Density Hypercubes: Interference and the Phase Group

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Presented in partial fulfilment for the degree of
Master of Research, MRes

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Abstract

Quantum interference is a well-known phenomenon which has no classical analogue. Bizarrely, quantum theory is somewhat limited in the amount of interference it exhibits. Only second order interference can be achieved, the lowest order beyond classical physics. Why quantum theory is limited in this way is not known.

The theory of density hypercubes is of interest because it exhibits higher order interference than quantum theory. In this report we study this theory and its place in the fields of categorical quantum mechanics and generalised probability theories. We extend the result of higher order interference to iterated density hypercube constructions and search for an understanding of interference from the phase group.

Density hypercubes are also of interest because they have an explicit hyperdecoherence map. Using this we define a hyperphase group and calculate it for density hypercubes. This leads us to consider entanglement in density hypercubes, define a new higher order hyperentanglement and consider some simple protocols that use hyperentanglement.

Summary of Originality

The results of section 4.2 of higher order interference in the iterated CPM construction are new results. They are a direct extension of the results of Gogioso and Scandolo [14].

The results of section 5.1 showing that the invertible maps of hypercubes are the unitaries of quantum theory and that the phase group is the same as quantum theory are new results. We also define a new hyperphase group, note that it is trivially a subgroup of the phase group for any theory with a decoherence map which decomposes into a hyperquantum-to-quantum map and a quantum-to-classical map and calculate this group for density hypercubes.

Section 6 studies the hyperdecoherence map and defines a new higher order form of entanglement we call hyperentanglement. We show that hyperdecoherence destroys hyperentanglement much like decoherence destroys entanglement in quantum theory. We then give a use of hyperentanglement by extending the quantum teleportation protocol to density hypercubes, showing that they can be teleported with the same amount of classical data transfer as quantum states. Finally, we consider dense coding, show that it can be achieved using density hypercubes but find no improvement over quantum theory, although no proof that it cannot be done.

Acknowledgements

Many thanks to Stefano Gogioso for his time and for pointing me in the direction of interesting questions. Thanks also to Bob Coecke for a place in the Quantum Group at Oxford. My best wishes to the other 19 members of my cohort at UCL. I have had a smashing year with all of you. A particular thanks to Matt Wilson, without whom I would not have discovered this fascinating area of research. Here's to three more years of the snake factory!

Contents

1 Categorical Quantum Mechanics

The study of categorical quantum mechanics (CQM) was initiated in 2004 by Abramsky and Coecke [1]. CQM does away with the Hilbert space formalism of quantum mechanics and instead recasts the theory in terms of processes; mappings from one space into another, capturing the notion of the evolution of a quantum system. Category theory provides the necessary structure to do this and thus we start with a few definitions that underpin the entire field of study.

Definition 1. *Category.* A category $\mathbf{C}$ is a collection of objects $A \in \mathrm{Ob}(\mathbf{C})$ and morphisms $f : A \to B$ between these objects, this collection of morphisms being notated $\mathrm{Hom}(\mathbf{C})$. The morphisms can be composed if their codomain and domains coincide. If we have morphisms $f : A \to B$ and $g : B \to C$ then we can form $g \circ f : A \to C$. The category is closed under such compositions and the composition is associative. Each object A has an identity morphism $\mathrm{id}_A : A \to A$ which is the unit of composition: $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

The objects in a category can be thought of as systems and the morphisms as ways of turning one system into another. For instance, we can take the category of Hilbert spaces with bounded linear maps as morphisms, known as **Hilb**. Composition in this category is just the standard composition of functions. If we only consider finite dimensional Hilbert spaces then we have the category **FdHilb**.

Once one has a series of categories they may ask if it is possible to move between them. Indeed it is, and this leads to the definition of a functor.

Definition 2. *Functor.* A functor $F : \mathbf{C} \to \mathbf{D}$ between categories maps each object $A \in \mathrm{Ob}(\mathbf{C})$ to an object $F(A) \in \mathrm{Ob}(\mathbf{D})$ and each morphism $f : A \to B$ in $\mathrm{Hom}(\mathbf{C})$ to a morphism $F(f) : F(A) \to F(B)$ in $\mathrm{Hom}(\mathbf{D})$ such that $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

As a clarifying, if trivial, example we can take the category with the natural numbers as objects and complex $m \times n$ matrices $M : n \to m$ as morphisms and construct a functor into the category of finite dimensional vector spaces and linear maps. We simply send each natural number to the vector space of that dimension and note that matrices are linear maps. In fact this functor is *full*, *faithful* and *essentially surjective on objects* which means it is an *equivalence of categories*. This is unsurprising since we already know that matrices represent linear maps on vector spaces given a choice of basis.

Definition 3. *Equivalence of Categories.* A functor $F : \mathbf{C} \to \mathbf{D}$ is an equivalence of categories if it is full, faithful and essentially surjective on objects.

- Full - for all $A, B \in \mathrm{Ob}(\mathbf{C})$ the function $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is surjective.
- Faithful - for all $A, B \in \mathrm{Ob}(\mathbf{C})$ the function $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is injective.
- Essentially surjective on objects - for $B \in \mathrm{Ob}(\mathbf{D})$ there exists an object $A \in \mathrm{Ob}(\mathbf{C})$ such that $F(A) \cong B$.

A less trivial example is that of a group representation. Consider a category with one object A where all morphisms $f : A \to A$ are isomorphisms. Then this category contains all the structure of a group: the morphisms are the group elements, composition in $\mathbf{C}$ is the group operation, the identity map on A is the unit e of the group and we have inverses because all morphisms are isomorphisms. Now recall that a group representation is a homomorphism $\rho : G \to GL(V)$. Thus we can view a group representation as a functor from the one element category into the category of vector spaces and linear maps. The functor just picks a vector space and sends all the morphisms to automorphisms of the vector space. In fact we can now generalise the notion of a group representation by picking any arbitrary category instead of just that of vector spaces. The functor now picks an object B in this category and sends morphisms to automorphisms of B.

A natural question to ask now is whether one can map between functors? The answer is yes and to do so we need to define a natural transformation.

Definition 4. *Natural Transformation.* A natural transformation between functors $F, G : \mathbf{C} \to \mathbf{D}$ assigns for each object A in $\mathrm{Ob}(\mathbf{C})$ a morphism η_A in $\mathrm{Hom}(\mathbf{D})$ such that the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

If for every object A, the morphism η_A is an isomorphism then we call the construction a *natural isomorphism.*

To really speak about quantum mechanics we need additional structure. Currently there is no way to discuss processes that occur on more than one system at once or two processes that happen at the same time. To this end we introduce a monoidal category.

Definition 5. *Monoidal Category.* A monoidal category is a category $\mathbf{C}$ equipped with a bifunctor $- \otimes - : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. There are natural isomorphisms $A \otimes I \xrightarrow{\rho_A} A$, $I \otimes A \xrightarrow{\lambda_A} A$ and $(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$ for all objects A, B and C.

These natural isomorphisms satisfy some coherence equations known as the triangle and pentagon equations which we will not state here for brevity but they can be found in any good category theory textbook [24, 18].

The domain of the functor $\otimes$ is the product category $\mathbf{C} \times \mathbf{C}$ where the objects are pairs of objects from $\mathbf{C}$ and the morphisms are pairs of morphisms from $\mathbf{C}$. The composition in the product category works element-wise as one would expect.

Essentially the $\otimes$ functor gives us a way of "tensoring" objects and morphisms together so we can consider composite systems. Any pair of objects (or morphisms) from $\mathbf{C}$ is itself an object (morphism) in $\mathbf{C} \times \mathbf{C}$ and thus by $\otimes$ we can map this pair back to an object (morphism) in the original category $\mathbf{C}$. From now on we will refer to $\otimes$ as the tensor product. The natural isomorphisms ρ and λ ensure that I is the unit of the tensor product and α ensures that it is associative.

The category **Hilb** can be made monoidal by equipping it with the usual tensor product. The unit I is just the complex numbers $\mathbb{C}$ and the natural isomorphisms are given by the canonical maps.

There is a notion of a functor which preserves the monoidal structure. This will be useful later (see section 2.4) in recovering classical and quantum systems from the Karoubi envelope where we will want the tensor product to behave well under transitions between these subtheories.

Definition 6. *Monoidal Functor.* A functor $F : \mathbf{C} \to \mathbf{D}$ between monoidal categories is itself monoidal if it is equipped with a natural isomorphism $F(A) \otimes_{\mathbf{D}} F(B) \to F(A \otimes_{\mathbf{C}} B)$ and an isomorphism $I_{\mathbf{D}} \to F(I_{\mathbf{C}})$.

If a monoidal functor gives an equivalence of categories then we say that we have a *monoidal equivalence of categories.*

So now we can speak about compositions of quantum systems. How do we swap systems over? We can do this with a braided monoidal category.

Definition 7. *Braided Category.* A braided category is a monoidal category $\mathbf{C}$ equipped with natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$ for all objects A and B. If $\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$ then we say the category is *symmetric.*

Like a monoidal category, the braids must satisfy certain coherence equations which again for brevity we will not state here [24, 18]. The category **Hilb** can be given a braiding by taking $\sigma_{A,B}$ to be the canonical mapping $a \otimes b \mapsto b \otimes a$. In fact this makes **Hilb** a symmetric monoidal category.

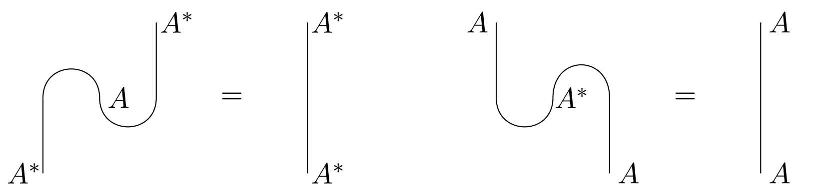
We also want to consider "adjoints" of maps and we do this with the dagger.

Definition 8. *Dagger Category.* A dagger category is a monoidal category $\mathbf{C}$ equipped with an involutive functor $\dagger : \mathbf{C}^{op} \to \mathbf{C}$ which is the identity on objects.

The dagger sends objects to themselves while reversing the direction of morphisms. By involutive we mean that $f^{\dagger^{\dagger}} = f$. The domain of $\dagger$ is the opposite category $\mathbf{C}^{op}$ in which objects are the same and we formally reverse the direction of morphisms. For **Hilb** we take the dagger to send linear maps $f : A \to B$ to their adjoints in the usual sense $f^{\dagger} : B \to A$.

The final demand we place on our category is that of dual objects and compact closure. Dual objects capture the notion of dual vector spaces.

Definition 9. *Duals.* An object A has dual A^* if there exist morphisms $\eta_A : I \to A \otimes A^*$ and $\epsilon_A : A^* \otimes A \to I$ (called the coevaluation and evaluation) such that the snake equations hold:



(1)

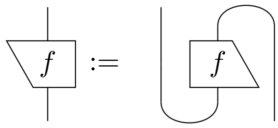
Here we have used the notation of string diagrams. The lines represent the identity morphism on an object and boxes represent non-identity morphisms. We have used special symbols for the evaluation and coevaluation maps which we will from now on call the cap and the cup.

There are at least several ways to interpret the snake equations on a more physical level. It turns out that in **FdHilb** the cup can be interpreted the Bell state and the cap as a Bell measurement, so the snake equations (as we will see later) capture some of the quantum teleportation protocol. Alternatively, one can imagine the cup to be the creation of a particle-antiparticle pair going forwards in time while the cap creates the same particles backwards in time. One particle-antiparticle pair annihilate in the middle of the snake, leaving the other pair to head towards the input and output wires.

It is worth noting that string diagrams are more than just some arbitrary way of drawing morphisms. Due to results by Joyal and Street [16, 17], these diagrams are totally rigorous and essentially an equation of morphisms follows from the axioms if and only if it is true in string diagrams up to spatial isotopy. This isotopy allows us to "bend" the wires and move boxes around. This makes the notation very versatile for proving results and we will use it frequently throughout this work.

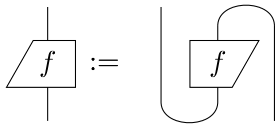
Using the cap and cup we can define the transpose of a morphism.

Definition 10. *Transpose.* The transpose of a morphism $f : A \to B$ is a morphism $f^T : B^* \to A^*$ given by:



Similarly we can define the conjugate of a morphism by first taking the dagger and then the transpose.

Definition 11. *Conjugate.* The conjugate of a morphism $f : A \to B$ is a morphism $f^* : A^* \to B^*$ given by:



1.1 Pure Quantum Theory

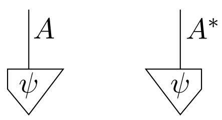
At this point it is worth spending a bit of time discussing basic pure quantum mechanics in this new framework. With all this additional category theory on top of Hilbert spaces it may not be clear that we are really talking about quantum theory. Where are the states and effects? What do maps look like?

The first question can be answered by a change of perspective. We fully embrace the process view of category theory and move away from thinking of states in a Hilbert space. Instead we define states and effects as follows:

Definition 12. *States and Effects.* A state is a morphism of the form $I \to A$ and an effect is a morphism of the form $A \to I$.

Working in **FdHilb** we consider states to be linear maps $\psi : \mathbb{C} \to H$. Once we pick $\psi(1) = \psi \in H$, the function ψ is completely fixed: $\psi(c) = c\psi(1) = c\psi$ for all $c \in \mathbb{C}$. Thus we see there is a bijection between elements of H and linear maps $\mathbb{C} \to H$ and we recover the usual concept of a state in quantum mechanics.

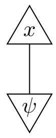
States and effects will be notated in string diagrams as triangles to differentiate them from other morphisms. When it is important to differentiate a state and its conjugate we will use triangles with the corners chopped off:



Given the definition of states and effects, the following definition of scalars may be unsurprising:

Definition 13. *Scalars.* A scalar is a morphism $s : I \to I$.

This means that taking a state ψ and composing with an effect x gives a scalar as we would expect $x \circ \psi : I \to I$. Given suitably normalised states and effects, we can interpret this morphism as a probability much like usual quantum mechanics:

$\mathrm{P}(x \mid \psi) \quad =$  $= \langle x|\psi \rangle$

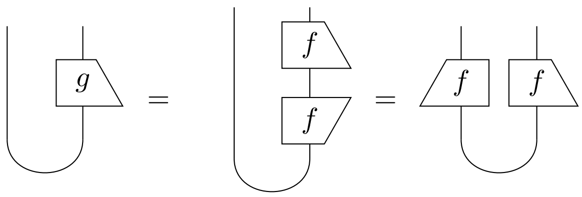
The choice of triangles for states and effects is at least in part as a nod to more conventional Dirac notation. It also points to the fact that the domain or codomain is the unit object I which we do not draw.

2 Classical vs Quantum

So far we have developed a categorical language of pure quantum mechanics. We can have pure states $I \to A$ and maps between these states. How do we capture classical data and how do we incorporate mixed quantum states? The key is similar to standard quantum theory, we essentially move from vectors to matrices in the CPM construction due to Selinger [21].

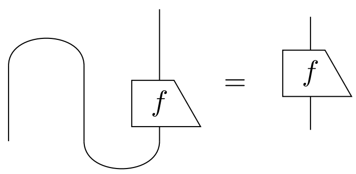
2.1 CPM Construction

Firstly recall that an operator g is positive if and only if there exists some operator f such that $g = f^\dagger \circ f$. We then consider taking the name of g which amounts to bending up the input wire:

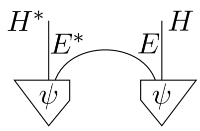


(2)

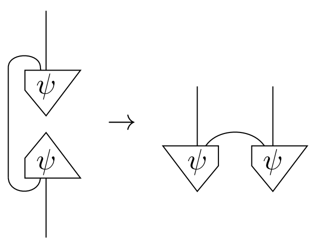
Taking names and conames (bending down output wires) is known in the quantum community as the channel-state duality or the Choi-Jamiolkowski isomorphism. There is a one-to-one correspondence between morphisms and their names and we can recover one from the other using the snake equations:



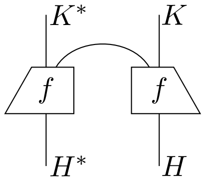
A positive operator (a density matrix for instance) takes the form of equation (2), taking names again to change the cup into a cap



The above diagram is equivalent to taking a density matrix on $H \otimes E$ and tracing out the environment E, $\mathrm{tr}_E(|\psi_{HE}\rangle \langle\psi_{HE}|)$ where the tracing out is captured by the cap. One way of thinking about what we have done is to transition from thinking of a density matrix as a morphism $H \to H$ to thinking of it as a state $I \to H^* \otimes H$. In diagrams this corresponds to:



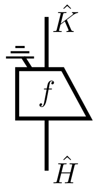
We can extend this to consider superoperators between density matrices. These take the form



(3)

If we consider the case where f is unitary then we see that this diagram captures the Stinespring dilation - a map between density matrices is given by a unitary on a larger system which we then trace out. Indeed, the Stinespring dilation tells us that if f is not unitary then we can dilate to a larger system where it will be unitary, including this extra part in the environment.

Note that these new maps we have written down are formed by "doubling" f - we reflect f in the vertical axis and use the cap to trace out the environment system. In general we can double maps from any category $\mathbf{C}$ to form a new category of completely positive maps $\mathbf{CPM}(\mathbf{C})$. This new category comes with its own string diagram notation in which we will use thick lines. The map (3) is drawn as:



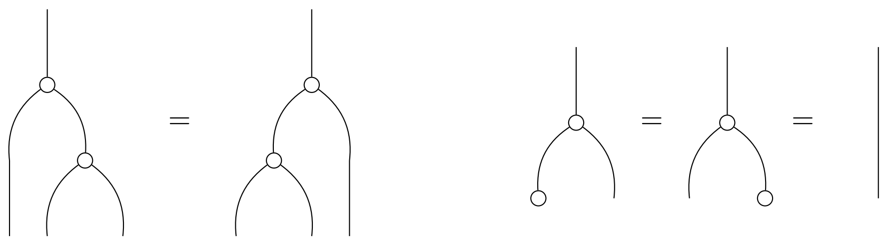
These thick lines represent doubled systems from the underlying category $\mathbf{C}$, i.e. $\hat{H} = H^* \otimes H$. We have used a special map called the discarding map to represent the trace between the two "halves" of f. This discarding map will be discussed further in section 2.3 but this new picture really captures our notion of what a CP map is - we apply some unitary to a larger system and then throw away the environment using the discarding map.

The CPM construction can be generalised to higher-order constructions which we will deal with in section 3. Before then we introduce another vital key in considering classical data.

2.2 Frobenius Structures

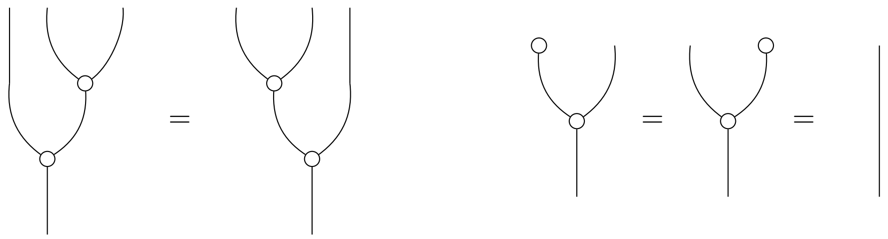
Frobenius structures were first introduced to capture classical data within the categorical framework. Underlying the definition of a Frobenius structure is that of a monoid, a set with associative multiplication and an identity element. We can lift this standard notion of a monoid into a category by the following definition

Definition 14. *Monoid.* A monoid on an object A is a multiplication $m : A \otimes A \to A$ and unit $e : I \to A$ that satisfy the following equations:



We have represented the map m as the dot with three legs and the map e as the dot with one leg. Similarly, we can define a comonoid by just reflecting all the diagrams in the horizontal axis.

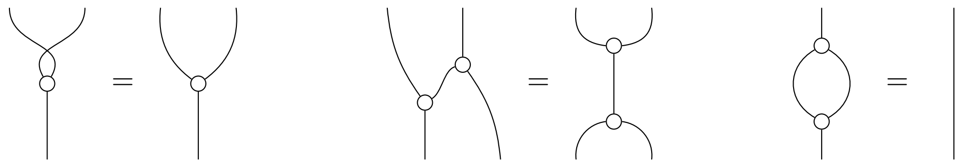
Definition 15. *Comonoid.* A comonoid on an object A is a comultiplication $\delta : A \to A \otimes A$ and counit $\epsilon : A \to I$ that satisfy the following equations:


(4)

It is clear that given a monoid, the daggers of the maps m and e give a comonoid. From now on we will make that association and set $m = \delta^\dagger$ and $e = \epsilon^\dagger$.

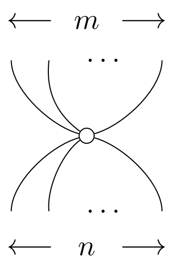
The language around Frobenius structures can get a bit confusing in the literature so to be explicit we will work with a special dagger commutative Frobenius structure which will we abbreviate to †-SCFA.

Definition 16. *†-SCFA.* A †-SCFA on an object A is a pair of a monoid and comonoid which are the dagger of each other and which are commutative, satisfy the Frobenius law and are special:



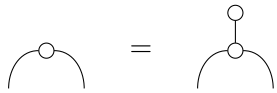
In the category **FdHilb** it was shown that there is a one-to-one correspondence between Frobenius structures on a Hilbert space H and bases of H [8]. We take $\delta : H \to H \otimes H$ to map $|i\rangle \mapsto |ii\rangle$ and $\epsilon : H \to I$ to map $|i\rangle \mapsto 1$ for a given basis $\{|i\rangle\}$ of H. δ can be seen as the map which copies a particular basis and ϵ as the map which erases that basis and in this sense, δ copies *classical* data.

By composing the maps δ, ϵ, the structure maps of the symmetric monoidal category and the daggers of all of these, we can form a class of maps $f : A^{\otimes n} \to A^{\otimes m}$ we will refer to as spiders and will draw as



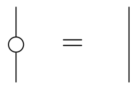
It was shown by Coecke and Duncan that all maps of this form can be written in a standard form and that a spider is entirely determined by the number of legs it has [4, 7]. As a result, they obey some pleasing equations. For instance, spiders that share legs fuse together to form one spider and swapping over the legs of a spider using the braid σ leaves the spider invariant. Tricks involving fusing and un-fusing spiders are common for proving results in the graphical language.

Using spiders we can also create compact structures on our category. A given †-SCFA induces a cap:



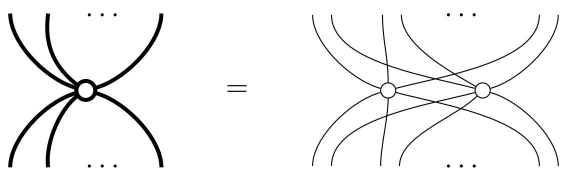
(5)

and a cup by taking the dagger of this. By the spider fusion rules, it is an immediate consequence that this choice of cap and cup do fulfil the snake equations (1). It is worth noticing that the $1 + 1$ legged spider is the identity map due to the counit equation (4):



A particularly pleasing use of spiders is in the ZX-calculus [4]. Here we take two †-SCFAs, one generated by the Z basis and the other by the X basis. Since these bases are complementary the two sets of spiders behave well together and obey a further series of equations. It is also the case that it is safe in this calculus to forget about the spiders in the induced compact structures (5) and draw them as the cap and cup. This is a result specific to the Z and X bases; they both induce the same compact structure. In general this is not true, so we will not make this simplification and will draw our induced compact structures explicitly. Caps and cups without spiders will be taken to be those due to the duality between A^* and A.

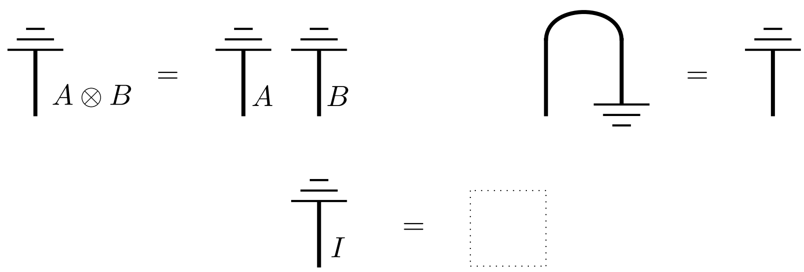
The spiders we have seen so far have been referred to as classical spiders in the literature. There is also a notion of a quantum spider, a spider in the category **CPM**(**FdHilb**). These spiders are the doubles of classical spiders:



2.3 Discarding

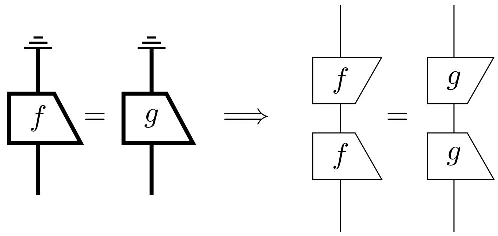
Earlier we used a special map which we called the discarding map and swept the details under the rug. We will now explain what this map is in more depth, starting with the definition of an environment structure.

Definition 17. *Environment Structure.* An environment structure $\{\bar{\bar{\top}}_A\}$ is a choice of morphism $\bar{\bar{\top}}_A : A \to I$ for each object A such that:

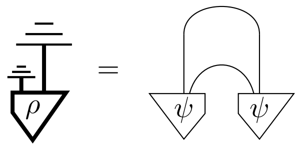


We also take the following axiom [3, 9]:

Definition 18. *CPM Axiom.* For pure maps f and g:



In **CPM**(**FdHilb**) the cap forms an environment structure. We see that discarding a state ρ corresponds to taking its trace:



Definition 19. *Normalised Process.* A process is normalised or *causal*, if and only if discarding it yields the empty diagram.

This is equivalent for states to saying that $\mathrm{tr}(\rho) = 1$ and thus we recover the usual notion from quantum mechanics.

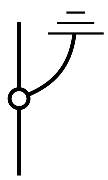
2.4 Karoubi Envelope and CP*

The Karoubi envelope is a way of splitting the idempotents of a category [13, 6, 10]. Recall that the idempotents are morphisms such that $f \circ f = f$. Given a category $\mathbf{C}$ we can form the Karoubi envelope by taking as objects (A, f) for objects A in $\mathrm{Hom}(\mathbf{C})$ and $f : A \to A$ an idempotent. The morphisms $x : (A, f) \to (B, g)$ in the Karoubi envelope are those $x : A \to B$ in $\mathrm{Hom}(\mathbf{C})$ that are invariant under the idempotents f and g, i.e. $g \circ x \circ f = x$. We call this new category **Split**($\mathbf{C}$).

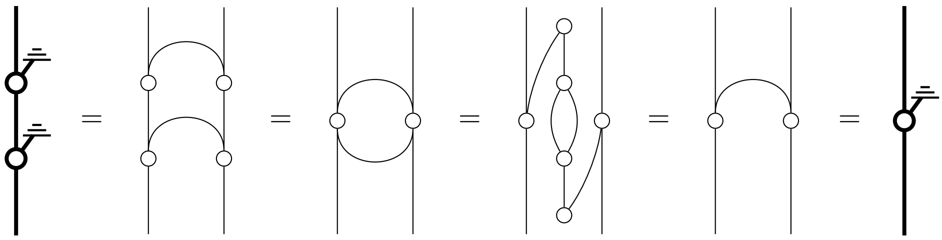
We can recover the category $\mathbf{C}$ from the Karoubi envelope by considering the full subcategory formed by objects of the form (A, id_A). It is clear that id_A is an idempotent and thus (A, id_A) is an object in the Karoubi envelope for all A. Further it is clear that for any $f : A \to B$, $\mathrm{id}_B \circ f \circ \mathrm{id}_A = f$. Thus we recover all morphisms in $\mathbf{C}$.

If we take the Karoubi envelope **Split**(**CPM**(**FdHilb**)) of the category of completely positive maps on Hilbert spaces then by the argument of the previous paragraph we can recover the quantum systems themselves.

We can also recover classical systems by considering the decoherence maps $\mathrm{dec}_\circ$. These take the form



and can be thought of as classical channels. They act to zero all non-diagonal entries of a density matrix, thus outputting a classical probability distribution. They are also idempotent:



The first equality follows by unfolding into **FdHilb**, the second and third by spider fusion and the fourth by the speciality of $\circ$. Thus we have objects of the form $(A, \mathrm{dec}_\circ)$ in $\mathbf{Split}(\mathbf{CPM}(\mathbf{FdHilb}))$ which can be thought of as the decohered version of A. There is a monoidal equivalence of categories between the full subcategory of these objects and $\mathbb{R}^+$-Mat, the category of classical systems. This category has matrices with positive real entries as morphisms. The normalised processes are the stochastic matrices.

It is common to take the full subcategory of $\mathbf{Split}(\mathbf{CPM}(\mathbf{FdHilb}))$ spanned by objects of the form (A, id_A) and $(A, \mathrm{dec}_\circ)$ so that we can capture both classical can quantum systems simultaneously.

3 Higher CPM Categories

In this section we will introduce the higher-order CPM constructions as developed by Gogioso [12]. This will allow us to consider a family of categories similar to the CPM category from the previous section.

The category $\mathbf{CPM}(\mathbf{C})$ was formed by taking morphisms $f \in \mathrm{Hom}(\mathbf{C})$ and doubling them. In this regard it is based on symmetries. One way of looking at what we did was to take a group and consider its action on the category. The CPM construction coincides with taking the group $\mathbb{Z}_2$ and the homomorphism into the automorphisms of $\mathbf{C}$ given by

$$\begin{aligned} \Phi : \mathbb{Z}_2 &\to \mathrm{Aut}(\mathbf{C}) \\ 0 &\mapsto \mathrm{id}_{\mathbf{C}} \\ 1 &\mapsto \mathrm{conj}_{\mathbf{C}} \end{aligned} \tag{6}$$

where $\mathrm{id}_{\mathbf{C}}$ is the identity functor and $\mathrm{conj}_{\mathbf{C}}$ is the conjugation functor (i.e. the functor that sends morphisms f to their conjugates f^* and objects A to their dual A^*). The doubling functor can then be expressed as

$$\mathrm{dbl}(\cdot) = \bigotimes_{g \in \mathbb{Z}_2} \Phi(g)(\cdot)$$

So we see that $\mathrm{dbl}(A) = A^* \otimes A$ and $\mathrm{dbl}(f) = f^* \otimes f$ as required. Note that we have chosen to tensor over the group elements in the order $1, 0$ and, as noted by Gogioso, this gives a functor which is naturally isomorphic to the other ordering and thus is essentially unimportant.

This notion of doubling can be extended to arbitrary folding given by the action of other groups. We take $\Phi : G \to \mathrm{Aut}(\mathbf{C})$ to be a homomorphism from an abelian group G into the automorphisms of $\mathbf{C}$. Then we can define the folding functor analogously to the doubling one.

Definition 20. *Folding Functor.* Given a homomorphism from an abelian group G into the automorphisms of $\mathbf{C}$, $\Phi : G \to \mathrm{Aut}(\mathbf{C})$, the folding functor is given by

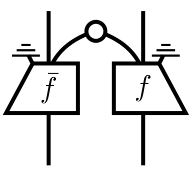
$$\mathrm{fld}(\cdot) = \bigotimes_{g \in G} \Phi(g)(\cdot)$$

3.1 Density Hypercubes

For the remainder of this work we will be interested in a particular case of the folding functor - that which allows us to generate higher-order CPM categories. In particular it is the functor generated by the iterated use of the action (6) and thus the action of $\mathbb{Z}_2^n$ for some $n \in \mathbb{N}$ defined as follows

$$\Phi^n(f_1, \ldots, f_n) = \Phi(f_1) \ldots \Phi(f_n)$$

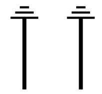
The case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ coincides with the double dilation construction of Zwart and Coecke [25]. This forms a category we will call $\mathbf{DD}(\mathbf{C})$ with morphisms of the form:



(7)

and objects $\mathrm{DD}(H) := H^* \otimes H \otimes H^* \otimes H$ for $H \in \mathrm{Ob}(\mathbf{C})$. By $\bar{f}$ we simply mean the morphism f with the outputs reversed in order, the diagrams are neater if drawn this way. We will draw morphisms in $\mathbf{DD}(\mathbf{C})$ in the string diagrams of either $\mathbf{CPM}(\mathbf{C})$ or $\mathbf{C}$ rather than directly using the diagrams of $\mathbf{DD}(\mathbf{C})$ itself in order to more easily see the underlying structure.

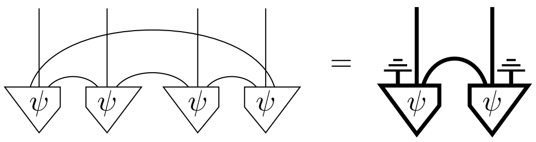
The discarding maps on $\mathbf{DD}(\mathbf{C})$ are the doubled discarding maps from quantum theory:



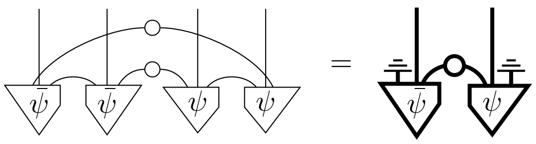
The states of density hypercubes first arose by considering the action of dilating (tracing-out) a system twice [25]. Taking a state on three subsystems $|\psi\rangle_{ABC}$ we can trace out once in the usual fashion: $|\rho\rangle = \mathrm{tr}_C(|\psi\rangle\langle\psi|) = \sum_i \langle e_i^C|\psi\rangle\langle\psi|e_i^C\rangle$ where $\{|e_i^C\rangle\}$ is a basis for C. We then perform a second trace in the following fashion:

$$\begin{aligned}
\mathrm{tr}_B(|\rho\rangle\langle\rho|) &= \sum_j \langle e_j^B|\rho\rangle\langle\rho|e_j^B\rangle \\
&= \sum_{ijkl} \langle e_k^B|\langle e_i^C|\psi\rangle\langle\psi|e_i^C\rangle|e_l^B\rangle\langle e_l^B|\langle e_j^C|\psi\rangle\langle\psi|e_j^C\rangle|e_k^B\rangle
\end{aligned}$$

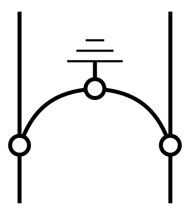
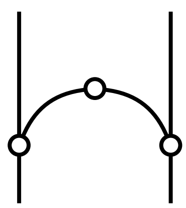
or in diagrams



To place this directly in the form of (7) one simply needs to fix an isomorphism between H and H^*:

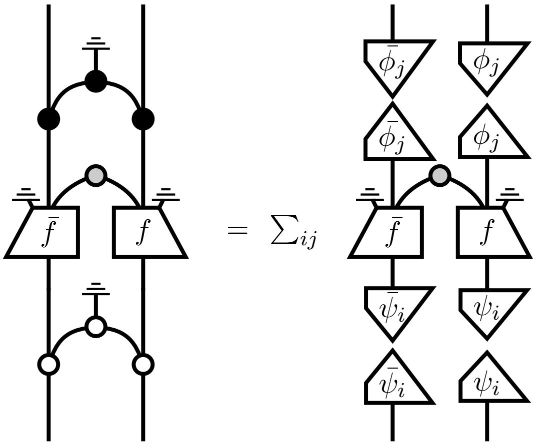


We will be particularly interested in the case $\mathbf{DD}(\mathbf{FdHilb})$ where the underlying category is that of quantum theory. It was shown that there is both a decoherence map $\mathrm{dec}_\circ$ and a *hyper*decoherence map $\mathrm{hypdec}_\circ$ for density hypercubes [14] the former is the tree-on-a-bridge and the latter is just the bridge.

$\mathrm{dec}_\circ \quad =$  $\qquad \mathrm{hypdec}_\circ =$ 

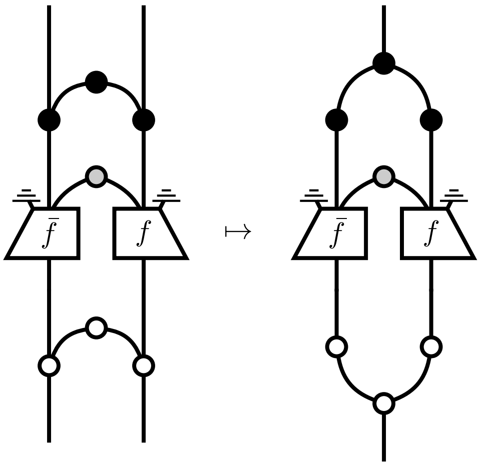
Both of these maps are idempotent (one can see this immediately by spider fusion) and thus the Karoubi envelope will contain objects of the forms $(\mathrm{DD}(H), \mathrm{dec}_\circ)$ and $(\mathrm{DD}(H), \mathrm{hypdec}_\circ)$.

There is a monoidal equivalence of categories between the full subcategory of the Karoubi envelope of $\mathbf{DD}(\mathbf{FdHilb})$ spanned by the decohered systems $(\mathrm{DD}(H), \mathrm{dec}_\circ)$ and the category $\mathbb{R}^+$-Mat. The equivalence can be witnessed by expanding $\mathrm{dec}_\circ$ in an orthonormal basis of the underlying Hilbert space [14]. The morphisms between decohered systems $(\mathrm{DD}(H), \mathrm{dec}_\circ) \to (\mathrm{DD}(K), \mathrm{dec}_\bullet)$ can be expanded as:



These are clearly matrices of non-negative real numbers. Thus we can recover classical systems.

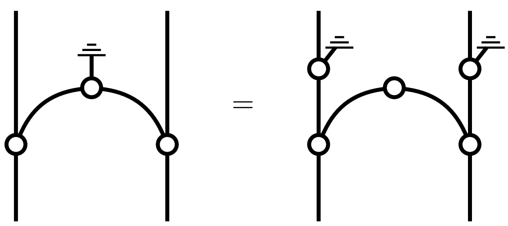
Similarly, there is a monoidal equivalence of categories between the full subcategory of the Karoubi envelope of $\mathbf{DD}(\mathbf{FdHilb})$ spanned by the hyperdecohered systems $(\mathrm{DD}(H), \mathrm{hypdec}_\circ)$ and the category $\mathbf{CPM}(\mathbf{FdHilb})$. This equivalence is witnessed by the following mapping:



Thus we can also recover quantum systems.

The theory of density hypercubes is formed by taking the full subcategory spanned by objects of the form $(\mathrm{DD}(H), \mathrm{id}_{\mathrm{DD}(H)})$, $(\mathrm{DD}(H), \mathrm{dec}_\circ)$ and $(\mathrm{DD}(H), \mathrm{hypdec}_\circ)$ thus recovering classical and quantum theory alongside the density hypercubes themselves.

The decoherence and hyperdecoherence maps play nicely together in the theory of density hypercubes. The decoherence map is equal to the composition of the hyperdecoherence map and the doubled decoherence map of quantum theory:



Thus it makes sense to think of density hypercubes as having three tiers of system which we can transition between as follows:

$$\begin{tikzcd} Hyperquantum \arrow[r, "\mathrm{hypdec}"] \arrow[rr, bend left, "\mathrm{dec}"] & Quantum \arrow[r, "\mathrm{dec}_Q"] & Classical \end{tikzcd}$$

where dec_Q is the doubled decoherence map of quantum theory.

4 Higher-Order Interference

The theory of density hypercubes is of interest to researchers because it has been shown to exhibit higher-order interference [14], specifically of fourth order in comparison to the second order interference of quantum theory. The order of interference in a theory is witnessed by the non-additivity of the probability measure associated with the theory [22].

The idea is to associate to an event A a non-negative real-valued measure $|A|$. For classical physics we know that probability distributions are linear over disjoint events A and B, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$, or in terms of the aforementioned measures $|A \cup B| = |A| + |B|$.

For quantum theory, on the other hand, this is not true. The measure becomes quadratic due to the Born rule and yields expressions like $|\psi_A + \psi_B|^2 = |\psi_A|^2 + |\psi_B|^2 + \psi_A^* \psi_B + \psi_A \psi_B^*$ which are no longer additive. Defining the expression $I_2 = |A \cup B| - |A| - |B|$ allows us to capture a notion of quantum-ness, indeed we saw for classical theory that $I_2 = 0$ but for quantum theory there exist states for which this is not true.

One can continue by defining a hierarchy:

$$
\begin{aligned}
I_1 &= |A| \\
I_2 &= |A \cup B| - |A| - |B| \\
I_3 &= |A \cup B \cup C| - |A \cup B| - |B \cup C| - |C \cup A| + |A| + |B| + |C| \\
&\vdots
\end{aligned}
\tag{8}
$$

We say that a theory exhibits n^{th} order interference if $I_n \neq 0$. In this regard classical physics exhibits first order interference and quantum physics exhibits second order interference. (Note that for quantum theory $I_3 = 0$ and it has been shown that $I_n = 0 \implies I_{n+1} = 0$ [22]).

But why does quantum theory only exhibit second order interference? One could argue that we just explained why - the Born rule! Yet on some level this is an insufficient explanation. Why the Born rule? What, if anything, goes awry in a theory with higher-order interference? What phenomena manifest that we simply could not allow?

Before diving into showing that density hypercubes exhibit fourth order interference, let us look in more detail at the quantum case. Barnum et. al. introduced a framework for studying higher-order interference in general probabilistic theories [2] which we will work within.

Definition 21. *Faces.* A face $\mathcal{F}_i$ of a theory is a set of states for which there exists an effect $\langle f_i|$ such that $\langle f_i|s\rangle = 1 \implies |s\rangle \in \mathcal{F}_i$.

In an n-slit experiment we have n disjoint faces $\mathcal{F}_i$. For quantum theory we can take $\mathcal{F}_i = \{|i\rangle\}$ since $\mathrm{tr}(P_i\,|s\rangle\langle s|) = |\langle i|s\rangle|^2 = 1 \implies |s\rangle = |i\rangle$ where P_i is the projector onto the subspace associated with $\mathcal{F}_i$.

We can take the union of faces, $\mathcal{F}_{ij} = \mathcal{F}_i \cup \mathcal{F}_j$ to form sets of states that are associated to either slit i, j or both. For any n slit experiment it is the case that $\mathcal{F}_{\{1,\ldots,n\}}$ is the full state space, since the projector $P_{1,\ldots,n}$ is just the identity operator.

We define an n-slit experiment in the following manner.

Definition 22. *n-slit Experiment.* An n-slit experiment is an effect $\langle E|$ representing the probability of finding a particle at a particular point on the screen and an effect $\{\langle e_I|\}$ for every subset $I \subset \{1, \dots, n\}$ such that

$$\begin{aligned} \langle e_I|s\rangle &= \langle E|s\rangle \qquad && \forall\, |s\rangle \in \mathcal{F}_I \\ \langle e_I|s\rangle &= 0 && \forall\, |s\rangle \perp \mathcal{F}_I \end{aligned}$$

In other words, if we close some slits and the state is in the face $\mathcal{F}_I$ associated with the remaining open slits then it should be unaffected by the closure and the outcome should be the same as the effect $\langle E|$. If, on the other hand, the state is in one of the other disjoint faces, it would be blocked by the slit closure and we should expect the zero effect.

A theory is said to exhibit n^{th} order interference, analogously to equation (8), if there exists a state $|s\rangle$ such that $I_n \neq 0$, where:

$$\begin{aligned} I_1 &= \langle E|s\rangle \\ I_2 &= \langle E|s\rangle - \langle e_0|s\rangle - \langle e_1|s\rangle \\ I_3 &= \langle E|s\rangle - \langle e_{01}|s\rangle - \langle e_{12}|s\rangle - \langle e_{20}|s\rangle + \langle e_0|s\rangle + \langle e_1|s\rangle + \langle e_2|s\rangle \\ &\ \vdots \end{aligned}$$

Recall that in quantum theory, the probability of a state ρ passing through the slits associated with face $\mathcal{F}_I$ can be written as $\mathrm{P}(\mathcal{F}_I \mid \rho) = \mathrm{tr}(P_I\rho)$. Suppose we now have a detector d with associated measurement operator D, the probability that ρ passes through slits $\mathcal{F}_I$ and is detected at D is given by [23]

$$\begin{aligned} \mathrm{P}(d \cup \mathcal{F}_I \mid \rho) &= \mathrm{P}(d \mid \mathcal{F}_I \cup \rho)\mathrm{P}(\mathcal{F}_I \mid \rho) \\ &= \frac{\mathrm{tr}(DP_I\rho P_I)\,\mathrm{tr}(P_I\rho)}{\mathrm{tr}(P_I\rho P_I)} \\ &= \mathrm{tr}(DP_I\rho P_I) \end{aligned}$$

The second equality follows by noting that the slits project ρ to $P_I\rho P_I$ with the denominator appearing due to normalisation. The final equality holds by using the cyclic property of the trace and noting that P_I is idempotent since it is a projection.

To see that quantum theory has second order interference, consider a 2-slit experiment and a detector which projects into the $|+\rangle$ state. Then we can take

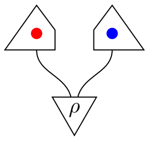
$$\begin{aligned} \langle E|\rho\rangle &= \mathrm{tr}(|+\rangle\langle+|\,\rho) \\ \langle e_0|\rho\rangle &= \mathrm{tr}(|+\rangle\langle+|\,P_0\rho P_0) \\ \langle e_1|\rho\rangle &= \mathrm{tr}(|+\rangle\langle+|\,P_1\rho P_1) \end{aligned}$$

Taking the input state $\rho = |+\rangle\langle+|$ we see that, $I_2 = \frac{1}{2} \neq 0$. Thus quantum theory exhibits second order interference.

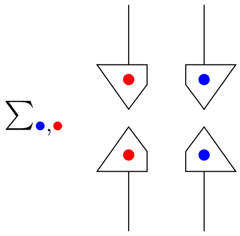
4.1 Geometric Approach

To explicitly do the calculations of the last section was a little tedious and difficult to follow, especially if we wanted to do something similar with density hypercubes and at higher than second order. Thankfully there is a lovely geometric approach to the same problem [14]. Let us see how this works for quantum theory before density hypercubes where it was originally used.

Showing that quantum theory exhibits second order interference essentially boiled down to calculating the value of $\mathrm{tr}(|+\rangle\langle+| P_I \rho P_I) = \langle+| P_I \rho P_I |+\rangle$. In this regard it depends on "how much" of ρ is allowed to get through the projectors P_I. More precisely, we can pick a basis $B = \{|\bullet\rangle\}$ where we represent each basis element by a colour for reasons that will become apparent and look at the matrix elements of ρ, $\langle\bullet| \rho |\bullet\rangle$. In diagram form:



Similarly, the projectors P_I can be expressed in the same basis:



We will suppress most of this notation and represent each piece of this sum by a shape e.g.

We note that there are two possible shapes:

 (9)

where on the left both indices are the same colour and on the right they are different colours.

If we take the basis B to be the computational basis, then our input state $|+\rangle\langle+|$ and detector measurement $|+\rangle\langle+|$ contain all possible shapes (9). As a result the probability depends entirely on the number of pieces in the projector P_I. Suppose the subset I has cardinality k. Then there are k pieces with shape of the first form of (9) and $\binom{k}{2} \cdot 2 = k(k-1)$ pieces of the second shape. Obviously there are k^2 pieces overall.

Let $\mathrm{P}(P_I) = \langle+| P_I \rho P_I |+\rangle$ be the outcome probability of the experiment given the projector P_I. Then $\mathrm{P}(P_I) = \frac{1}{d^2}|I|^2$. Looking at a 2-slit experiment we find:

$$\mathrm{P}(P_{01}) = \frac{1}{d^2}|\{0,1\}|^2 = \frac{4}{d^2}$$

while

$$\mathrm{P}(P_0) = \frac{1}{d^2}, \quad \mathrm{P}(P_1) = \frac{1}{d^2}$$

which gives us the inequality we found earlier:

$$I_2 = \mathrm{P}(P_{01}) - \mathrm{P}(P_0) - \mathrm{P}(P_1) = \frac{2}{d^2} \neq 0$$

Indeed for dimension $d = 2$ we get the same value as before.

Moving up to a 3-slit experiment we find

$$\begin{aligned} I_3 &= \frac{3^2}{d^2} - \binom{3}{2}\frac{2^2}{d^2} + \binom{3}{1}\frac{1}{d^2} \\ &= \frac{1}{d^2}(9 - 12 + 3) \\ &= 0 \end{aligned}$$

Showing that quantum theory does not exhibit higher than second order interference.

4.2 Density Hypercubes

In this section we aim to extend the proof that density hypercubes exhibit fourth order interference [14] to the iterated CPM construction, showing that they exhibit even higher order interference.

Theorem 1. *The iterated CPM construction corresponding to the folding functor generated by the group action of $\mathbb{Z}_2^n$ exhibits $\lambda = 2^n$ order interference.*

Proof. First note that the probability $\mathrm{P}(P_I)$ depends entirely on the number of pieces in the projector P_I for all the iterated CPM constructions. The number of pieces in P_I is $|I|^\lambda$ and thus:

$$\mathrm{P}(P_I) = \frac{1}{d^\lambda}|I|^\lambda \tag{10}$$

where d is the dimension of the input state (and the number of slits).

To show that we have $\lambda = 2^n$ order interference we need to show that

$$\mathrm{P}(P_{0,\ldots,\lambda}) \neq \sum_{\substack{I \subset \{0,\ldots,\lambda\} \\ |I| = \lambda - 1}} \mathrm{P}(P_I) - \sum_{\substack{I \subset \{0,\ldots,\lambda\} \\ |I| = \lambda - 2}} \mathrm{P}(P_I) + \cdots + \sum_{\substack{I \subset \{0,\ldots,\lambda\} \\ |I| = 1}} \mathrm{P}(P_I) \tag{11}$$

We can evaluate each side using equation (10). The left hand side is

$$\mathrm{P}(P_{0,\ldots,\lambda}) = \frac{1}{d^\lambda}\lambda^\lambda$$

while the right hand side is

$$\frac{1}{d^\lambda}\left[\binom{\lambda}{\lambda-1}(\lambda-1)^\lambda - \binom{\lambda}{\lambda-2}(\lambda-2)^\lambda + \cdots + \binom{\lambda}{1}1^\lambda\right] = -\frac{1}{d^\lambda}\sum_{k=1}^{\lambda-1}(-1)^k\binom{\lambda}{k}k^\lambda$$

The inequality (11) follows by the following series identity [26]

$$\begin{aligned} &\sum_{k=1}^{\lambda}(-1)^k\binom{\lambda}{k}k^\lambda = (-1)^\lambda\lambda! \\ \implies\ &(-1)^\lambda\lambda^\lambda + \sum_{k=1}^{\lambda-1}(-1)^k\binom{\lambda}{k}k^\lambda = (-1)^\lambda\lambda! \end{aligned}$$

Noting that λ is even we see:

$$-\sum_{k=1}^{\lambda-1}(-1)^k\binom{\lambda}{k}k^\lambda = \lambda^\lambda - \lambda! < \lambda^\lambda$$

So the iterated CPM construction exhibits $\lambda = 2^n$ order interference. □

Theorem 2. *The iterated CPM construction corresponding to the folding functor generated by the group action of $\mathbb{Z}_2^n$ exhibits no higher than $\lambda = 2^n$ order interference.*

Proof. We aim to show that the theory does not exhibit $\lambda + 1$ order interference. Then by the result $I_n = 0 \implies I_{n+1} = 0$ we have that the theory does not exhibit any higher order interference than λ.

To show that we do not have $\lambda + 1$ order interference we need to show that

$$\mathrm{P}(P_{0,\dots,\lambda+1}) = \sum_{\substack{I\subset\{0,\dots,\lambda+1\}\\|I|=\lambda}} \mathrm{P}(P_I) - \sum_{\substack{I\subset\{0,\dots,\lambda+1\}\\|I|=\lambda-1}} \mathrm{P}(P_I) + \dots - \sum_{\substack{I\subset\{0,\dots,\lambda+1\}\\|I|=1}} \mathrm{P}(P_I) \quad (12)$$

The left hand side can be evaluated to

$$\mathrm{P}(P_{0,\dots,\lambda+1}) = \frac{1}{d^\lambda}(\lambda+1)^\lambda$$

while the left hand side is

$$\frac{1}{d^\lambda}\left[\binom{\lambda+1}{\lambda}\lambda^\lambda - \binom{\lambda+1}{\lambda-1}(\lambda-1)^\lambda + \dots - \binom{\lambda+1}{1}1^\lambda\right] = \frac{1}{d^\lambda}\sum_{k=1}^{\lambda}(-1)^k\binom{\lambda+1}{k}k^\lambda$$

We use another series identity [26]

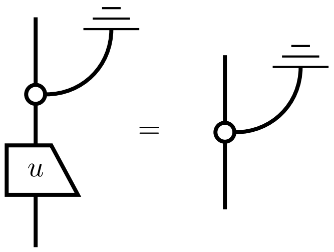
$$\sum_{k=1}^{\lambda+1}(-1)^k\binom{\lambda+1}{k}k^\lambda = 0$$
$$\implies \sum_{k=1}^{\lambda}(-1)^k\binom{\lambda+1}{k}k^\lambda = (\lambda+1)^\lambda$$

which shows that the equality (12) holds. □

5 The Phase Group

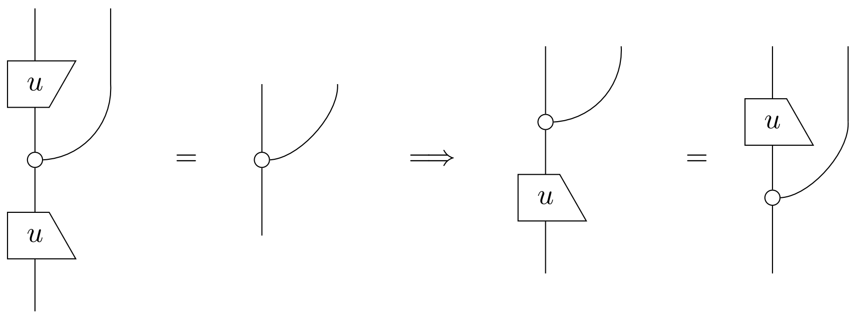
The phase group was first discussed by Coecke and Duncan in their development of the ZX-calculus [4]. It captures a few notions from quantum mechanics including phase gates and unbiased states and has more recently been shown to underpin the locality of a quantum-like theory [5, 15]. We define the group in the following way [11]:

Definition 23. *Phase Group.* The phase group is the group formed by invertible maps u under composition which satisfy the following equality

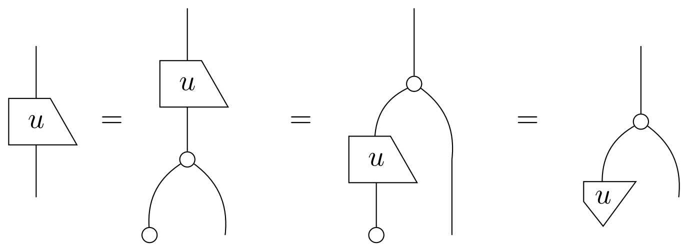


(13)

It is clear that this forms a group. We have a unit element given by the identity map, all maps have an inverse by definition and the set is closed under composition. We know that all completely positive trace preserving maps whose inverse is also completely positive and trace preserving must be unitary [20], writing equality (13) as a diagram in **FdHilb** gives



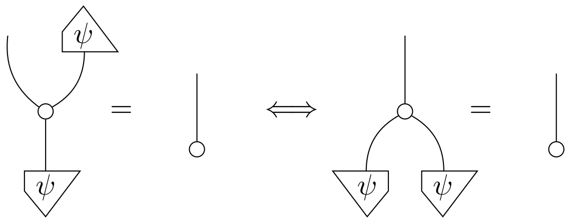
where we have used the fact that u is unitary. It is possible to derive an explicit expression for the unitaries u that are elements of the phase group:



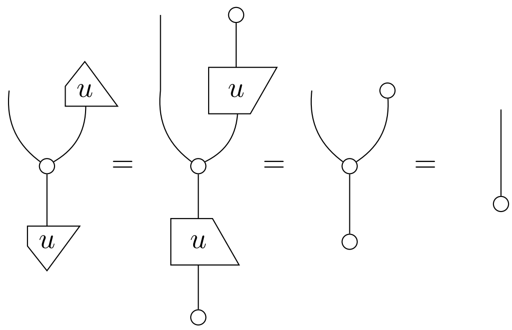
(14)

where the first equality follows by the unit of the †-SCFA, the second by equation (13) and the final by defining the state u in the obvious way.

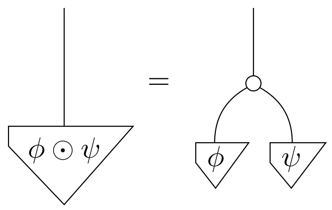
Definition 24. *Unbiased State.* A state $\psi : I \to A$ is unbiased for the †-SCFA $\circ$ if and only if



We can see that the state u is unbiased for $\circ$:



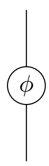
Thus we can view the phase group in two ways. Either we can say that is is the set of unitaries given by equation (14) for unbiased u under composition of morphisms, or we can view it as the states u themselves, under a multiplication $\odot$ given by



Either way we see that the group for qubits is the circle group S^1. For the †-SCFA generated by the computational basis $\{|0\rangle, |1\rangle\}$ it is the equator of the Bloch sphere, the states $|0\rangle + e^{i\phi}|1\rangle$. Equivalently it is the unitaries

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

It is typical for qubits to write these unitaries as spiders notated by the phase ϕ



Since the group multiplication $\odot$ is just addition of phases this notation is nice as it allows for a more general set of spider rules where fusing spiders sum their phases.

In general, for qudits we get the unitaries $\mathrm{diag}(1, e^{i\phi_1}, \ldots, e^{i\phi_{d-1}})$ forming the torus group $T^{d-1} = S^1 \times \ldots \times S^1$. One could again notate this as a phased spider but one can imagine that as the dimension increases and number of phases gets large this becomes unwieldy.

The phase group gives us the maps which are "erased" by decoherence, it tells us what information we lose by decohering to a classical system. Imagine a qubit and take the decoherence map associated with the †-SCFA due to the Z basis. Decohering a pure state $|\psi\rangle$ is equivalent to the following:

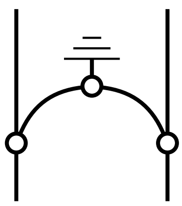
$$\begin{aligned}
\operatorname{tr}\Big(\delta\,|\psi\rangle\,\langle\psi|\,\delta^\dagger\Big) &= \operatorname{tr}\Big(\delta(\alpha\,|0\rangle + \beta\,|1\rangle)(\alpha^*\,\langle 0| + \beta^*\,\langle 1|)\delta^\dagger\Big) \\
&= \operatorname{tr}((\alpha\,|00\rangle + \beta\,|11\rangle)(\alpha^*\,\langle 00| + \beta^*\,\langle 11|)) \\
&= |\alpha|^2\,|0\rangle\,\langle 0| + |\beta|^2\,|1\rangle\,\langle 1| \\
&= \cos^2\left(\frac{\theta}{2}\right)|0\rangle\,\langle 0| + \sin^2\left(\frac{\theta}{2}\right)|1\rangle\,\langle 1|
\end{aligned}$$

As expected we have a classical probability distribution over the computational basis. We also note that we have lost all information about the angle ϕ that the original state had on the Bloch sphere. This coincides with what we found earlier - it is the phase gates that are erased by the decoherence map.

5.1 Density Hypercubes

What is the phase group of density hypercubes and is there a connection to the higher order interference exhibited by this theory? Since the theory is a candidate for hyper-quantum phenomena we might expect the group to be larger than normal quantum mechanics, with the usual torus group as a subgroup. This turns out to be false.

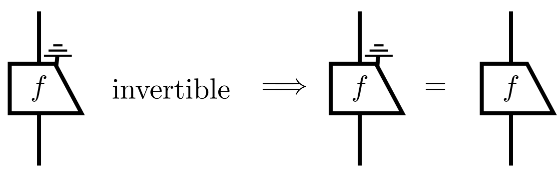
We saw earlier that the decoherence map $\mathrm{dec}_\circ$ for density hypercubes is the following tree-on-a-bridge:



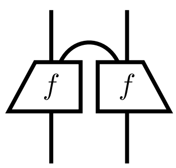
The phase group is, as before, the set of *invertible* maps which are "erased" by $\mathrm{dec}_\circ$. This leads to the following claim:

Claim 1. *The invertible maps of density hypercubes on* ***FdHilb*** *are the double dilated maps corresponding to unitaries on* ***FdHilb****.*

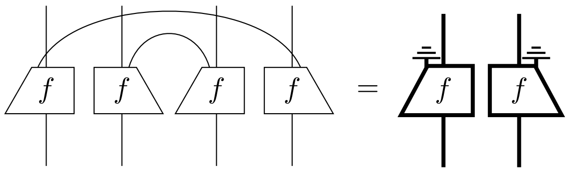
Proof. Firstly recall that a completely positive map has an inverse which is also completely positive if and only if it is a unitary. Thus



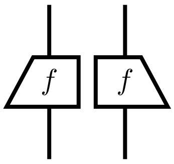
An invertible map of density hypercubes must therefore take the form:



noting the lack of discarding. Now, unfolding this morphism and pairing maps in the other order yields:



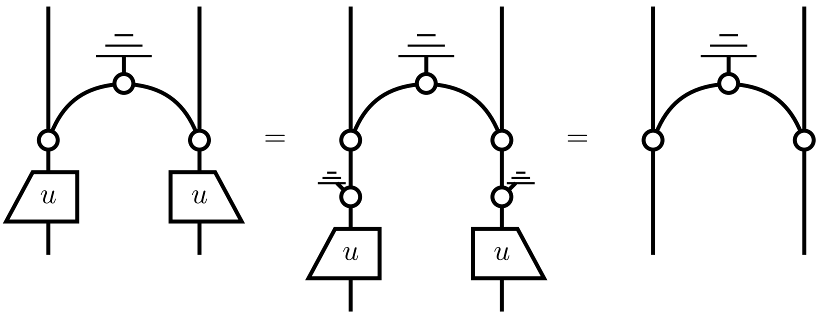
so we have a doubled completely positive map here too. Thus we cannot have the bridge if we want the map to be invertible. We conclude that invertible maps of density hypercubes take the form:



where f is a unitary on the Hilbert space H. □

Claim 2. *The phase group of density hypercubes is the same as for quantum theory.*

Proof. This follows by claim 1. The only invertible maps on **DD**(**FdHilb**) correspond to unitaries on **FdHilb**. The elements of the phase group for quantum theory are clearly of this form and are erased by $\mathrm{dec}_\circ$:

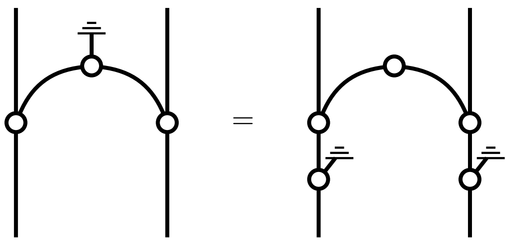


The first equality follows by un-fusing some spiders and the second because u is erased by the decoherence map of quantum theory. □

That the phase group of density hypercubes is the same as quantum theory is surprising. We do not lose anything extra when we decohere and the higher order interference cannot be explained by the phase group.

The result also implies that density hypercubes have the same locality as quantum theory in the sense defined in [5, 15].

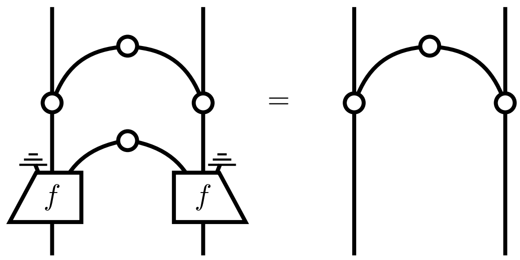
A further interesting question to consider is what maps are erased by hyper-decoherence alone. Given the previous result we might expect that nothing is lost because of the following decomposition of the decoherence map:



Thus anything that is erased by density hypercube decoherence includes that which is erased by the quantum decoherence. We already know that density hypercubes and quantum theory have the same phase group, thus everything in the phase group is erased by quantum decoherence and the hyperdecoherence map should erase nothing more.

In fact, the hyperdecoherence map erases a subgroup of the phase group as we will now demonstrate.

Definition 25. *Hyperphase Group.* The hyperphase group is the group formed by the invertible maps f which are erased by the hyperdecoherence map:

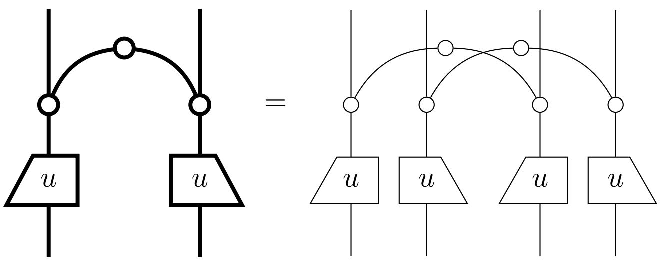


Claim 3. *Take a theory with a decoherence map $dec_{HQ\to C}$ from hyperquantum to classical, a hyperdecoherence map $hypdec_{HQ\to Q}$ from hyperquantum to quantum and a quantum decoherence map $dec_{Q\to C}$ from quantum to classical. If these maps are such that $dec_{Q\to C} \circ hypdec_{HQ\to Q} = dec_{HQ\to C}$ then the hyperphase group is a subgroup of the phase group.*

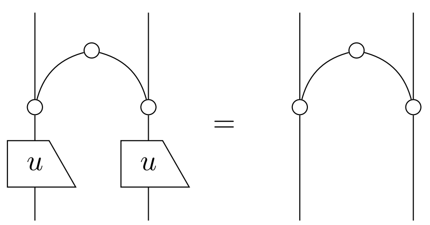
Proof. Let f be an element of the hyperphase group. Then $\mathrm{dec}_{HQ\to C} \circ f = \mathrm{dec}_{Q\to C} \circ (\mathrm{hypdec}_{HQ\to Q} \circ f) = \mathrm{dec}_{Q\to C} \circ \mathrm{hypdec}_{HQ\to Q} = \mathrm{dec}_{HQ\to C}$ □

Claim 4. *The hyperphase group for density hypercubes is $\mathbb{Z}_2^{d-1}$ where d is the dimension of the underlying Hilbert space.*

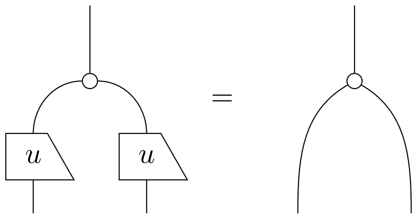
Proof. As we showed earlier the invertible maps on density hypercubes take the form of the unitaries on quantum systems. We have:



Thus we are looking for unitaries u which satisfy



Fusing spiders and using the speciality of the †-SCFA we can see that this is equivalent to



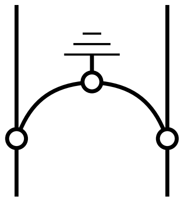
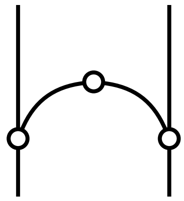
(15)

which looks very familiar - it is almost identical to the expression for the phase group of quantum theory. The important difference is the lack of u^* in this expression. Most of the rest of the proof for the phase group of quantum theory holds and we find that u can be expressed in the same form in terms of an unbiased state for $\circ$. It is vitally important to note that in order to satisfy equation (15) the state u must be self-conjugate. This means that the only valid states are those where the phases are ± 1, i.e. $|0\rangle \pm |1\rangle$ in two dimensions. Thus we see we have the group $\mathbb{Z}_2^{d-1}$ which is a subgroup of the torus phase group T^{d-1}. $\square$

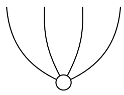
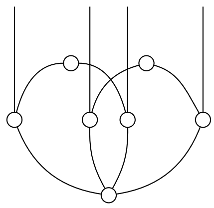
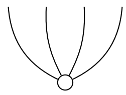
While the hyperphase group for density hypercubes is will always be a subgroup of the phase group it is worth noting that it is comparatively "small". It is only a finite group in comparison to the infinite full phase group. This may stem from the fact that the invertible maps are the doubled unitaries from quantum theory. Thus there is no set of invertible uniquely hyperquantum maps which could be erased, let alone a continuous family.

6 Hyperdecoherence and Hyperentanglement

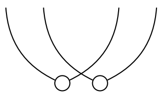
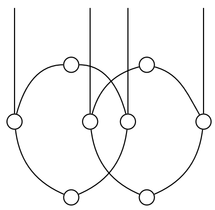
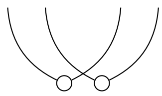
Earlier we met the decoherence and hyperdecoherence maps for density hypercubes:

$\mathrm{dec}_\circ \quad =$  $\mathrm{hypdec}_\circ \; =$ 

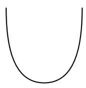
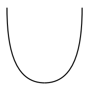
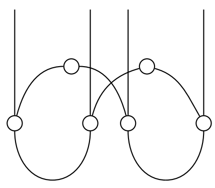
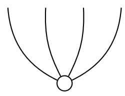
The hyperdecoherence map is not understood very well yet. Since we did not find the any connection between the phase group and higher-order interference, this map seems like a good place to look next. Let us see how it behaves on three normalised density hypercube states.

$\frac{1}{d}$  $\mapsto$ $\frac{1}{d}$  $= \frac{1}{d}$ 

This is unsurprising. We started with a completely classical state - the maximally mixed state on a density hypercube. The hyperdecoherence map leaves it invariant as we would hope.

$\frac{1}{d}$  $\mapsto$ $\frac{1}{d}$  $= \frac{1}{d}$ 

Again, this state is left invariant implying that it already was either quantum or classical.

$\frac{1}{d^2}$   $\mapsto$ $\frac{1}{d^2}$  $= \frac{1}{d^2}$ 

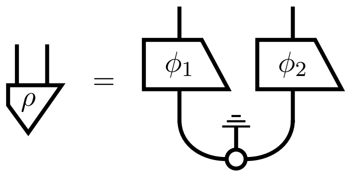
This final state is one that is acted on non-trivially by $\mathrm{hypdec}_\circ$. The first thing to note is that the final state is now sub-normalised. We will come back to this.

It is clear that the behaviour of the hyperdecoherence map is a bit strange and it is not immediately obvious what it does or how to decide if a state will be affected by the hyperdecoherence map. In the next sections we hope to shed a bit more light on this problem starting with considering what entanglement looks like for density hypercubes.

6.1 Entanglement

We can study the behaviour of entangled hypercube states under the decoherence and hyperdecoherence maps. Recall the definition of an entangled state in normal quantum theory [7]:

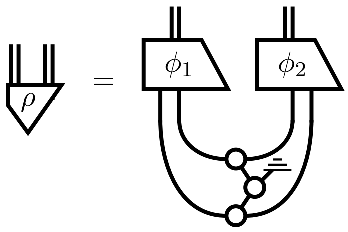
Definition 26. *Entangled Quantum State.* A quantum state is entangled if it cannot be written in the form:



The point of this definition is that the two quantum particles are disentangled if they are either separable (there is no link between them) or if they are only connected by a classical channel. Thus they can only be classically correlated and not quantumly.

We can extend this to density hypercubes

Definition 27. *Entangled Density Hypercube State.* A density hypercube state is entangled if it cannot be written in the form:

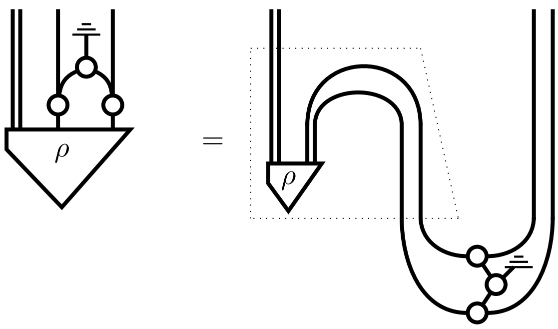


This definition has precisely the same idea behind it as for quantum theory: two hypercube states ought to be disentangled if they are either disconnected or connected by a classical channel - which now takes the form of the tree-on-a-bridge.

By a very similar proof to Coecke and Kissinger [7] we can show that, as we would hope, decoherence destroys entanglement just like in quantum theory

Claim 5. *Decoherence destroys entanglement.*

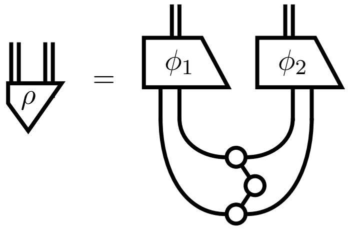
Proof.



□

We can now define a higher order version of entanglement we call *hyper*entangement

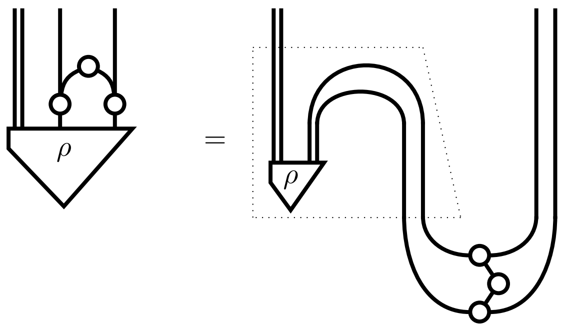
Definition 28. *Hyperentangled Density Hypercube State.* A density hypercube state is hyperentangled if it cannot be written in the form:



The idea here is to allow quantum communication between the hypercubes but not hyperquantum communication. By essentially the same proof as before, the hyperdecoherence map destroys hyperentanglement:

Claim 6. *Hyperdecoherence destroys hyperentanglement.*

Proof.



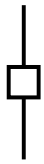
□

Obviously decoherence also destroys hyperentanglement.

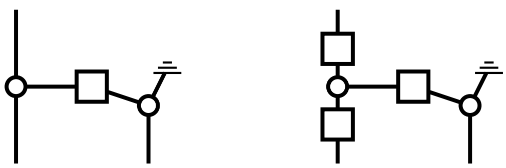
6.2 Teleportation

We now look at a canonical example of a quantum phenomenon - teleportation. We will start by recapping its exposition in diagrammatic calculus [9] working in two dimensions.

A key ingredient will be that of the Hadamard box which we will represent by

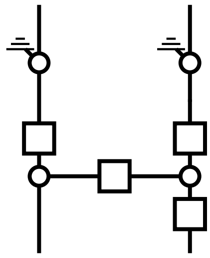


This is just the standard Hadamard gate which we recall is unitary and Hermitian. Using this box and the †-SCFA corresponding to the Z basis we can write down diagrams for controlled Z and controlled X gates:

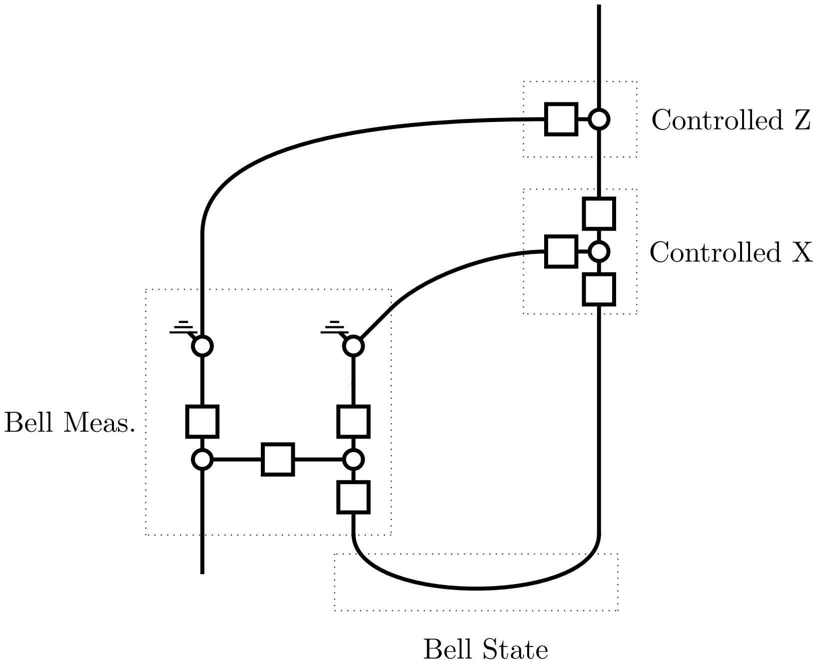


Controlled Z   Controlled X

One can check by inputting the $|0\rangle$ and $|1\rangle$ as the control, for instance, that these do give the correct maps. We will also require the Bell state measurement. It is a well known fact that a Bell state measurement can be achieved by implementing a controlled NOT gate, a Hadamard on the control qubit and then measuring in the Z basis. In diagram form:



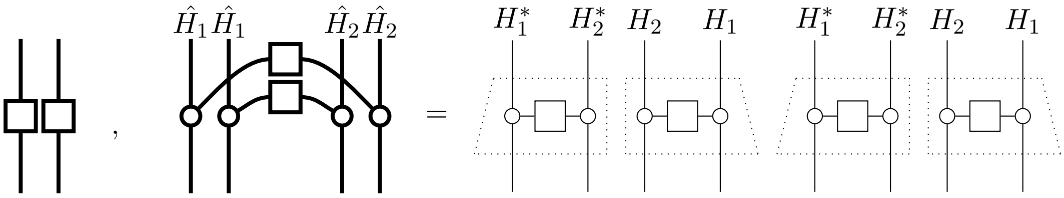
Coecke and Perdrix gave a diagrammatic proof of correctness of the teleportation protocol by putting together the pieces described above [9]. The diagram takes the following form



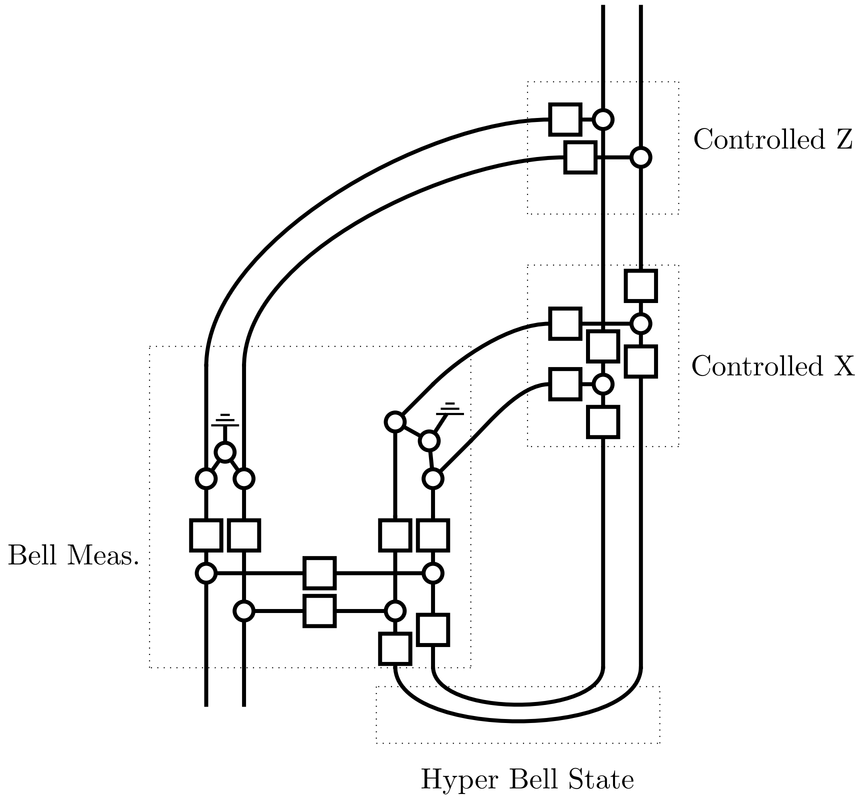
(16)

The proof of correctness can be found in their paper [9]. We note two things: the two parties looking to teleport a state needed to share an entangled Bell state and they were required to communicate two classical bits, shown by the two classical channels connecting the Bell measurement and controlled gates.

Can a similar protocol be given for density hypercubes? We now demonstrate that teleportation can be achieved and only requires the same amount of classical data transfer as quantum teleportation. The protocol takes the form of equation (17) where each pair of wires should be interpreted as a single density hypercube. In places, we have drawn the two parts of a density hypercube map slightly vertically displaced for ease of reading. The two maps beyond decoherence that we require for the protocol are:



The doubled Hadamard is clearly a valid map on a single density hypercube. The other two-density hypercube map can be constructed in the fashion presented.



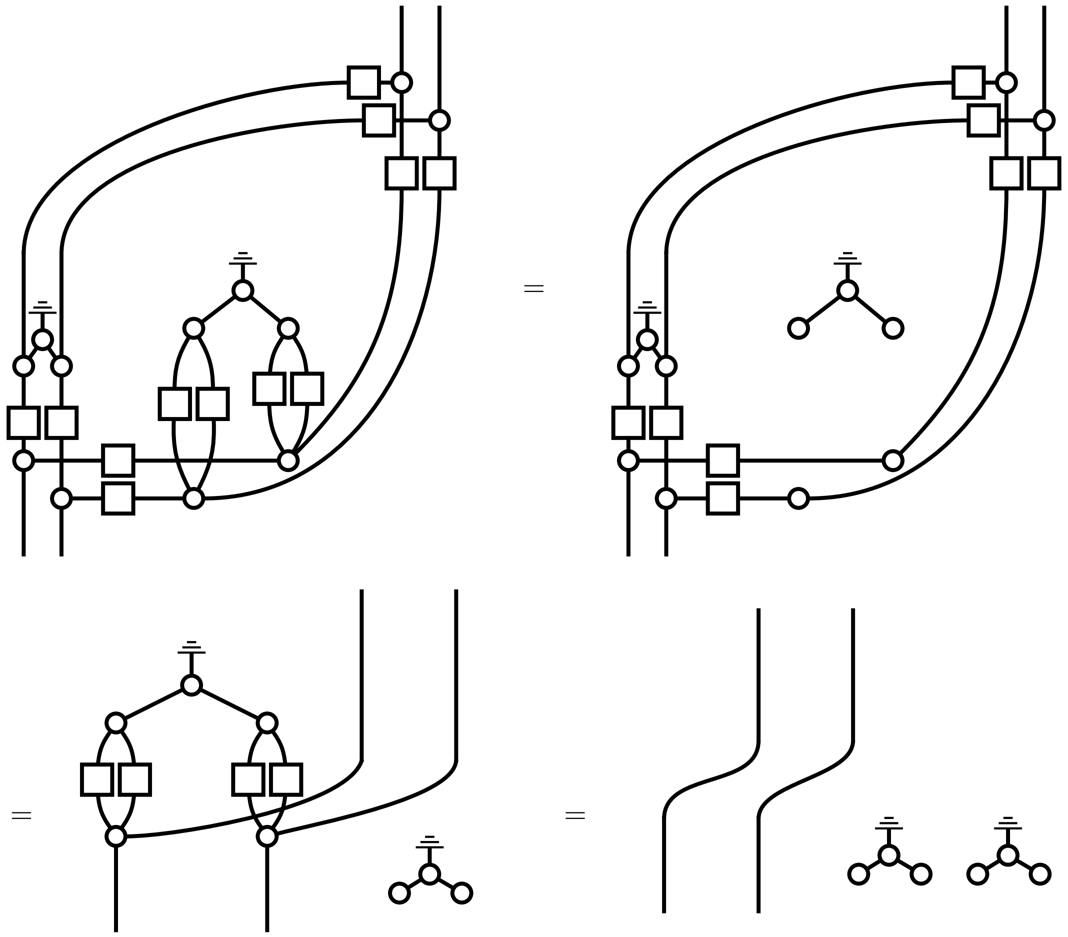
(17)

The first thing to note is that at its core this protocol is the doubled quantum teleportation protocol (16). The main differences are the need for the two parties to share a hyperentangled state and the use of the density hypercube decoherence maps to garner two classical channels (as opposed to the four channels that we would have by naively doubling).

The proof of correctness follows very similarly to [9]:

Theorem 3. *Protocol (17) allows for the teleportation of a density hypercube.*

Proof. Direction application of the diagram rewrite rules yields



$\square$

So although it looked like we might have connected the two lines of the input density hypercube together via the classical channel, because the discarding maps disconnect it turned out to be okay. Thus we have given an example of a protocol which utilises hyperentanglement. Furthermore, it is of interest because it can be achieved using only two classical bits of information just like quantum teleportation.

We now aim to give an algebraic argument for the correctness of the teleportation protocol. This serves several purposes: it offers further evidence that the protocol works and it highlights some details that may not be immediately apparent in the graphical language.

Suppose we take a pure density hypercube state $|\,|\psi\rangle\rangle = |\psi\rangle\langle\psi|\ \ |\psi\rangle\langle\psi|$ corresponding to taking the state $|\psi\rangle\langle\psi|$ in $\mathbf{CPM}(\mathbf{FdHilb})$ and doubling it. From now on we will not explicitly write the part on the dual space H^*, instead representing the density hypercube state as $|\,|\psi\rangle\rangle = |\psi\rangle\,|\psi\rangle$.

At the start of the protocol the entire state space consists of the density hypercube we want to teleport and the hyperentangled Bell state:

$$|\psi\rangle\,|\psi\rangle\,\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Recall though that this is a state on *three* density hypercubes, $|\,|\psi\rangle\rangle$ is one of those with the second represented as the first entry in the two Bell states, and the final density hypercube as the second entry in the two Bell states [1].

Taking $|\psi\rangle = \psi_0 |0\rangle + \psi_1 |1\rangle$ and applying the doubled CNOT gate yields:

$$\begin{aligned}
&\psi_0 |0\rangle \psi_0 |0\rangle \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
+&\psi_0 |0\rangle \psi_1 |1\rangle \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\
+&\psi_1 |1\rangle \psi_0 |0\rangle \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
+&\psi_1 |1\rangle \psi_1 |1\rangle \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)
\end{aligned}$$

The rest of the Bell measurement consists of performing a Hadamard on the first density hypercube and then measuring both density hypercubes in the computational basis. We do the measurement on the second density hypercube first. Recall it is the first entry in the two Bell states. The computational basis for two dimensional density hypercubes is $\{|00\rangle, |11\rangle\}$ corresponding to the outcomes 0 and 1 (recall the symmetries of the density hypercube states). This measurement yields one of

$$\begin{aligned}
0: \quad & \psi_0 |0\rangle \psi_0 |0\rangle \frac{1}{2} |00\rangle + \psi_0 |0\rangle \psi_1 |1\rangle \frac{1}{2} |01\rangle + \psi_1 |1\rangle \psi_0 |0\rangle \frac{1}{2} |10\rangle + \psi_1 |1\rangle \psi_1 |1\rangle \frac{1}{2} |11\rangle \\
1: \quad & \psi_0 |0\rangle \psi_0 |0\rangle \frac{1}{2} |11\rangle + \psi_0 |0\rangle \psi_1 |1\rangle \frac{1}{2} |10\rangle + \psi_1 |1\rangle \psi_0 |0\rangle \frac{1}{2} |01\rangle + \psi_1 |1\rangle \psi_1 |1\rangle \frac{1}{2} |00\rangle
\end{aligned}$$

Performing the doubled Hadamard on the first density hypercube and measuring it in the computational basis yields up to normalisation:

$$\begin{aligned}
00: \quad & \psi_0^2 |00\rangle + \psi_0\psi_1 |01\rangle + \psi_1\psi_0 |10\rangle + \psi_1^2 |11\rangle \\
10: \quad & \psi_0^2 |00\rangle - \psi_0\psi_1 |01\rangle - \psi_1\psi_0 |10\rangle + \psi_1^2 |11\rangle \\
01: \quad & \psi_0^2 |11\rangle + \psi_0\psi_1 |10\rangle + \psi_1\psi_0 |01\rangle + \psi_1^2 |00\rangle \\
11: \quad & \psi_0^2 |11\rangle - \psi_0\psi_1 |10\rangle - \psi_1\psi_0 |01\rangle + \psi_1^2 |00\rangle
\end{aligned}$$

where the first number on the left is the outcome of this second measurement and the second number is the outcome from the previous measurement.

We see that we have recovered the input density hypercube state $|| \psi \rangle\rangle$ when the measurement outcome is 00. For the other cases, just like quantum teleportation, we need to apply corrections given by doubled controlled X and Z. One can check that these do indeed return the output to the desired state.

While this algebraic approach has only shown the validity of the protocol in the case of a pure density hypercube state, the earlier graphical proof of correctness confirms that it works for mixed states.

One thing that becomes apparent from the algebraic method is the reliance on only having the two measurement outcomes $|00\rangle$ and $|11\rangle$ as opposed to the four outcomes that would be available if a similar doubled quantum teleportation protocol was performed on two qubits. This works out because of the symmetry of a density hypercubes state.

[1] We now see why we did not use Dirac notation for most of this report!

6.3 Dense Coding

It is a well known fact that it is possible to transmit two classical bits by transferring only one qubit as long as the two parties share an entangled state. This protocol is known as dense coding. We now show that the same can be done using density hypercubes.

The protocol for dense coding for density hypercubes can be realised by simply doubling the dense coding protocol given in [9] and replacing doubled quantum to classical decoherence maps with the hypercube decoherence map, just like we did for teleportation. The proof of its correctness follows along very similar lines to the aforementioned paper and we will not reiterate that here.

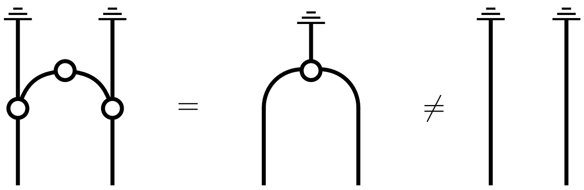
There is one major difference between this dense coding protocol and that given for teleportation; we have not gained anything. With density hypercube teleportation we were able to teleport a hyperquantum state using the same amount of classical data as for quantum teleportation. Here, we are using a hyperentangled state and a hyperquantum channel and are still only able to transmit two classical bits, just like in quantum theory.

We have not been able to find a way to transmit more than two classical bits by the transmission of one density hypercubes. It seems unlikely that it is possible because a measurement of a two dimensional density hypercube still yields only two possible outcomes, yet we do not have a proof of this.

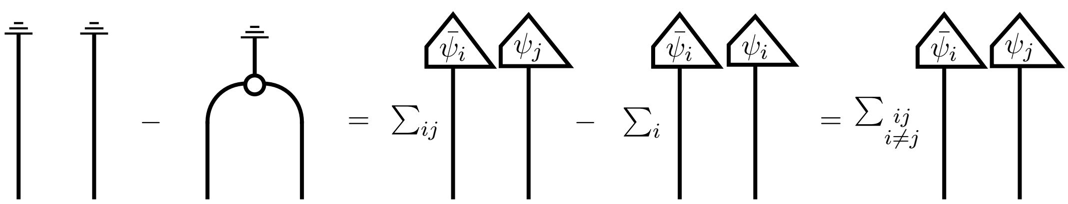
It must also be noted that since quantum theory is a subtheory of density hypercubes, the usual teleportation and dense coding protocols should hold true in the subtheory. The precise forms of these protocols on density hypercubes is not known but an educated guess is to simply hyperdecohere the hyperentangled state and the hyperquantum channel in the density hypercube protocols. Unfortunately a proof of the correctness of these protocols has not yet been achieved.

6.4 Issues with Normalisation

One issue with the hyperdecoherence map at present is that it is subnormalised. Discarding the hyperdecoherence map does not give the full discarding map on DD(H):



but it can be completed to this map by the addition of the following effect which does exist within **CPM**(**FdHilb**):



There is a recent paper in which a no-go theorem for theories which decohere to quantum theory is proven [19]. A theory which possess this hyperquantum to quantum transition must break either causality or purification. All assumptions of that paper hold in the theory of density hypercubes (restricted to the normalised states and processes) apart from the normalisation of the hyperdecoherence map. As argued in the previous paragraph the hyperdecoherence map can be completed to a normalised one, so it is interesting to ask which, causality or purification, must be broken. As we are taking the normalised subtheory of density hypercubes, we have by assumption that causality holds. In such a case, it seems that purification must be broken but unfortunately we do not have a proof of this claim.

7 Summary and Outlook

The theory of density hypercubes has been shown to be rich in interesting questions about the interplay of quantum theory with possible hyperquantum theory, in whatever form that may take. We covered the result of Gogioso and Scandolo [14] that density hypercubes show fourth order interference - higher than quantum theory suggesting that this theory has some hyperquantum nature to it. We extended this result to show that Gogioso's family of iterated CPM constructions [12] exhibit even higher order interference.

We also studied the phase group of density hypercubes in the hope that this might explain the origin of the higher order interference. We found that the phase group was exactly the same as quantum theory essentially because the invertible maps are the doubled unitaries from **CPM**(**FdHilb**). This was unexpected since one might imagine a hyperquantum theory which hyperdecoheres to quantum theory to erase the quantum phase group as just a subgroup of some larger group under full decoherence. This may suggest that this theory is not as hyperquantum as first thought or that the theory is only partially complete. The fact that the hyperdecoherence map is subnormalised lends further credence to the incompleteness of the theory.

We then went on to define a new hyperphase group by looking for invertible maps which are erased by hyperdecoherence. We showed that this group is a subgroup of the phase group for any theory where full decoherence decomposes into a hyperdecoherence map followed by a decoherence map. For density hypercubes we found that the hyperphase group is $\mathbb{Z}_2^{d-1}$ showing that hyperdecoherence is only erasing a finite group as opposed to the infinite group of the full decoherence.

In a final section we looked at the hyperdecoherence map and defined a set of hyperentangled states - those in which the density hypercubes have a hyperquantum connection as opposed to the quantum connection allowed by usual entangled states and the classical connection allowed by disentangled states. With a very minor alteration to the proof of Coecke and Kissinger [7] we were able to confirm that decoherence destroys entanglement and hyperentanglement and hyperdecoherence destroys hyperentanglement. We then demonstrated a use for hyperentanglement by extending the well-known teleportation protocol to density hypercubes. This protocol requires a hyperentangled state but only two classical bits of data transfer just like quantum teleportation.

A key milestone for future work is to complete the theory of density hypercubes so that the hyperdecoherence map is normalised. It would also be good to extend density hypercubes to a theory where the set of the invertible maps is larger than quantum theory. One might hope then to find a larger hyperphase group and perhaps find interesting structural connections between the phase group and hyperphase group which may provide insights into the behaviour of hyperdecoherence.

In a version of density hypercubes with a normalised hyperdecoherence map we will also have to concern ourselves with the no-go theorem of [19] and find whether causality or purifications are broken. Such a result will be highly informative about the structure of our theory alongside confirming the no-go theorem itself. It is plausible that the theory of density hypercubes could act as a playground not only for higher order interference but for investigations into whichever of the assumptions of the no-go theorem it breaks. While there are more abstract paths to investigating the lack of causality or purifications, an explicit theory in which one or both do not hold would be invaluable.

Further exploration of hyperentanglement is a goal for future research. It would be interesting to search for further uses of hyperentanglement and study it as a resource. It seems that hyperquantum theories may possess a hierarchy of entanglement and perhaps a hierarchy of entanglement resources. The implications of this for computational tasks is an outstanding question.

An adequate explanation of higher-order interference has still not been found and this will be the focus of future research. A deeper understanding of hyperdecoherence may assist in this, as well as searching for other theories with higher-order interference.

References

[1] S. Abramsky and B. Coecke. A categorical semantics of quantum protocols. *Proc. IEEE*, pages 415–425, 2004.

[2] H. Barnum, Müller M, and C. Ududec. Higher-order interference and single-system postulates characterizing quantum theory. *New Jour. Phys.*, 16, 2014.

[3] B. Coecke. Axiomatic description of mixed states from selinger's cpm-construction. *Electronic Notes in Theoretical Computer Science*, 210:3 – 13, 2008.

[4] B. Coecke and R. Duncan. Interacting quantum observables: categorical algebra and diagrammatics. *New Journal of Physics*, 13, 2011.

[5] B. Coecke, B. Edwards, and R. Spekkens. Phase groups and the origin of non-locality for qubits. *arXiv preprint*, arXiv:1003.5005v1, 2010.

[6] B. Coecke, C. Heunen, and A. Kissinger. Categories of quantum and classical channels. *Quantum Information Processing*, 15:5179–5209, 2016.

[7] B. Coecke and A. Kissinger. *Picturing Quantum Processes.* Cambridge University Press, 2017.

[8] B. Coecke, D. Pavlovic, and J. Vicary. A new description of orthogonal bases. *Mathematical Structures in Computer Science*, 23(3):555–567, 2013.

[9] B. Coecke and S. Perdrix. Environment and classical channels in categorical quantum mechanics. *Logical Methods in Computer Science*, 8, 2012.

[10] B. Coecke, J. Selby, and S. Tull. Two roads to classicality. *EPTCS*, 266:104–118, 2018.

[11] S. Gogioso. *Categorical Quantum Dynamics.* DPhil thesis, Trinity College, University of Oxford, 2016.

[12] S. Gogioso. Higher-order cpm constructions. *EPTCS*, 287:145–162, 2019.

[13] S. Gogioso and C. Scandolo. Categorical probabilistic theories. *EPTCS*, 266:367–385, 2018.

[14] S. Gogioso and C. Scandolo. Density hypercubes, higher order interference and hyper-decoherence: a categorical approach. *arXiv preprint*, arXiv:1806.00915v1, 2018.

[15] S. Gogioso and W. Zeng. Generalised mermin-type non-locality arguments. *Logical Methods in Computer Science*, 15, 2019.

[16] A. Joyal and R. Street. The geometry of tensor calculus, i. *Adv. in Math.*, 88:55–112, 1991.

[17] A. Joyal and R. Street. Braided tensor categories. *Adv. in Math.*, 102:20–78, 1993.

[18] S. Lane. *Categories for the Working Mathematician.* Springer Science, 1971.

[19] C. Lee and J. Selby. A no-go theorem for theories that decohere to quantum mechanics. *Proc. R. Soc. A*, 474(20170732), 2018.

[20] J. Preskill. Quantum computation. `http://www.theory.caltech.edu/people/preskill/ph229/`.

[21] P. Selinger. Dagger compact closed categories and completely positive maps. *Electronic Notes in Theoretical Computer Science*, 170:139–163, 2007.

[22] R. Sorkin. Quantum mechanics as quantum measure theory. *Mod. Phys. Lett. A*, 9(33):3119–3127, 1994.

[23] C. Ududec, H. Barnum, and J. Emerson. Three slit experiments and the structure of quantum theory. *Found. of Phys.*, 41:396–405, 2011.

[24] J. Vicary and C. Heunen. Categorical quantum mechanics - an introduction. 2019.

[25] M. Zwart and B. Coecke. Double dilation $\neq$ double mixing. *EPTCS*, 266:133–146, 2017.

[26] D. Zwillinger. *Table of Integrals, Series and Products.* Academic Press, 8 edition, 2015.