

A categorical semantics for causal structure

Aleks Kissinger and Sander Uijlen

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PICTURING QUANTUM PROCESSES

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Symmetric monoidal categories

$$f : A \rightarrow B := \begin{array}{c} | \\ \boxed{f} \\ | \end{array} \begin{array}{l} B \\ A \end{array}$$

$$g \circ f := \begin{array}{c} | \\ \boxed{g} \\ | \\ \boxed{f} \\ | \end{array} \quad f \otimes g := \begin{array}{c} | \\ \boxed{f} \\ | \end{array} \quad \begin{array}{c} | \\ \boxed{g} \\ | \end{array}$$

$$1_A := \begin{array}{c} | \\ A \end{array} \quad 1_I := \begin{array}{c} \\ \end{array} \quad \sigma_{A,B} := \begin{array}{c} \\ \end{array} \begin{array}{c} B \\ A \\ A \\ B \end{array}$$

States, effects, numbers

Morphisms in/out of the monoidal unit get special names:

$$\mathit{state} := \left(\rho : I \rightarrow A \right)$$

$$\mathit{effect} := \left(\pi : A \rightarrow I \right)$$

$$\mathit{number} := \left(\lambda : I \rightarrow I \right)$$

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$$\begin{array}{c} \overline{\top}_B \\ | \\ \boxed{\Phi} \\ | \\ A \end{array} = \overline{\top}_A$$

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“If the output of a process is discarded,
it doesn't matter which process happened.”

The classical case

$\mathbf{Mat}(\mathbb{R}_+)$ is the category whose objects are natural numbers and morphisms are *matrices of positive numbers*. Then:

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Causal states = probability distributions

Causal processes = stochastic maps

The quantum case

CPM is the category whose objects are Hilbert spaces and morphisms are *completely positive maps*. Then:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \text{Tr}(-) \qquad \begin{array}{c} \text{---} \\ | \\ \triangle \\ \rho \end{array} = \text{Tr}(\rho) = 1$$

The quantum case

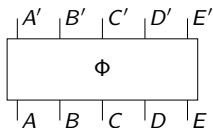
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Causal states = density operators

Causal processes = CPTPs

Causal structure of a process



A **causal structure** on Φ associates input/output pairs with a set of ordered *events*:

$$\mathcal{G} := \left\{ \begin{array}{l} (A, A') \leftrightarrow A \\ (B, B') \leftrightarrow B \\ (C, C') \leftrightarrow C \\ (D, D') \leftrightarrow D \\ (E, E') \leftrightarrow E \end{array} \right. \left. \begin{array}{c} \text{E} \\ \diagdown \quad \diagup \\ \text{B} \quad \text{D} \\ \diagup \quad \diagdown \\ \text{A} \quad \text{C} \end{array} \right\}$$

Causal structure of a process

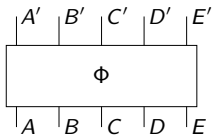
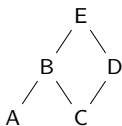
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Φ admits causal structure \mathcal{G} , written $\Phi \models \mathcal{G}$ if the output of each event only depends on the inputs of itself and its causal ancestors.

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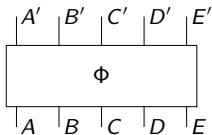
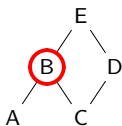
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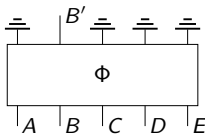
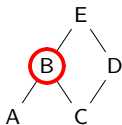
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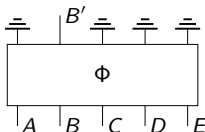
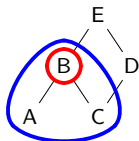
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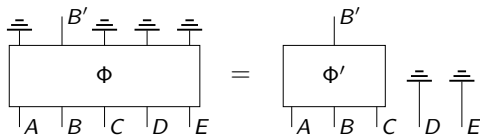
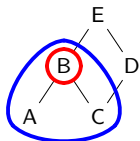
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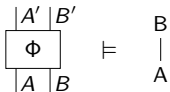
Causal structure of a process

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Example: one-way signalling

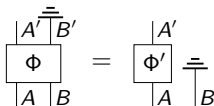
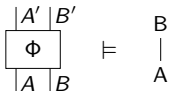


Example: one-way signalling

$$\begin{array}{|c|c|} \hline A' & B' \\ \hline \Phi \\ \hline A & B \\ \hline \end{array} \vDash \begin{array}{c} B \\ | \\ A \end{array}$$

$$\begin{array}{|c|c|} \hline \overline{\overline{A'}} & \overline{\overline{B'}} \\ \hline \Phi \\ \hline A & B \\ \hline \end{array} = \begin{array}{|c|} \hline A' \\ \hline \Phi' \\ \hline A \\ \hline \end{array} \overline{\overline{B}}$$

Example: one-way signalling

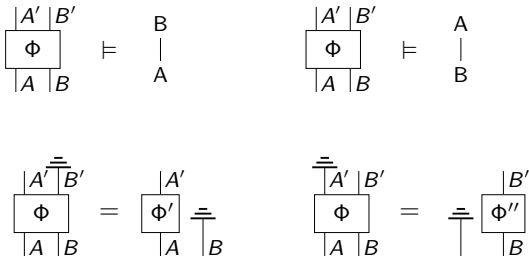


$$P(A'|AB) = P(A'|A)$$

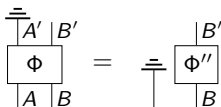
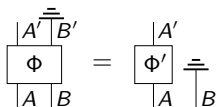
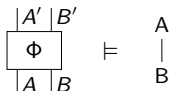
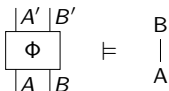
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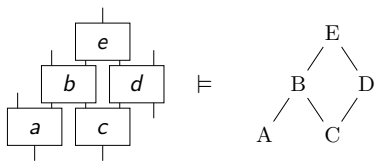
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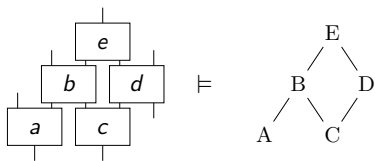
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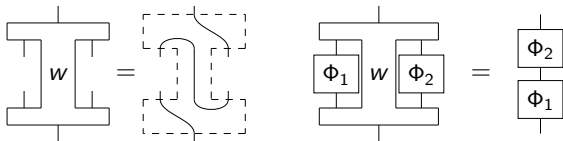


Theorem

All acyclic diagrams of processes admit their associated causal structure if and only if all processes are causal.

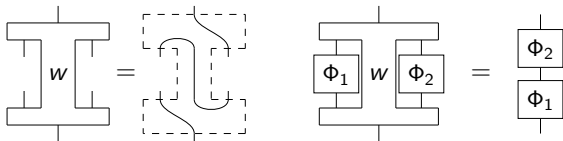
Higher-order causal structure

We can also define (super-)processes with *higher-order causal structure*:

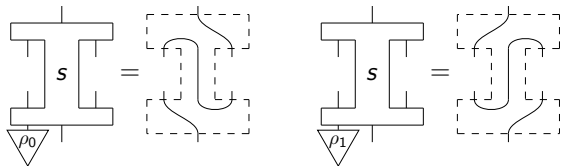


Higher-order causal structure

We can also define (super-)processes with *higher-order causal structure*:



These can introduce definite, or **indefinite** causal structure:



e.g. Quantum Switch, OCB W -matrix, ...

The questions

Q1: Can we define a category whose *types* express causal structure?

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It turns out answering **Q2** gives the answer to **Q1**.

Compact closed categories

An easy way to get higher-order processes is to use compact closed categories:

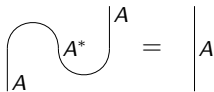
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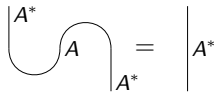
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Definition

An SMC \mathcal{C} is *compact closed* if every object A has a *dual* object A^* , i.e. there exists $\eta_A : I \rightarrow A^* \otimes A$ and $\epsilon_A : A \otimes A^* \rightarrow I$, satisfying:

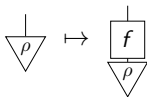
$$(\epsilon_A \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A \quad (1_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes 1_{A^*}) = 1_{A^*}$$





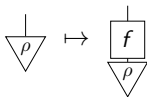
Higher-order processes

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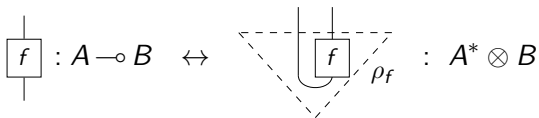


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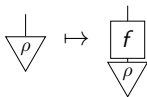


In compact closed categories, everything is a state, thanks to *process-state duality*:

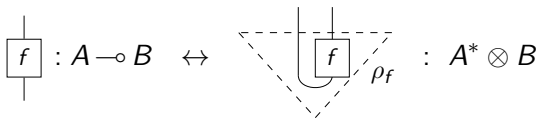


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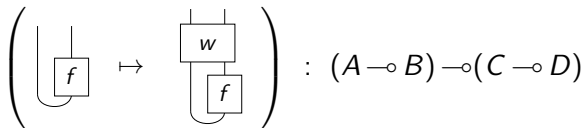
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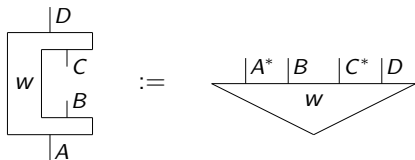


\Rightarrow **higher order processes** are the same as **first-order processes**:



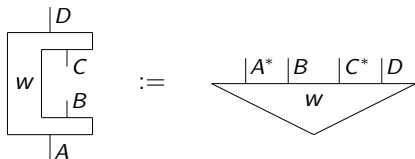
Some handy notation

We can treat *everything* as a state, and write states in any shape we like:

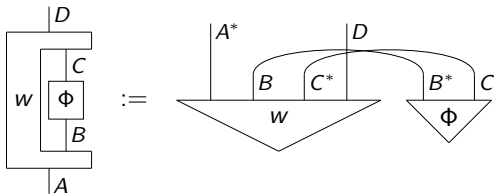


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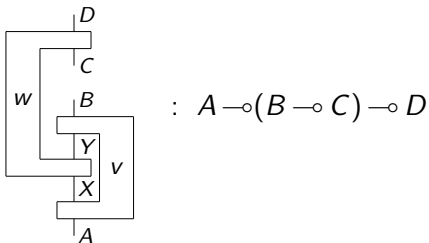


Then plugging shapes together means composing the appropriate caps:



Some handy notation

It looks like we can now freely work with higher-order causal processes:



...but theres a problem.

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In a compact closed category:

$$(A \otimes B)^* = A^* \otimes B^*$$

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\Rightarrow everything collapses to first order!

Definition

A **-autonomous category* is a symmetric monoidal category equipped with a full and faithful functor $(-)^* : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ such that, by letting:

$$A \multimap B := (A \otimes B^*)^* \quad (1)$$

there exists a natural isomorphism:

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, B \multimap C) \quad (2)$$

The recipe

Precausal category \mathcal{C} \mapsto

$\text{Caus}[\mathcal{C}]$

*compact closed category
of 'raw materials'*

**-autonomous category
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$\text{Mat}(\mathbb{R}_+)$
CPM

\mapsto

higher-order stochastic maps

\mapsto

higher-order quantum channels

Precausal categories

Precausal categories give 'good' raw materials, i.e. discarding behaves well w.r.t. the categorical structure. The standard examples are $\mathbf{Mat}(\mathbb{R}_+)$ and \mathbf{CPM} .

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Definition

A *precausal category* is a compact closed category \mathcal{C} such that:

- (C1) \mathcal{C} has discarding processes for every system
- (C2) For every (non-zero) system A , the *dimension* of A :

$$d_A := \text{discarding } A$$

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- (C3) \mathcal{C} has *enough causal states*
- (C4) *Second-order causal* processes factorise

Enough causal states

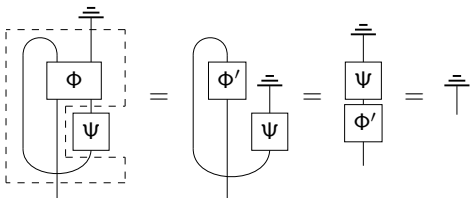
$$\left(\forall \rho \text{ causal} . \begin{array}{c} \boxed{f} \\ \downarrow \\ \triangle \rho \end{array} = \begin{array}{c} \boxed{g} \\ \downarrow \\ \triangle \rho \end{array} \right) \Rightarrow \begin{array}{c} \boxed{f} \\ | \end{array} = \begin{array}{c} \boxed{g} \\ | \end{array}$$

Theorem

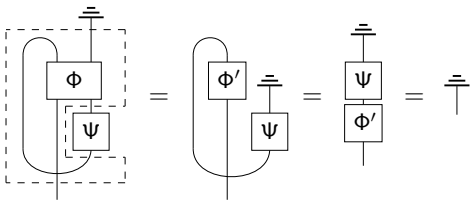
In a pre-causal category, one-way signalling processes factorise:

$$\left(\begin{array}{c} \exists \Phi' \text{ causal .} \\ \begin{array}{c} \text{---} \\ \text{---} \\ \square \Phi \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \square \Phi' \\ \text{---} \\ \text{---} \end{array} \end{array} \right) \Rightarrow \left(\begin{array}{c} \exists \Phi_1, \Phi_2 \text{ causal .} \\ \begin{array}{c} \text{---} \\ \square \Phi \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \square \Phi_2 \\ \text{---} \\ \square \Phi_1 \\ \text{---} \\ \text{---} \end{array} \end{array} \right)$$

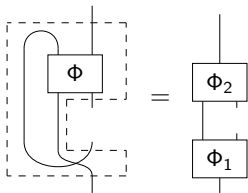
Proof. Treat Φ as a second-order process by bending wires. Then for any causal Ψ , we have:



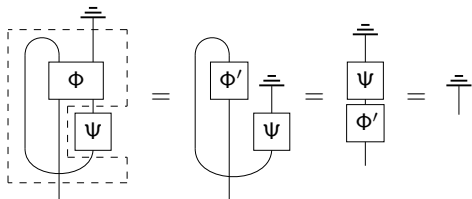
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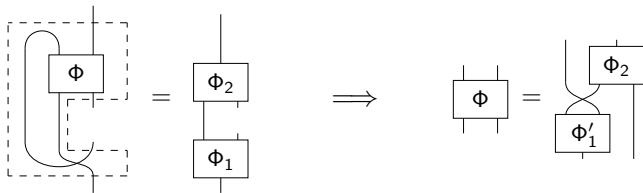
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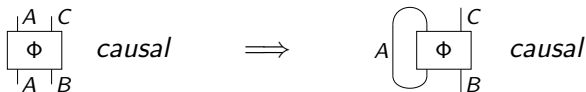


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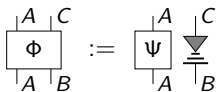
Theorem (No time-travel)

No non-trivial system A in a precausal category \mathcal{C} admits time travel. That is, if there exist systems B and C such that:



then $A \cong I$.

Proof. For any causal process Ψ and causal state \Downarrow :



is causal.

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$$\begin{array}{c} A \quad C \\ \hline \boxed{\Phi} \\ \hline A \quad B \end{array} := \begin{array}{c} A \quad C \\ \hline \boxed{\Psi} \quad \Downarrow \\ \hline A \quad B \end{array}$$

is causal. So:

$$\begin{array}{c} A \\ \hline \boxed{\Psi} \\ \hline A \end{array} = \begin{array}{c} A \\ \hline \boxed{\Phi} \\ \hline A \quad B \quad \Downarrow \\ \hline C \quad \Downarrow \end{array} = \begin{array}{c} \Downarrow \\ \hline B \\ \hline \Downarrow \end{array} = 1$$

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$$A \begin{array}{c} \text{loop} \\ \square \Psi \end{array} = A \begin{array}{c} \text{loop} \\ \square \Phi \\ \Downarrow C \\ \Downarrow B \end{array} = \Downarrow B = 1$$

Applying (C4):

$$\begin{array}{c} \text{loop} \\ \square \Psi \end{array} = \begin{array}{c} \Downarrow A \\ \Downarrow \rho \end{array} \Rightarrow \left| A \right. = \begin{array}{c} \Downarrow \rho \\ \Downarrow A \end{array}$$

for some ρ causal.

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is causal. So:

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for some ρ causal. So $\rho \circ \ddagger = 1_A$ and $\ddagger \circ \rho = 1_I$ is causality.

Causal states

A process is causal, a.k.a. *first order causal*, if and only if it preserves the set of causal states:

$$\begin{array}{c} | \\ \triangle \\ \rho \end{array} \text{ causal} \implies \begin{array}{c} | \\ \boxed{f} \\ \triangle \\ \rho \end{array} \text{ causal}$$

Duals and closure

Duals and closure

Note *any* set of states $c \subseteq \mathcal{C}(I, A)$ admits a *dual*, which is a set of effects:

$$c^* := \left\{ \pi : A^* \mid \forall \rho \in c . \begin{array}{c} \triangle \pi \\ | \\ \rho \\ \nabla \end{array} = 1 \right\}$$

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The double-dual c^{**} is a set of states again.

Definition

A set of states $c \subseteq \mathcal{C}(I, A)$ is *closed* if $c = c^{**}$.

Flatness

If c is the set of causal states, discarding $\in c^*$, and up to some rescaling, discarding-transpose:

$$\frac{1}{D} \perp_{\equiv}$$

i.e. the maximally mixed state $\in c$.

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i.e. the maximally mixed state $\in c$.

We make this symmetric $c \leftrightarrow c^*$, and call this property flatness:

Definition

A set of states $c \subseteq \mathcal{C}(I, A)$ is *flat* if there exist invertible numbers λ, μ such that:

$$\lambda \perp\!\!\!\perp \in c \qquad \mu \overline{\perp\!\!\!\perp} \in c^*$$

The main definition

Definition

For a precausal category \mathcal{C} , the category $\text{Caus}[\mathcal{C}]$ has as objects pairs:

$$\mathbf{A} := (A, c_{\mathbf{A}} \subseteq \mathcal{C}(I, A))$$

where $c_{\mathbf{A}}$ is closed and flat. A morphism $f : \mathbf{A} \rightarrow \mathbf{B}$ is a morphism $f : A \rightarrow B$ in \mathcal{C} such that:

$$\rho \in c_{\mathbf{A}} \implies f \circ \rho \in c_{\mathbf{B}}$$

The main theorem

Theorem

$\text{Caus}[\mathcal{C}]$ is a $*$ -autonomous category, where:

$$\mathbf{A} \otimes \mathbf{B} := (A \otimes B, (c_{\mathbf{A}} \otimes c_{\mathbf{B}})^{**}) \qquad \mathbf{I} := (I, \{1_I\})$$

$$\mathbf{A}^* := (A^*, c_{\mathbf{A}}^*)$$

Connectives

One connective \otimes becomes 3 interrelated ones:

$$\mathbf{A} \otimes \mathbf{B}$$

$$\mathbf{A} \wp \mathbf{B} := (\mathbf{A}^* \otimes \mathbf{B}^*)^*$$

$$\mathbf{A} \multimap \mathbf{B} := \mathbf{A}^* \wp \mathbf{B} \cong (\mathbf{A} \otimes \mathbf{B}^*)^*$$

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- \otimes is the smallest joint state space that contains all product states
- \wp is the biggest joint state space normalised on all product effects:

$$c_{\mathbf{A} \wp \mathbf{B}} = \left\{ \rho : \mathbf{A} \otimes \mathbf{B} \mid \forall \pi \in c_{\mathbf{A}}^*, \xi \in c_{\mathbf{B}}^* . \begin{array}{c} \triangle \pi \quad \triangle \xi \\ | \quad | \\ \rho \\ \triangle \end{array} = 1 \right\}$$

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- \multimap is the space of causal-state-preserving maps

Example: first-order systems

First order := systems of the form $\mathbf{A} = (A, \{\overline{\top}\}^*)$

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Theorem

For first order systems, $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A} \wp \mathbf{B}$.

When $\otimes \neq \mathfrak{A}$

When $\otimes \neq \wp$

For f.o. A, A', B, B' :

$$(A \multimap A') \wp (B \multimap B')$$

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$$(A \multimap A') \wp (B \multimap B') \cong A^* \wp A' \wp B^* \wp B'$$

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$(A \multimap A') \wp (B \multimap B') =$ all causal processes

Theorem

$(A \multimap A') \otimes (B \multimap B') = \text{causal, non-signalling processes}$

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Proof. (idea) The causal states for $(A \multimap A') \otimes (B \multimap B')$ are:

$$\left\{ \begin{array}{|c|} \hline \Phi_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline \Phi_2 \\ \hline \end{array} \right\}^{**}$$

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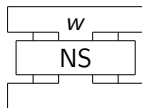
$$\left\{ \begin{array}{|c|} \hline \Phi_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline \Phi_2 \\ \hline \end{array} \right\}^{**}$$

We show:

The diagram shows a process with two input wires labeled A and A' and two output wires labeled B and B' . A wire labeled W connects the top of A and A' to the top of B and B' . This process is equated to a set of causal states, represented by two boxes labeled Φ_1 and Φ_2 in sequence, with a superscript $*$.

$$\begin{array}{|c|} \hline W \\ \hline \end{array} \begin{array}{|c|} \hline A' \\ \hline \end{array} \begin{array}{|c|} \hline B' \\ \hline \end{array} \in \left\{ \begin{array}{|c|} \hline \Phi_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline \Phi_2 \\ \hline \end{array} \right\}^*$$

is also normalised for all non-signalling processes:



Theorem

$(A \multimap A') \otimes (B \multimap B') = \text{causal, non-signalling processes}$

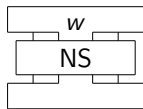
Proof. (idea) The causal states for $(A \multimap A') \otimes (B \multimap B')$ are:

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We show:

$$\begin{array}{c} \text{---} \\ \boxed{w} \\ \text{---} \\ \begin{array}{c} A' \\ | \\ A \end{array} \quad \begin{array}{c} B' \\ | \\ B \end{array} \\ \text{---} \end{array} \in \left\{ \begin{array}{c} \text{---} \\ | \\ \boxed{\Phi_1} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \boxed{\Phi_2} \\ | \\ \text{---} \end{array} \right\}^*$$

is also normalised for all non-signalling processes:



This follows from a graphical proof using all 4 pre-causal axioms.

Refining causal structure

Since $I \cong I^* = (I, \{1\})$, a standard theorem of $*$ -autonomous gives a canonical embedding:

$$(A \multimap A') \otimes (B \multimap B') \hookrightarrow (A \multimap A') \wp (B \multimap B')$$

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What about in between?

$$(A \multimap A') \otimes (B \multimap B') \hookrightarrow \dots \hookrightarrow (A \multimap A') \wp (B \multimap B')$$

One-way signalling

Theorem

One-way signalling processes are processes of the form:

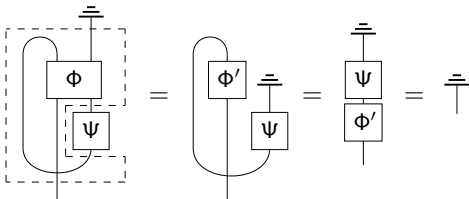
$$\begin{array}{|c|c|} \hline A' & B' \\ \hline \Phi & \\ \hline A & B \\ \hline \end{array} : \mathbf{A} \multimap (\mathbf{A}' \multimap \mathbf{B}) \multimap \mathbf{B}'$$

One-way signalling

Proof.

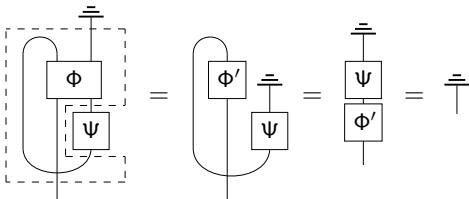
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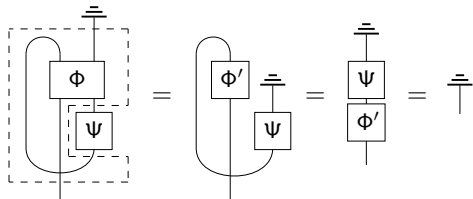


we have:

$$\begin{array}{c} A' \quad B' \\ \boxed{\Phi} \\ A \quad B \end{array} : (A' \multimap B) \multimap (A \multimap B')$$

One-way signalling

Proof. Exploiting the relationship between one-way signalling and second-order causal:



we have:

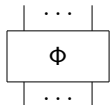
$$\begin{array}{c}
 A' \quad B' \\
 \hline
 \Phi \\
 \hline
 A \quad B
 \end{array}
 : (A' \multimap B) \multimap (A \multimap B')$$

Then $*$ -autonomous structure gives a canonical iso:

$$(A' \multimap B) \multimap (A \multimap B') \cong A \multimap (A' \multimap B) \multimap B'$$

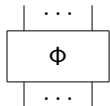
Further examples

- n -party non-signalling:

A rectangular box labeled with the Greek letter Φ in the center. It has two vertical lines on the left side and two vertical lines on the right side. Above the top two lines is an ellipsis \dots , and below the bottom two lines is another ellipsis \dots .
$$\Phi : (\mathbf{A}_1 \multimap \mathbf{A}'_1) \otimes \cdots \otimes (\mathbf{A}_n \multimap \mathbf{A}'_n)$$

Further examples

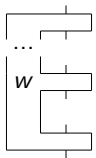
- n -party non-signalling:



A rectangular box labeled with the Greek letter Φ . It has two vertical lines on the left side and two on the right side. Each side has three dots between the two lines, indicating multiple inputs and outputs.

$$: (\mathbf{A}_1 \multimap \mathbf{A}'_1) \otimes \cdots \otimes (\mathbf{A}_n \multimap \mathbf{A}'_n)$$

- Quantum n -combs:

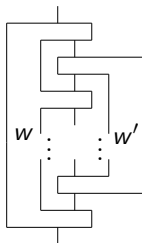


A diagram of a quantum comb. It consists of a large vertical line on the left labeled with the letter w . From the top of this line, a horizontal line goes right and then down, forming a U-shape. This U-shape is repeated n times, with three dots between the first and second U-shapes. The final U-shape at the bottom has a horizontal line that goes right and then down, ending in a vertical line on the right.

$$: \mathbf{A}_1 \multimap (\mathbf{A}'_1 \multimap (\cdots) \multimap \mathbf{A}_n) \multimap \mathbf{A}'_n$$

Further examples

- Compositions of those things:



Further examples

- Indefinite causal structures (e.g. quantum switch, OCB W -process, Baumeler-Wolf):

$$\begin{aligned}
 & \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) + \frac{1}{4\sqrt{2}} \left(\begin{array}{cc} \text{Diagram 3} & \text{Diagram 4} \\ \text{Diagram 5} & \text{Diagram 6} \end{array} \right) \\
 & \frac{1}{8} \left(\begin{array}{cccc} \text{Diagram 7} & \text{Diagram 8} & \text{Diagram 9} & \text{Diagram 10} \\ \text{Diagram 11} & \text{Diagram 12} & \text{Diagram 13} & \text{Diagram 14} \end{array} \right)
 \end{aligned}$$

The diagrams consist of nodes (circles with horizontal lines) and triangles (pointing up or down) connected by lines. Some nodes and triangles are enclosed in dashed boxes, representing different causal structures. The first diagram shows two parallel paths of nodes, each ending in a downward-pointing triangle. The second diagram shows a similar structure but with a single downward-pointing triangle at the bottom. The third and fourth diagrams in the second row show paths with σ_z triangles. The fifth and sixth diagrams show paths with σ_z and σ_x triangles. The seven diagrams in the third row show more complex causal structures with multiple paths and triangles.

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 & \left[(\mathbf{A}_1 \multimap \mathbf{A}'_1) \otimes \dots \otimes (\mathbf{A}_n \multimap \mathbf{A}'_n) \right]^*
 \end{aligned}$$

Automation

The internal logic of $*$ -autonomous categories is multiplicative linear logic (MLL):

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}$$

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$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}$$

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\Rightarrow use off-the-shelf theorem provers to prove causality theorems.

Automation

For example, we can show using `llprover` that:

$$(A \multimap A') \otimes (B \multimap B')$$

$$\Downarrow$$

$$A \multimap (A' \multimap B) \multimap B'$$

$$\Downarrow$$

$$(A \multimap A') \wp (B \multimap B')$$

Thanks

...and some refs:

- **A categorical semantics for causal structure.** [arXiv:1701.04732](#)
- *Causal structures and the classification of higher order quantum computation.* Paulo Perinotti. [arXiv:1612.05099](#)