Quantum Picturalism

Bob Coecke[1], Chris Heunen[2], and **Aleks Kissinger**[3]

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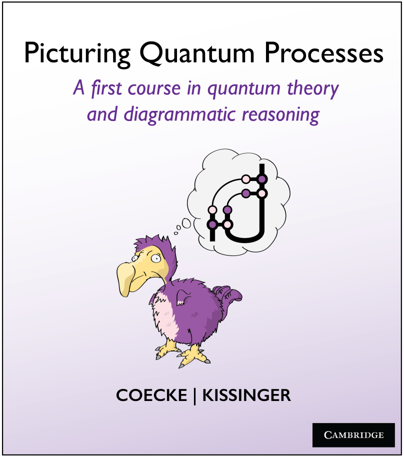
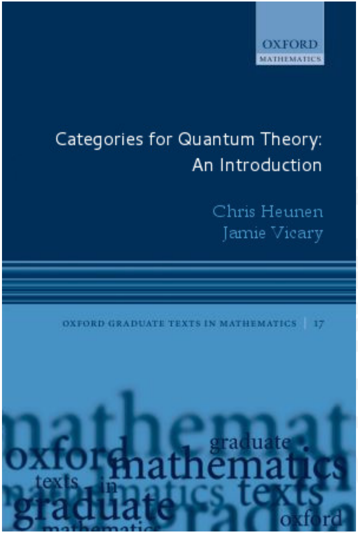
[2]University of Edinburgh
[3]**Radboud University Nijmegen**

Foundations 2016, LSE



CUP 2016



OUP 2016

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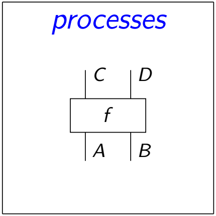

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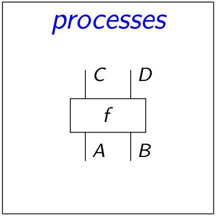

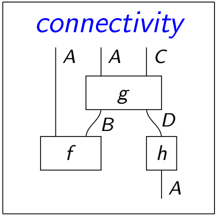

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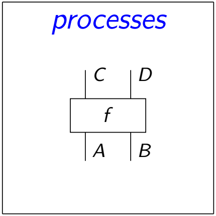

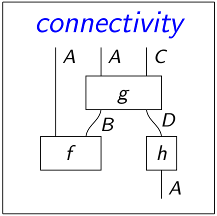

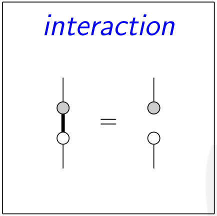

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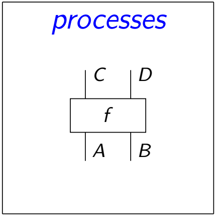

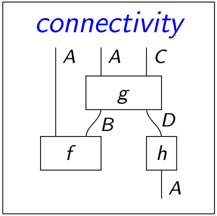

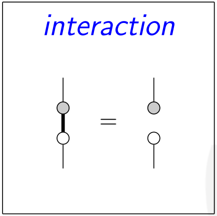

Not in terms of:

- Hilbert space
- self-adjoint operators, unitary transformations
- calculations with matrices/complex numbers
- ....

(though some may be emergent notions)

Why?

- Simpler!

$$(1 \otimes \sigma \otimes k) \circ (\sigma \otimes 1 \otimes 1 \otimes 1) \circ (f \otimes g \otimes 1 \otimes 1) \circ (h \otimes 1) = (g \otimes f) \circ (1 \otimes k) \circ (h \otimes 1)$$

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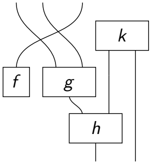

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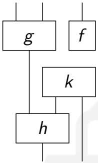

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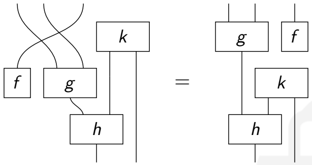

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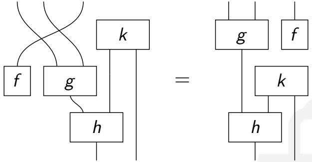

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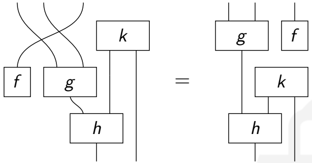

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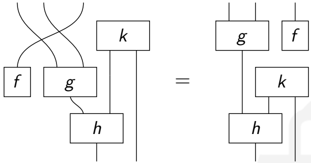

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- New calculational tools, applications in quantum info/computation

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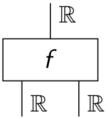

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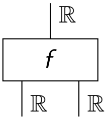

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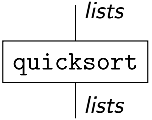

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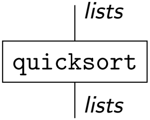
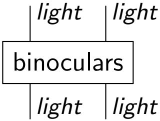
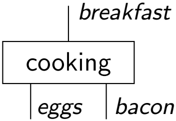
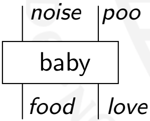


- These are also perfectly good processes:

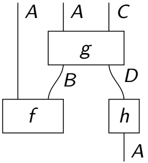






Diagrams

- We can combine simple processes to make more complicted ones, described by diagrams:



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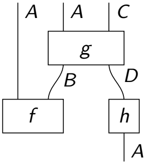
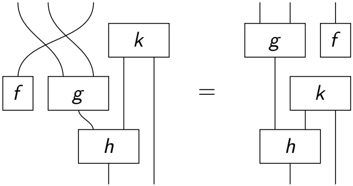
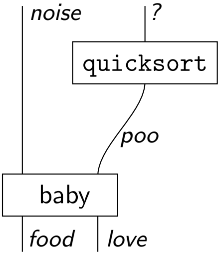


- The golden rule: only connectivity matters!



Types and Process Theories

- Connections are only allowed where the types match
- Ill-typed diagrams are undefined:



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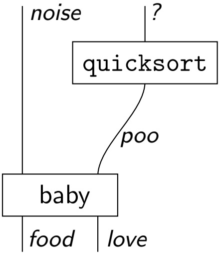

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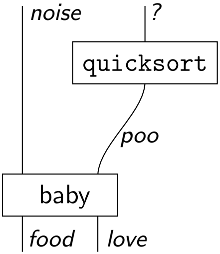


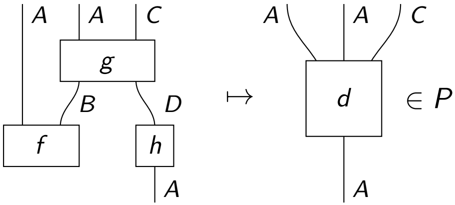
- In fact, these processes don't ever make sense to plug together
- A family of processes which do make sense together is called a process theory

A *process theory* consists of:

- a set T of *system-types*,
- a set P of *processes*

which are:

- closed under forming diagrams:



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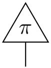

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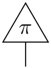

Interpret as: testing for a property π, where we don't care what happens after.

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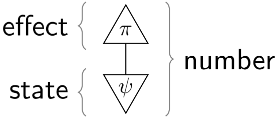

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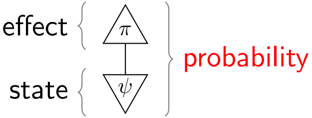

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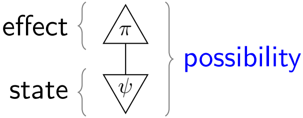

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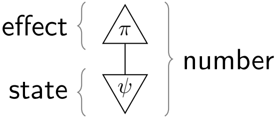

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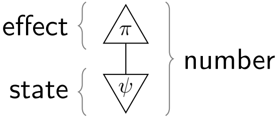

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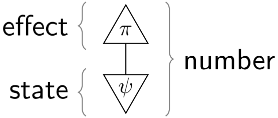

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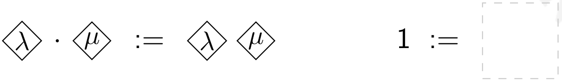

Q: What kinds of behaviour can we study using just diagrams, and nothing else?

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A: (Non-)separability

Separability for states

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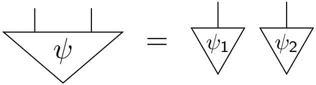

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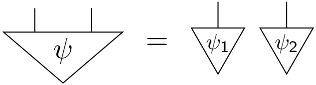
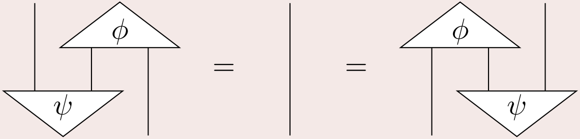


- vs. 'completely non-separable':

Definition

A state ψ is called *cup-state* if there exists an effect ϕ, called a *cap-effect*, such that:



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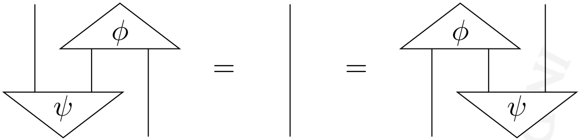

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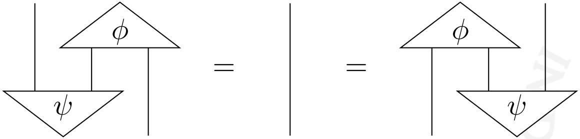

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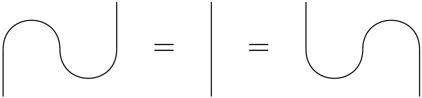

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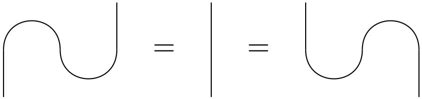

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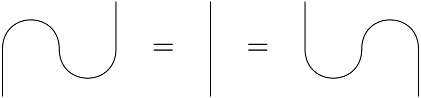

A no-go theorem for separability

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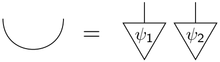

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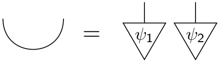

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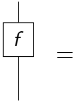

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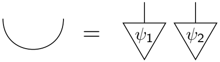

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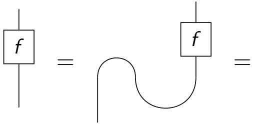

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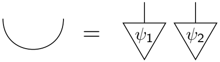

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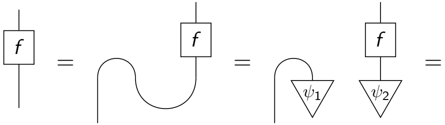


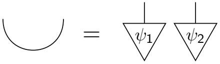
□

A no-go theorem for separability

Theorem

If a process theory (i) has cup-states for every type and (ii) every state separates, then it has trivial dynamics.

Proof. Suppose a cup-state separates:



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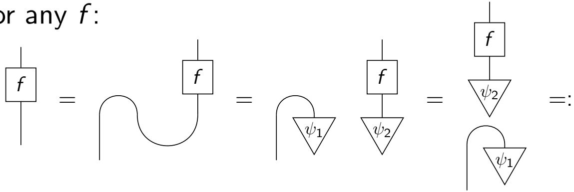

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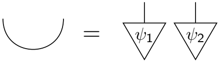

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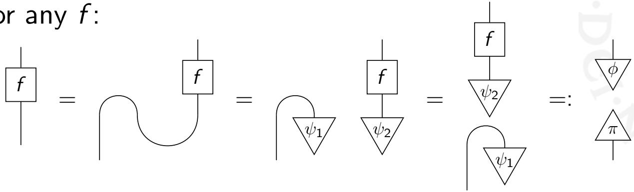
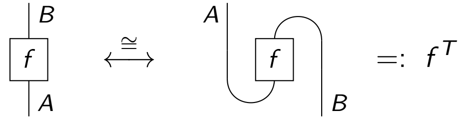
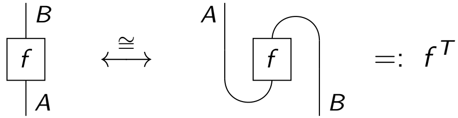
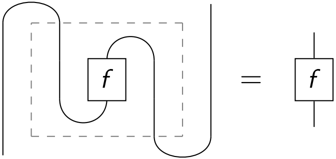


Transpose





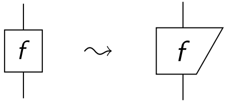


i.e.

$$(f^T)^T = f$$

Tranpose = rotation

A bit of a deformation:



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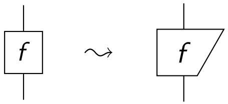
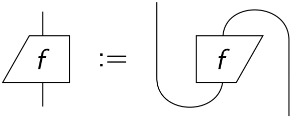


allows some clever notation:



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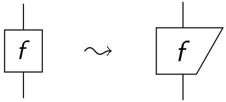

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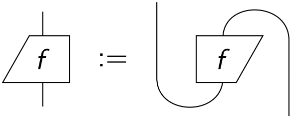

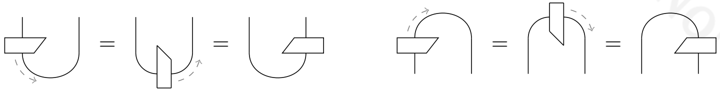

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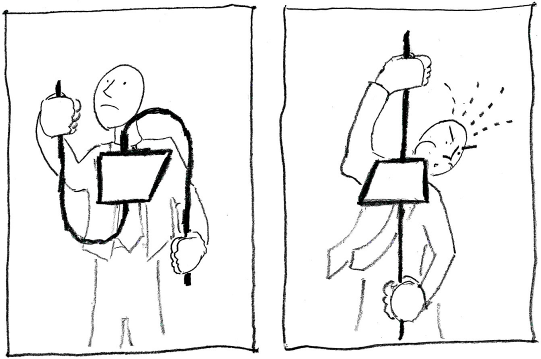

Adjoint = reflection

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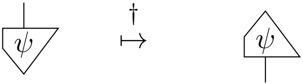
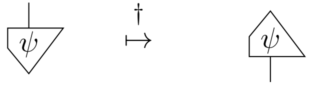


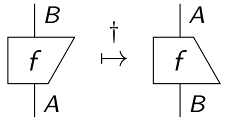
state ψ **testing for** ψ

Adjoint = reflection

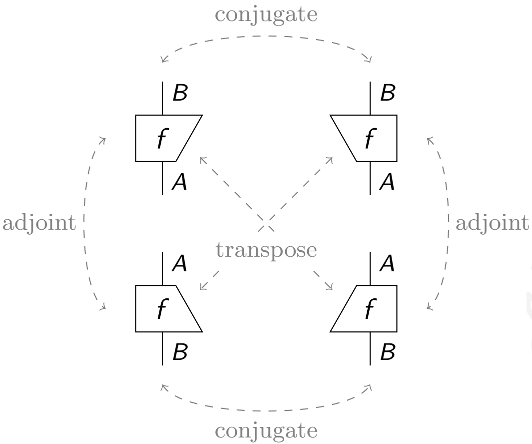


state ψ **testing for** ψ

Extends from states/effects to all processes:



4 kinds of box



Doubling

Doubling

If the ‘numbers’ of our process theory are complex numbers (e.g. as in **linear maps**), then we have a problem:

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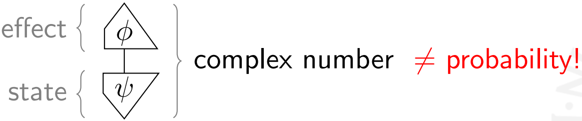
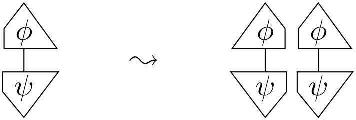


Doubling

Solution: multiply by the conjugate:



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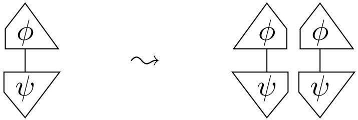
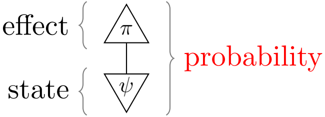


(i.e. use the 'plain old' Born rule: $\overline{\langle\phi|\psi\rangle}\langle\phi|\psi\rangle = |\langle\phi|\psi\rangle|^2$)

New problem: We lost this:



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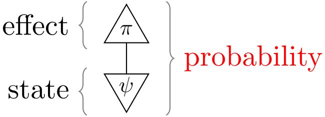

...which was the basis of our interpretation for states, effects, and numbers.

Doubling

Solution: Make a new process theory with doubling 'baked in':

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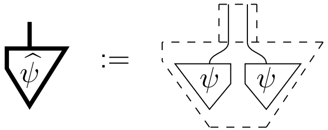

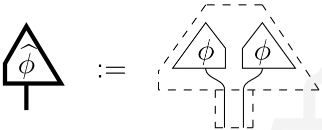

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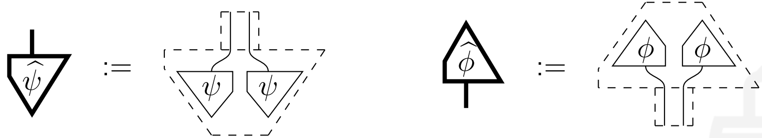
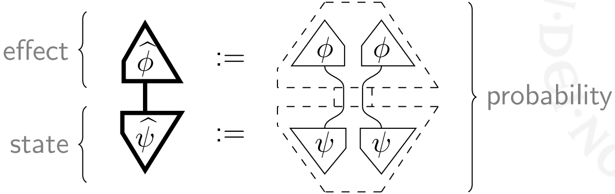
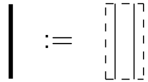


Then:



Doubling

The new process theory has doubled systems $\widehat{H} := H \otimes H$:



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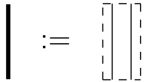
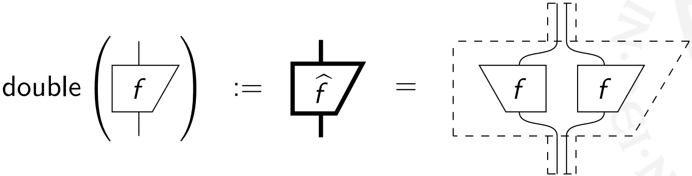
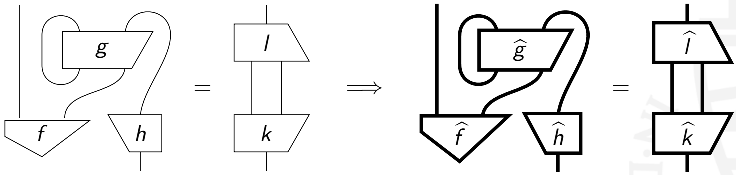
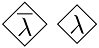


and processes:



Doubling preserves diagrams

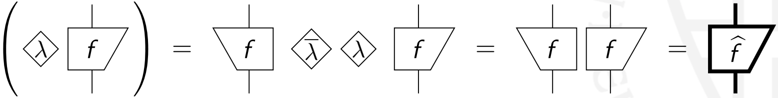




$= 1$ (i.e. $\lambda = e^{i\alpha}$)

$\Longrightarrow$

double



Discarding

Doubling also lets us do something we couldn't do before:

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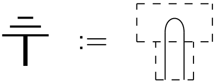
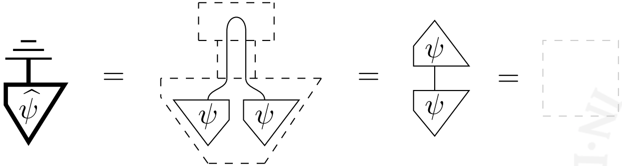
How? Like this:

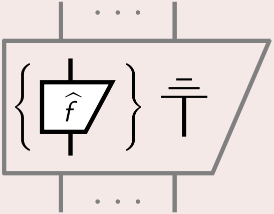


For normalised ψ, the two copies annihilate:



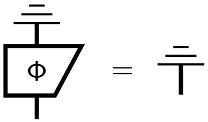
Definition

The process theory of **quantum maps** has as types (doubled) Hilbert spaces $\widehat{H}$ and as processes:



Causality

A quantum map is called *causal* if:



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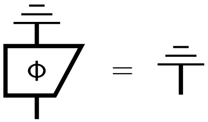

If we discard the output of a process,
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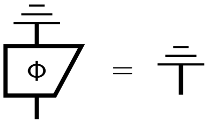
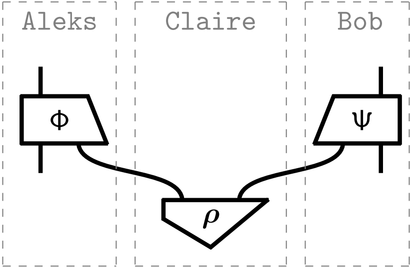


If we discard the output of a process,
it doesn't matter which process happened.

causal $\iff$ *deterministically physically realisable*

Consequence: no signalling



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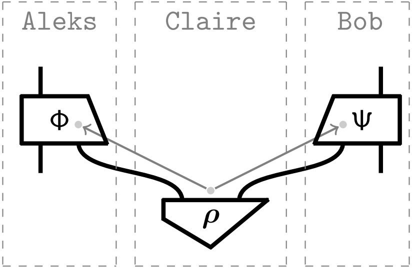

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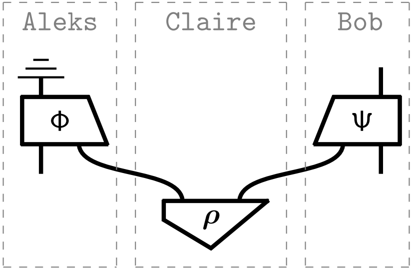

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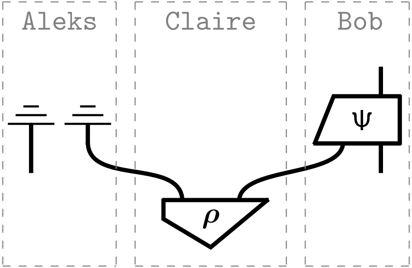

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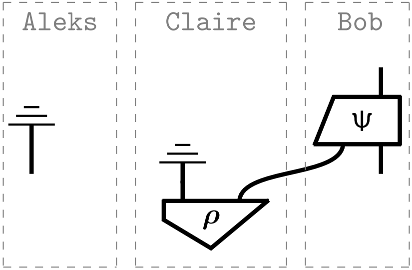
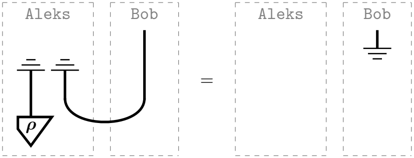
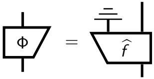


Consequences of doubling + causality

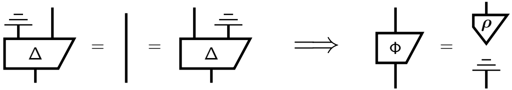
- Impossibility of deterministic teleporation:



- Purification/Stinespring dilation



- Quantum no-broadcasting theorem



Classical and quantum interaction

Classical and quantum interaction

$$\left(\text{quantum} \quad := \quad \Big| \right)$$

Classical and quantum interaction

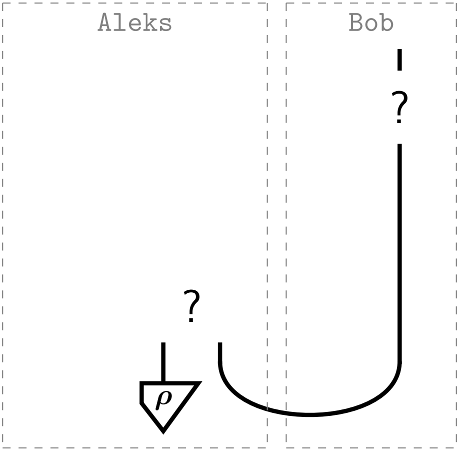
$$\left(\text{quantum} \;\; := \;\; \rule{3pt}{2em} \right) \;\neq\; \left(\text{classical} \;\; := \;\; \Big| \right)$$

Classical and quantum interaction

encode $:=$ 

measure $:=$ 

Quantum teleportation: take 2



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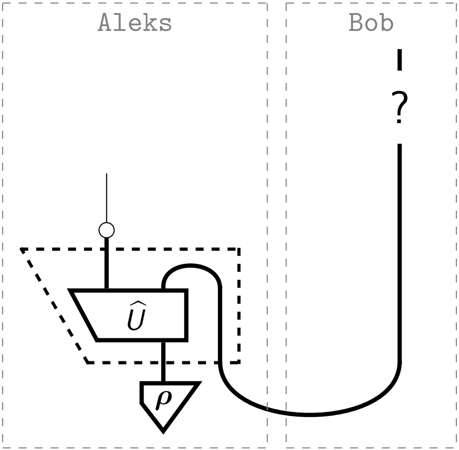

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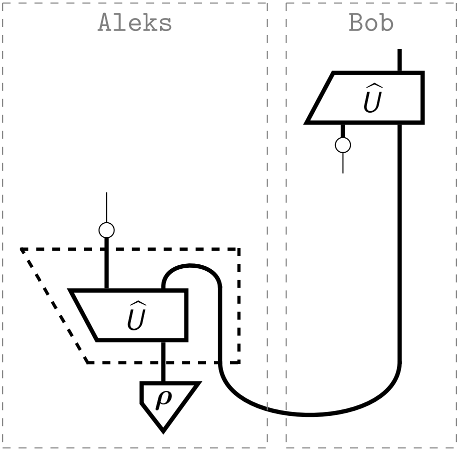
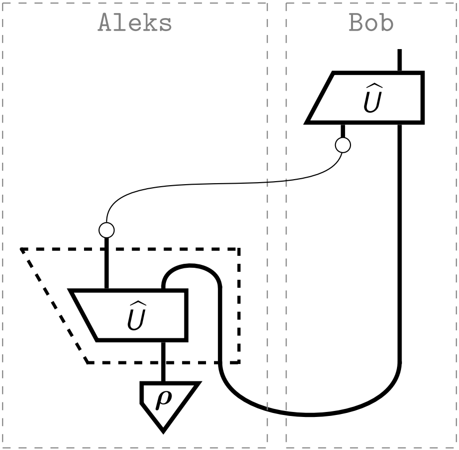
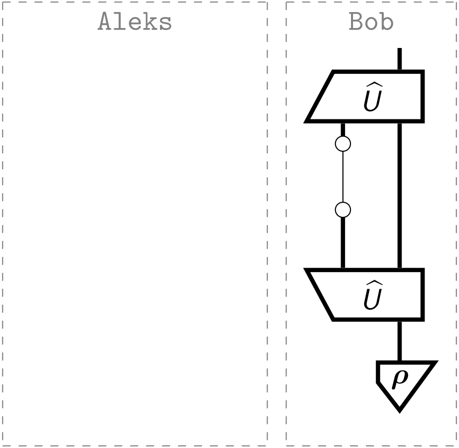



Aleks
Bob
$\widehat{U}$
$\widehat{U}$
ρ

Quantum teleportation: take 2



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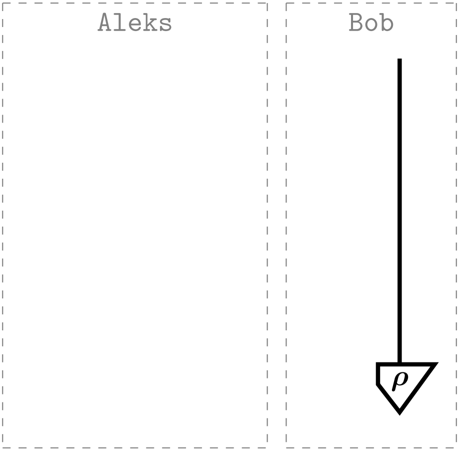
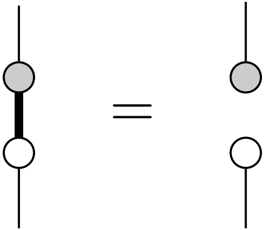
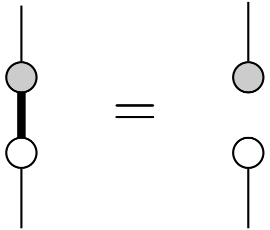


Complementarity

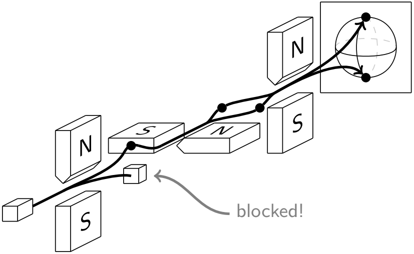


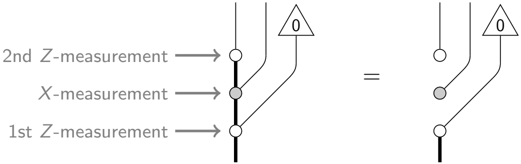


Interpretation:

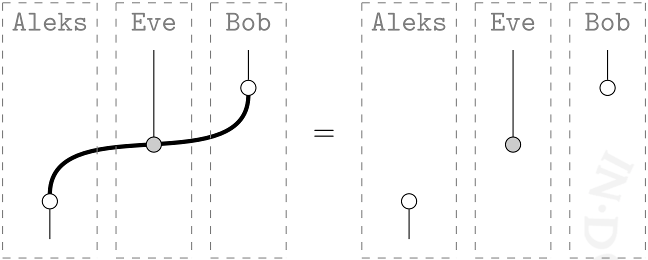
(encode in ○) THEN (measure in ◉) = (no data flow)

e.g. Stern-Gerlach



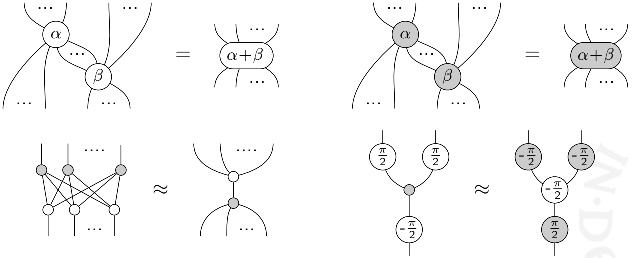


e.g. Quantum Key Distribution



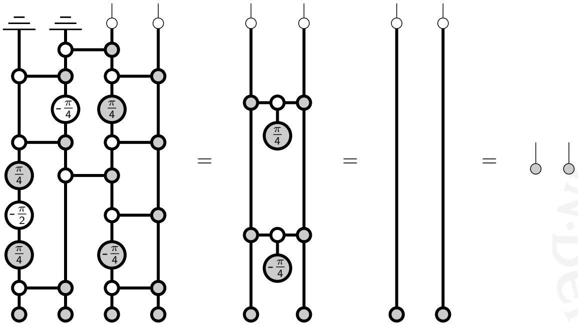
Graphical calculus

Complementarity + group structure = **ZX-calculus**:

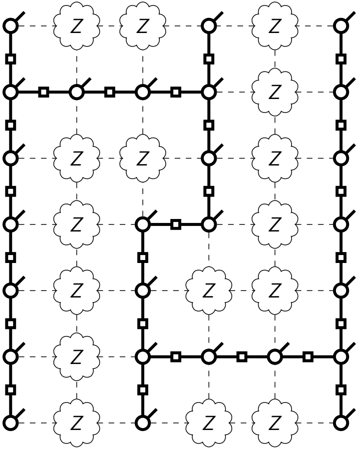


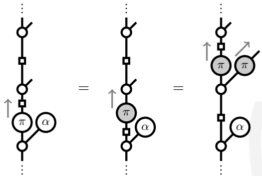
A **sound and complete** equational theory for stabilizer quantum mechanics.

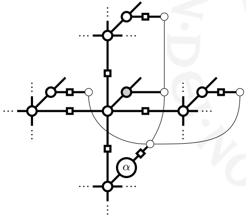
Quantum circuit simplification



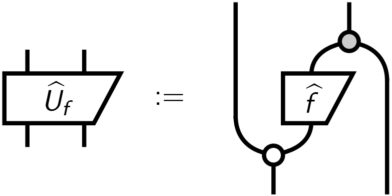
Measurement-based quantum computation





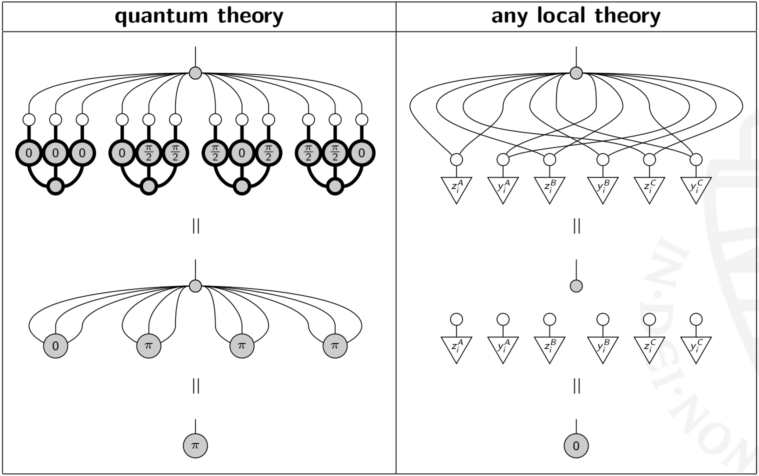


Quantum algorithms

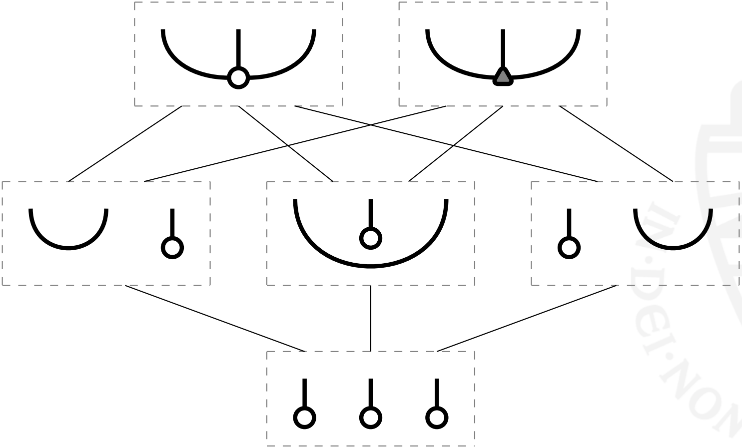


⇒ simple derivations of **Deutsch-Jozsa**, **quantum seach**, and **hidden subgroup** algorithms.

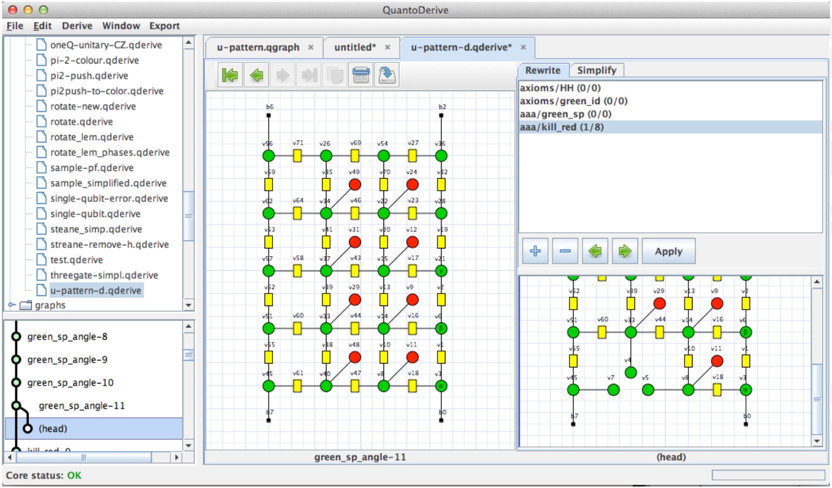
GHZ/Mermin non-locality

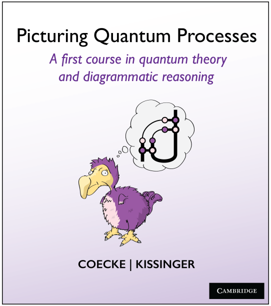


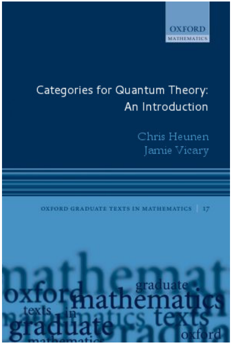
SLOCC-classification of 3 qubits:



Quantomatic:







- Categorical Quantum Mechanics I: Causal Quantum Processes. Coecke and Kissinger. arXiv:1510.05468
- Categorical Quantum Mechanics II: Classical-Quantum Interaction. Coecke and Kissinger. arXiv:1605.08617
- Categories of Quantum and Classical Channels. Coecke, Kissinger, Heunen. arXiv:1305.3821

Thanks! Joint work with:

...

Abramsky, Backens, Coecke, Duncan, Edwards, Gogioso, Hadzihasanovic, Heunen, Lal, Merry, Pavlovic, Paquette, Perdrix, Quick, Selinger, Vicary, Wang, Zamdzhiev, ...and many more!

http://quantomatic.github.io