Strategy Improvement for Parity Games

A Combinatorial Perspective

Paul Hunter

Outline

- Parity Games
- Strategy Improvement Algorithm
- Completely Unimodal Hypercubes
- Results – Old and New
- Conclusion

Parity Games

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- Two player, zero-sum, non-cooperative, infinite game.
- Played on a finite, directed graph (V, E).
  - Bi-partite
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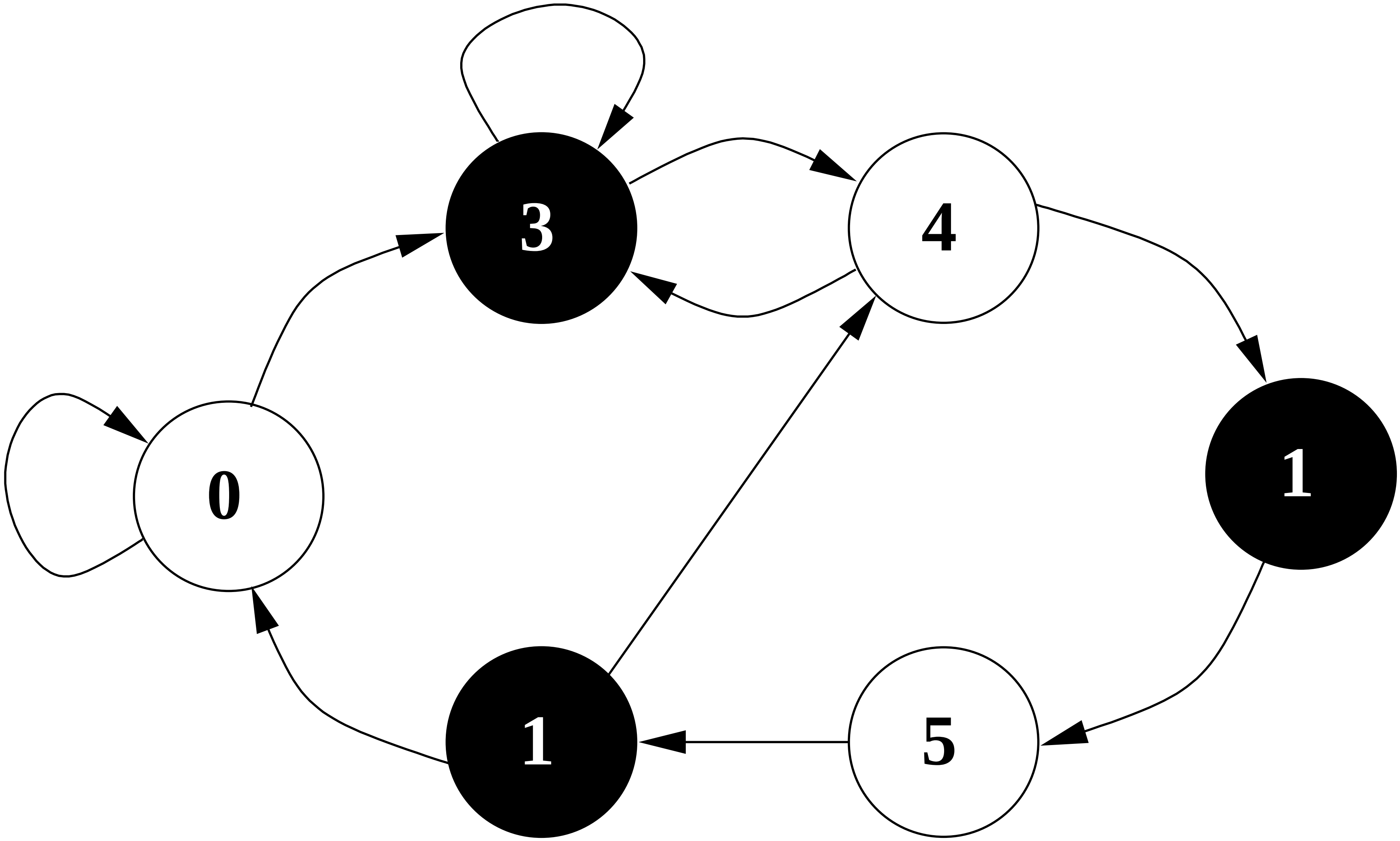
Players (Player 0 and Player 1) alternately move a token around the graph for an infinite number of turns, generating an infinite sequence S of vertices visited. Winner is determined by a <u>parity condition</u>:

- Priority function $\chi : V \to \mathbb{P} \qquad (\mathbb{P} \leq \omega)$
- Player 0 wins if and only if $\max_{v \in S} \chi(v)$ is even.

Parity Games – Example



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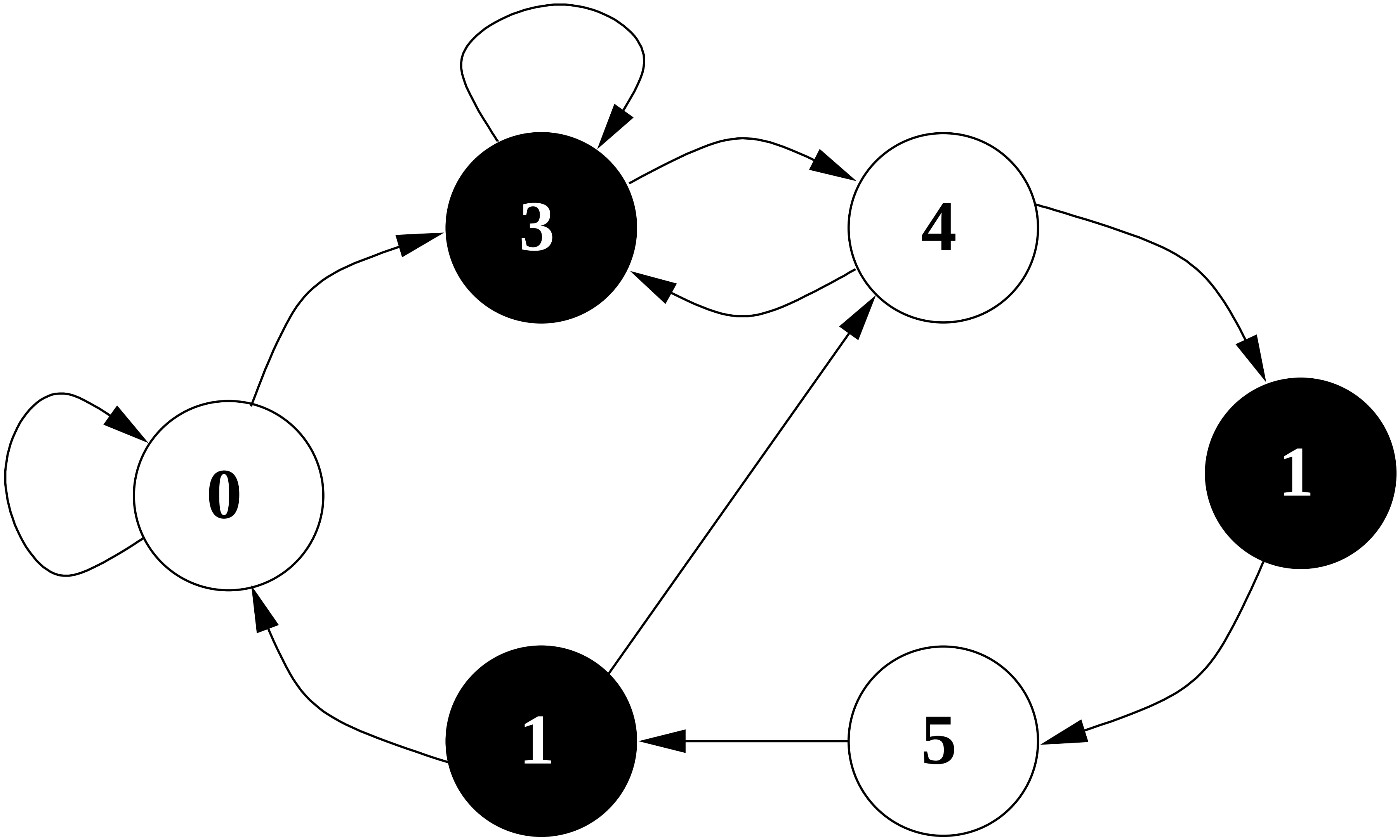
- Polynomially equivalent to μ-calculus model checking
- Whichever player has a winning strategy has a positional (memoryless) winning strategy

Parity Games – Winning strategy



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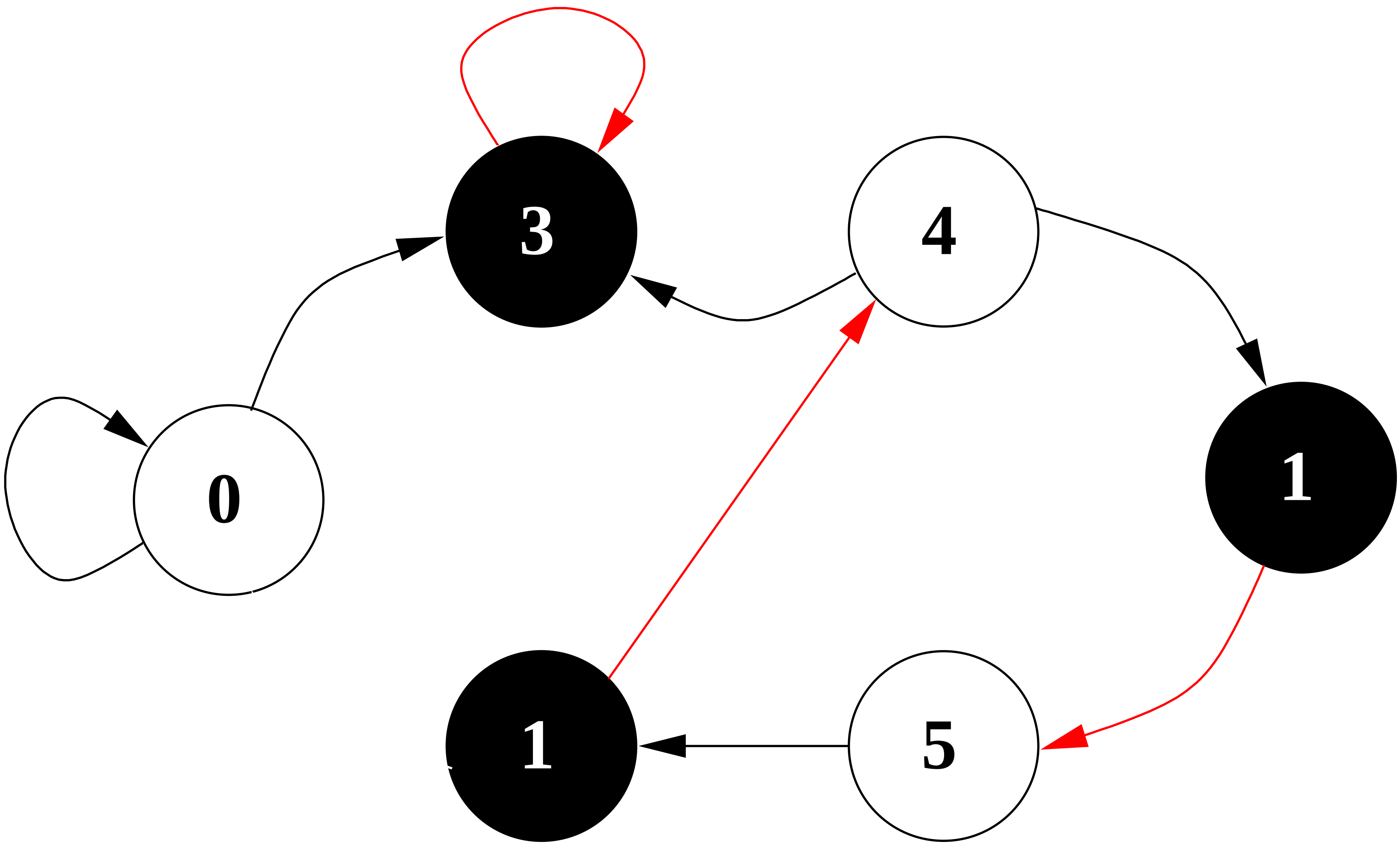

Parity Games – Complexity

Memoryless strategies imply that deciding Parity games is in NP $\cap$ co-NP.

Open problem: *Is deciding Parity games in P?*

Best known algorithm (Jurdziński 2000)

$$O\left(d|E|\left(\frac{|V|}{\lfloor d/2 \rfloor}\right)^{\lfloor d/2 \rfloor}\right)$$

where d is the number of priorities.

Recent approach is <u>strategy improvement</u>.

Strategy Improvement

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Introduced by Vöge and Jurdziński, 2000.

Works by "improving" memoryless strategies until optimum is reached.

Naïve time complexity analysis gives $O(|V||E|2^{|V|})$ upper bound, but no known example worse than linear time has been found!

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Question: *What is the exact complexity of this algorithm?*

Strategy Improvement – Valuations

A valuation is a function

$$\varphi : V \to \mathbb{P} \times \mathcal{P}(\mathbb{P}) \times \omega$$

which assigns to each vertex:

- A loop priority
- A set of priorities, and
- A natural number

Intuitively, a valuation corresponds to a “best-play” counter-strategy.

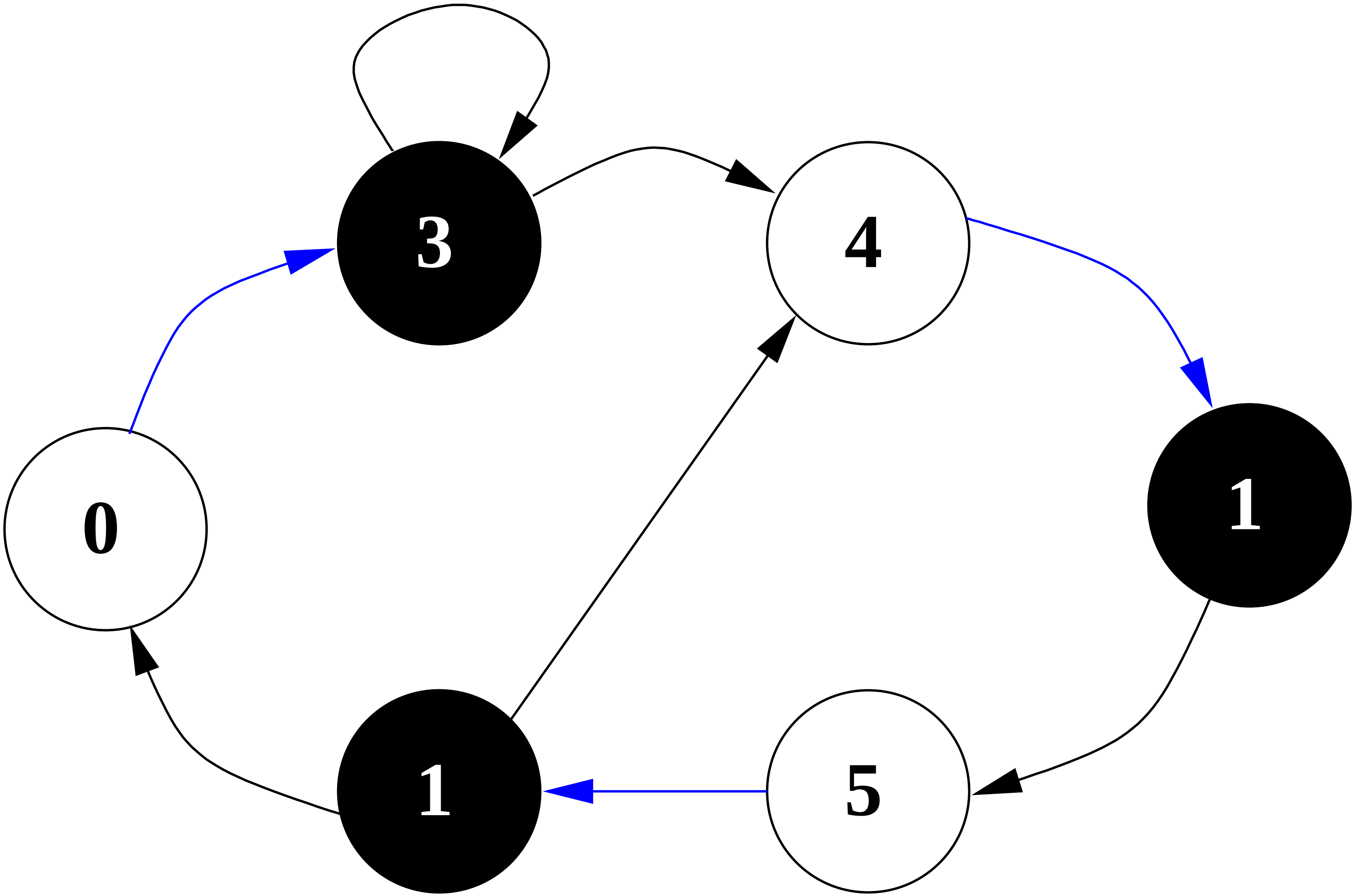
Strategy Improvement – Valuations

We can partially order valuations lexicographically according to what is best for Player 1

- High even priorities $\preceq$ Low even $\preceq$ Low odd $\preceq$ High odd
- For sets P and Q, $P \prec Q$ if $\max(P\Delta Q)$ is odd and in Q or even and in P
- Path lengths depend on the loop priority – short paths are better if the loop priority is odd

A $\preceq$-maximal valuation is 1-optimal.

Valuation example



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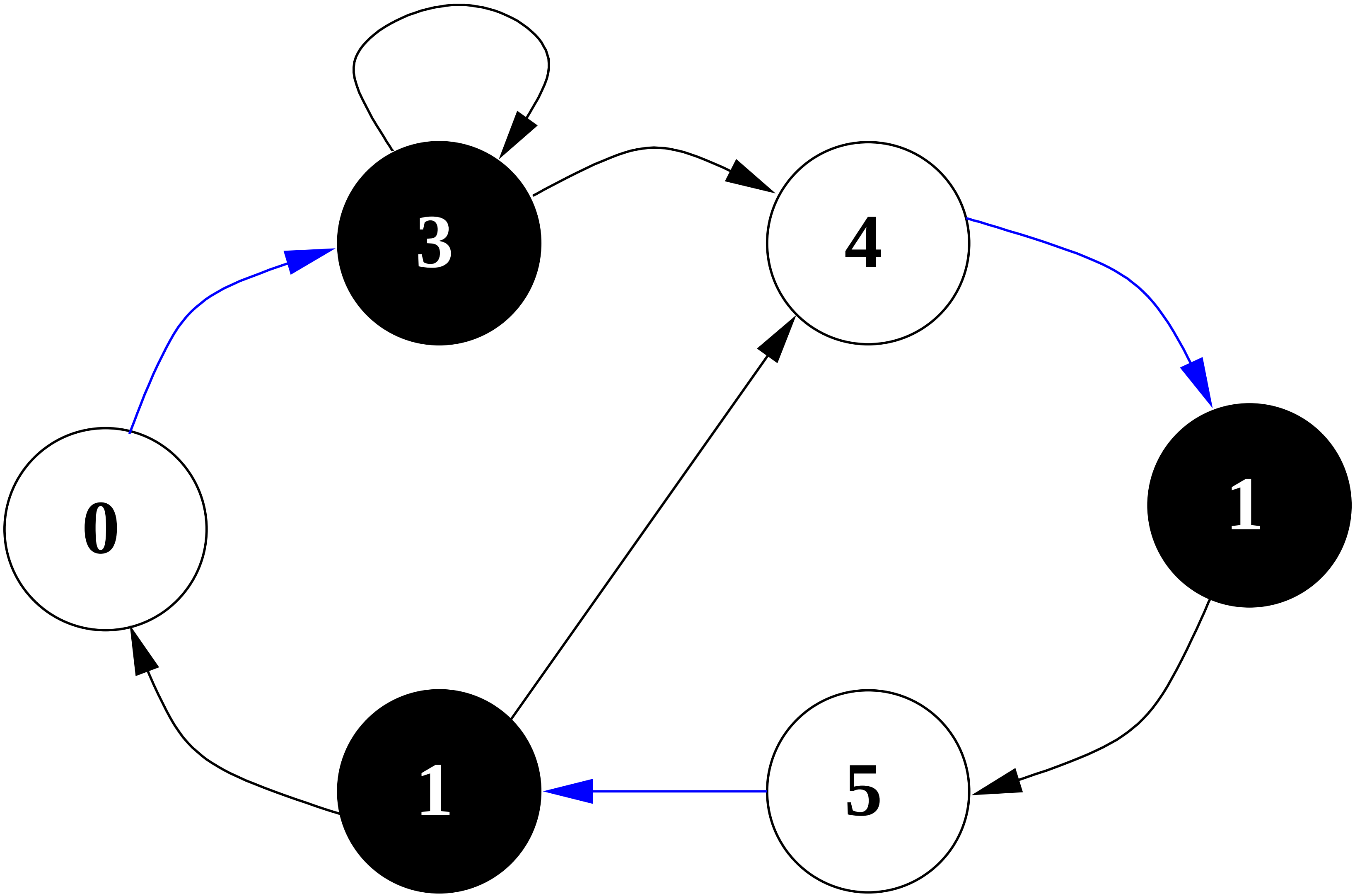
At this point σ is the best Player 0 can do, so it is straightforward to determine each player's winning sets.

Note that we are changing the strategy at different vertices simultaneously.

Strategy Improvement – Example



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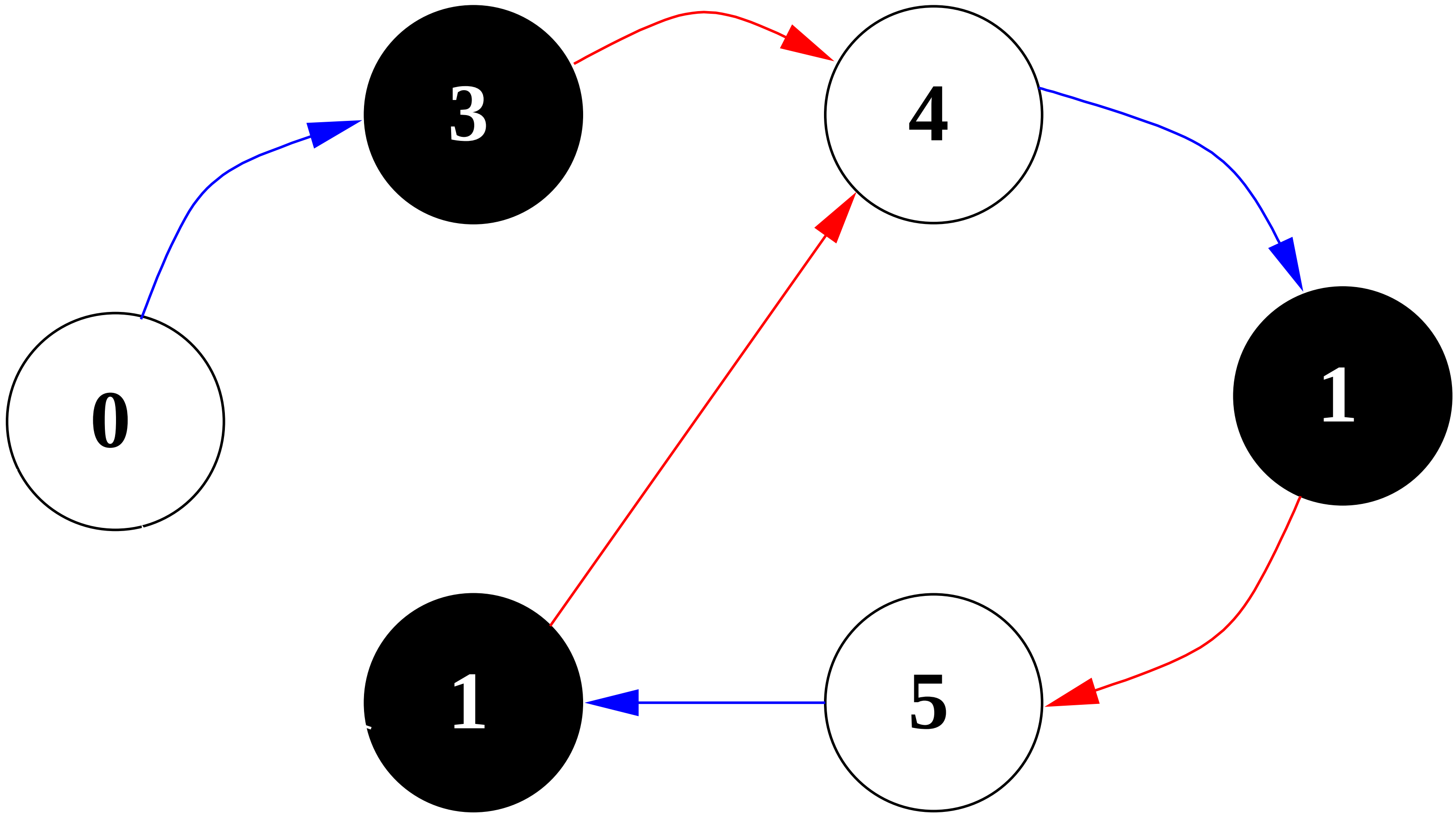

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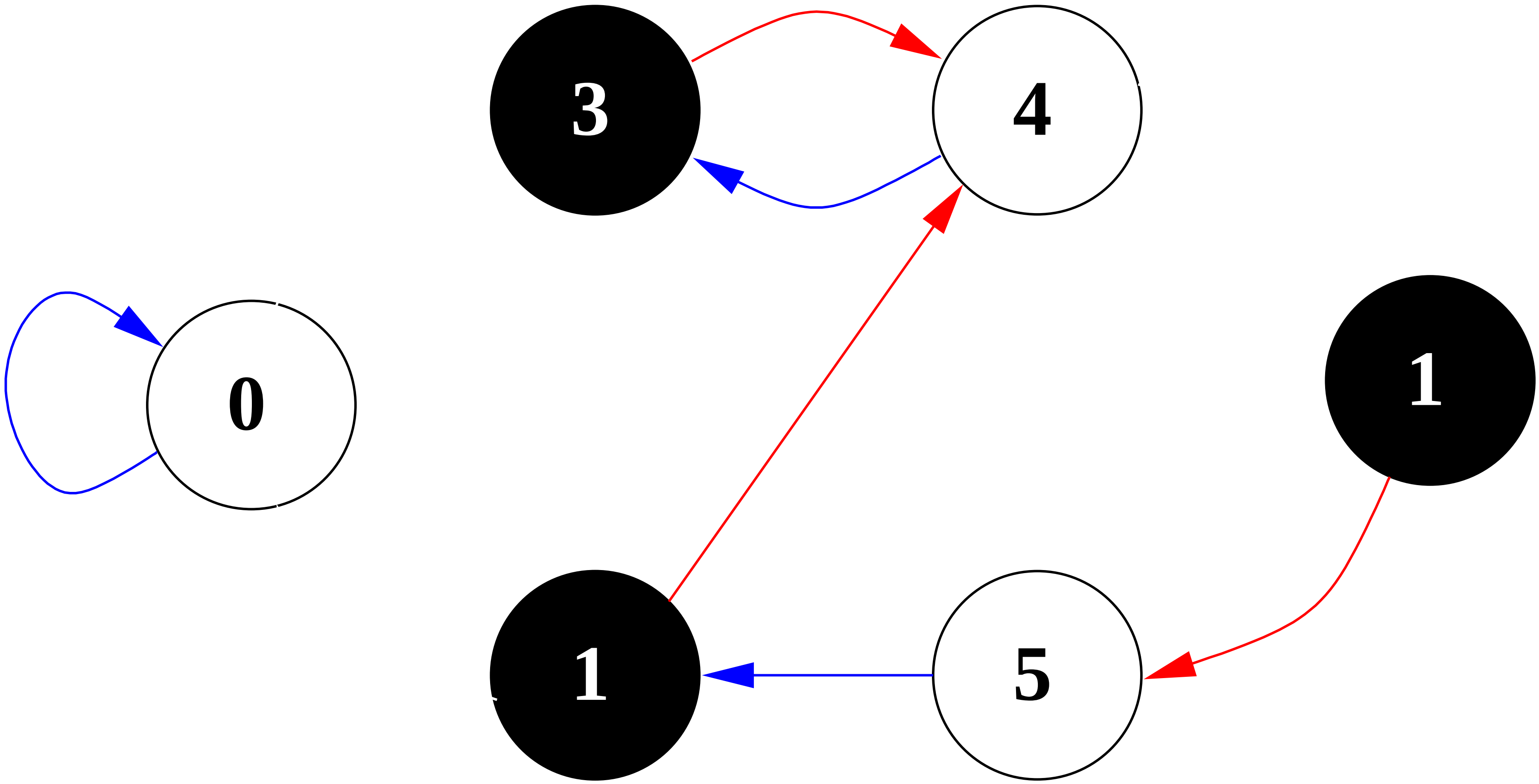

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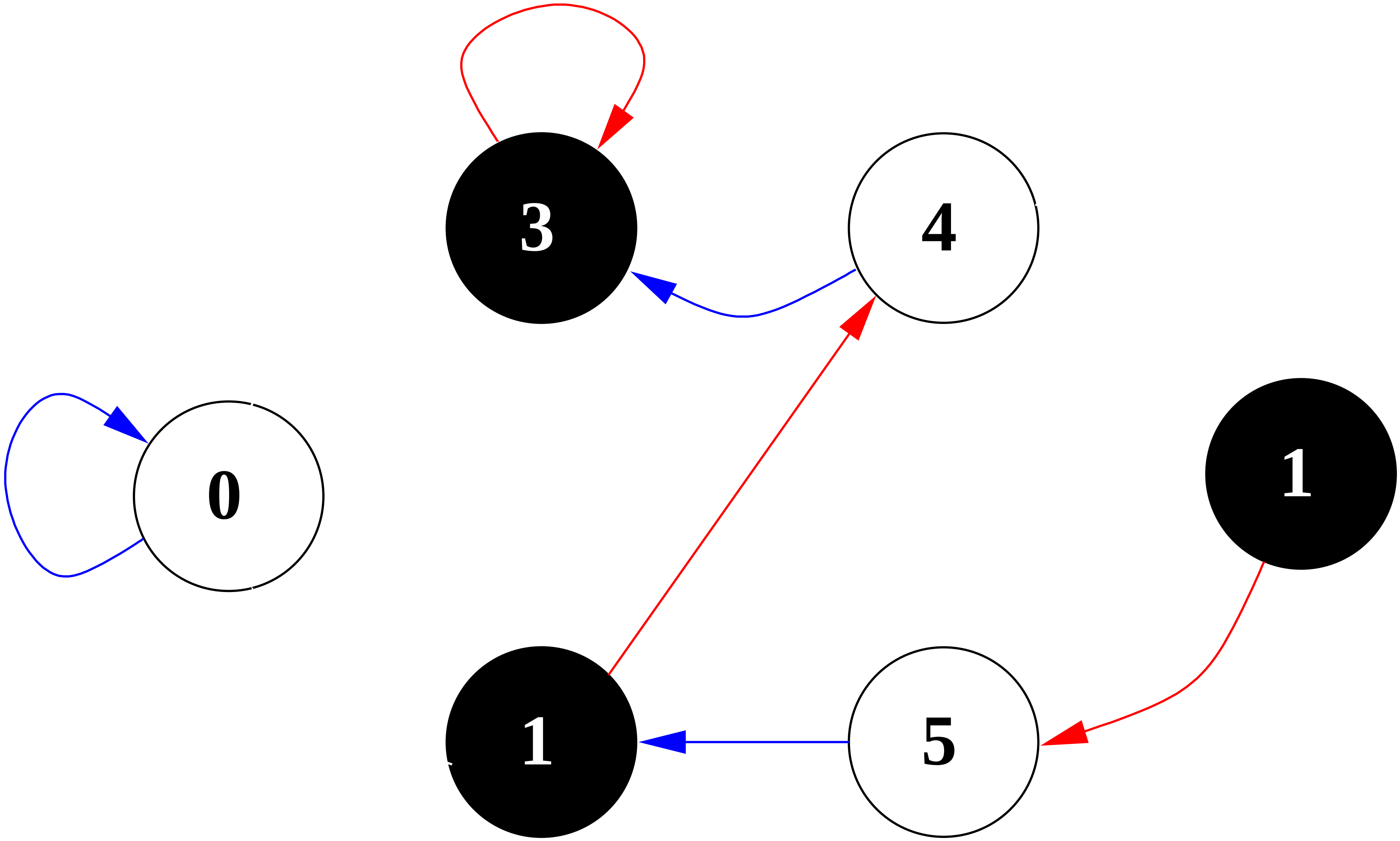
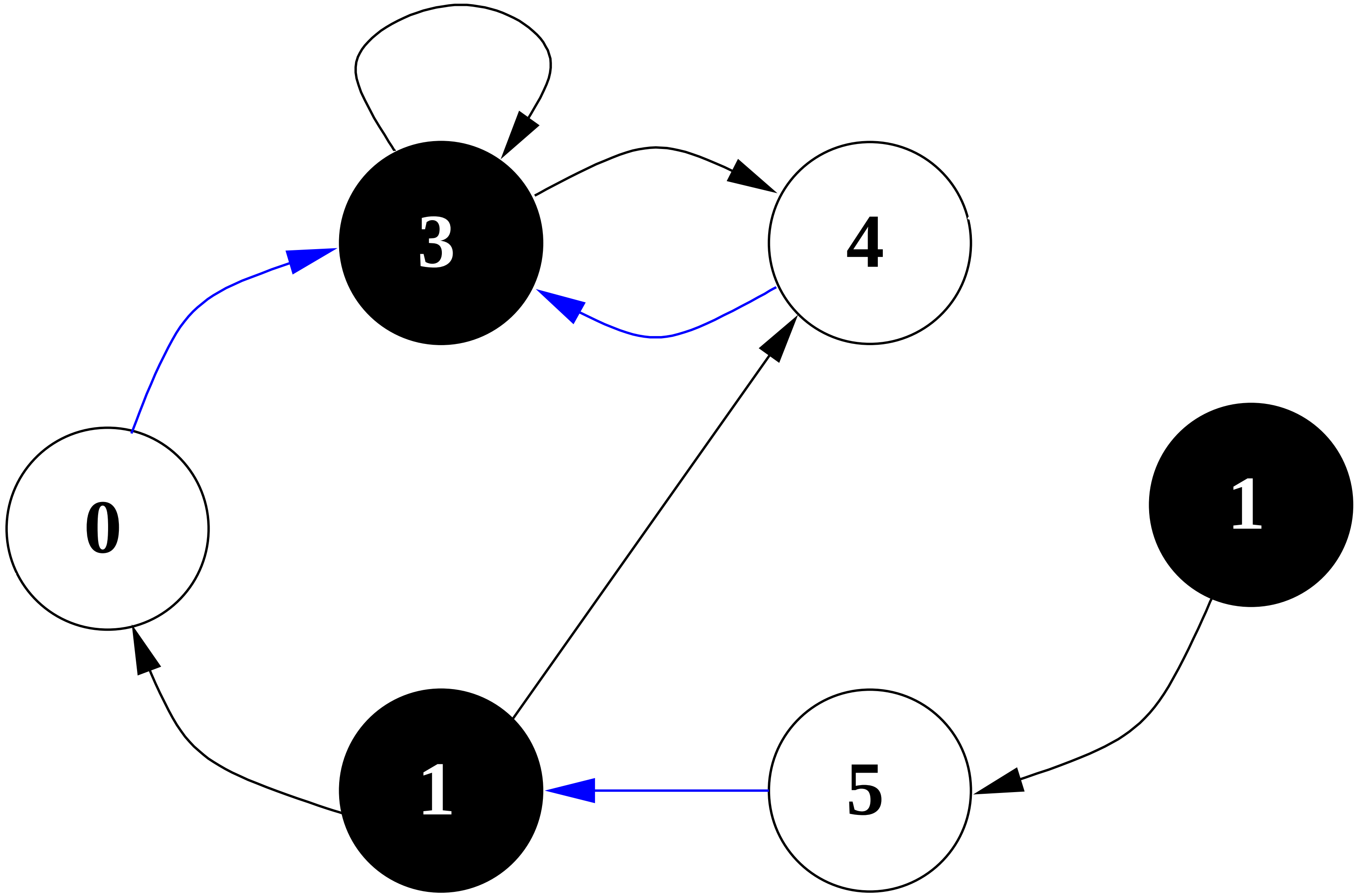


Strategy Improvement – Comments

Inherent asymmetry in algorithm. We can extract a strategy from a valuation, so why not compute a 0-optimal valuation and use this to improve σ?

Strategy Improvement – Asymmetry



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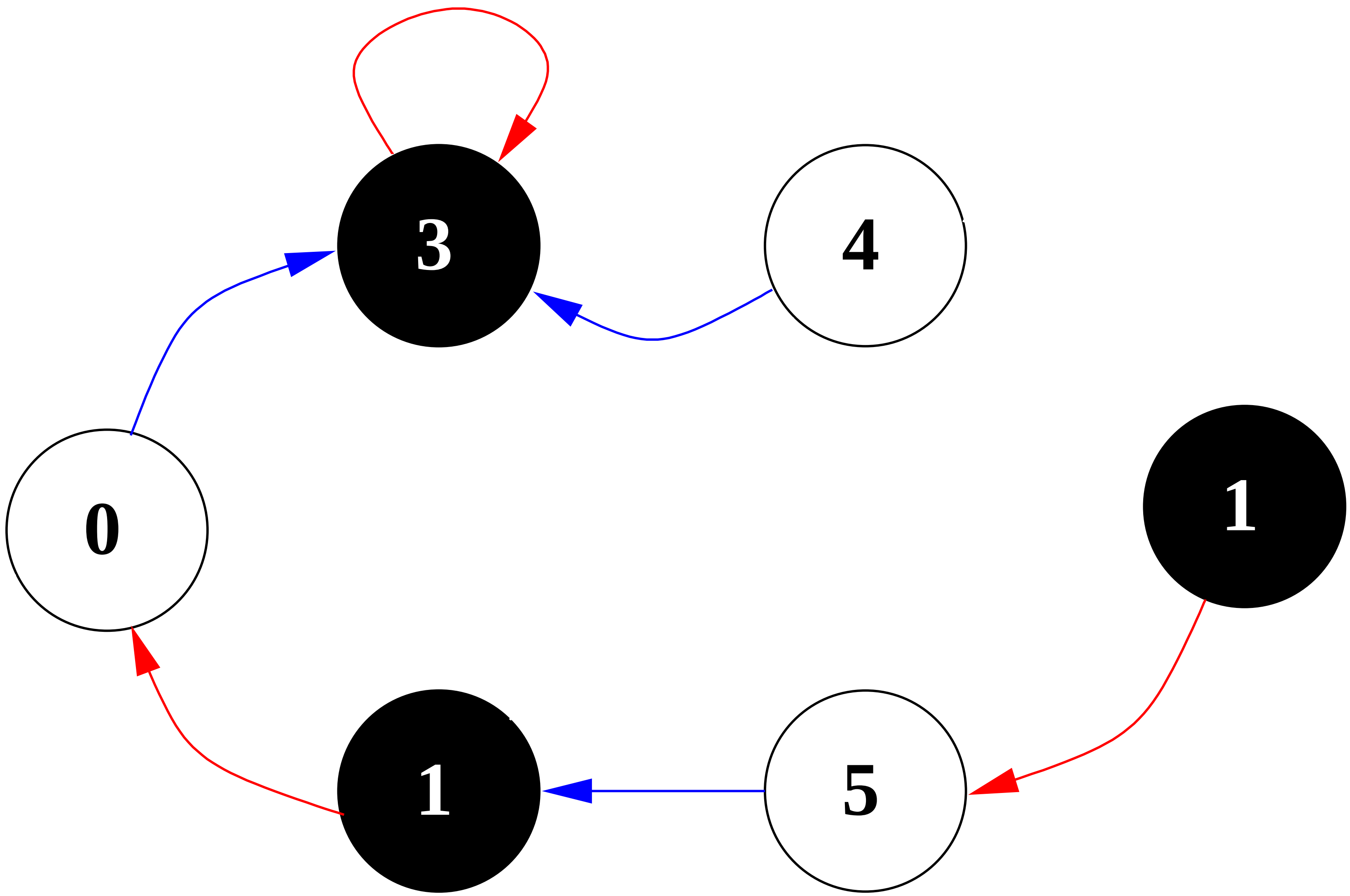

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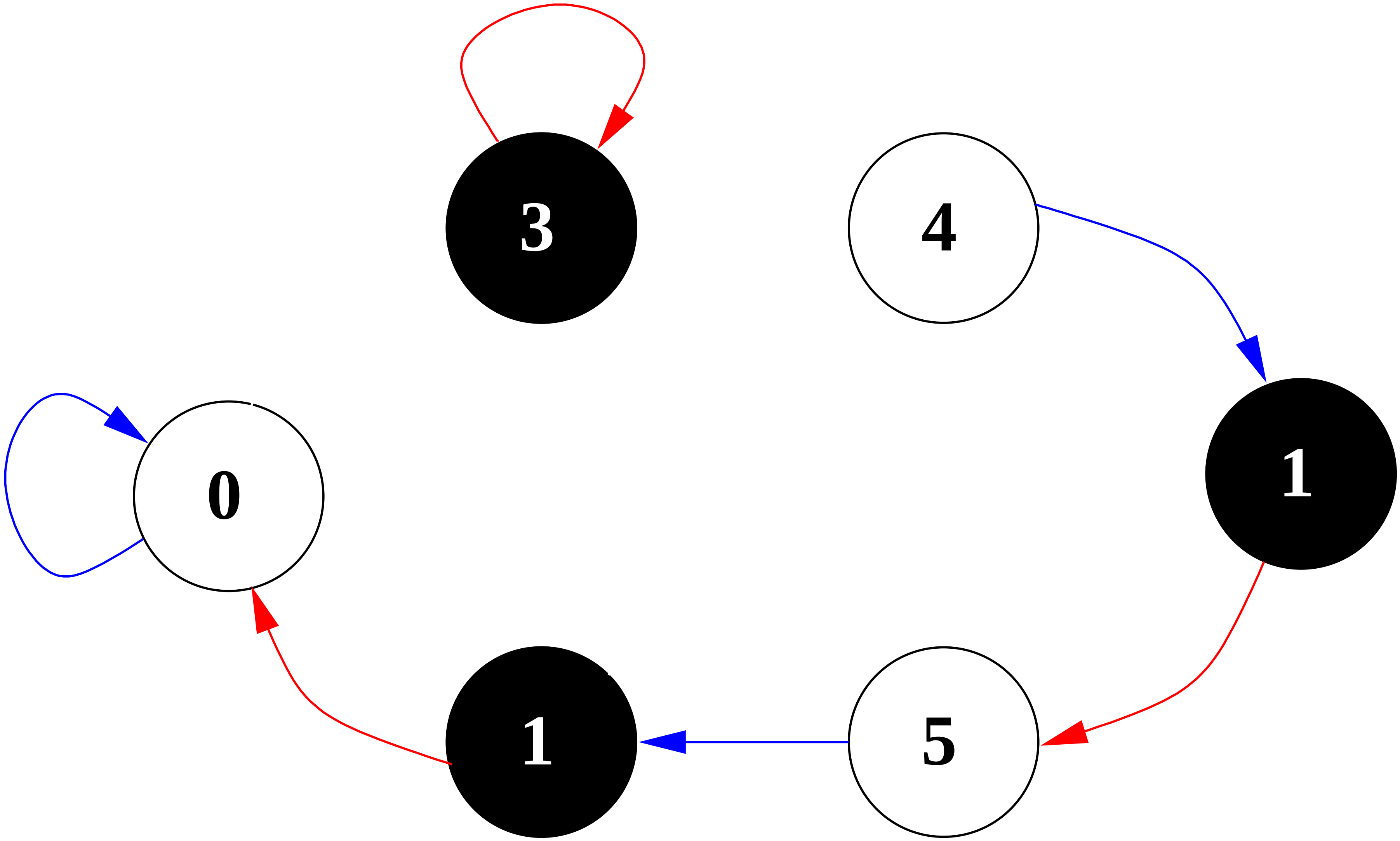

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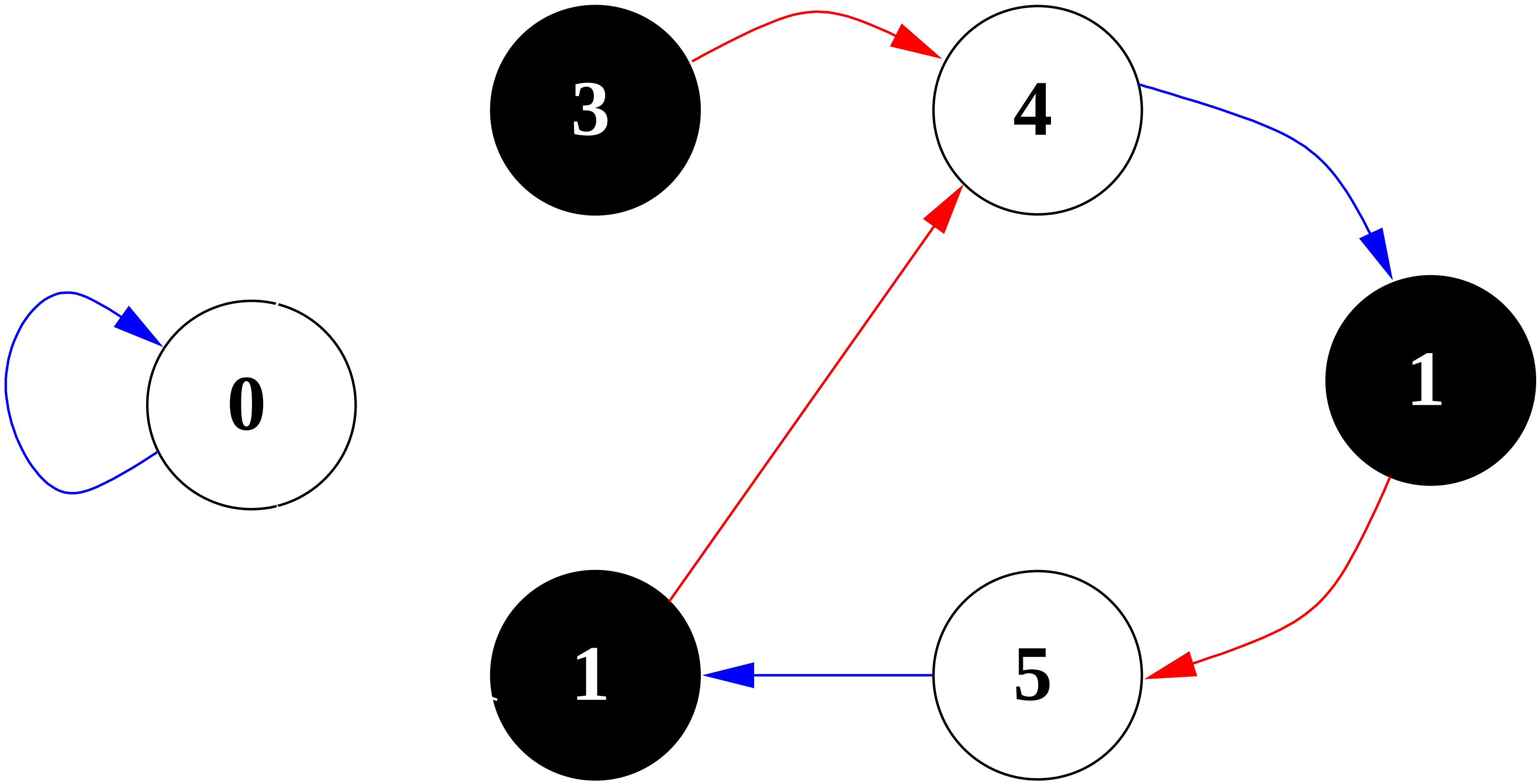

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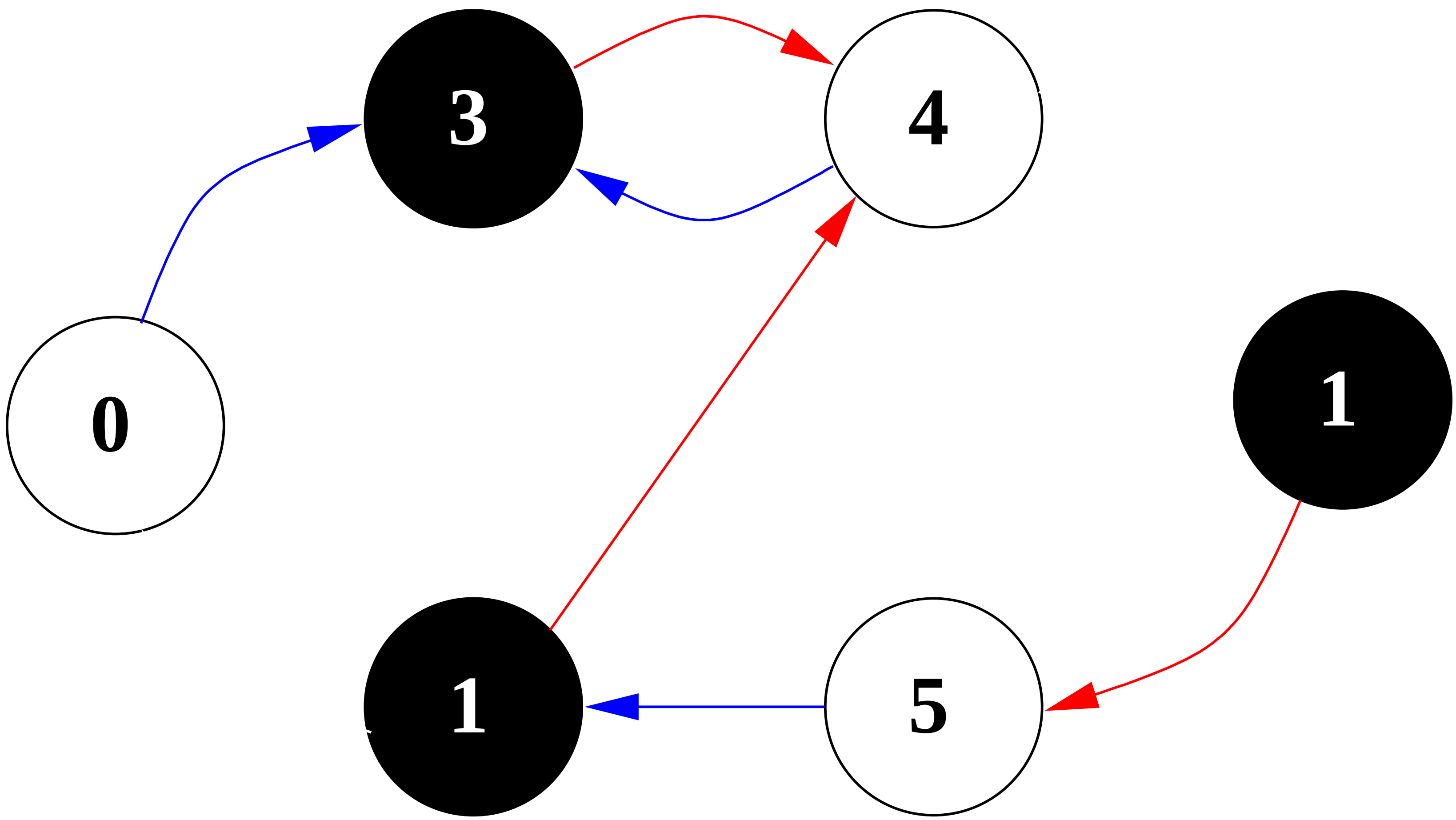

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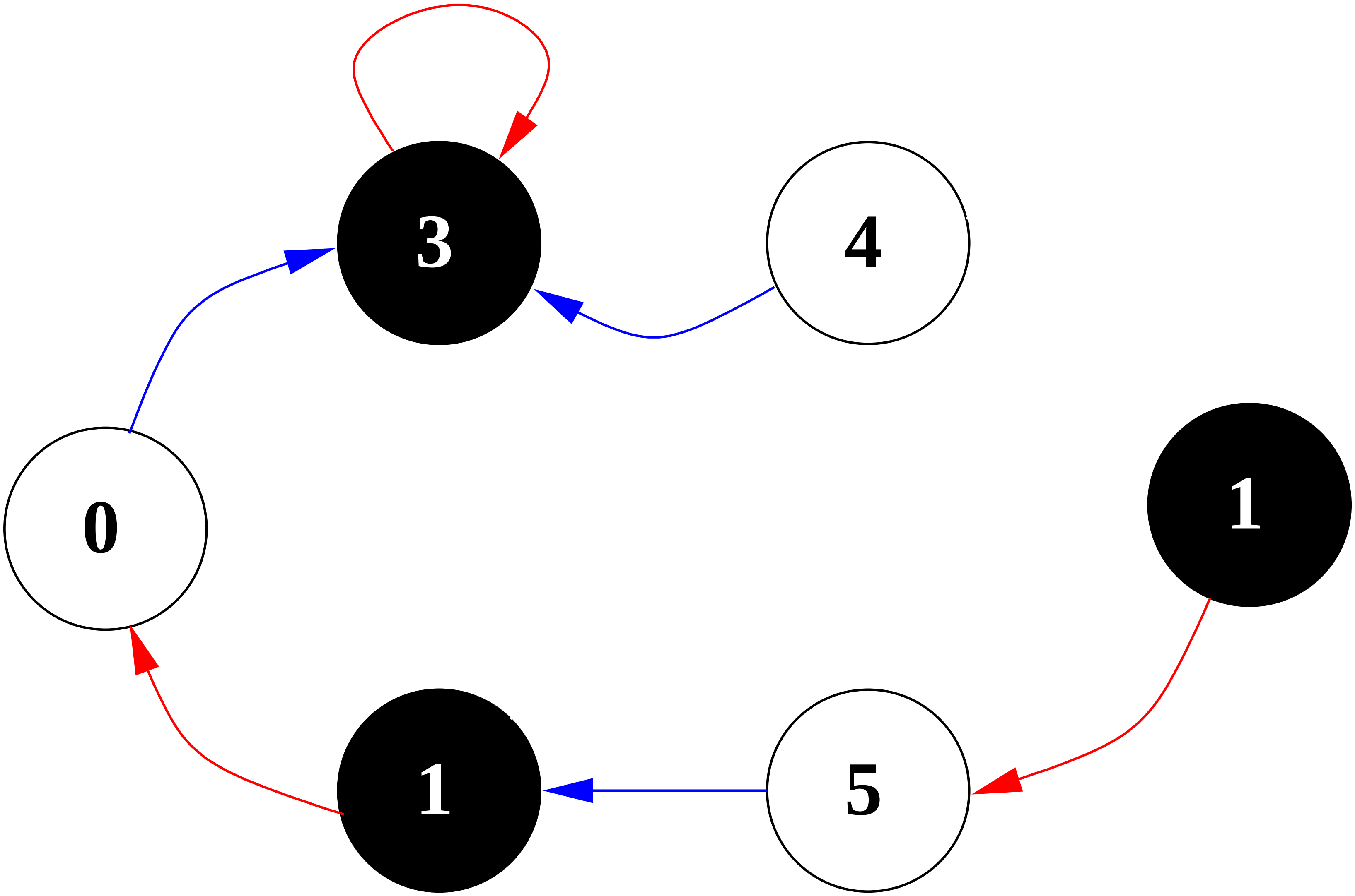


Completely Unimodal Hypercubes

Completely Unimodal Hypercubes

A psuedo-boolean function (PBF) of dimension n is a function from the n-dimensional boolean hypercube $\{0,1\}^n$ to ω.

Standard problem: *Find a local/global minimum/maximum*

This problem motivated the Polynomial Local Search (PLS) complexity class.

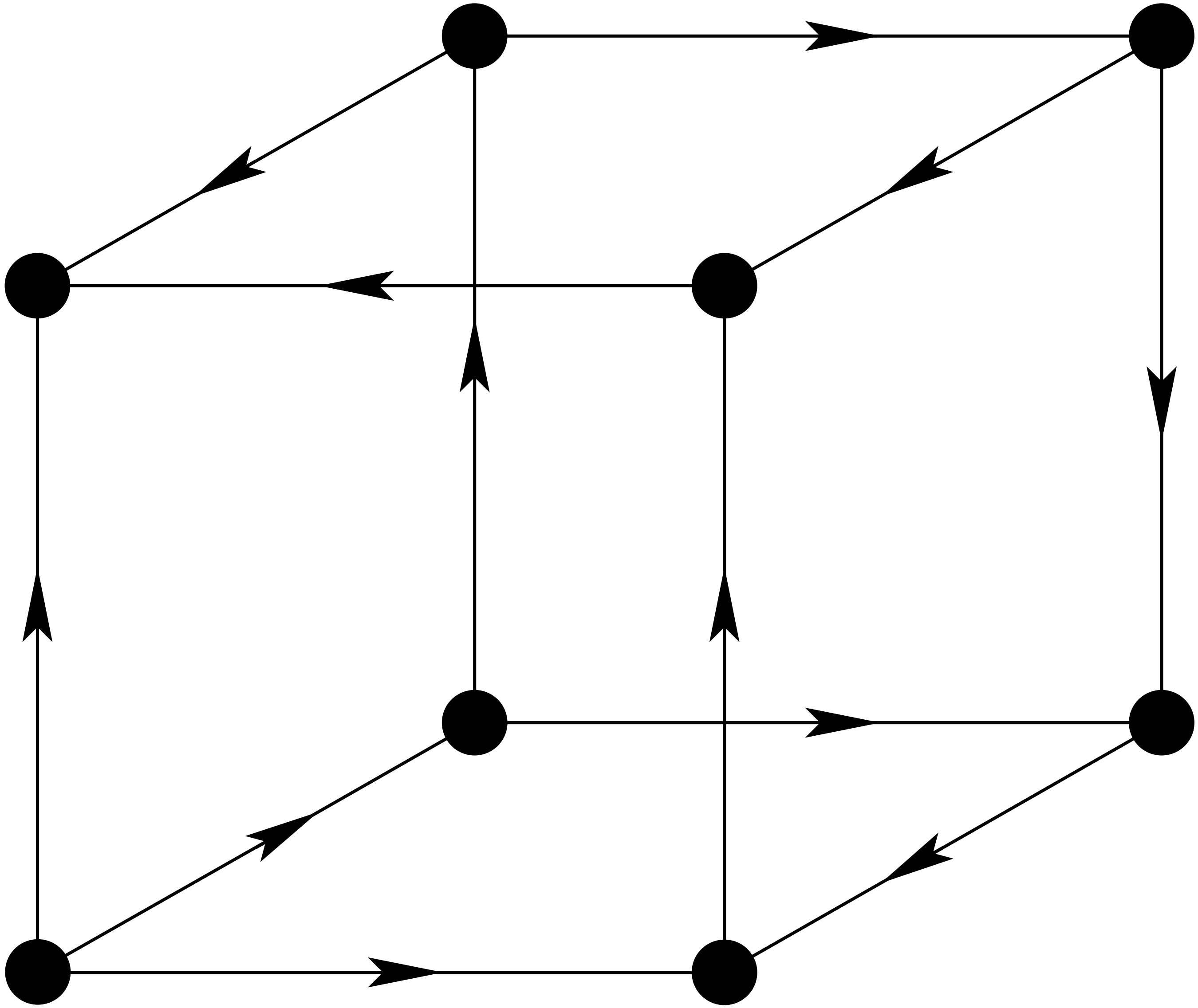
Completely Unimodal Hypercubes

A PBF is <u>completely unimodal</u> (CU) if it has exactly one maximum on every face of the hypercube.

Completely unimodal functions are also known as

- Completely Unimodal numberings, and
- Acyclic Unique Sink Orders.

CU Hypercubes – Example



CU Hypercubes – Properties

- All local optima are global
- A sufficient condition is for all 2-faces to be Completely Unimodal
- A CU numbering corresponds to a shelling of the dual polytope
- An n-dimensional CU Hypercube satisfies the Hirsch Conjecture. That is, from every vertex there is a path of length $\leq n$ to the global maximum.
- The Vector of Improving Directions is injective.

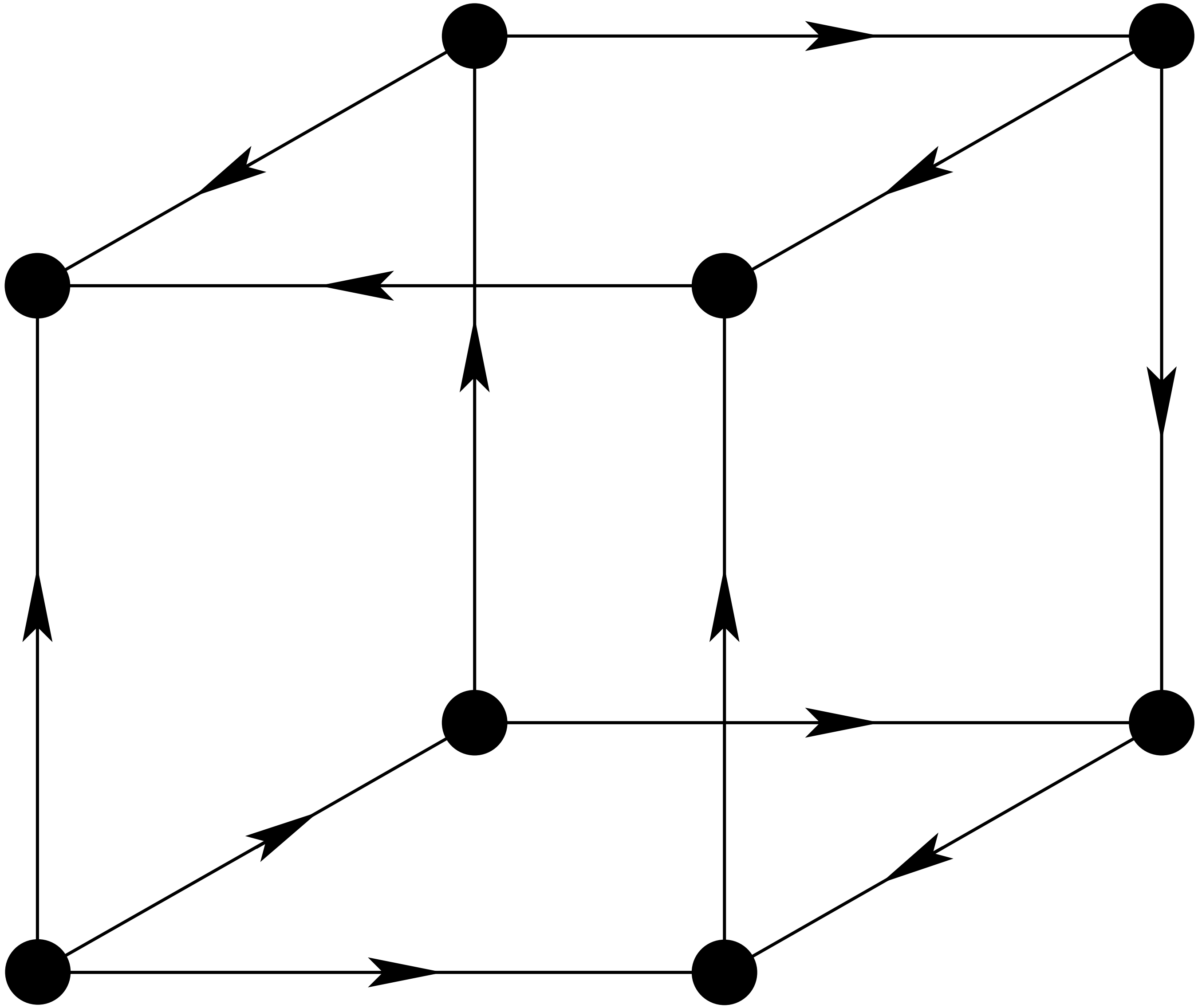
CU Hypercubes – Algorithms

Algorithms to find the global maximum:

Greedy Local Improvement (GLI): While there are better neighbours of the current position, change in all co-ordinates that are improving.
The complete unimodality condition guarantees that every change results in an improved position.

Fibonacci See-Saw (FSS): Store the maxima of opposite i-faces as i goes from 0 to n. To proceed from i to $i+1$ choose a direction which is improving for only one maximum (such a direction exists by the injectivity of the VID).

CU Hypercubes – GLI Example



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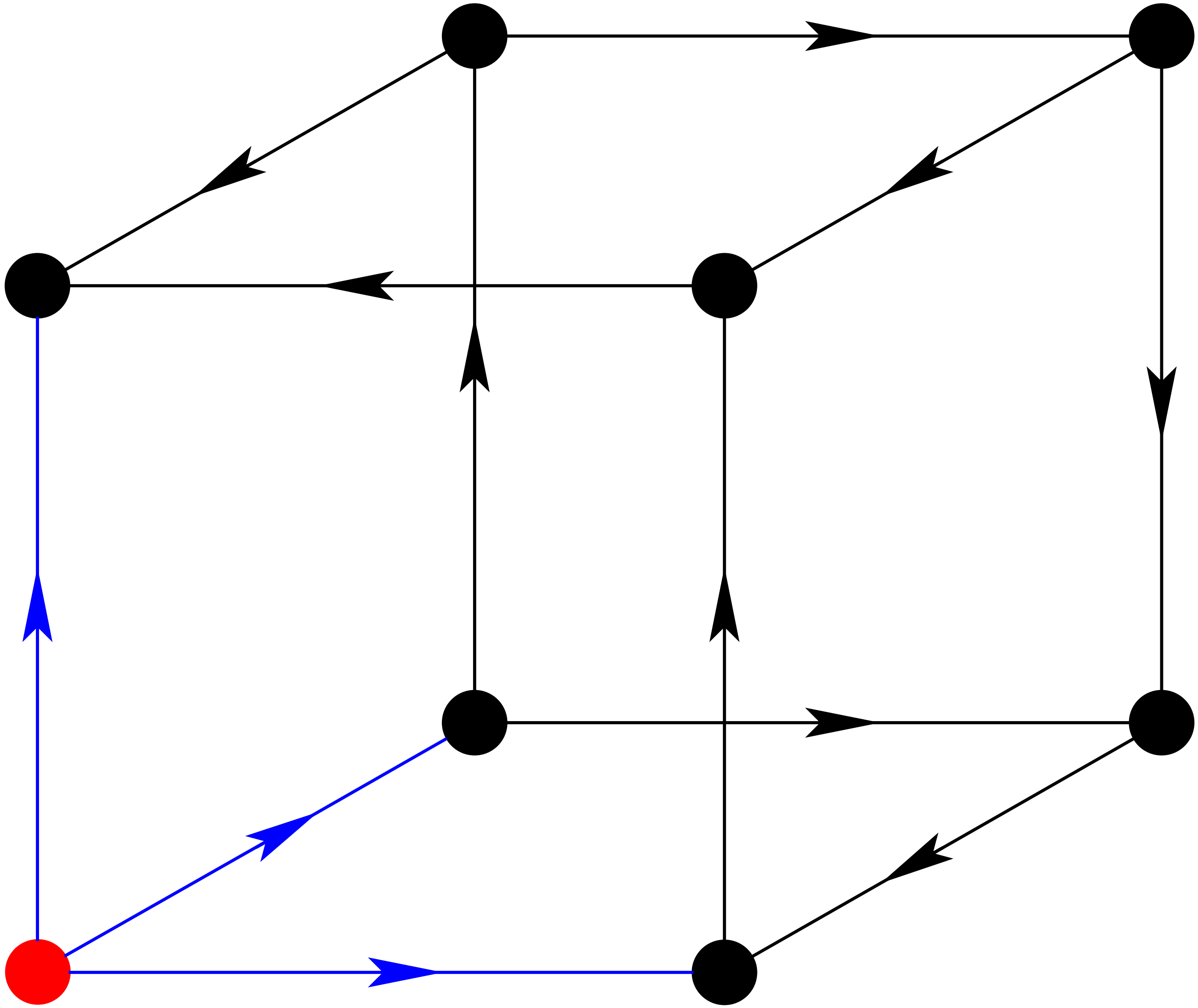

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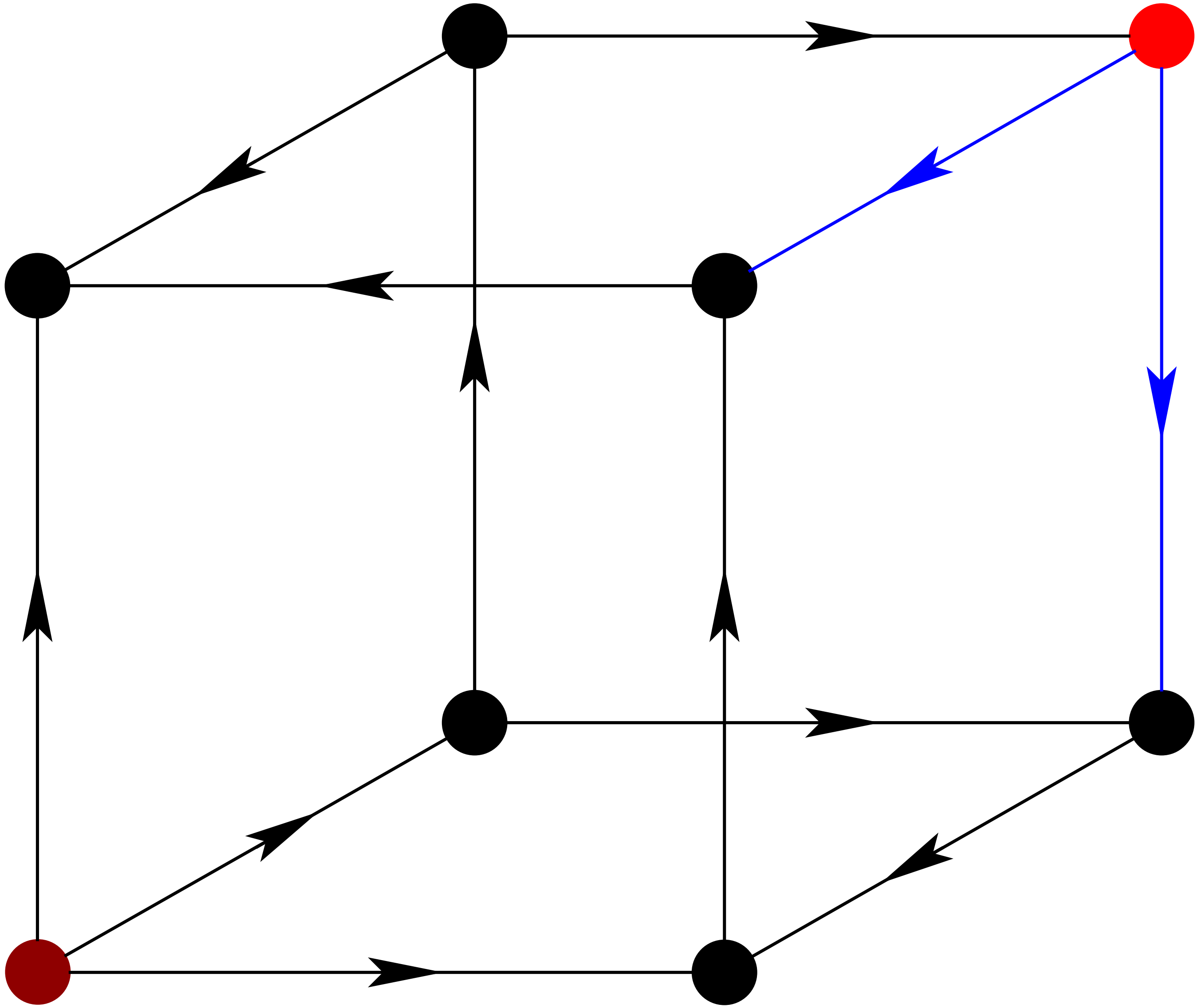

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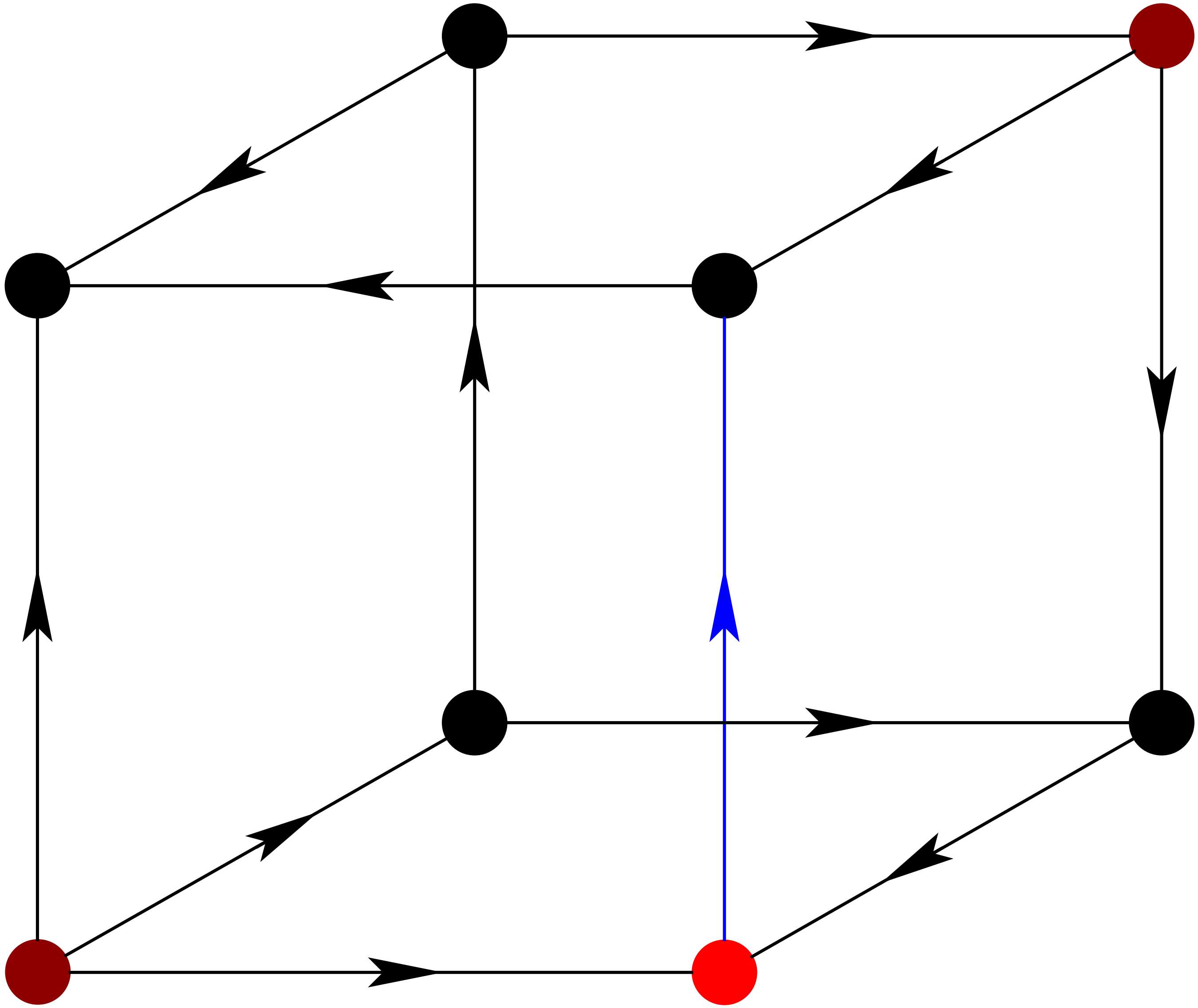

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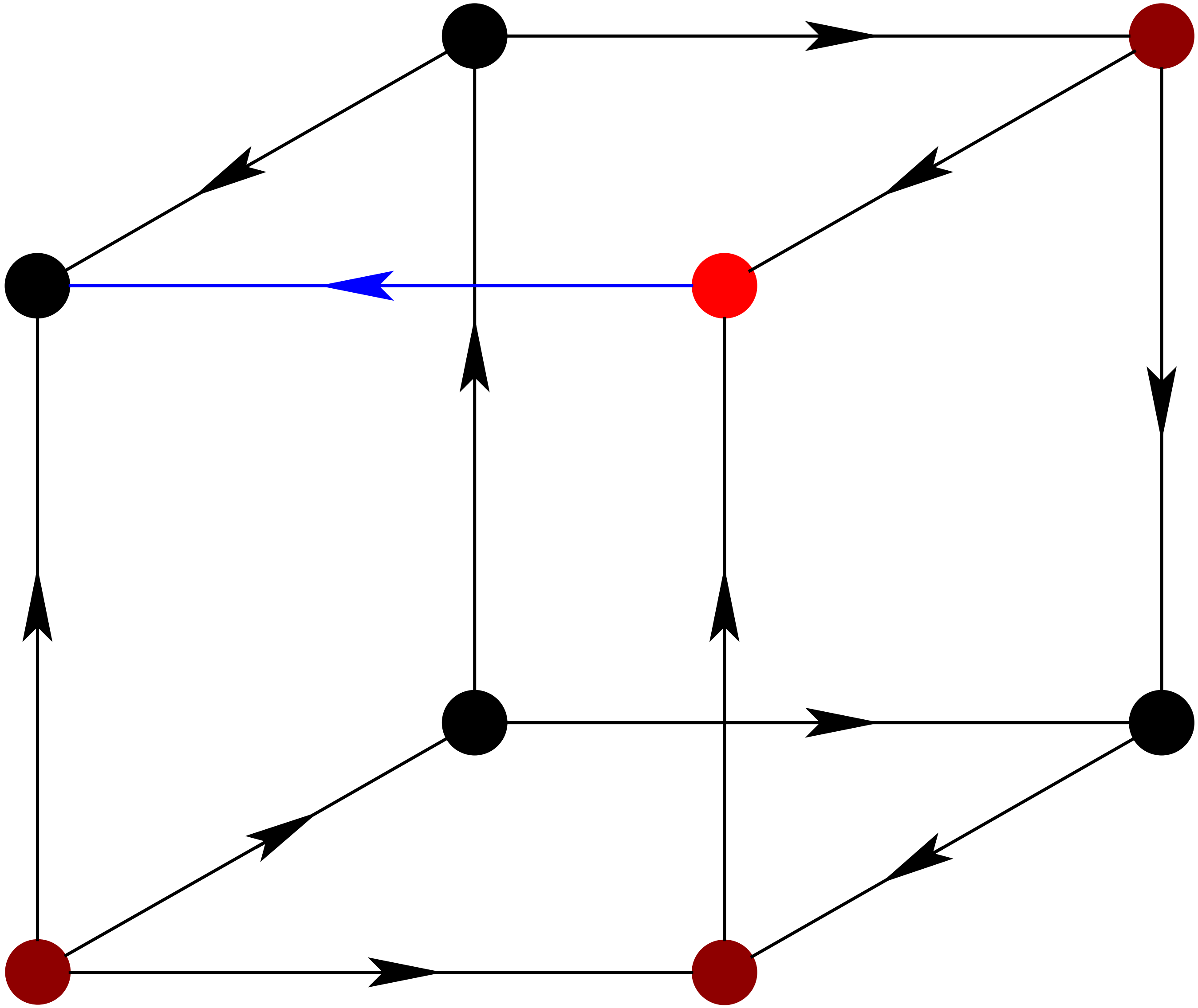

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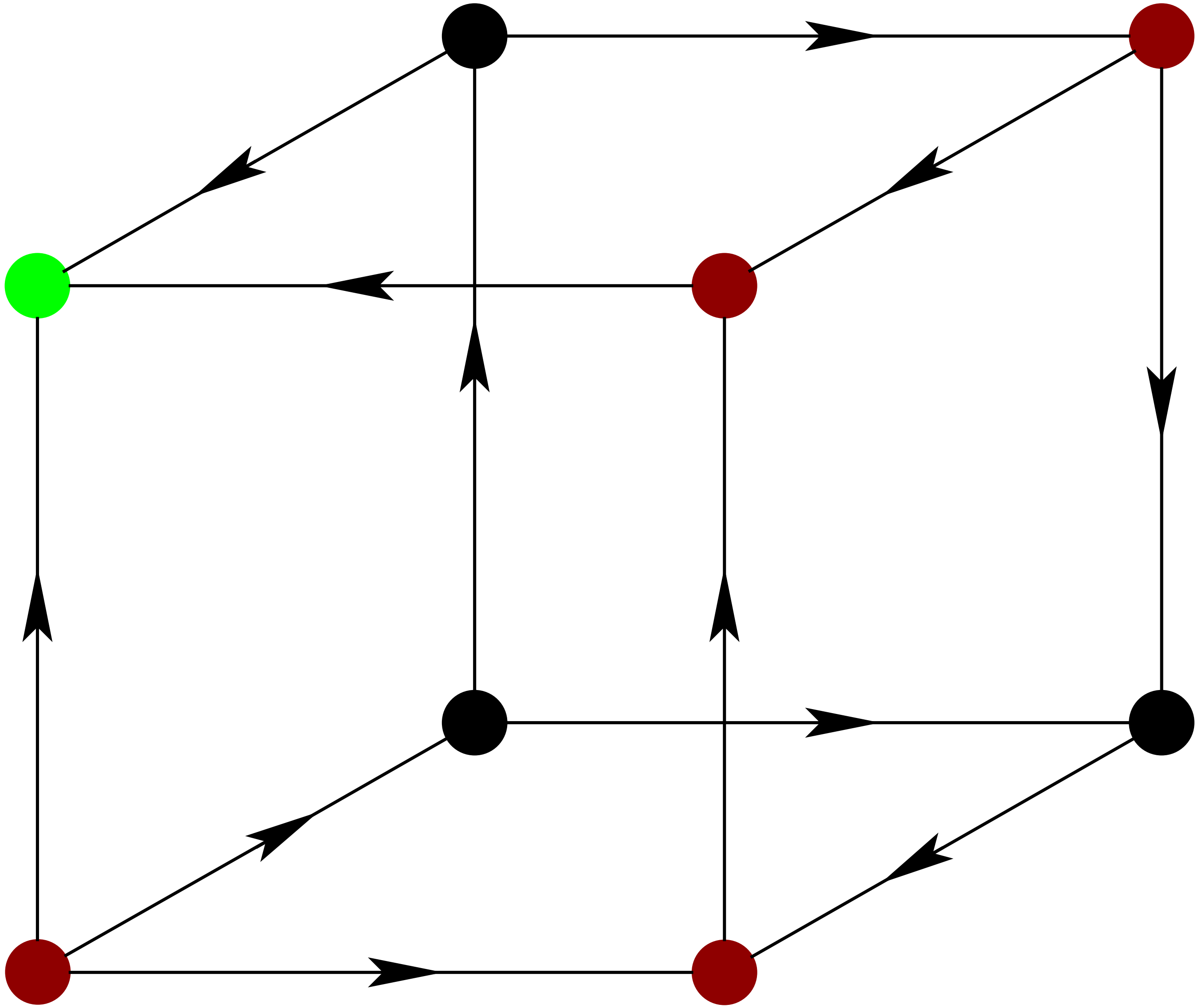

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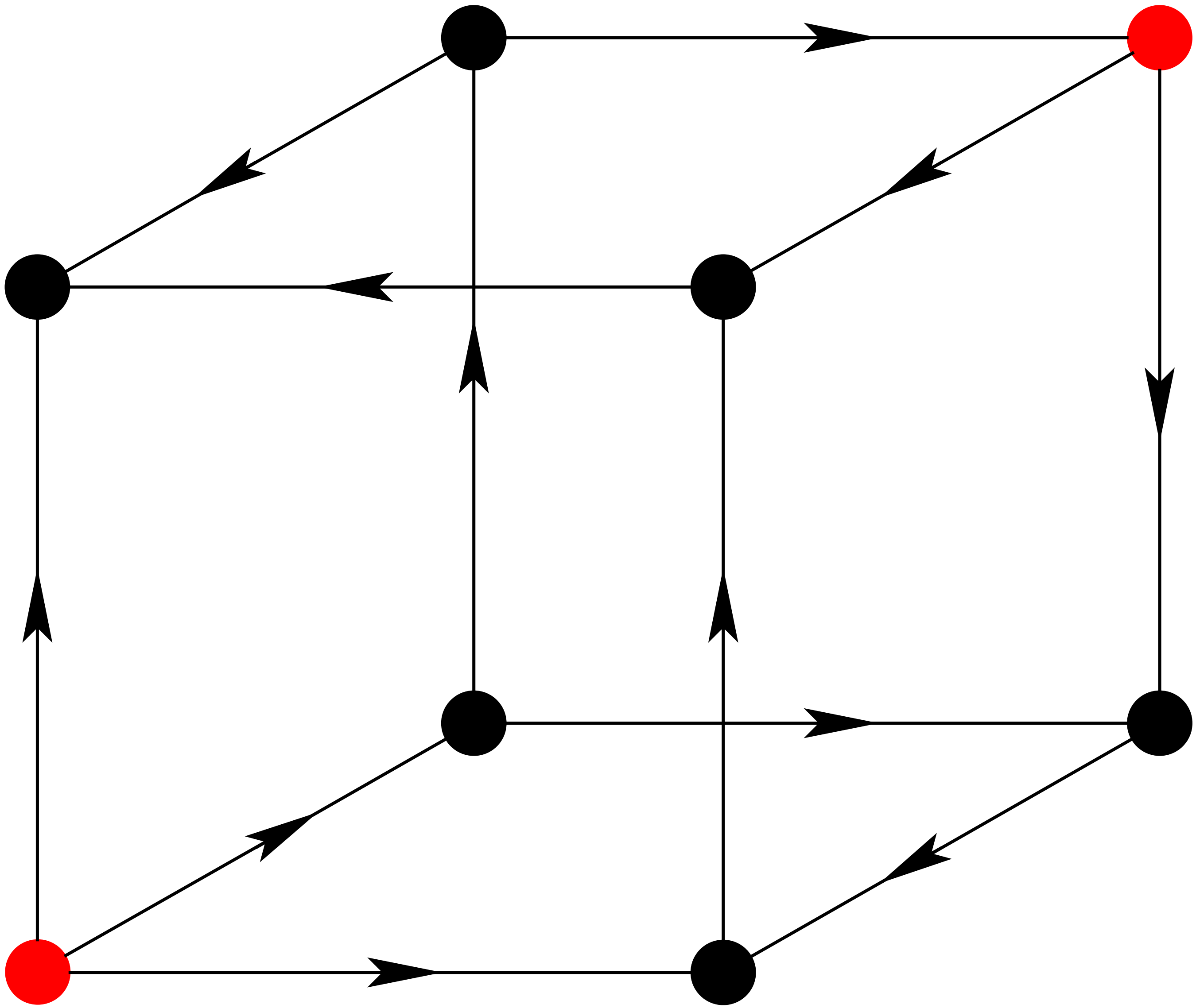

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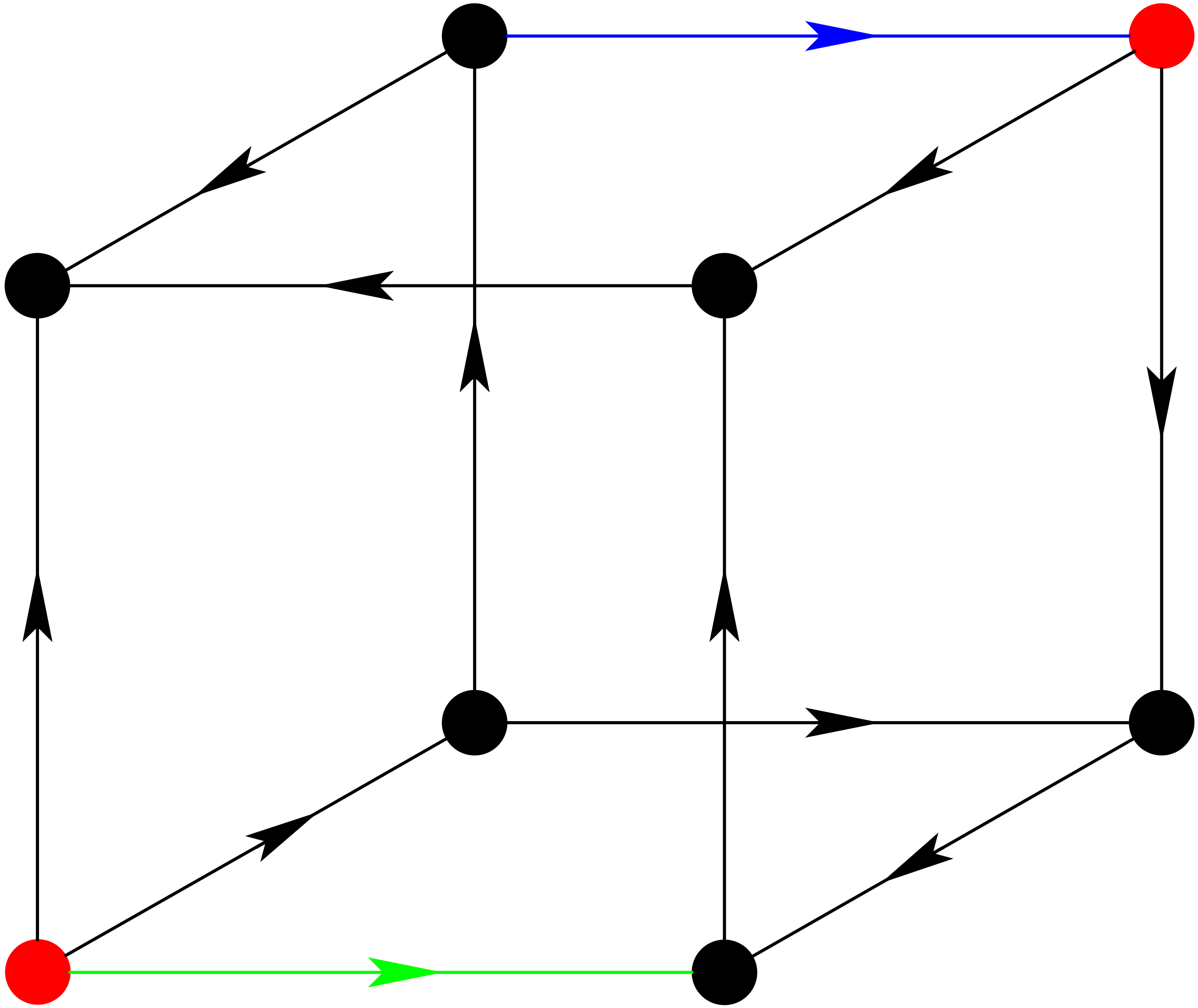

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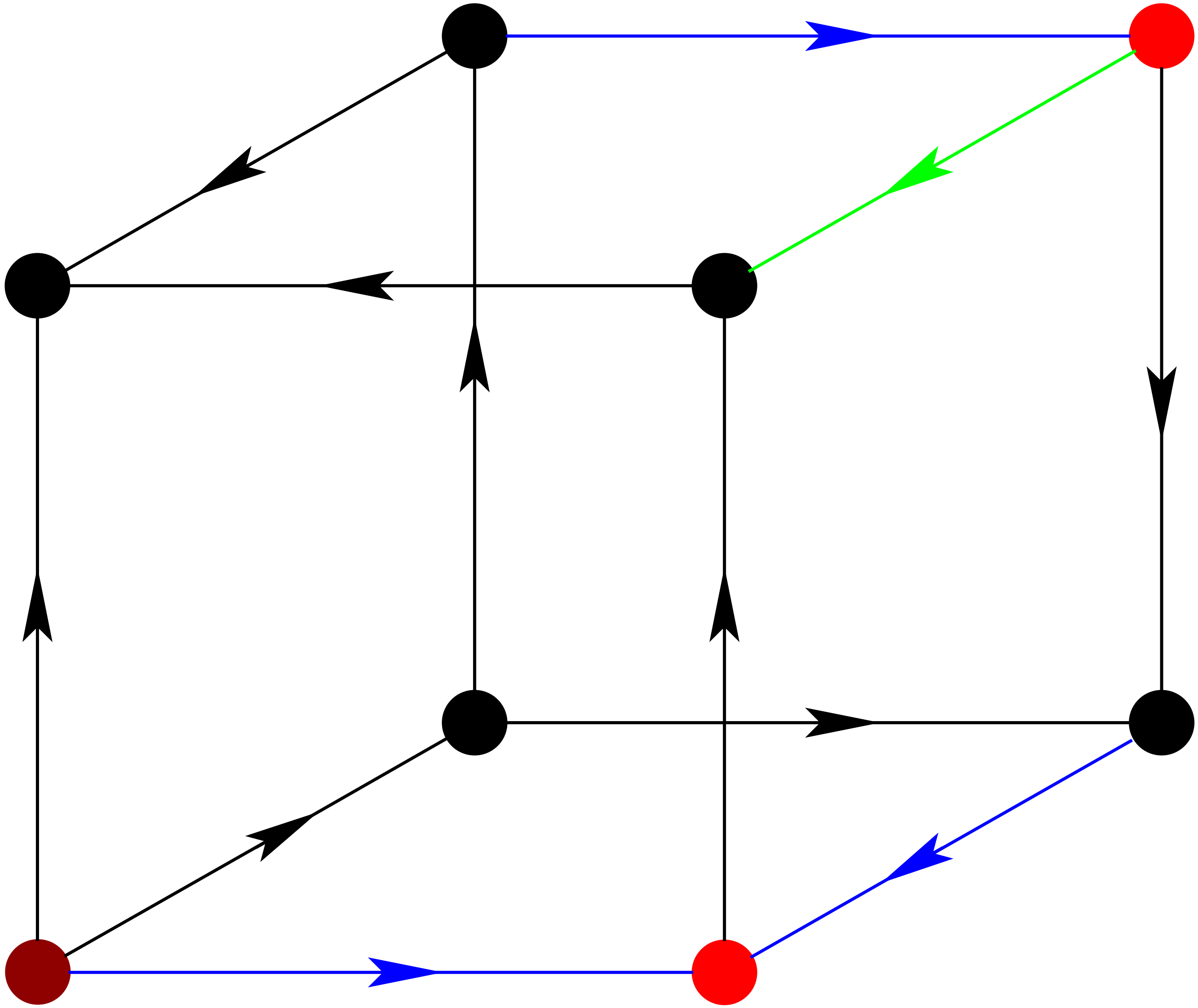

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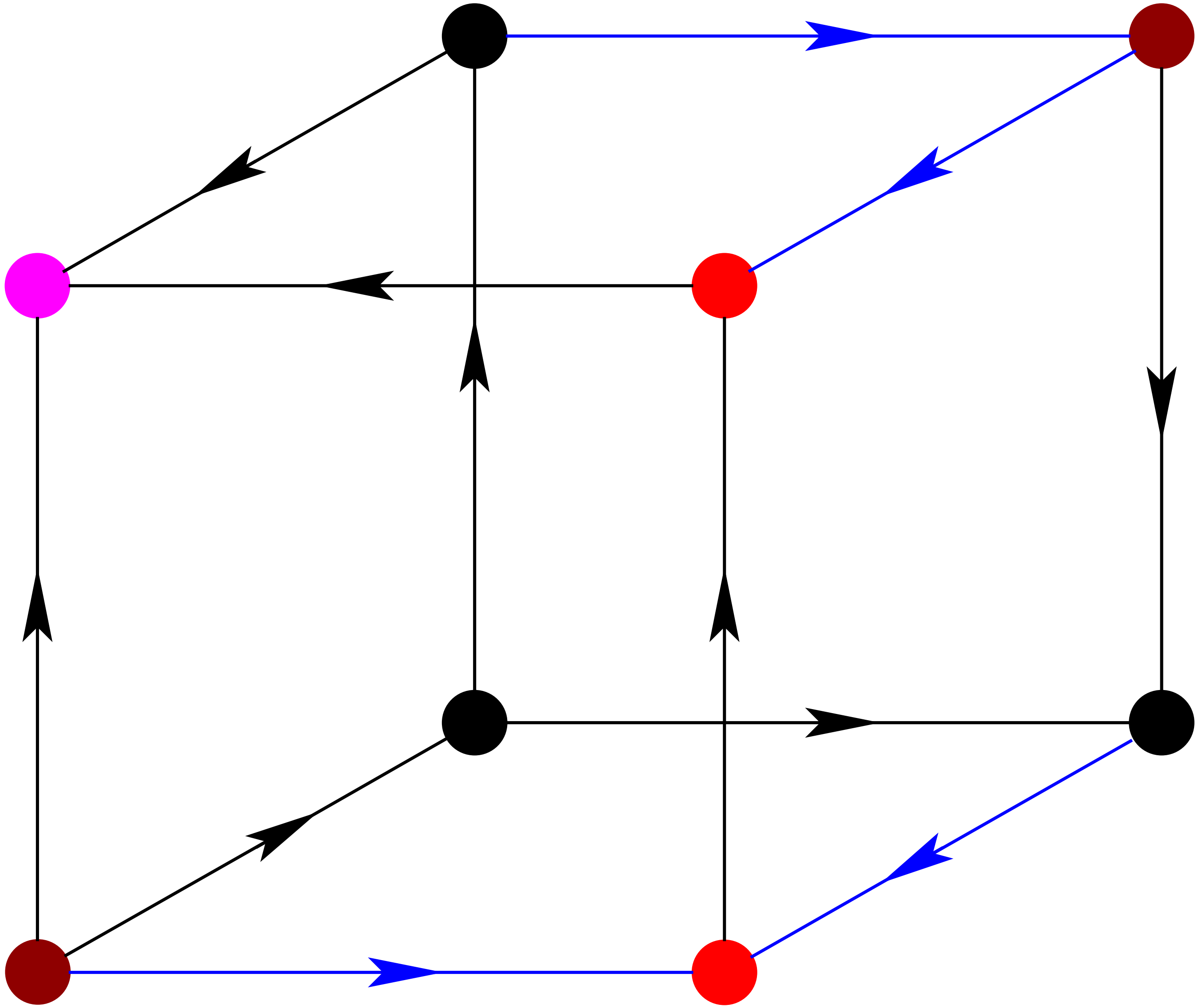

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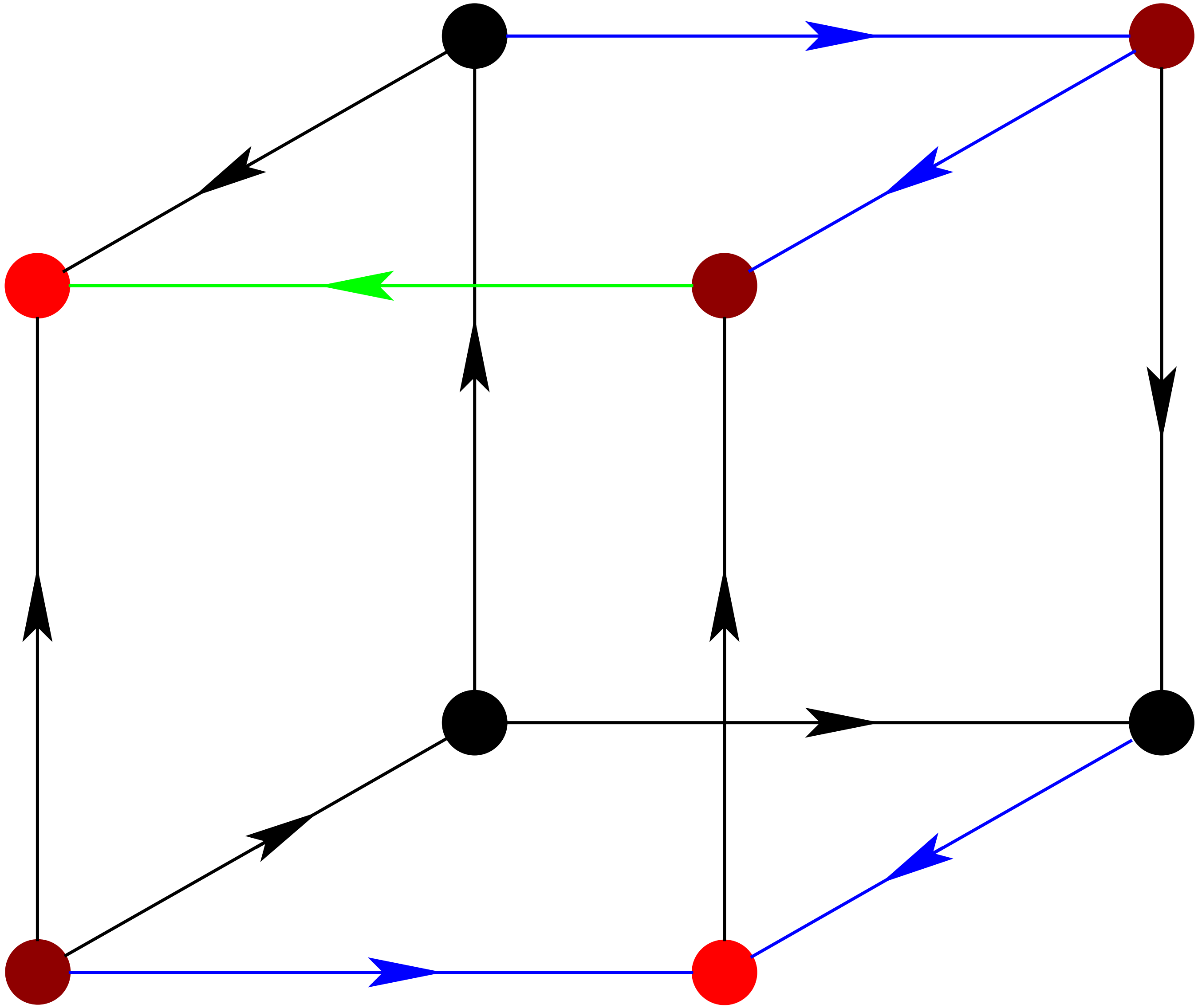

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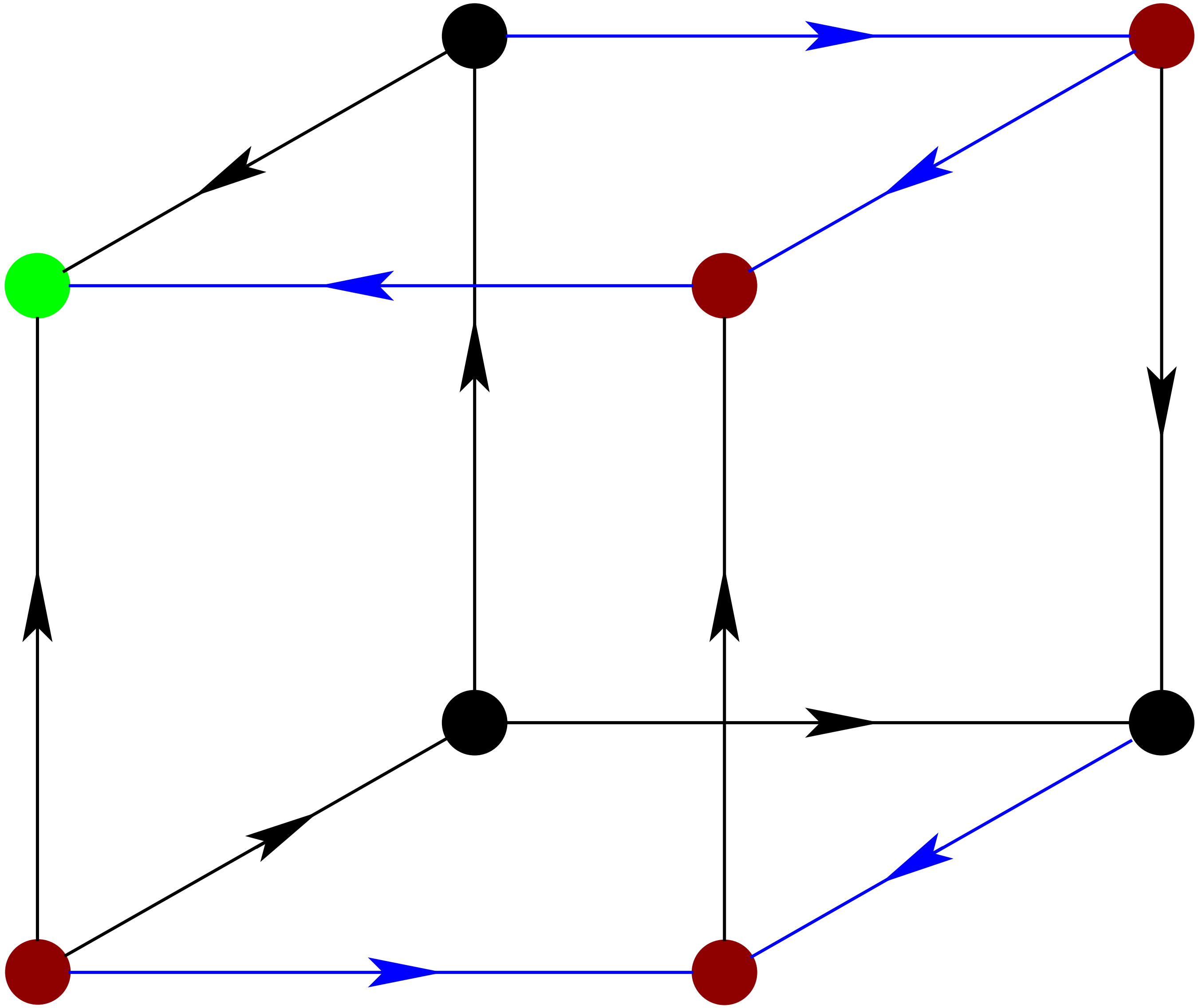

CU Hypercubes – Parity Games

The strategy space of Player 0's strategies forms a hypercube.

Björklund, Sandberg and Vorobyov (2004) showed that the valuation of Vöge and Jurdziński is a CU function on this hypercube.

Their algorithm is then an instance of a GLI.

CU Hypercubes – Problems

Question: *What are the bounds for a GLI?*

Question: *Does every GLI arise from an instance of the Strategy Improvement algorithm?*

Results

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Upper bounds: Find necessary conditions for GLI

Lower bounds: Find sufficient conditions for GLI

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Lower bounds: Find sufficient conditions for GLI

Notation: If $x_0, x_1, \ldots$ is a sequence of hypercube vertices,

- Δ_{ij} is the set of co-ordinates on which x_i and x_j differ
- $\Delta_i := \Delta_{i(i+1)}$

Results – Necessary conditions

Mansour and Singh (1999):

- $\Delta_i \not\subseteq \Delta_j$ for $i < j$
- There are at least $|\Delta_i|$ hypercube vertices valued between x_i and x_{i+1}

These imply that a GLI has at most $O(\frac{2^n}{n})$ steps.

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Madani (1999), H. (2004):

- For $i < j$, x_j is not in the face defined at x_i by the directions not improving at x_i ($\Delta_{ij} \not\subseteq \overline{\Delta_i}$)

This implies a GLI has at most 2^{n-1} steps.

Results – Necessary conditions

H. (2004):

PI: For $i < j$, $\Delta_i \cap \Delta_{ij} \not\subseteq \Delta_j$

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Question: *What are the bounds for a PI sequence?*

Dimension	1	2	3	4	5	6	7
Longest PI sequence	2	3	5	8	13	21	$\geq$ 26

Conjecture: n*-dimensional PI sequences are bounded by* F_{n+1} *and this bound is attained.*

Results – Sufficient conditions

Conjecture: *PI is sufficient for GLI.*

Results – Other

FSS has worst case running time F_{n+1} (Szabó and Welzl, 2001)

Question: *Is this bound attained?*

Question: *Does this worst case coincide with that of PI?*

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- Identified several algorithms (Strategy Improvement, GLI, PI) for which upper and lower bounds remain elusive
- Improved bound on Strategy Improvement algorithm to $O(|E|2^{|V|})$
- Can improve Strategy Improvement algorithm to $O(|V||E|F_{|V|}) = O(|V||E|(1.62)^{|V|})$ by using FSS
- Conjectured that the complexity of the Strategy Improvement algorithm is $O(|V||E|F_{|V|})$, and this bound is attained.

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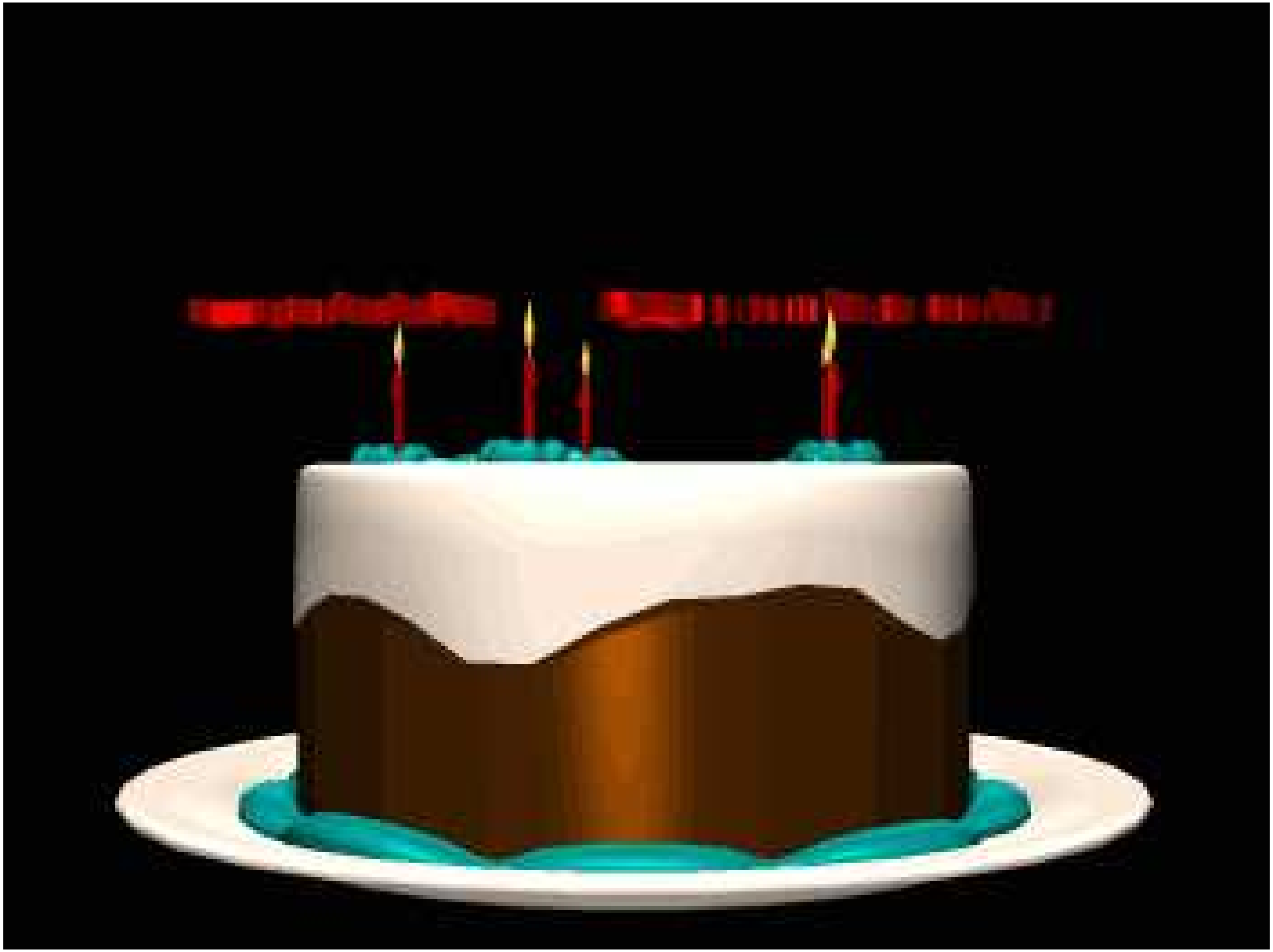

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