DAG-width and Parity Games

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Joint work with
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Motivation

Tree-width introduced by Robertson and Seymour

- Tree decompositions provide for recursive algorithms
- Bounding tree-width gives polynomial time execution

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Problem

Tree-width ignores direction

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Find a natural extension of tree-width to directed graphs that is algorithmically useful.

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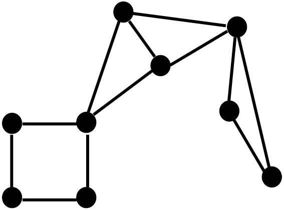
Overview

- Review tree-width
- Cops and robber game
- DAG-decompositions and DAG-width
- An algorithm for parity games
- Further work

Tree-width

The tree-width of a graph measures its similarity to a tree.

A graph has tree-width $\leq k$ if it can be covered by sub-graphs of size $\leq (k+1)$ in a tree-like fashion.



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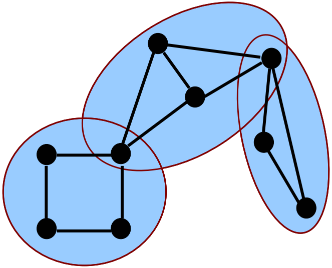

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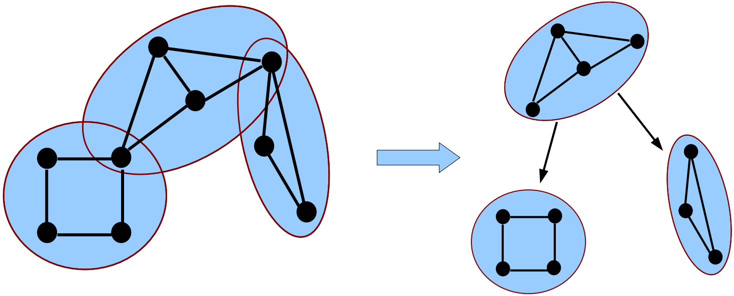

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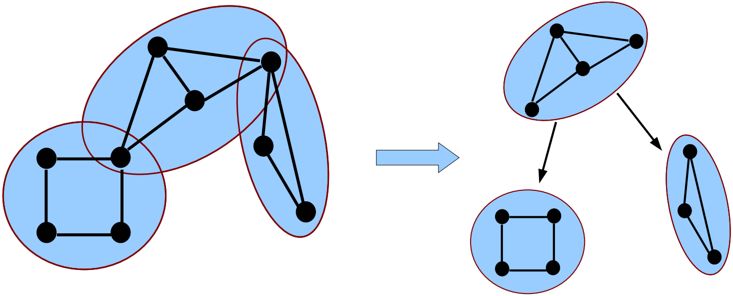

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Cops and robber game

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k Cops

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Robber

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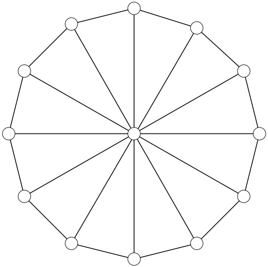

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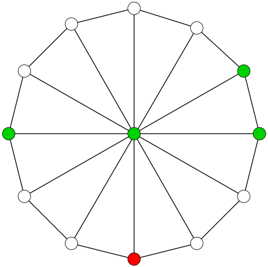

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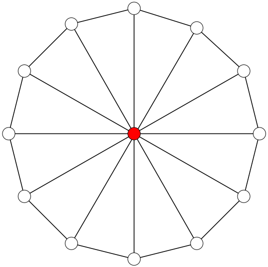

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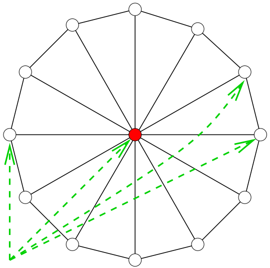

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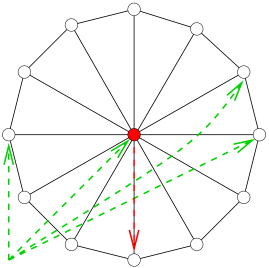

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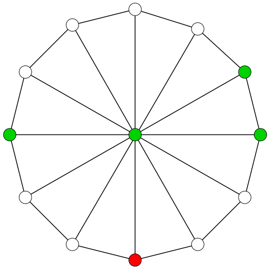

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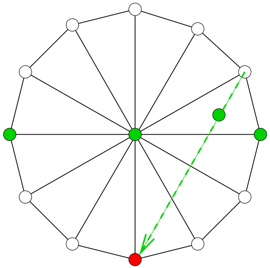

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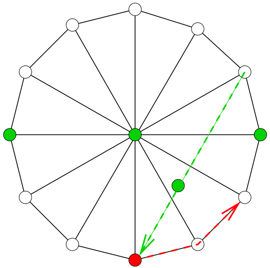

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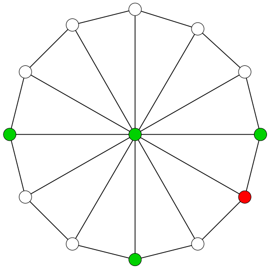

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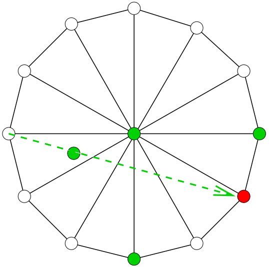

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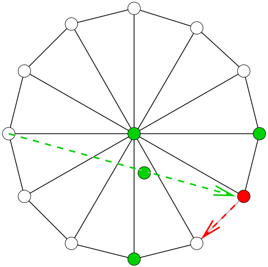

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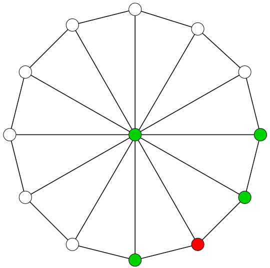

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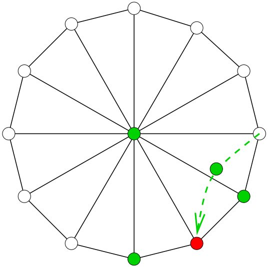

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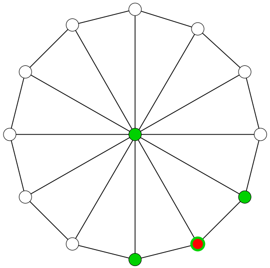

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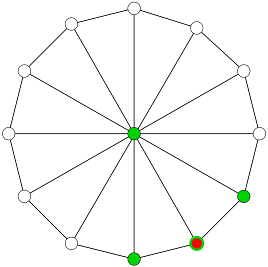

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Cops, robbers and tree-width

Theorem (Seymour and Thomas 1993)

$\mathcal{G}$ has tree-width $\leq k$ if, and only if $k+1$ cops have a winning strategy

Question

What about directed graphs?

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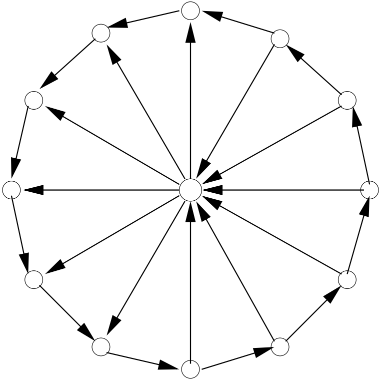

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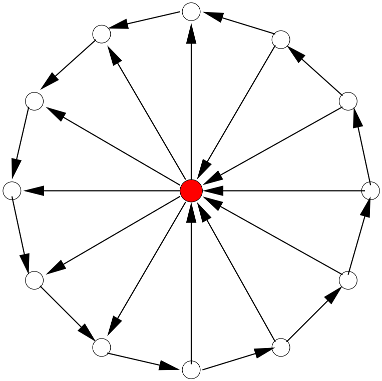

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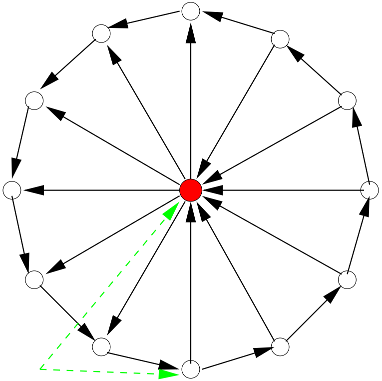

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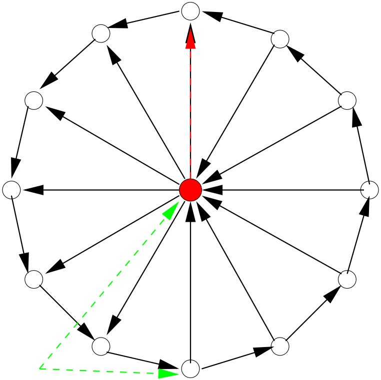

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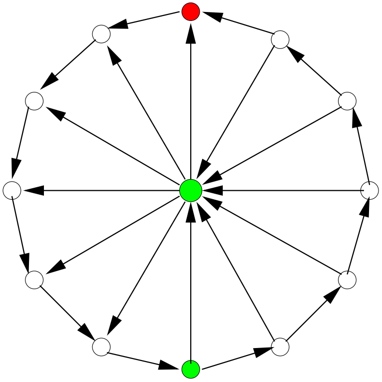

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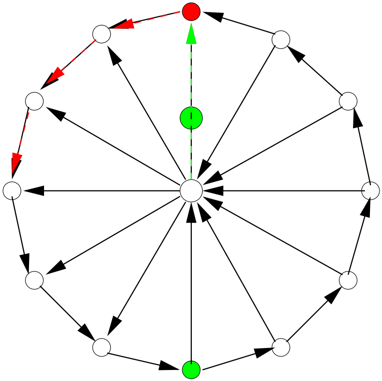

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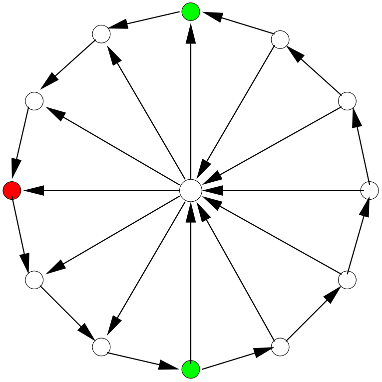

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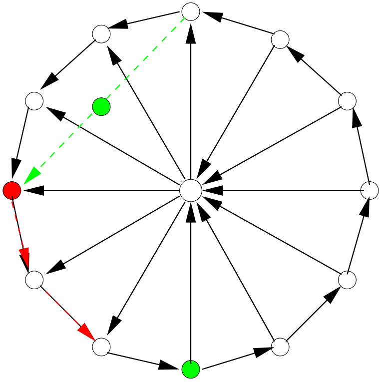

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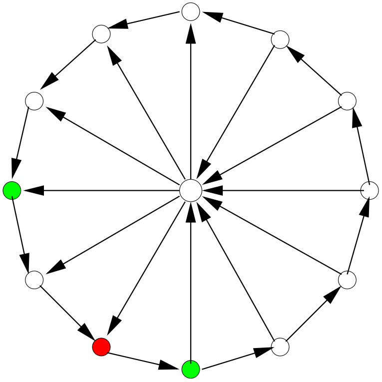

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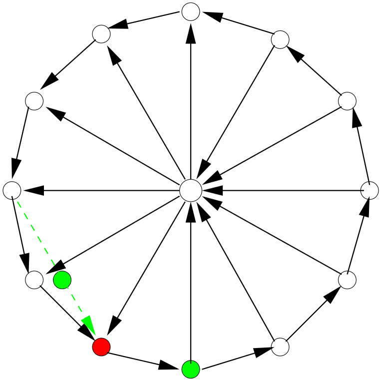

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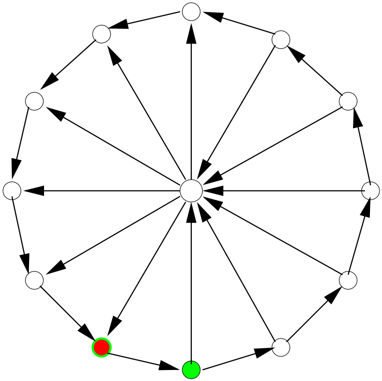

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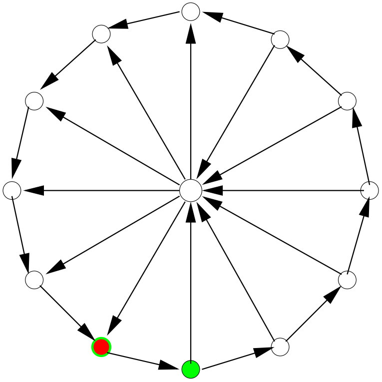

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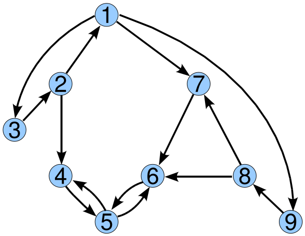
A graph has DAG-width $\leq k$ if it can be covered by **subsets** of size $\leq k$ in a DAG-like fashion such that an edge only leaves a sub-DAG through its (root's) connection with the rest of the DAG



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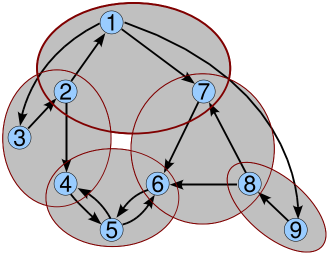

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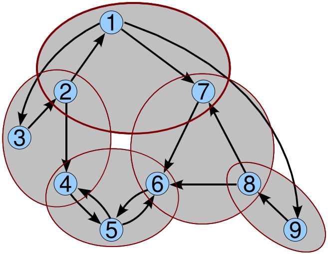

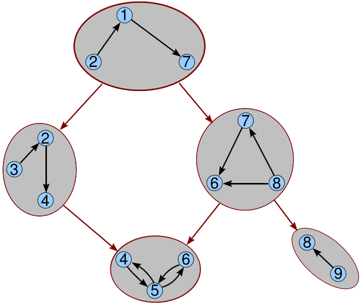

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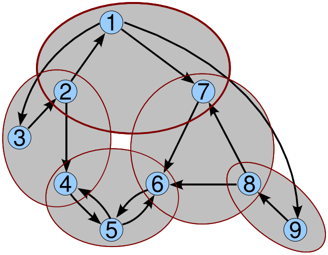

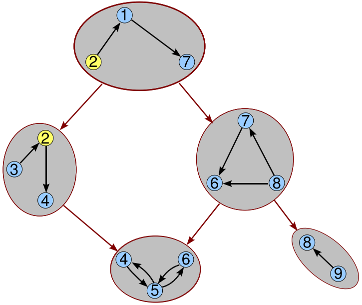

DAG-decompositions and DAG-width

A DAG-decomposition of a directed graph $\mathcal{G}$ is a tuple $(\mathcal{D}, (X_d)_{d \in V(\mathcal{D})})$ such that:

- $\mathcal{D}$ is a DAG
- X_d cover $V(\mathcal{G})$
- For every d' on the path from d to d'' ($d \preceq_{\mathcal{D}} d' \preceq_{\mathcal{D}} d''$), $X_d \cap X_{d''} \subseteq X_{d'}$
- For every $(c, d) \in E(\mathcal{D})$, $X_c \cap X_d$ guards $\left(\bigcup_{d \preceq_{\mathcal{D}} d'} X_{d'}\right) \setminus X_c$. If d is a root of $\mathcal{D}$, we replace X_c with $\emptyset$.

The width of a DAG-decomposition is $\max_{d \in V(\mathcal{D})} |X_d|$.
The DAG-width of a directed graph is the minimal width of all its DAG-decompositions.

Results

Theorem

$\mathcal{G}$ has DAG-width k if and only if k cops have a monotone winning strategy on $\mathcal{G}$

A monotone strategy is one where every vertex is visited by a cop at most once.

Theorem (Complexity Issues)

- *For fixed k, deciding if $\mathcal{G}$ has DAG-width $\leq k$ is in* $\text{P}\small{\text{TIME}}$
- *Given $\mathcal{G}$ and k, deciding if $\mathcal{G}$ has DAG-width $\leq k$ is* NP*-hard*

Results

Theorem

$\mathcal{G}$ has DAG-width k if and only if k cops have a monotone winning strategy on $\mathcal{G}$

A monotone strategy is one where every vertex is visited by a cop at most once.

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More results...

- $\mathrm{dtw}(\mathcal{G}) \leq \text{DAG-width}(\mathcal{G}) \leq \mathrm{tw}(\mathcal{G})$
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Theorem

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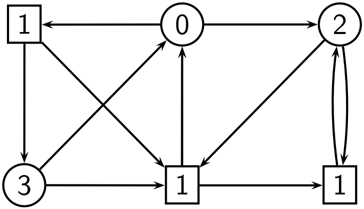
Parity games

Players: *Player 0* and *Player 1*
Arena: (V, E, V_0, V_1, Ω), where

- (V, E) is a directed graph
- V_0 and V_1 partition V
- $\Omega : V \to \mathbb{N}$ priority function

Players move a token around the graph for possibly infinitely many moves

Winner is determined by minimum priority seen infinitely often



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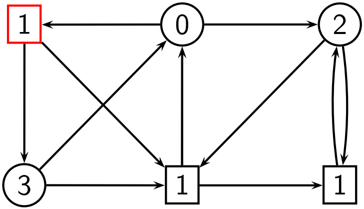

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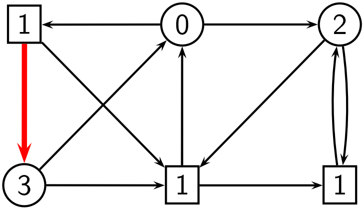

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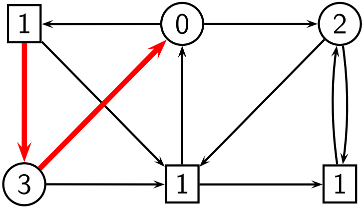

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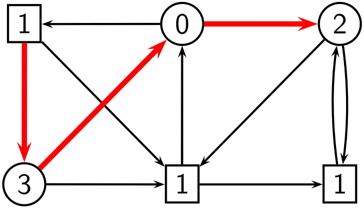

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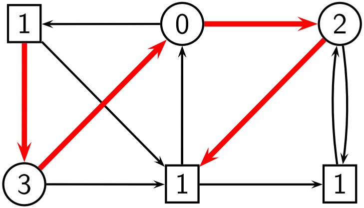

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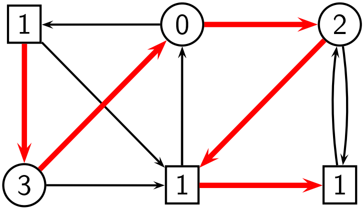

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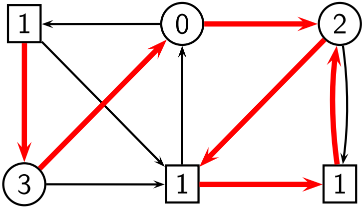

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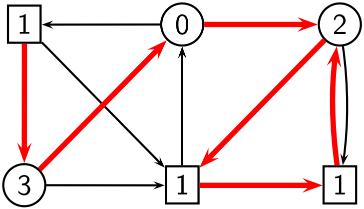

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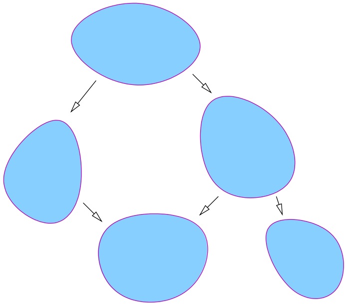
Parity game results

Polynomial-time equivalent to μ-calculus model checking

Decidable in $\mathrm{NP} \cap \textsc{co}\text{-}\mathrm{NP}$

Decidability in $\textsc{Ptime}$ an open problem

Parity games algorithm



1. **Compute DAG-Decomposition**
2. **Use structure to succinctly represent all plays in subgraphs**
   - $\text{result}_f(U, v)$ is all possible outcomes when Player 0 plays f from v in U
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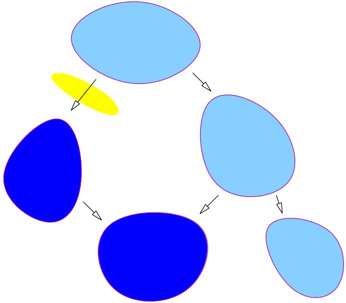

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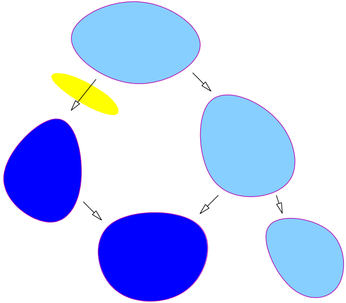

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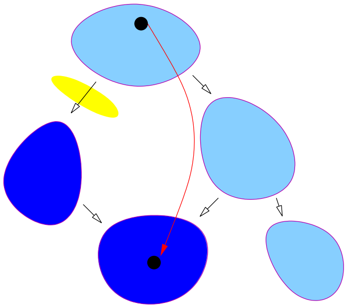

Major problem: Edges entering the sub-DAG

- Cannot "forget" vertices
- Exponential number of strategies generated

Solutions:

- Consider functions from sub-DAG to border
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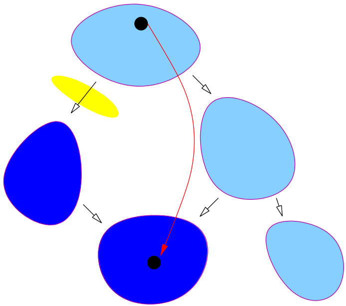

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Conclusions and further work

- Introduced a natural extension of tree-width to directed graphs.
- Provided a polynomial-time algorithm for parity games on graphs of bounded DAG-width – subsuming other results such as bounded tree-width.

- Are monotone strategies sufficient?
- Generalisation of havens, brambles, minors, separators?
- Generalisation of Courcelle's theorem?

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