DAG-width: Cops and Robbers on Directed Graphs

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Joint work with
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GAMES Workshop, 2005

Motivation

- Tree-width introduced by Robertson and Seymour
  - Tree decompositions provide for recursive algorithms
  - Bounding tree-width gives polynomial time execution
- Directed tree-width by Johnson, Robertson, Seymour and Thomas
  - Not an obvious extension of tree-width
  - Complicated definition does not lend itself to algorithms

Aim

Find a natural extension of tree-width to directed graphs that is algorithmically useful.

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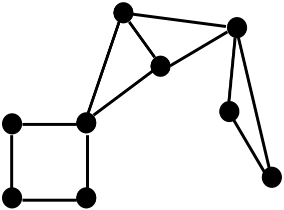
Overview

- Review tree-width
- Cops and robber game
- DAG-decompositions and DAG-width
- An algorithm for parity games
- Further work

Recall...

The tree-width of a graph measures its similarity to a tree.

A graph has tree-width $\leq k$ if it can be covered by sub-graphs of size $\leq (k+1)$ in a tree-like fashion.



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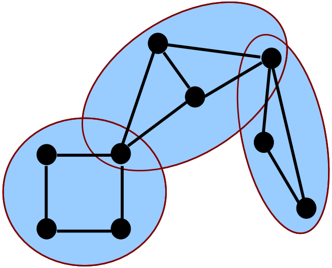

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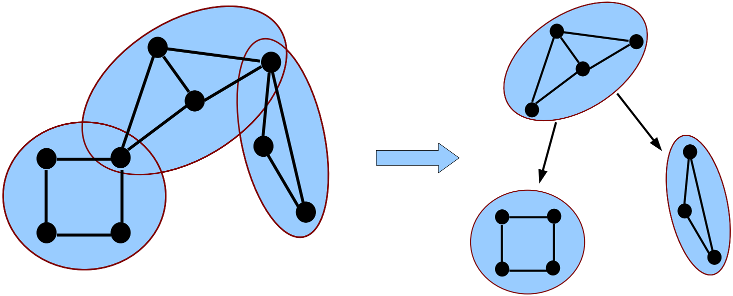

Tree-width

A tree decomposition of a graph $\mathcal{G}$ is a tuple $(\mathcal{T}, (X_t)_{t \in V(\mathcal{T})})$ such that:

- $\mathcal{T}$ is a tree
- X_t cover $V(\mathcal{G})$
- For every edge $(u, v) \in E(\mathcal{G})$, there is a $t \in V(\mathcal{T})$ with $\{u, v\} \subseteq X_t$
- For every t' on the path from t to t'', $X_t \cap X_{t''} \subseteq X_{t'}$

The width of a tree decomposition is $\max_{t \in V(t)} |X_t| - 1$.

The tree-width of a graph is the minimal width of all its tree decompositions.

Tree-width can be characterised by a cops and robber game.

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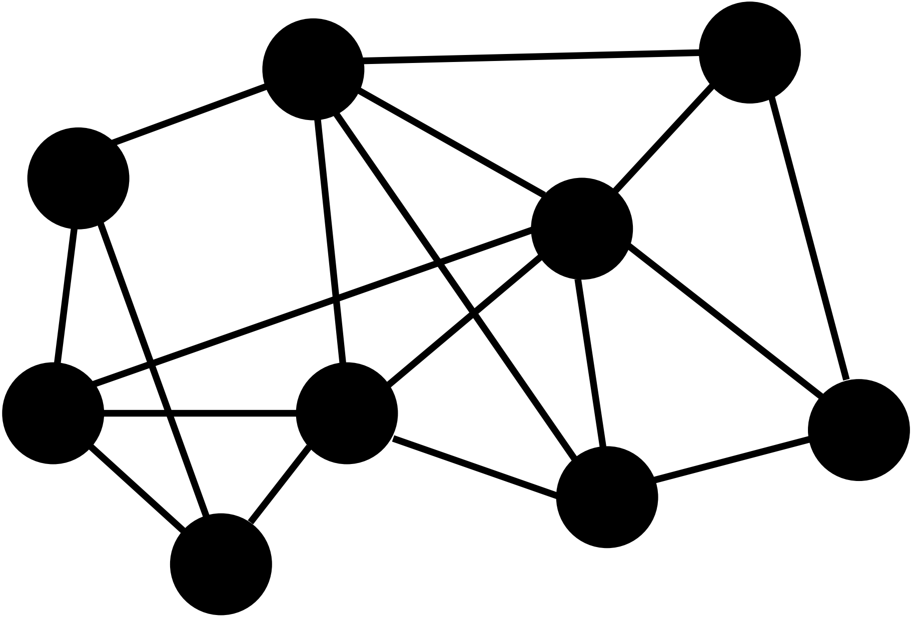
Cops and robber game



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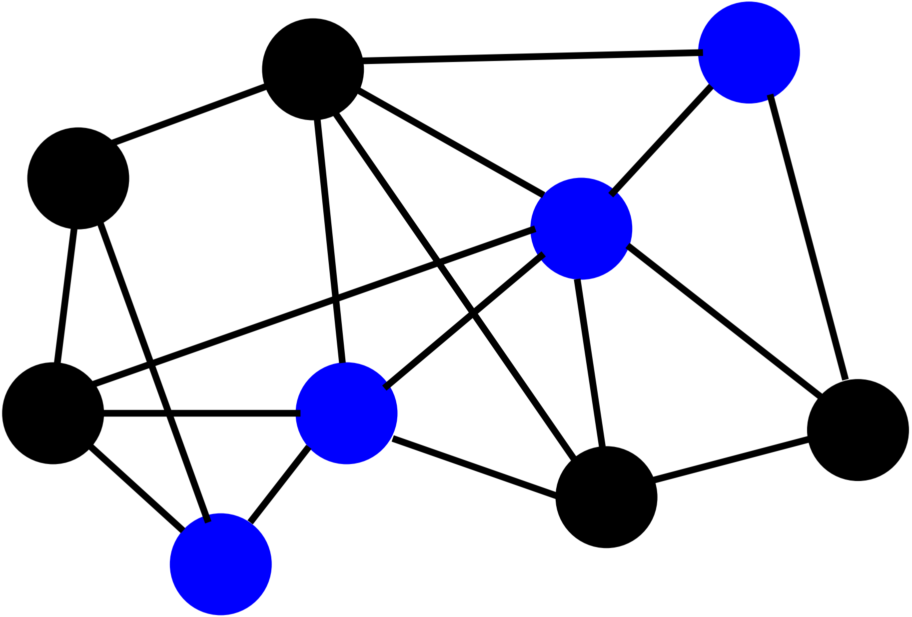

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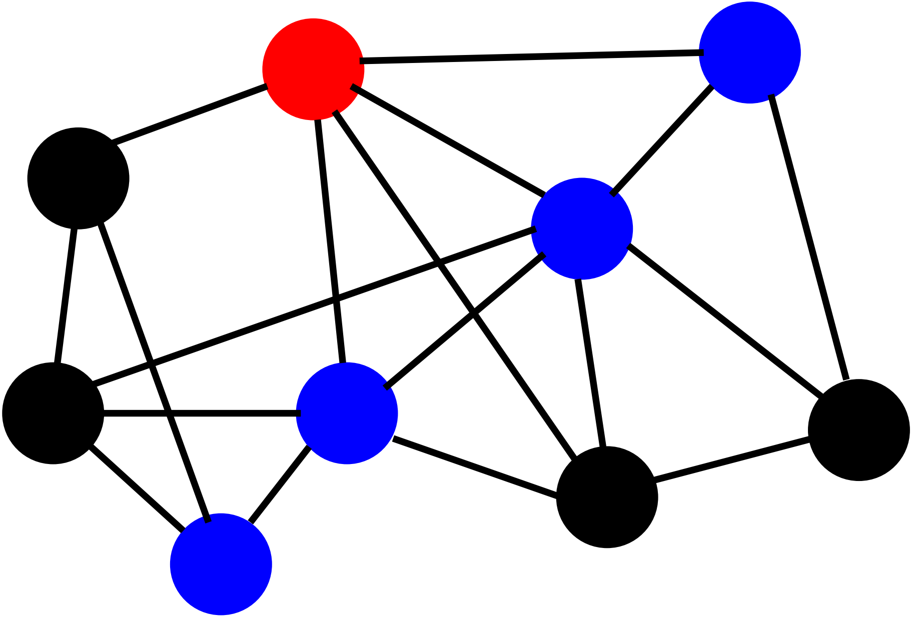

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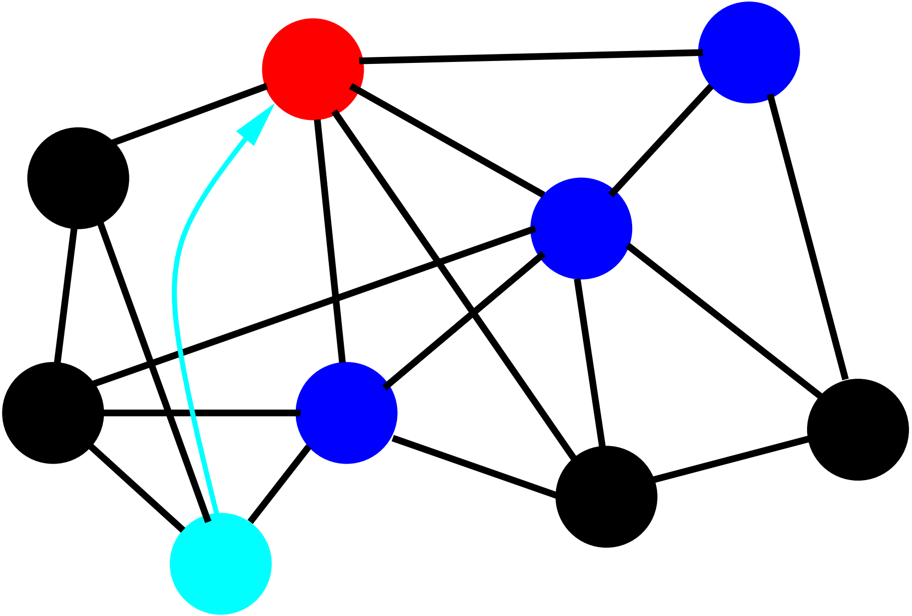

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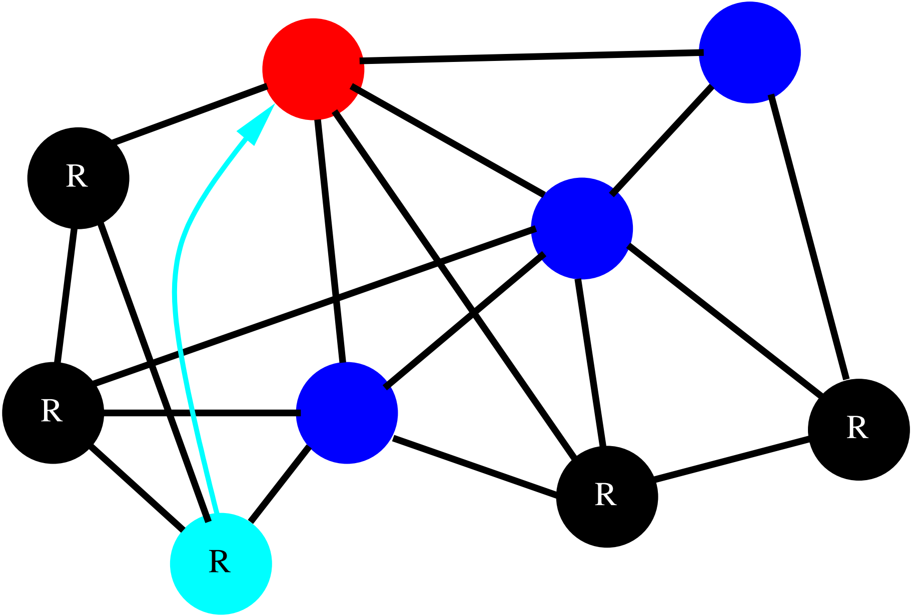

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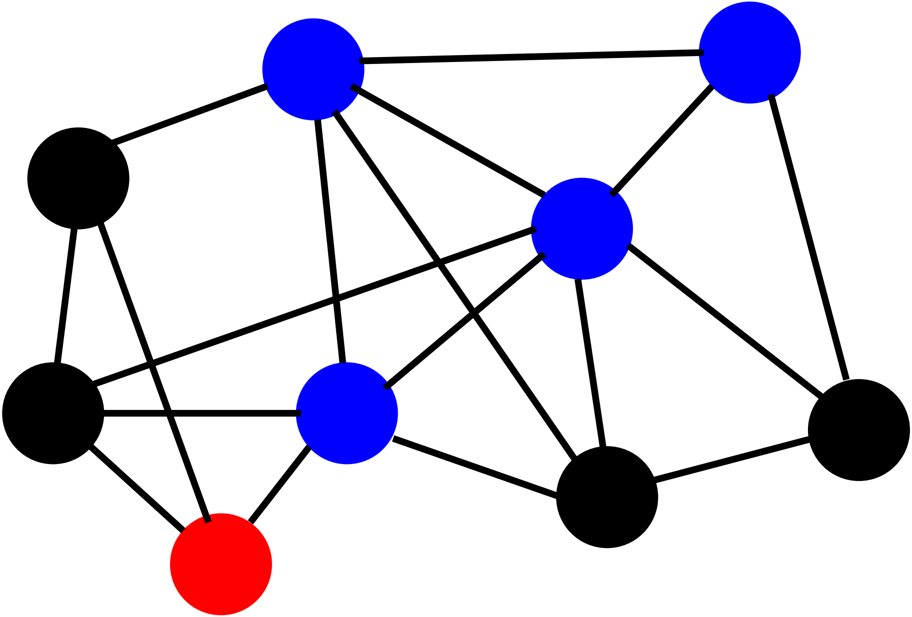

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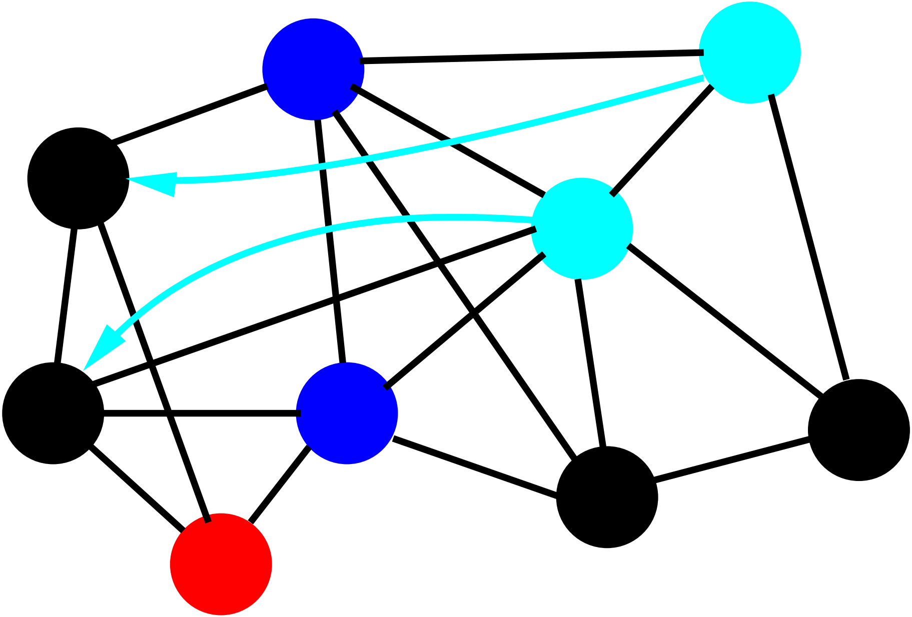

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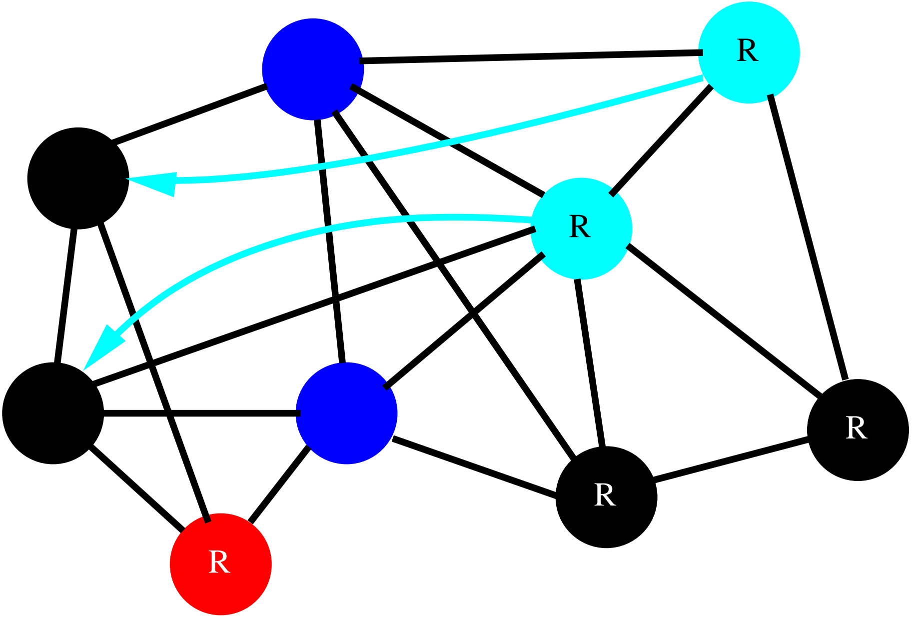

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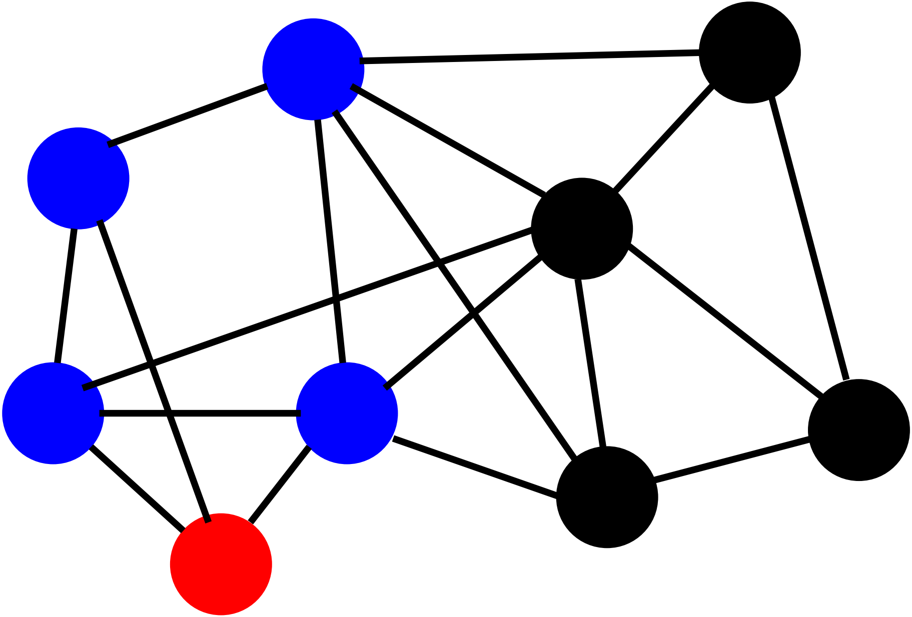

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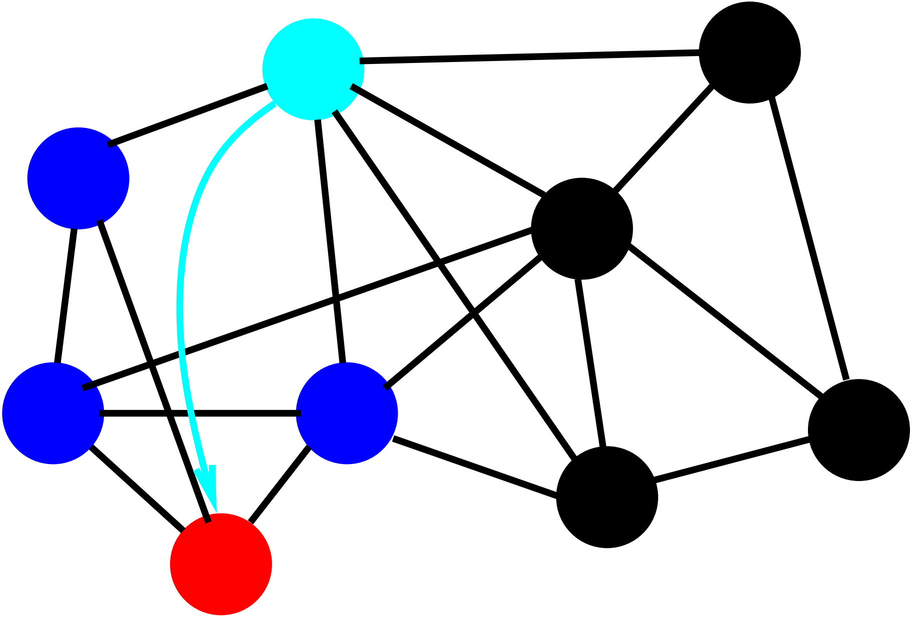

Cops and robber game

Theorem (Seymour and Thomas 1993)

$\mathcal{G}$ has tree-width $\leq k$ if, and only if $k+1$ cops have a winning strategy

Question

What about directed graphs?

Cops and robber game

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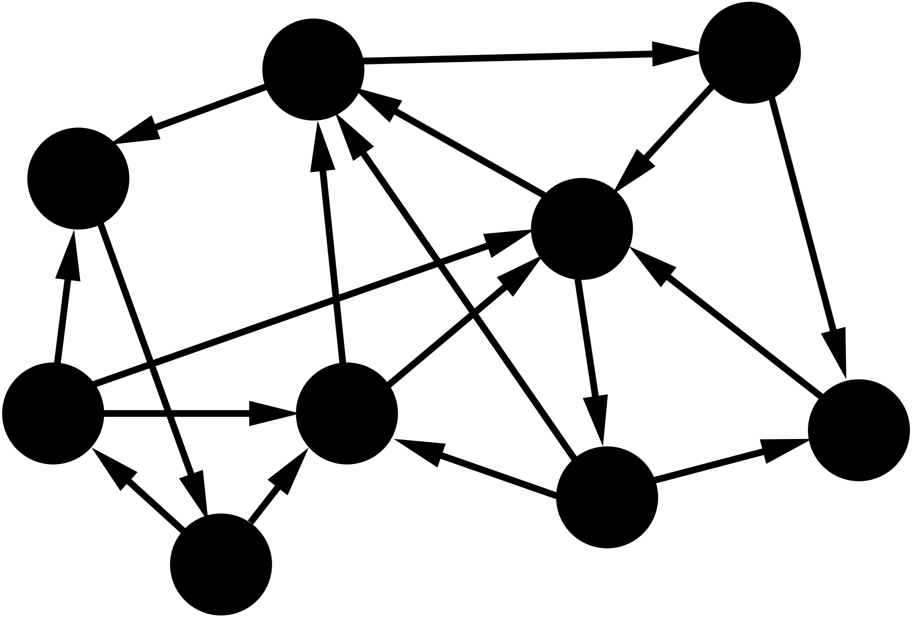

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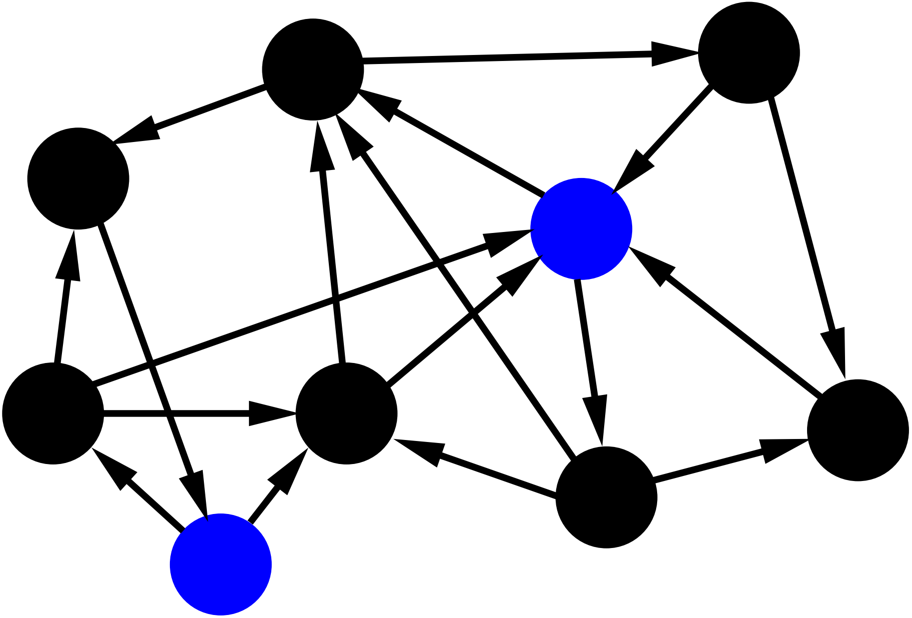

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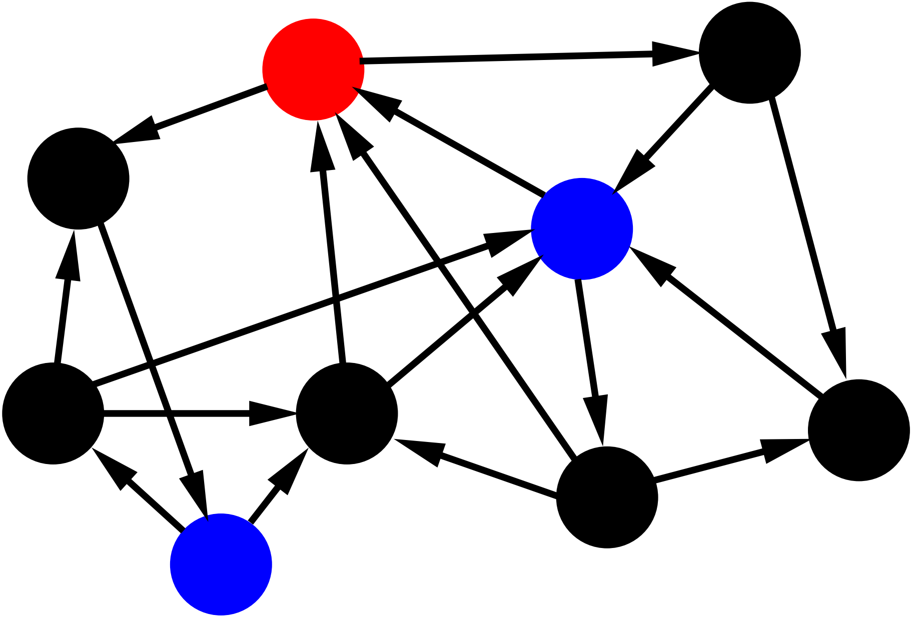

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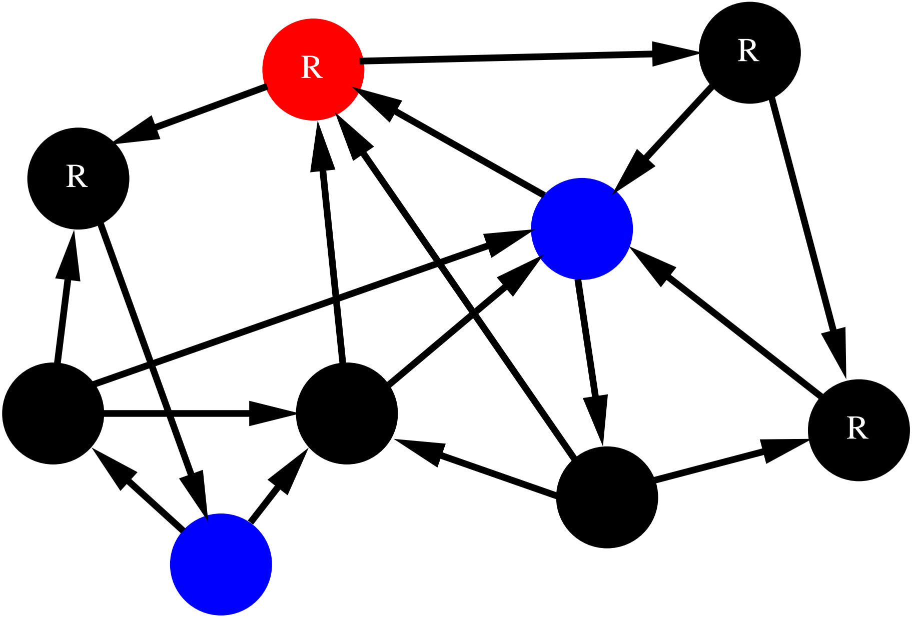

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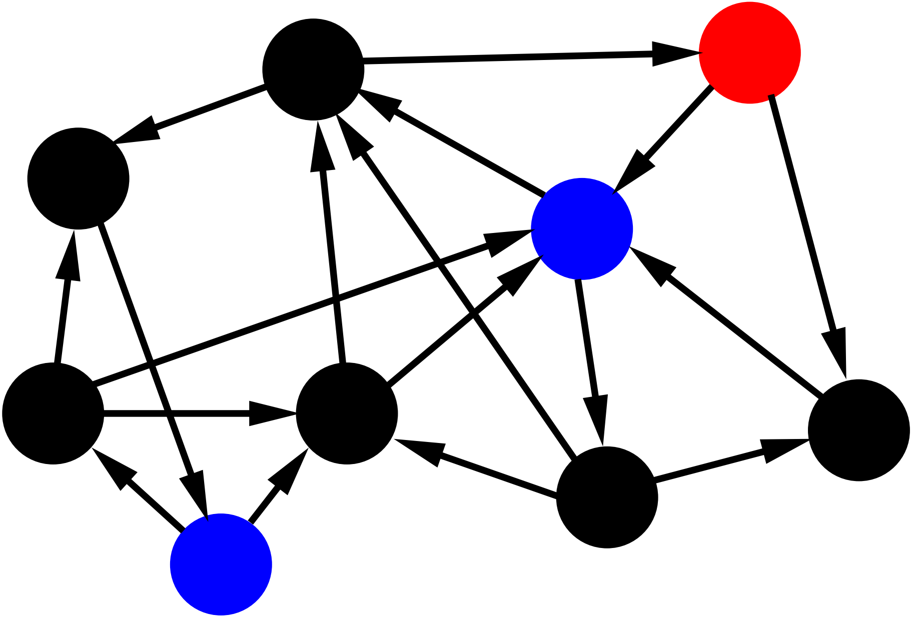

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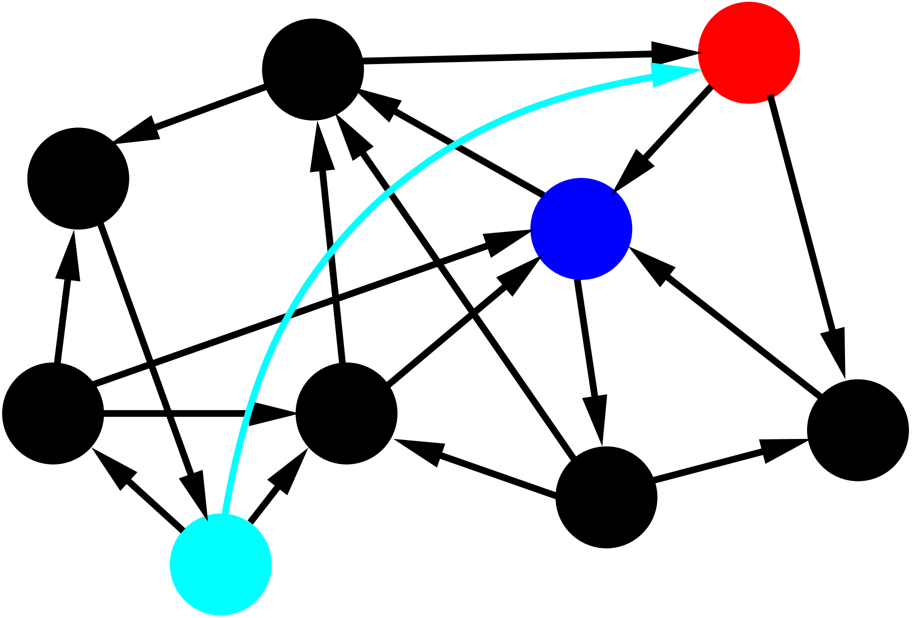

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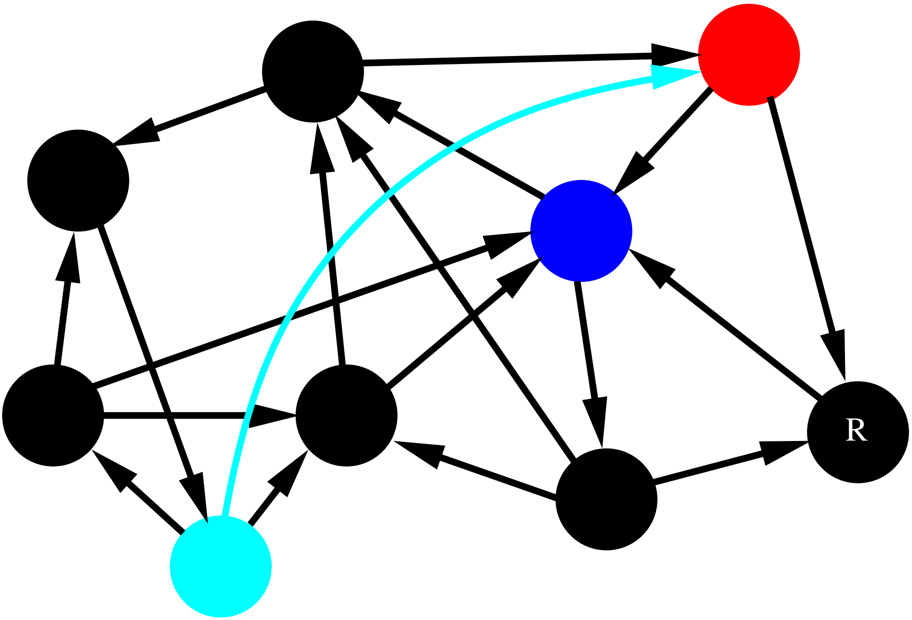

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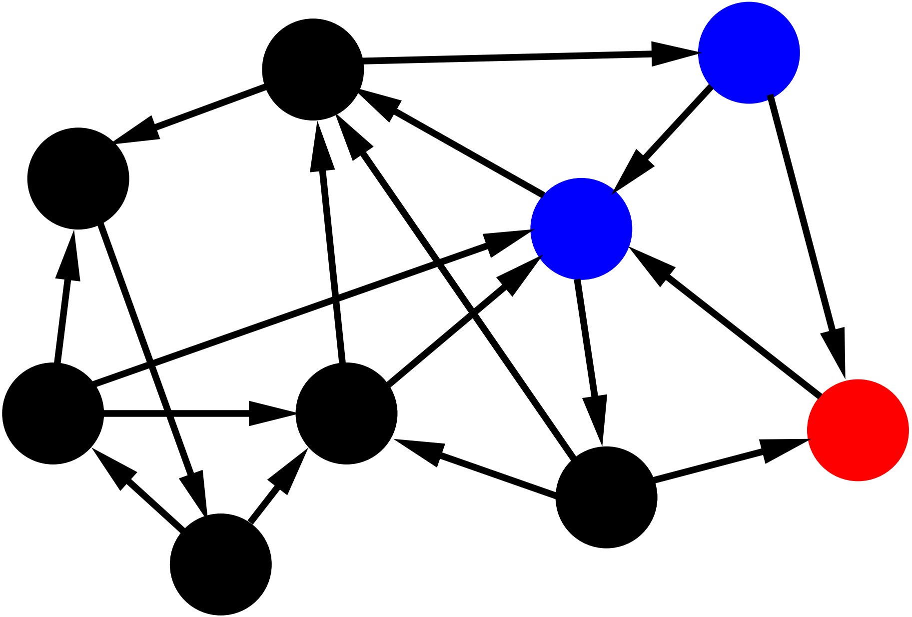

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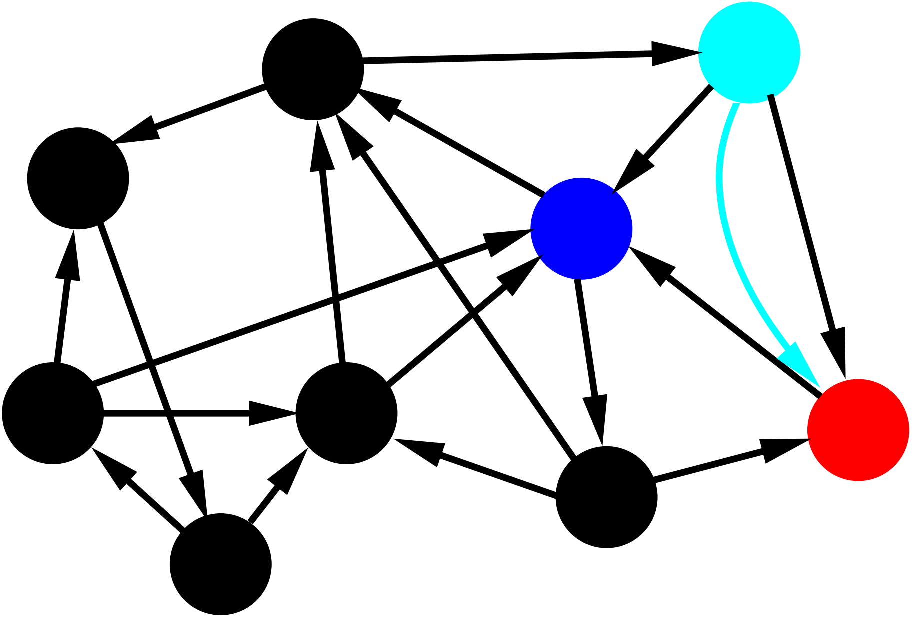

Observations

Let game-width($\mathcal{G}$) be the minimal number of cops required to catch a robber on $\mathcal{G}$.

- directed tree-width($\mathcal{G}$) $\leq$ game-width($\mathcal{G}$) $\leq$ tree-width($\mathcal{G}$)
- game-width($\mathcal{G}$) $= 1$ iff $\mathcal{G}$ is a DAG
- game-width of directed union is maximum width of components
- game-width is **not** preserved under edge reversal

Problem

Find a decomposition that corresponds to game-width

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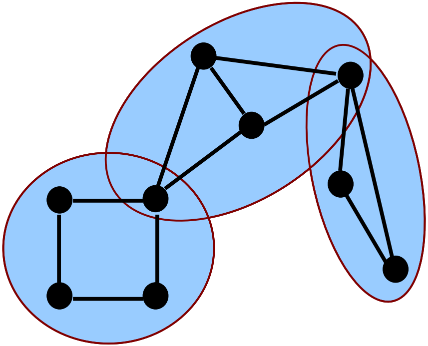
Another observation...

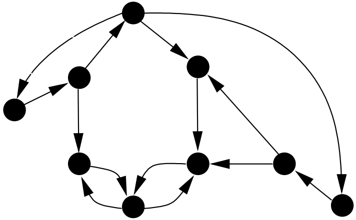
In a tree decomposition, an edge only leaves a subtree through its connection with the rest of the tree



DAG-width

The DAG-width of a directed graph measures its similarity to a DAG.

A graph has DAG-width $\leq k$ if it can be covered by **subsets** of size $\leq k$ in a DAG-like fashion such that an edge only leaves a sub-DAG through its (root's) connection with the rest of the DAG



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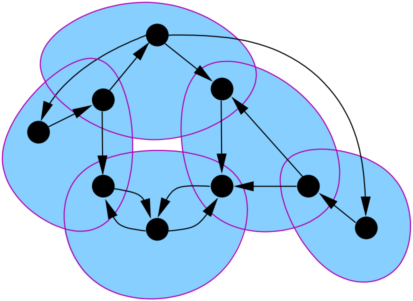

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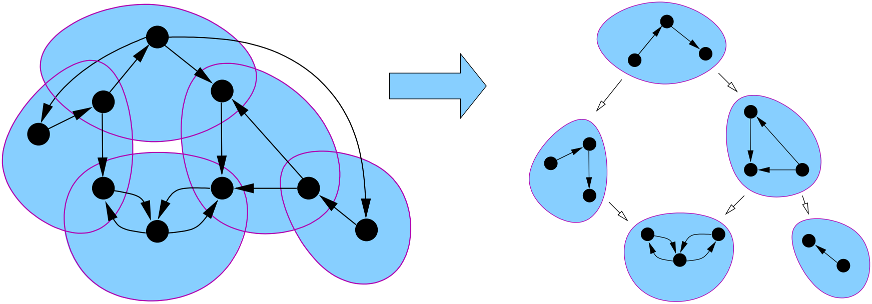

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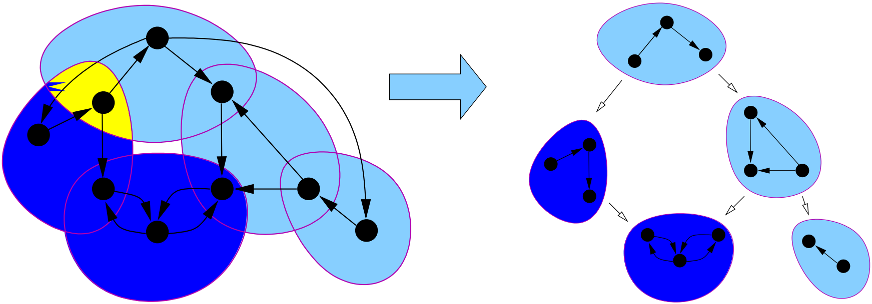

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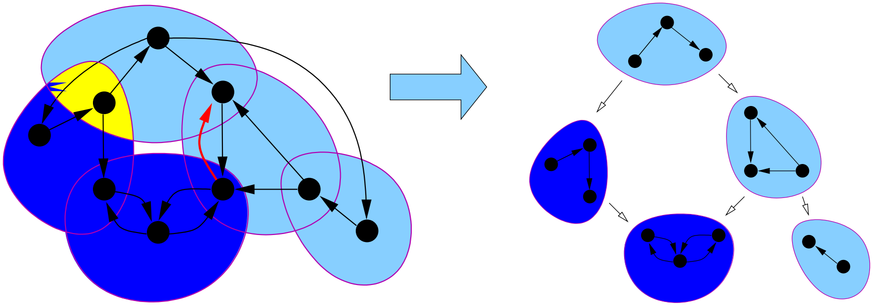
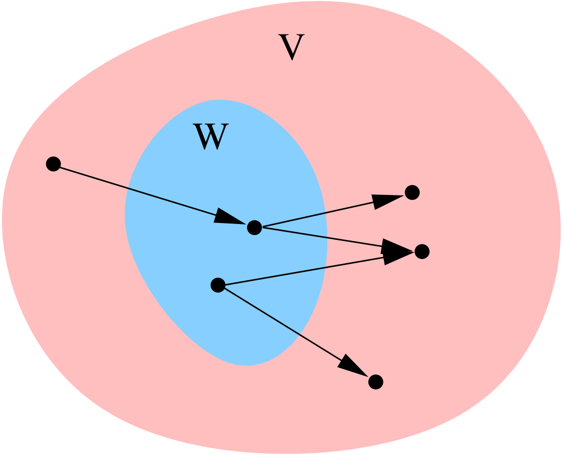


Guarding

Definition

If $\mathcal{G}$ is a directed graph, $W, X \subseteq V(\mathcal{G})$, we say X guards W if every edge which leaves W ends in X.



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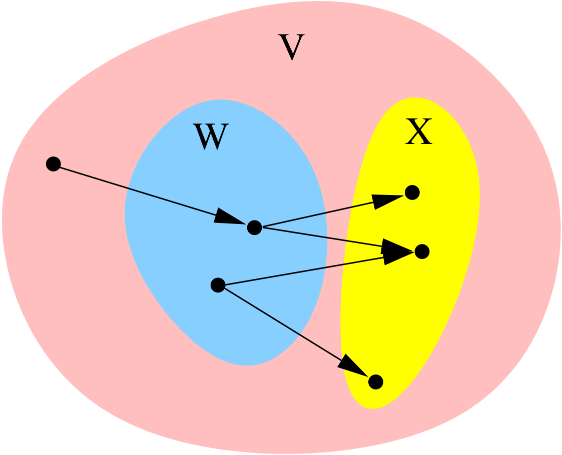

DAG-decompositions and DAG-width

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The DAG-width of a directed graph is the minimal width of all its DAG-decompositions.

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Results

Theorem

$\mathcal{G}$ has DAG-width k if and only if k cops have a monotone winning strategy on $\mathcal{G}$

A monotone strategy is one where every vertex is visited by a cop at most once.

Theorem (Complexity Issues)

- *For fixed k, deciding if $\mathcal{G}$ has DAG-width $\leq k$ is in* PTIME
- *Given $\mathcal{G}$ and k, deciding if $\mathcal{G}$ has DAG-width $\leq k$ is* NP*-complete*

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More results...

- Results from game-width carry over to DAG-width
  - $\text{dtw}(\mathcal{G}) \leq \text{game-width}(\mathcal{G}) \leq \text{DAG-width}(\mathcal{G}) \leq \text{tw}(\mathcal{G})$
  - Directed unions
- $\text{DAG-width}(\mathcal{G}) \leq \text{entanglement}(\mathcal{G}) + 1$
- $\text{DAG-width}(\mathcal{G}) \leq \text{directed path-width}(\mathcal{G})$

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Parity games on graphs of bounded DAG-width can be decided in polynomial time

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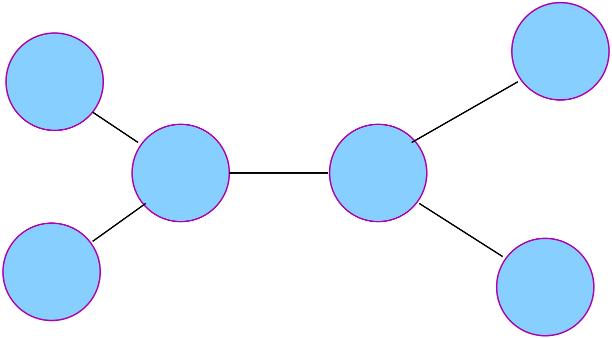
Parity games algorithm

Similar to Obdržálek's algorithm for bounded tree-width

1. An edge (or node) of a tree decomposition separates the graph
2. Positional strategies can then be represented as functions from the interface to itself (border)
3. Compute borders in a bottom-up manner



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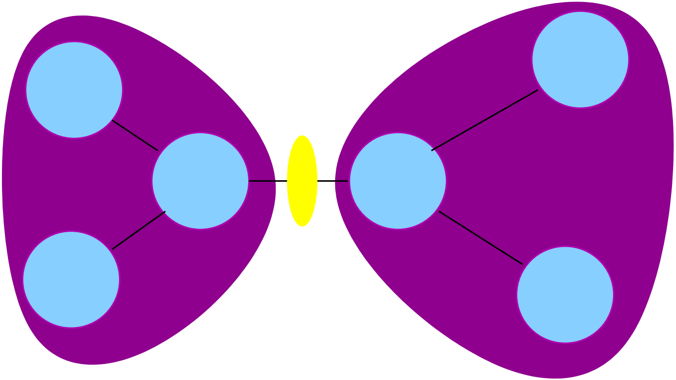

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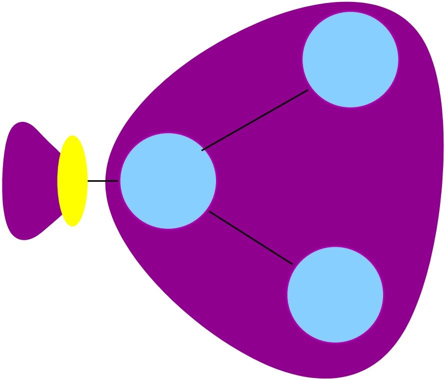
1. An edge (or node) of a tree decomposition separates the graph
2. Positional strategies can then be represented as functions from the interface to itself (border)
3. Compute borders in a bottom-up manner



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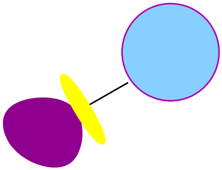
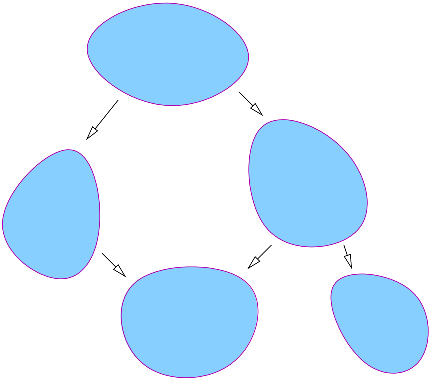


Extension to DAG-decompositions

- Interface covers edges **leaving** sub-DAG
- Problem 1: Handling edges entering sub-DAG
  - Solution: Use functions from sub-DAG to interface (frontier)
- Problem 2: Adding vertices to frontiers



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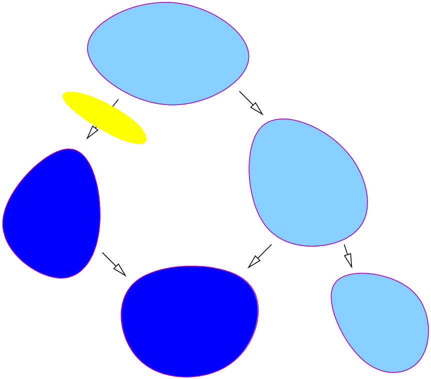

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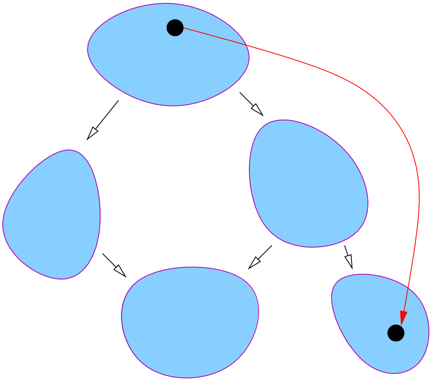

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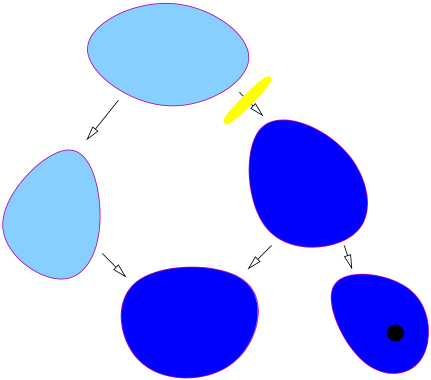

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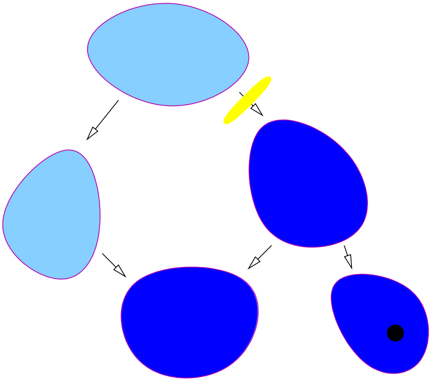

Conclusions and further work

- Introduced a natural extension of tree-width to directed graphs.
- Provided a polynomial-time algorithm for parity games on graphs of bounded DAG-width – subsuming results on bounded tree-width and entanglement.

- Are monotone strategies sufficient?
- Generalisation of havens, brambles, minors, separators?
- Generalisation of Courcelle's theorem?

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