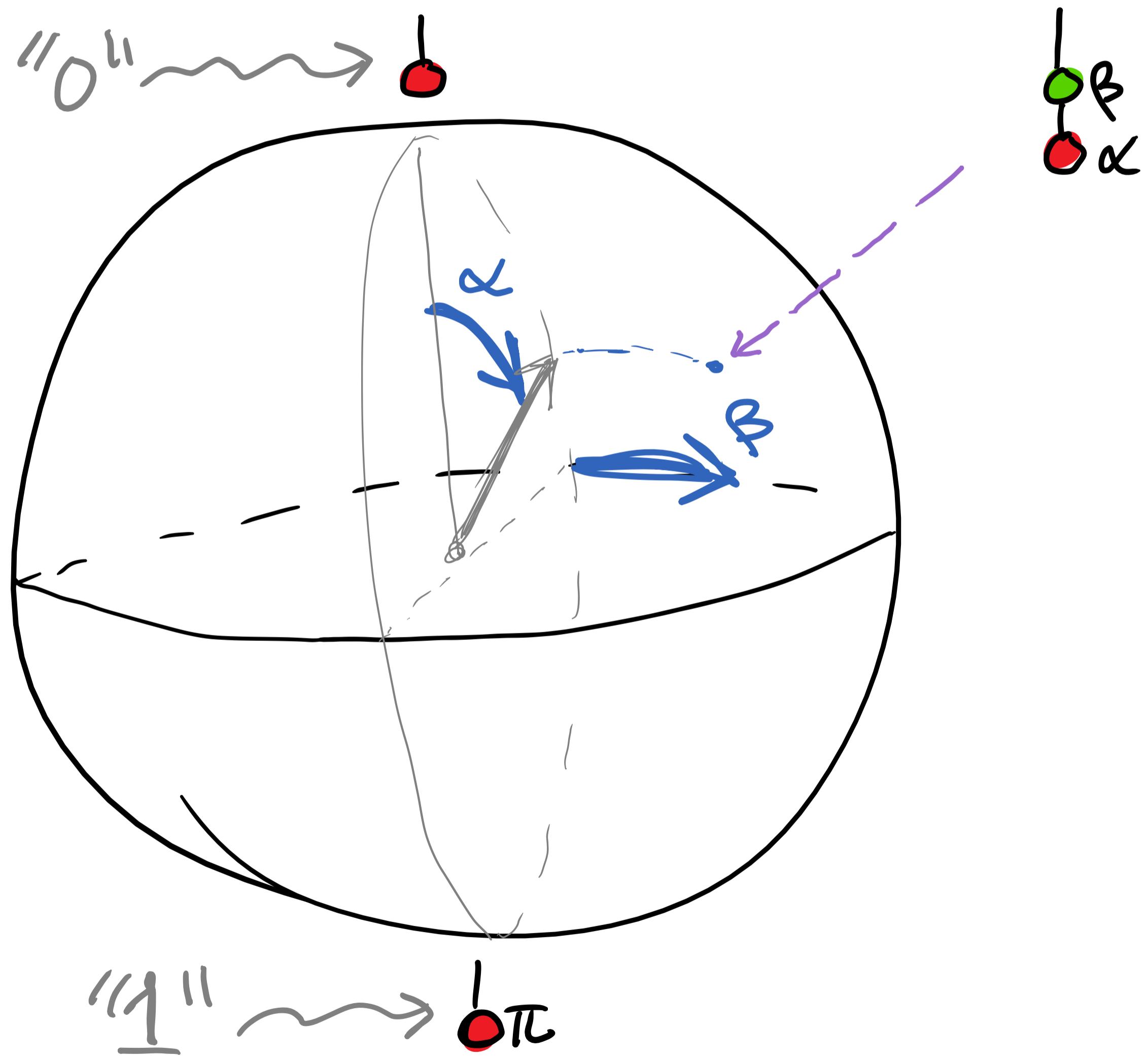
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BITS

QUBITS

BITS

2 States:

• 0

• 1

QUBITS

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QUBITS

a whole sphere of states:



BITS

2 States:

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logic gates :=

Any function $\{0,1\}^n \rightarrow \{0,1\}^m$

QUBITS

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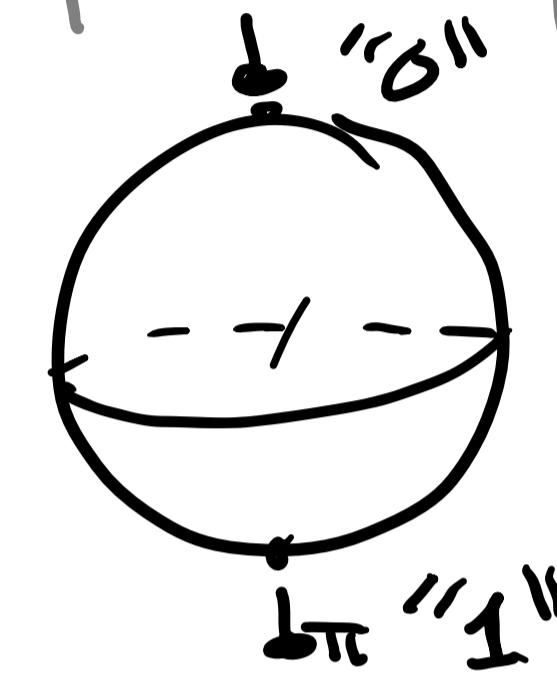
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QUBITS

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unitary matrices \leftrightarrow rotations

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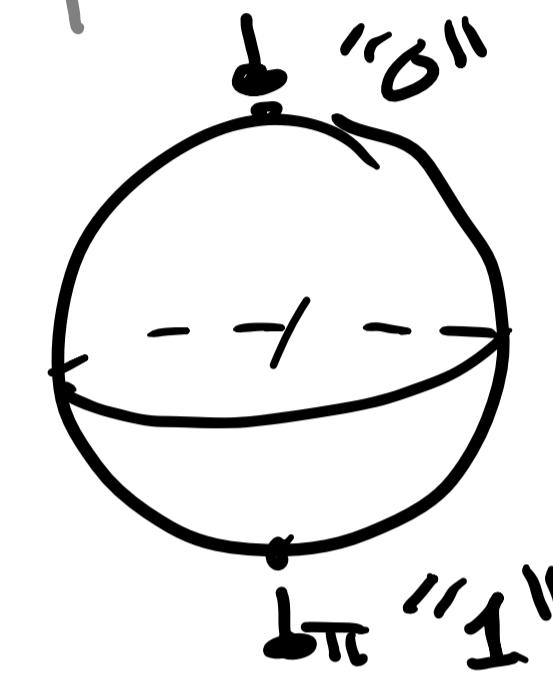
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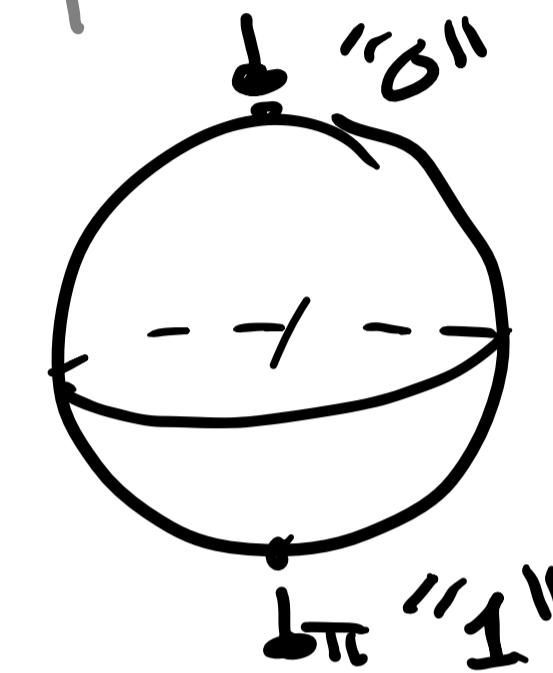
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CAN ONLY BE READ
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BITS

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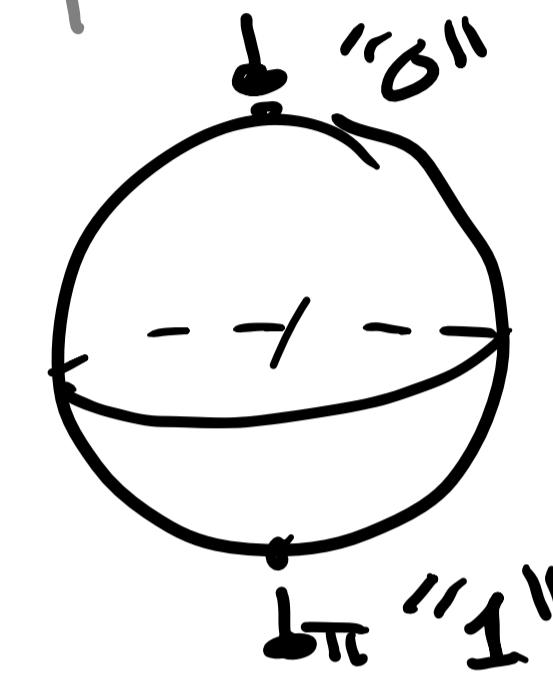
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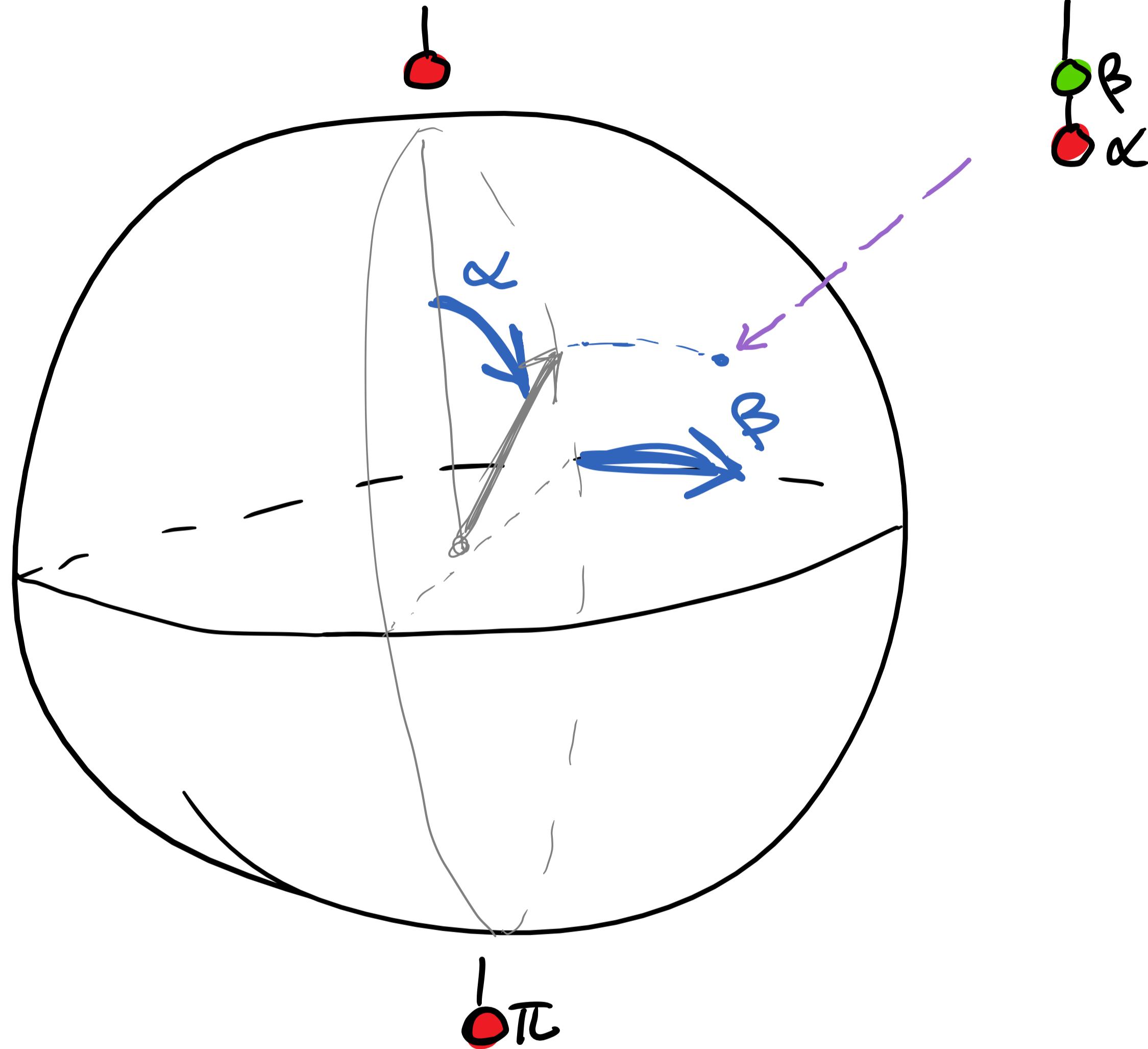
QUBITS

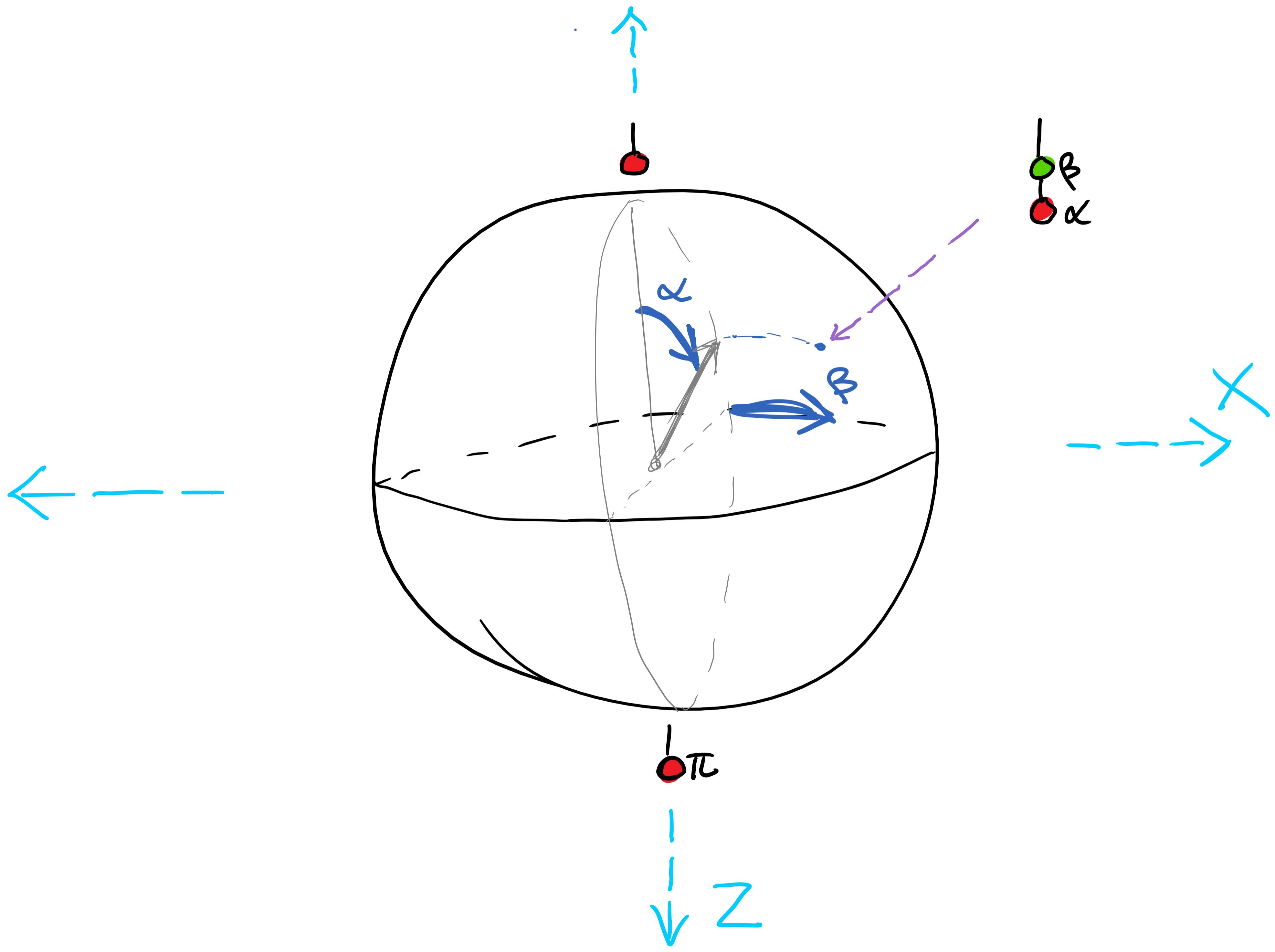
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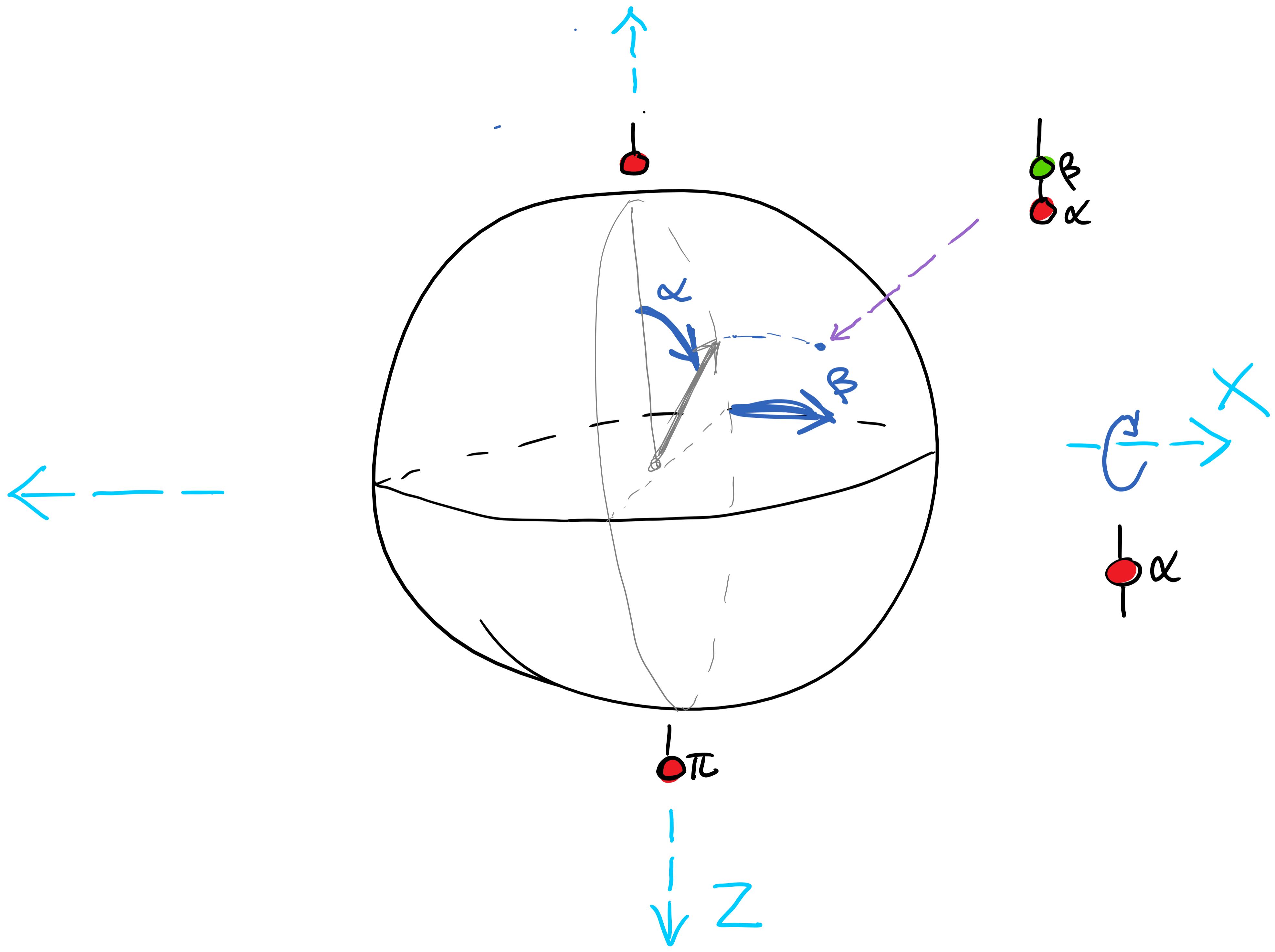


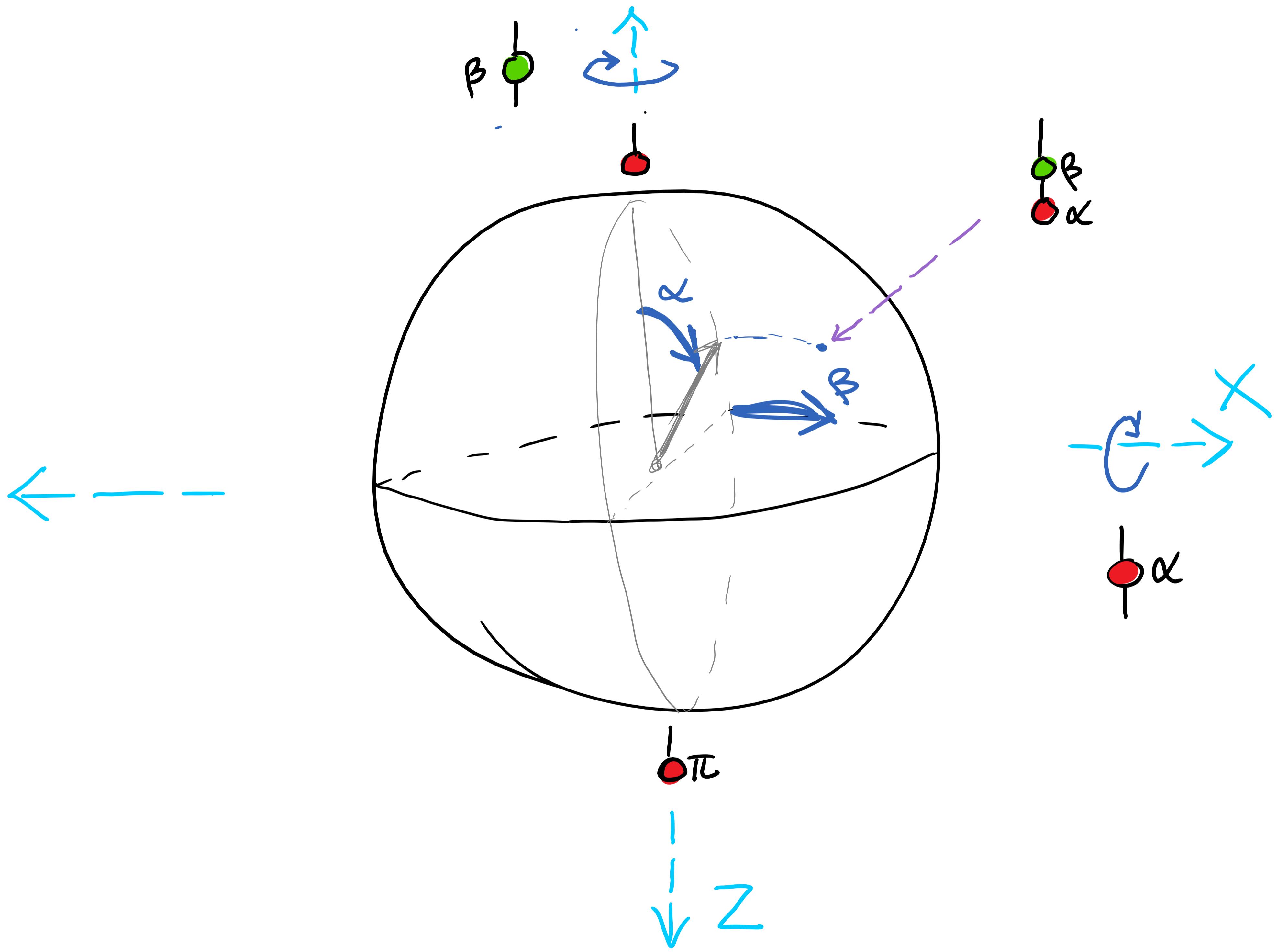
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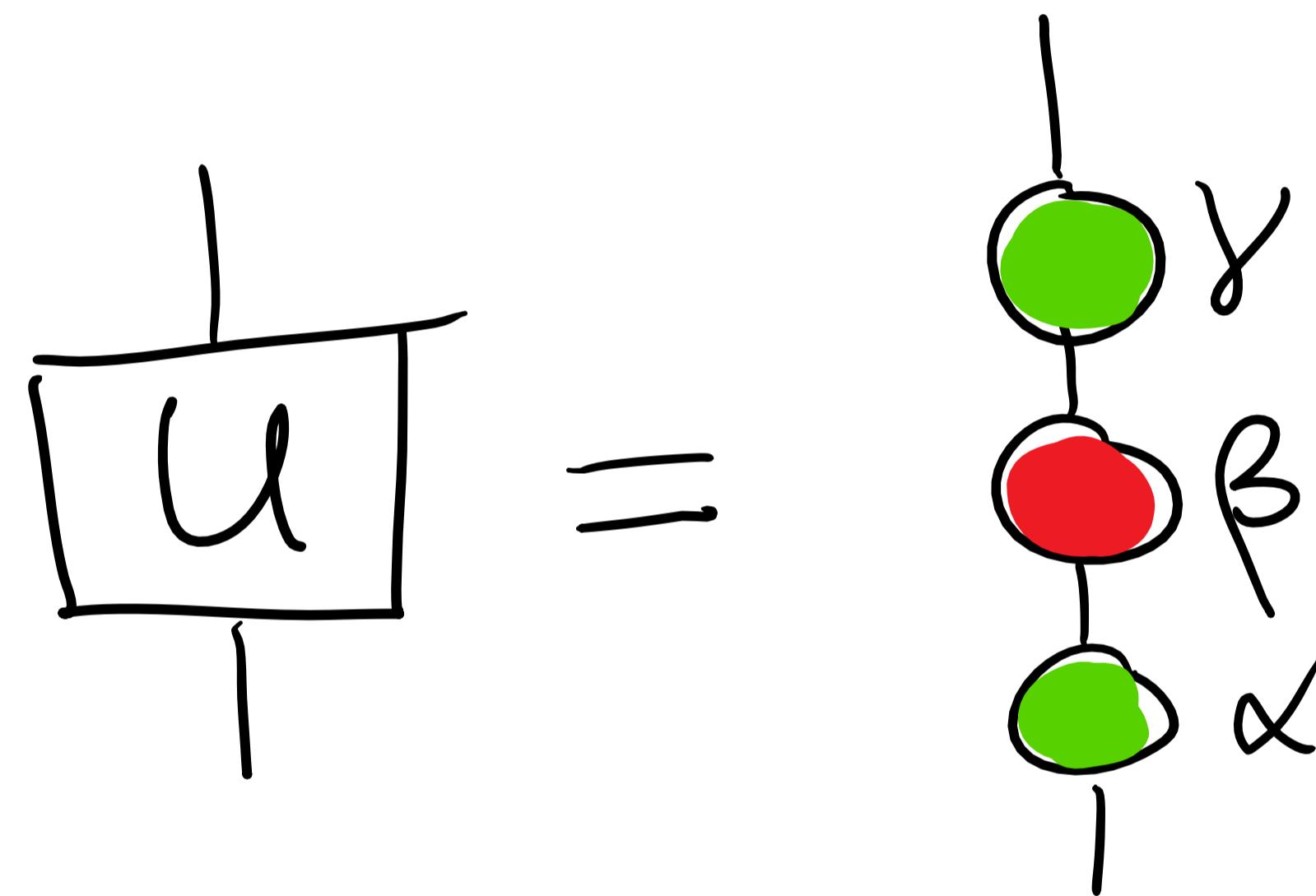
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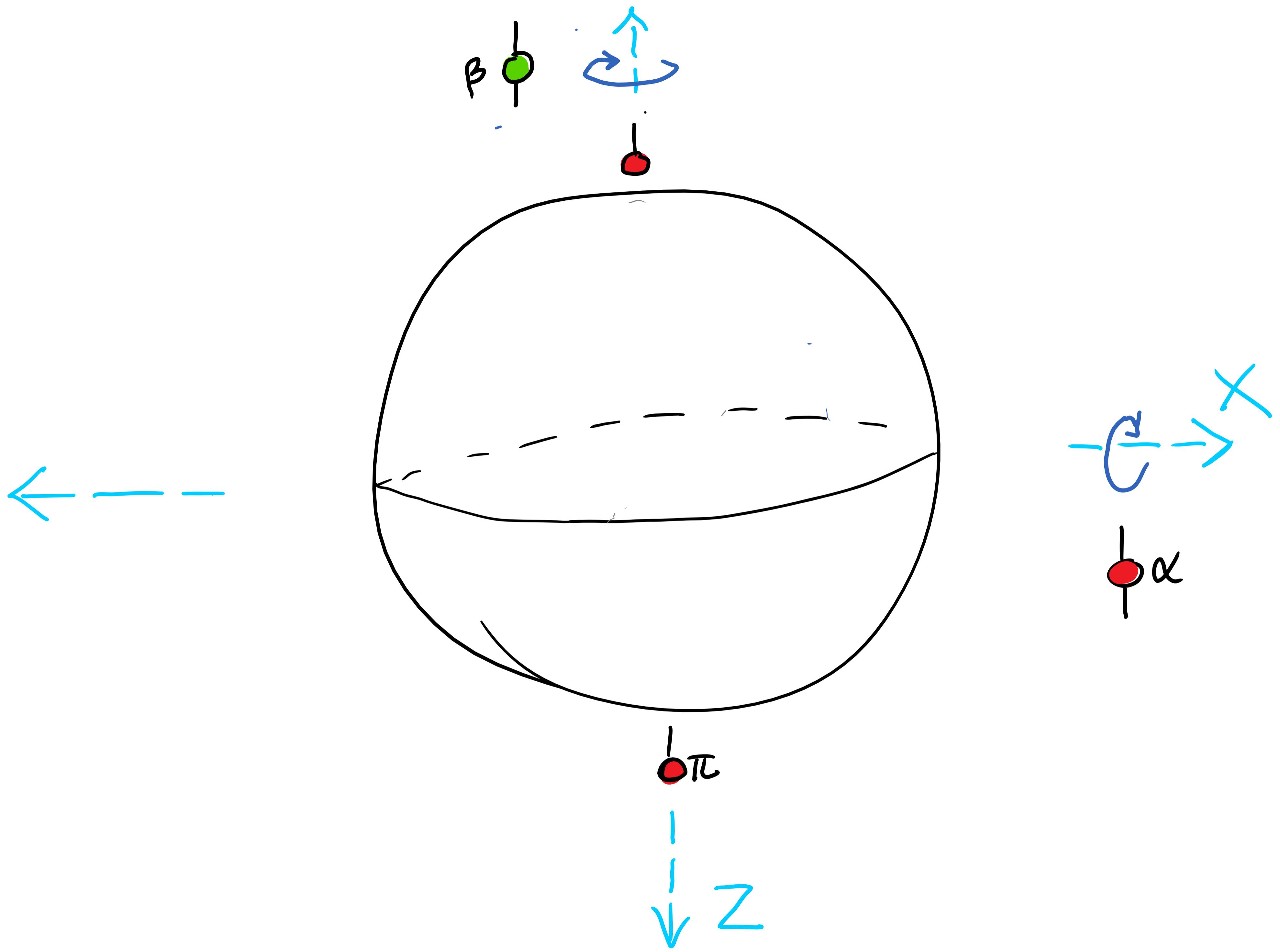


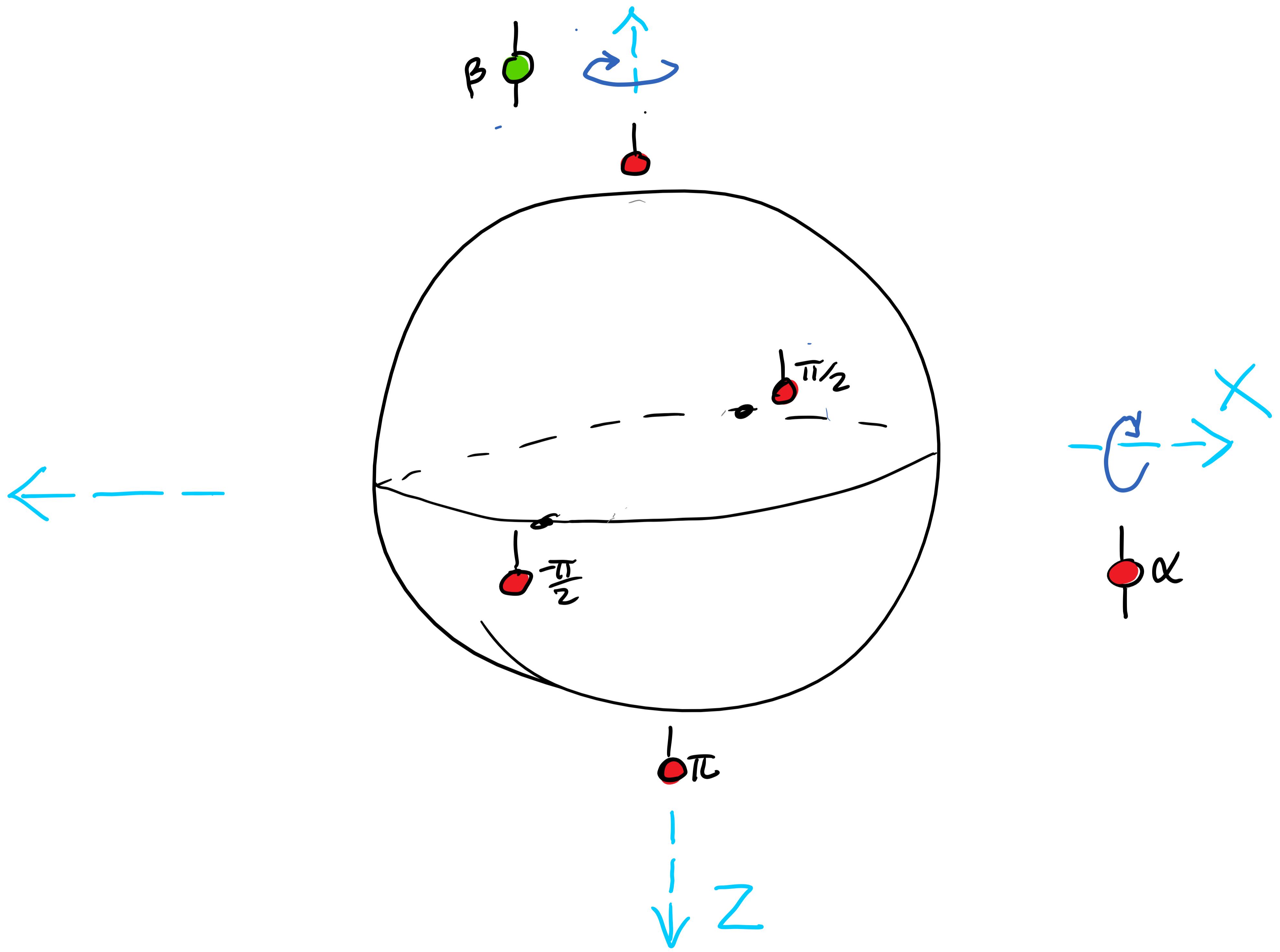


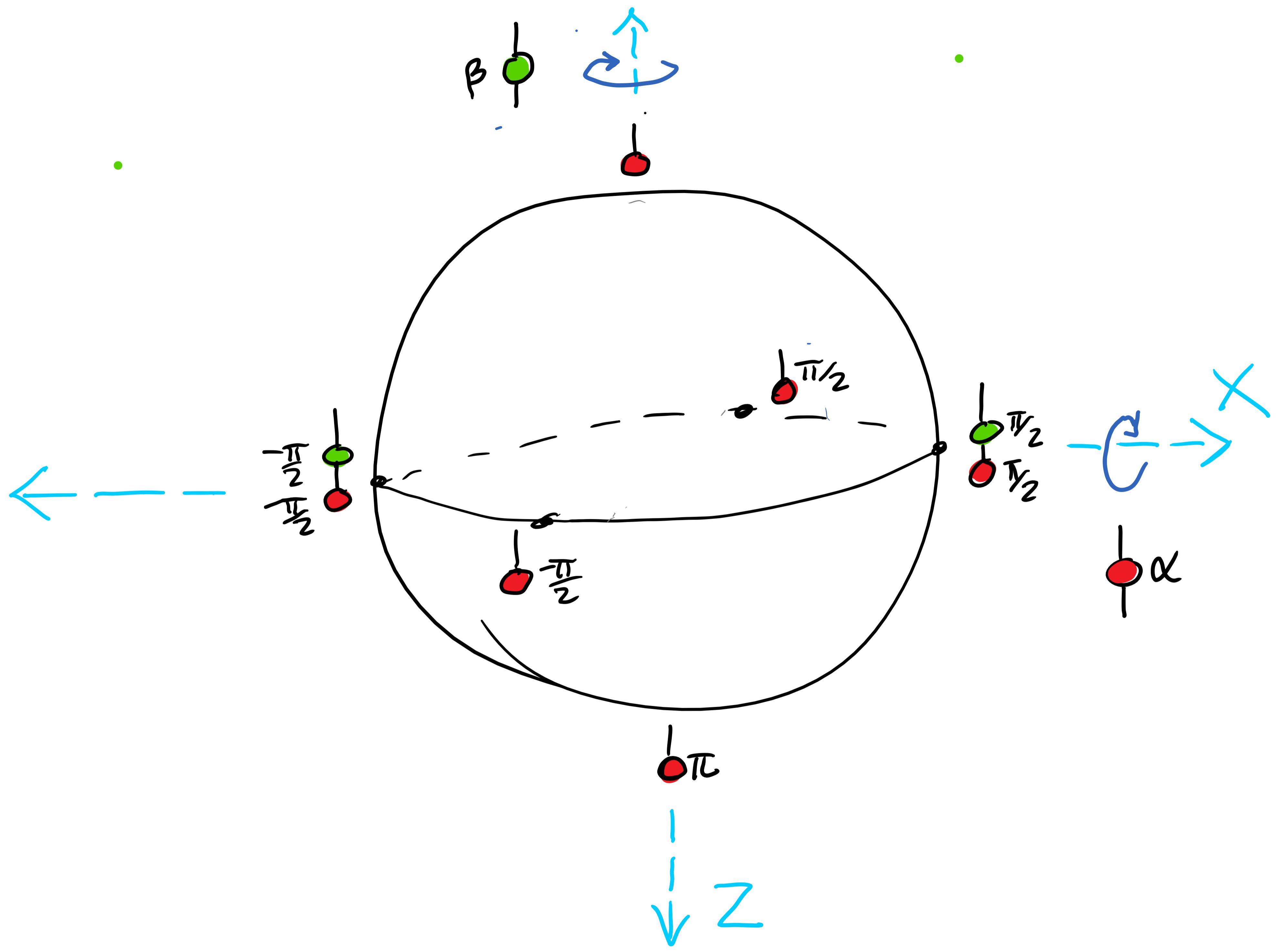


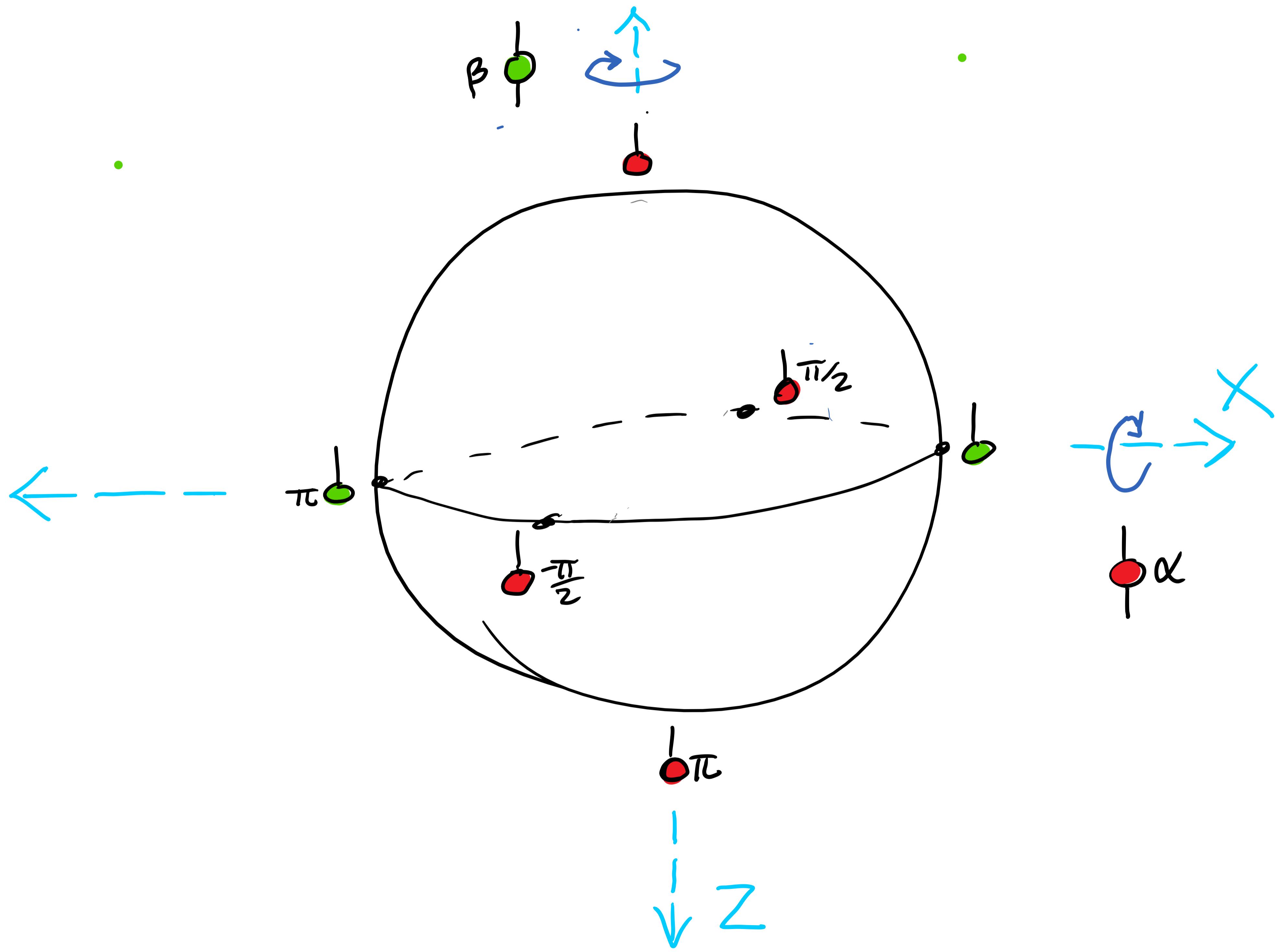


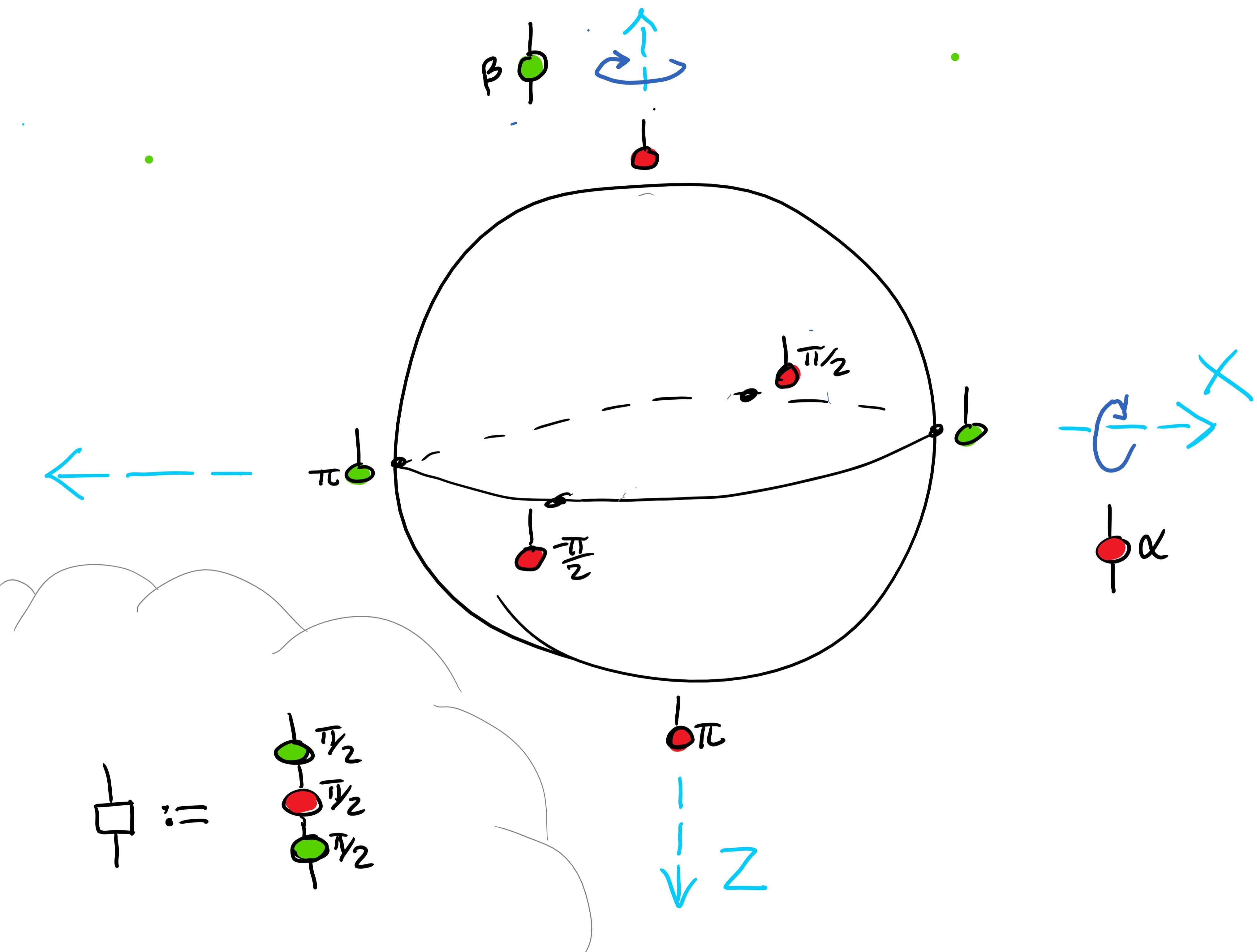












MULTI-QUBIT LOGIC

①

②

BIGGER UNITARIES

e.g.-

SWAP \rightarrow

CNOT \rightarrow

DEF A quantum circuit is a unitary built from a fixed set of basic gates.

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Thm The set $\{ \text{ } \text{ } \text{ } \}$ is universal for unitaries.

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Thm The set $\left\{ \begin{array}{c} \text{green circle} \\ | \frac{\pi}{4} \end{array}, \begin{array}{c} \text{red circle} \\ | \frac{\pi}{4} \end{array}, \begin{array}{c} \text{green circle} \\ \text{---} \\ \text{red circle} \end{array} \right\}$ is ^{approx} universal for unitaries.

BITS

2 States:

• 0

• 1

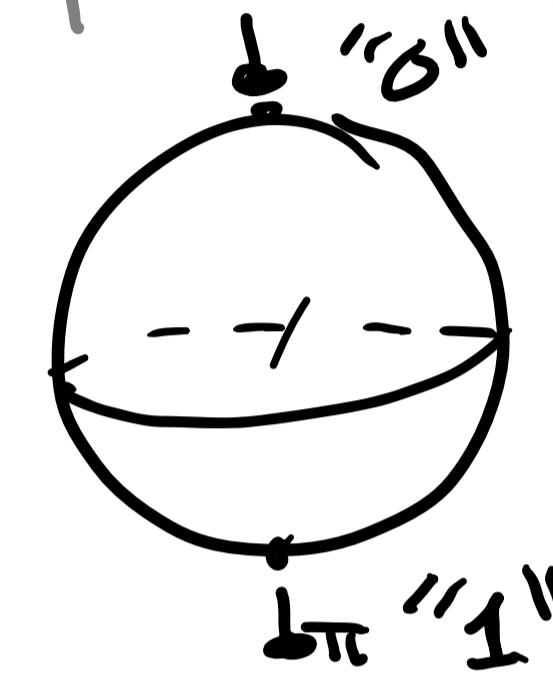
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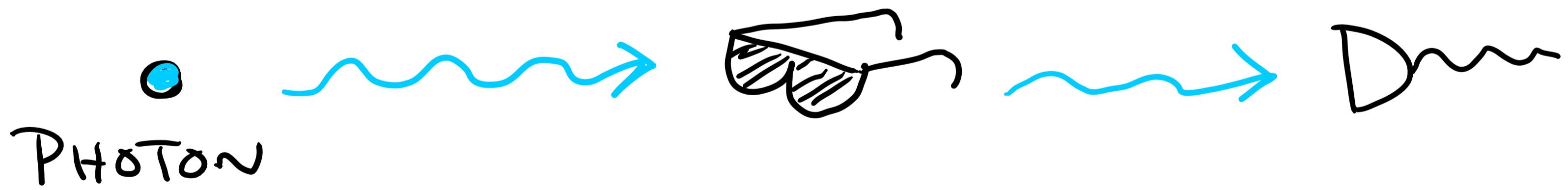
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- * get classical data out of a system, probabilistically.
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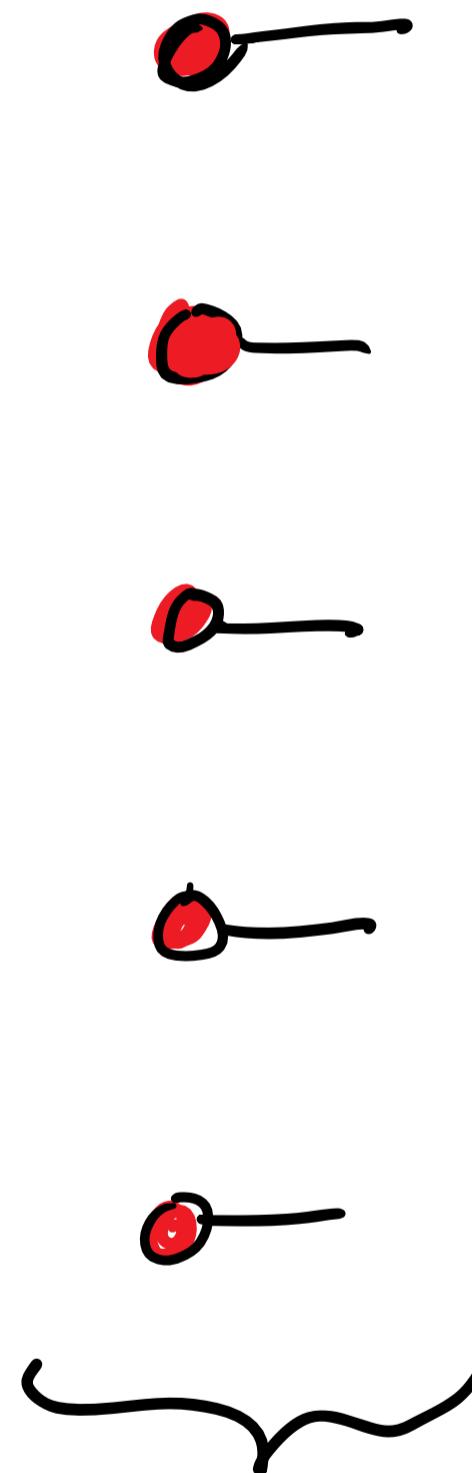
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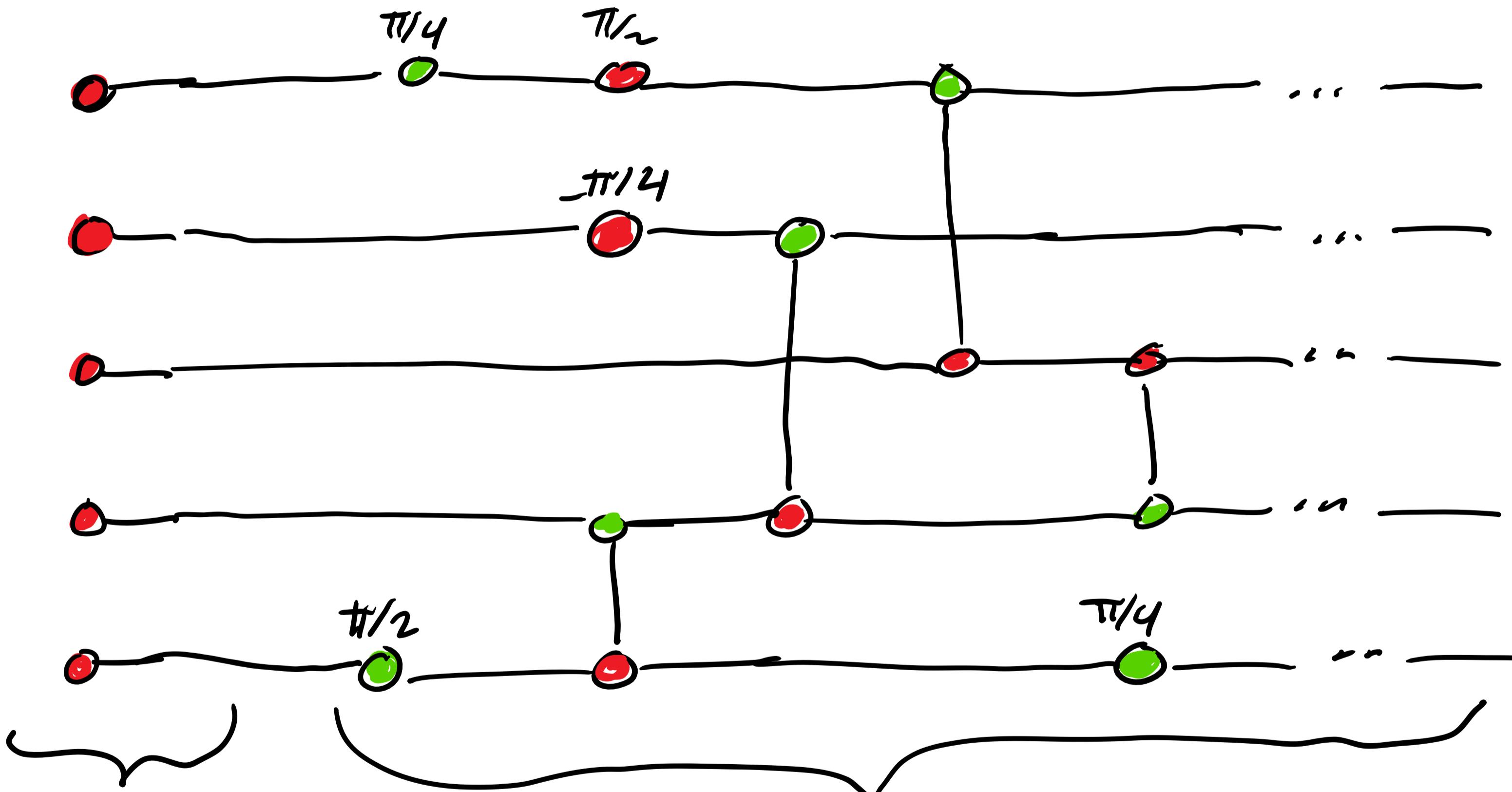
QUANTUM CIRCUIT MODEL

QUANTUM CIRCUIT MODEL



1. prepare a
fixed state

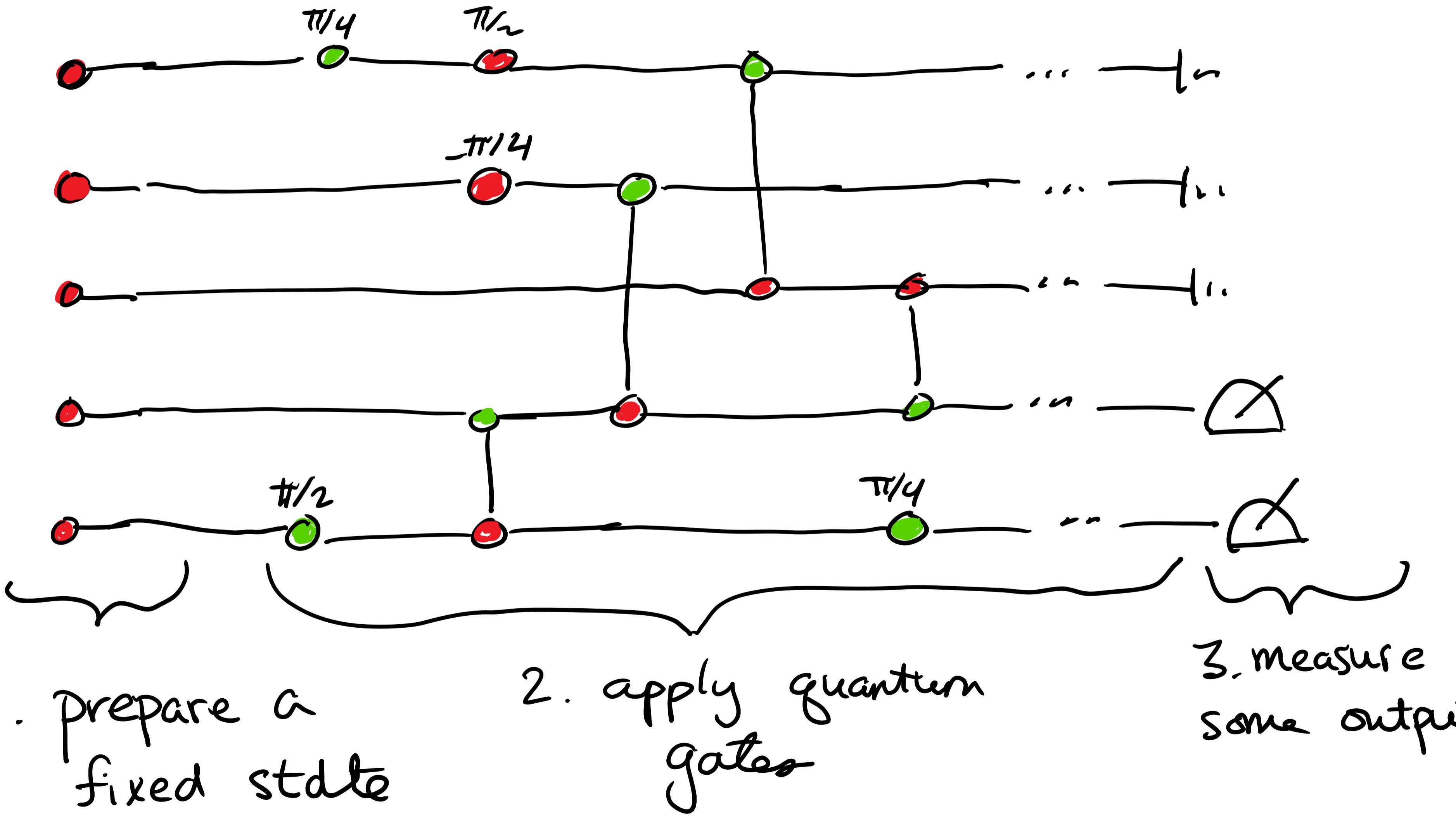
QUANTUM CIRCUIT MODEL



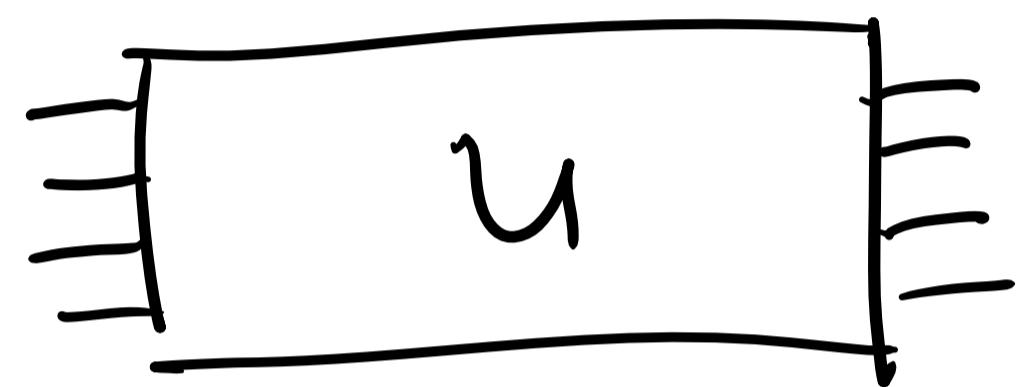
1. prepare a
fixed state

2. apply quantum
gates

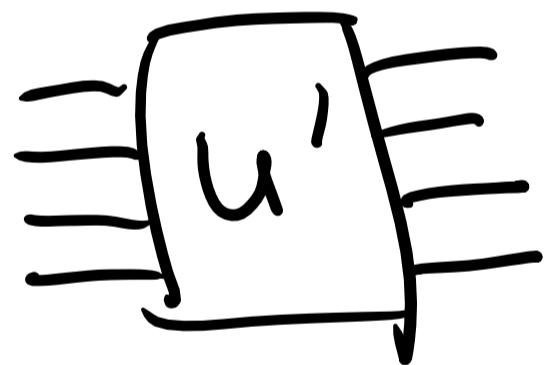
QUANTUM CIRCUIT MODEL



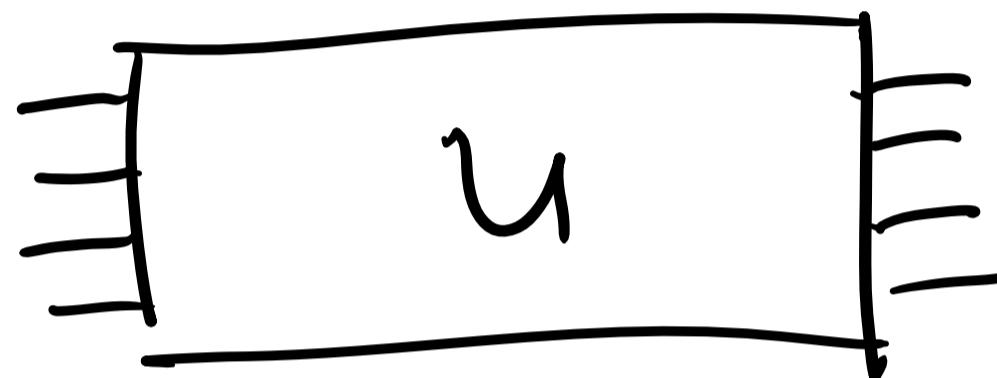
QUANTUM CIRCUIT OPTIMISATION



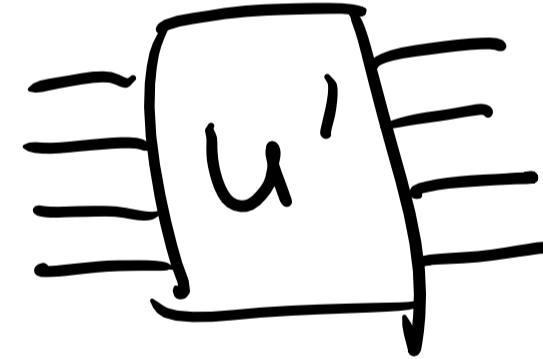
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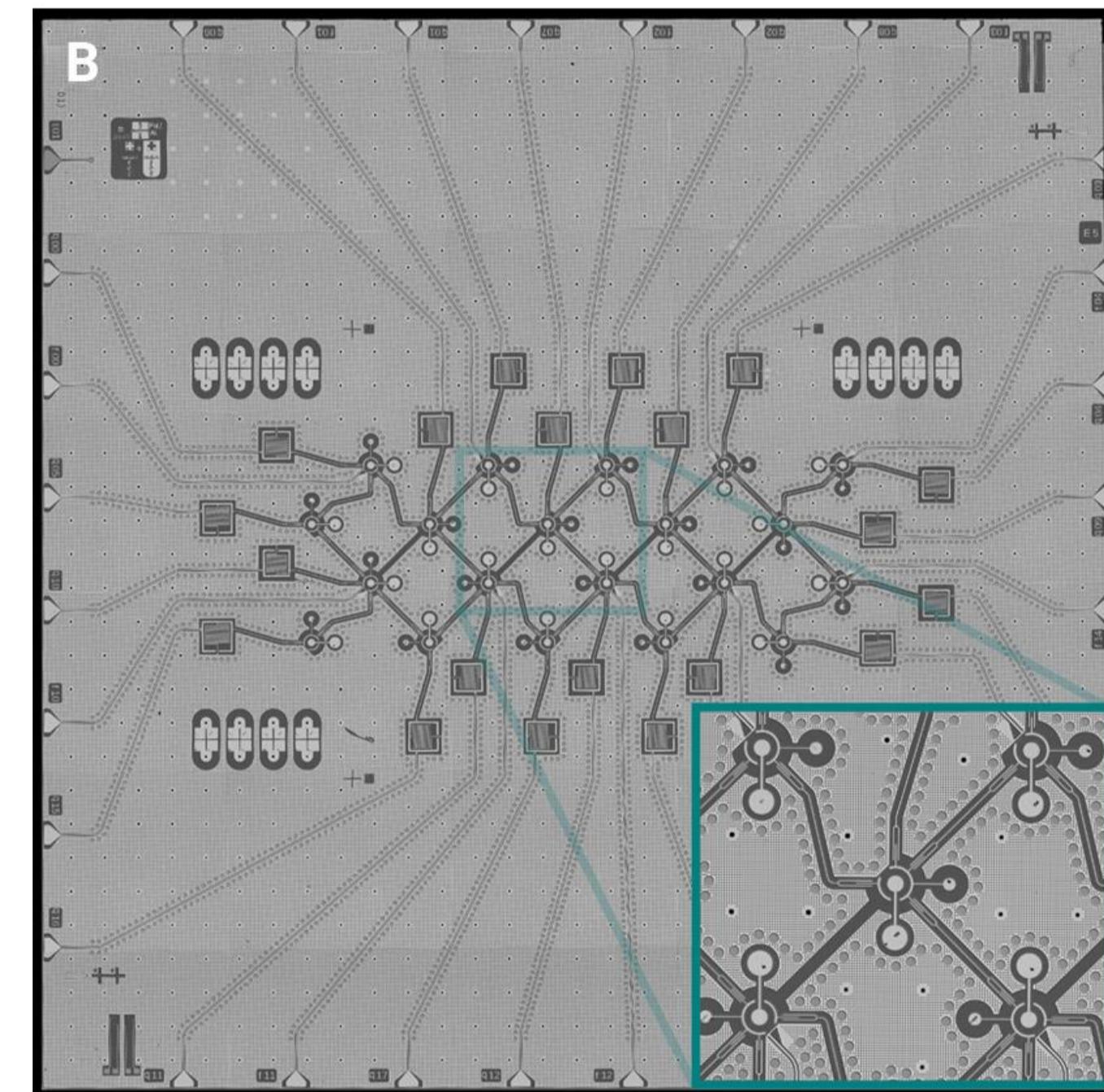
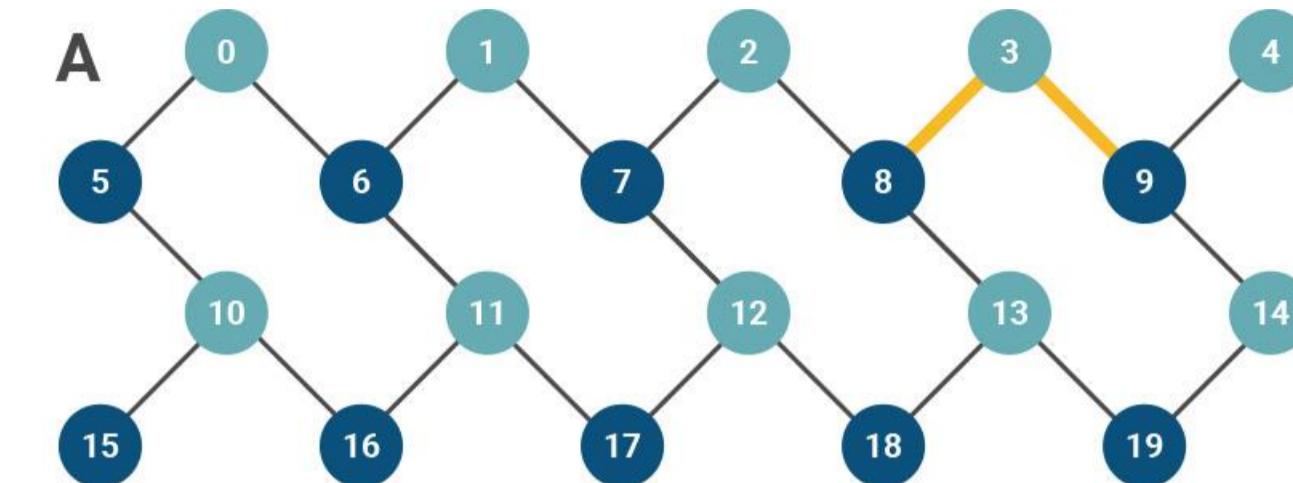
Quantum Circuit Optimisation



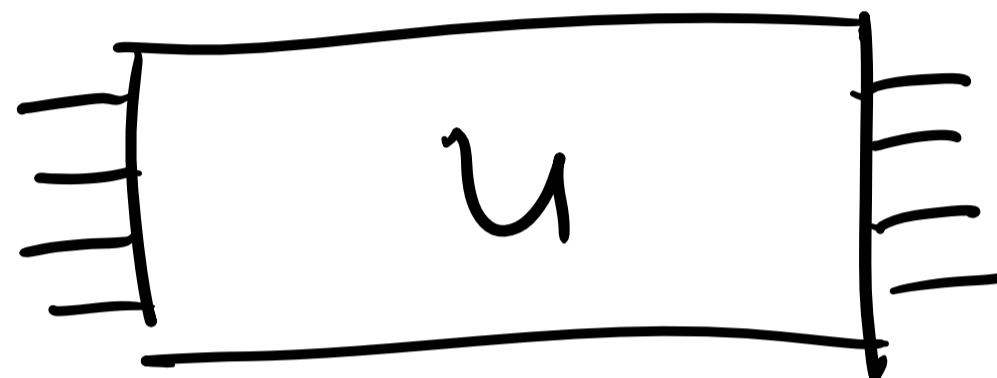
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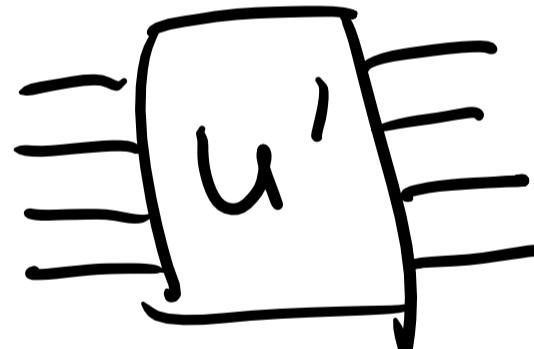
Routing



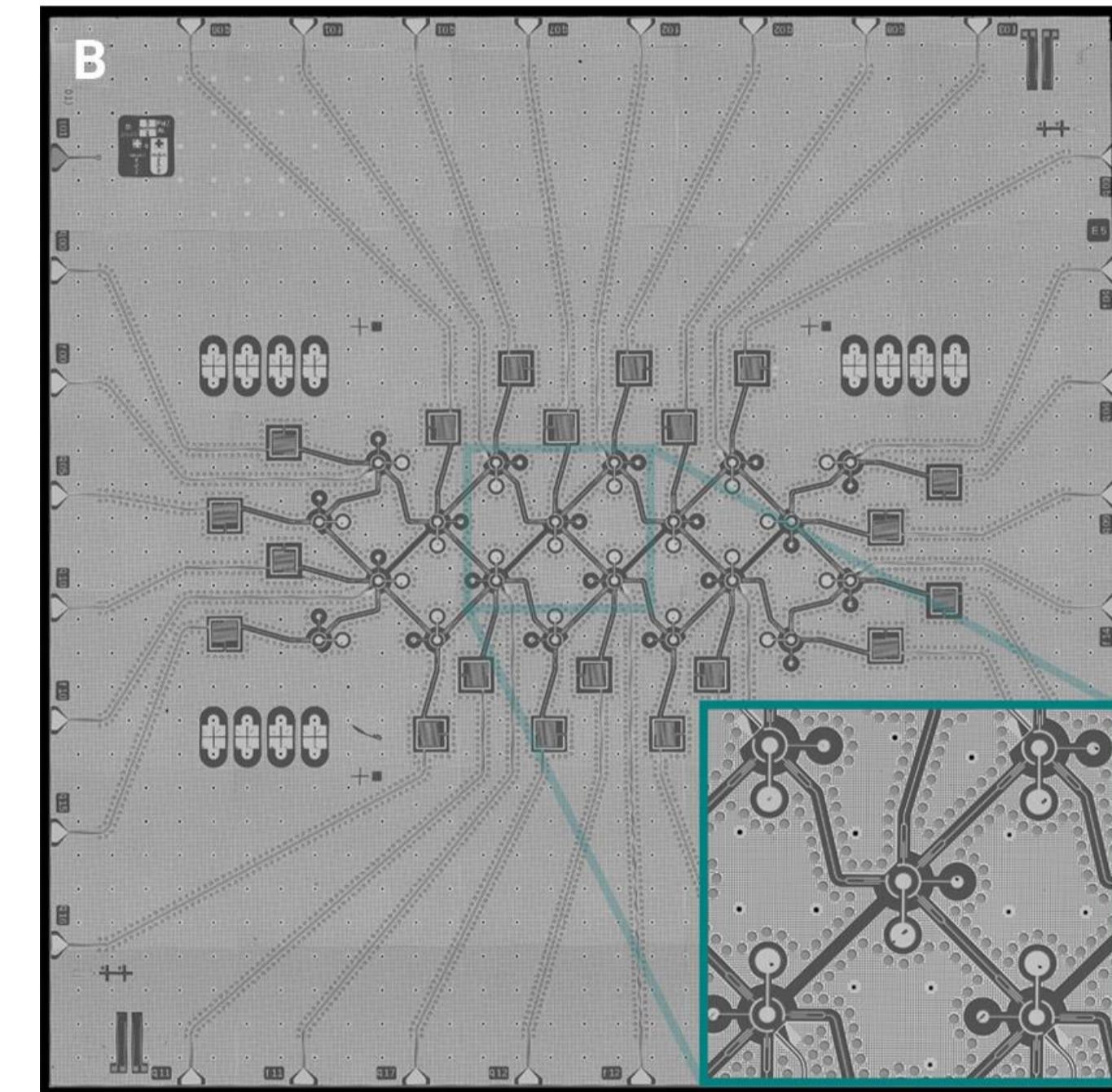
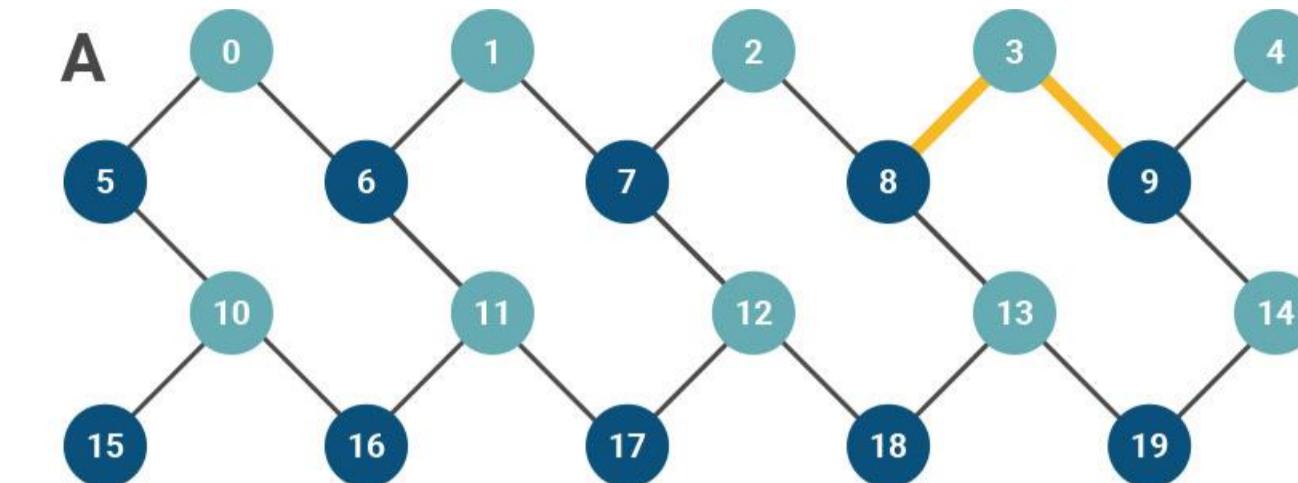
Quantum Circuit Optimisation



=



Routing



Thanks!

- *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Coecke & Kissinger. CUP (2017)
- ZX completeness:
 - *The ZX-calculus is complete for stabilizer quantum mechanics*, Backens, NJP (2014).
 - *A Complete Axiomatisation of the ZX-Calculus for Clifford+T Quantum Mechanics*. Jeandel, Perdrix, Vilmart. LICS (2018).
 - *A universal completion of the ZX-calculus*. Ng and Wang. arXiv:1706.09877 (2017).
 - *A Near-Optimal Axiomatisation of ZX-Calculus for Pure Qubit Quantum Mechanics*. Vilmart. arXiv:1812.09114 (2018).
- Simplification & circuit optimisation with ZX:
 - *Graph-theoretic Simplification of Quantum Circuits with the ZX-calculus*, Duncan, Kissinger, Perdrix, van de Wetering, arXiv:1902.03178 (2019).
 - *Reducing T-count with the ZX-calculus*, Kissinger, van de Wetering, arXiv:1903.10477 (2019).
- Many more papers about ZX: <https://zxcalculus.com/publications.html>
- Formalisation of diagram rewriting as DPO:
 - *Open Graphs and Monoidal Theories*. Dixon, Kissinger, arXiv:1011.4114 (2010).
- <http://quantomatic.github.io>
- <http://github.com/Quantomatic/pyzx>