Expressive Equivalence of Least and Inflationary Fixed-Point Logic

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Abstract

We study the relationship between least and inflationary fixed-point logic. By results of Gurevich and Shelah from 1986, it has been known that on finite structures both logics have the same expressive power. On infinite structures however, the question whether there is a formula in IFP *not equivalent to any* LFP*-formula was still open.*

In this paper, we settle the question by showing that both logics are equally expressive on arbitrary structures. The proof will also establish the strictness of the nesting-depth hierarchy for IFP *on some infinite structures. Finally, we show that the alternation hierarchy for* IFP *collapses to the first level on all structures, i.e. the complement of an inflationary fixed-point is an inflationary fixed-point itself.*

1. Introduction

The study of inductive definitions has a long history in logic and computer science. Beginning with the work of Kleene and others on inductive definitions on the structure of arithmetic in recursion theory, inductive definitions on abstract structures have been studied since the early seventies, most notably by Moschovakis [9, 10], Aczel [1], and others.

Whereas in the seventies the study of inductive definitions focused on monotone or non-monotone inductions of first-order formulae on infinite structures, the rise of database and finite model theory in the eighties gave birth to a renewed interest in such kinds of definitions in the more coherent framework of fixed-point logics. But many of the methods proved important for fixed-point logics had already be obtained in the framework studied in the seventies.

Although different in scope and focus, finite model theory and database theory both concentrate on finite structures. In database theory, the interest in fixed-point logics results from the desire to have more expressive query languages for databases than the relational calculus. The interest in these logics in finite model theory originates from the close relationship between fixed-point logics and computational complexity.

This relationship is made precise in the remarkable results by Immerman [8] and Vardi [12] that, on finite ordered structures, *least fixed-point logic* (LFP) provides a logical characterisation of polynomial time computations, in the sense that a class of finite ordered structures is decidable in polynomial time if, and only if, it is definable in least fixed-point logic. Other complexity classes such as polynomial or logarithmic space can also be characterised in this way, using different fixed-point logics. Since the discovery of these results, fixed-point logics play a fundamental role in finite model theory, arguably even more important than first-order logic itself. One result of this development is that today a lot more is known about fixed-point logics on finite than on infinite structures.

The best known of these logics is the least fixed-point logic (LFP), already mentioned above, which extends first-order logic by an operator to form least fixed-points of positive formulae (which define monotone operators.) But there are other fixed-point logics. Besides fragments of LFP, such as transitive closure logic and existential or stratified fixed-point logic, which all have in common that they form fixed points of monotone operators, there are also fixed-point logics that allow the use of non-monotone operators. The simplest of these logics is *inflationary fixed-point logic* (IFP), which allows the definition of inflationary fixed-points of arbitrary formulae. We give precise definitions in Section 2. See [5] for an extensive study of fixed-point logics on finite structures. A survey that also treats infinite structures can be found in [4].

It follows from results by Knaster and Tarski that for formulae defining monotone operators, the least and inflationary fixed-points coincide. Thus, we immediately get that LFP $\subseteq$ IFP. On the other hand, IFP provides a much more liberal syntax than LFP and one might conjecture that IFP also provides more expressive power. For the case of monadic inductions, inductions defining unary relations, i.e.

sets, this is true indeed. This case has been studied in [3] for fixed-point extensions of modal logics where it has been shown that inflationary fixed-point inductions have very different properties and provide much more expressive power than least fixed-point inductions. To some extent, these results generalise to monadic fixed-point logics based on first-order logic as well.

However, for formulae without arity restrictions, Gurevich and Shelah [7] showed that least and inflationary fixed-point logic have the same expressive power on finite structures.

The most basic question concerning LFP and IFP on arbitrary structures is whether IFP is more expressive than LFP, a question that has was left open since the study of inflationary inductions in the seventies. As noted by Dawar and Gurevich [4] this question comes in two forms:

> **Question:** Is there a formula φ of IFP and a structure $\mathfrak{A}$ such that for every formula ψ of LFP, $\mathfrak{A} \not\models (\varphi \leftrightarrow \psi)$?
>
> Is there a formula φ of IFP such that for every formula ψ of LFP, there is a structure $\mathfrak{A}$ such that $\mathfrak{A} \not\models (\varphi \leftrightarrow \psi)$?

It is clear that the method used by Gurevich and Shelah on finite structures does not carry over to infinite structures as it relies crucially on the fact that on finite structures every fixed-point induction is itself finite and thus only successor stages occur in the induction.

The main contribution of this paper is to answer both questions negatively, i.e. to show that for every formula in IFP there is a formula in LFP equivalent to it on all structures.

As some simple implications of the method used to show this, we can also settle some hierarchy questions for IFP. In particular, we will show that there is a negation normal form, i.e. every formula of IFP is equivalent to a formula where negation occurs only in front of atoms. Thus, the alternation hierarchy for IFP, i.e. the hierarchy obtained from the number of alternations between fixed-points and negation in the formulae, collapses to the first level. This is contrary to least fixed-point logic as it follows from results by Moschovakis [9, Chapter 5D] and Bradfield [2] that the alternation hierarchy for LFP is strict.

On the other hand, we will show that the nesting depth hierarchy, i.e. the hierarchy obtained from the number of fixed points nested inside each other, is strict in general. To be precise, this hierarchy is infinite on a structure $\mathfrak{A}$, if, and only if, the alternation hierarchy for LFP is infinite on $\mathfrak{A}$. This also differs from least fixed-point logic as in LFP nested positive applications of fixed-points can always be eliminated.

Organisation. In the next section we give precise definitions of the fixed-point logics under consideration. In Section 3 we explain the stage comparison relations and theorems. Stage comparison is the basis of the most important method for reasoning about fixed-point logics and provides the main tool used in the sections thereafter. In Section 4 we present our main result that least and inflationary fixed-point logic have the same expressive power. Finally, we obtain as corollaries of this result that in general the nesting depth hierarchy of IFP does not collapse whereas the alternation hierarchy for IFP collapses on all structures. This will be explained in Section 5.

2. Fixed-Point Logics

In this section we present the basic definitions for the explorations in the later sections. Let τ be a signature and $\mathfrak{A} := (A, \tau)$ a τ-structure. Let $\varphi(R, \overline{x})$ be a first-order formula with free variables $\overline{x}$ and a free relation symbol R not occurring in τ. The formula φ defines an operator

$$\begin{array}{rccc} F_\varphi : & \mathcal{P}(A) & \longrightarrow & \mathcal{P}(A) \\ & R & \longmapsto & \{\overline{a} : (\mathfrak{A}, R) \models \varphi[\overline{a}]\}. \end{array}$$

A fixed point of the operator F_φ is any set R such that $F_\varphi(R) = R$. Clearly, as φ is arbitrary, the corresponding operator F_φ need not to have any fixed points at all. For instance, the formula $\varphi(R, \overline{x}) := \neg\forall\overline{y}\, R\overline{y}$ defines the operator F_φ mapping any set $R \subsetneq A^k$ to A^k and the set A^k itself to the empty set. Thus F_φ has no fixed points.

However, if the class of admissible formulae φ is restricted, then the existence of fixed points can be guaranteed. A formula $\varphi(R, \overline{x})$ is *monotone in* R, if for all τ-structures $\mathfrak{A} := (A, \tau)$ and all sets $R, R' \subseteq A^k$,

$$R \subseteq R' \text{ implies } F_\varphi(R) \subseteq F_\varphi(R').$$

For a monotone operator F_φ it can be shown that fixed points always exist and in fact even a least fixed point exists which can be defined as

$$\mathbf{lfp}(F_\varphi) := \bigcap\{R : F_\varphi(R) = R\}.$$

This gives rise to the first of our inductive logics, the *monotone fixed-point logic*.

2.1 Definition (Monotone Fixed-Point Logic). Monotone fixed-point logic (MFP) *is defined as the extension of first-order logic by the following formula building rule. If* $\varphi(R, \overline{x})$ *is a formula with free first-order variables* $\overline{x} := x_1, \ldots, x_k$ *and a free second-order variable* R *of arity* k *such that the corresponding operator* F_φ *is monotone on all structures, then*

$$\psi := [\mathbf{lfp}_{R,\overline{x}}\, \varphi](\overline{t})$$

is also a formula, where $\overline{t}$ is a tuple of terms of the same length as $\overline{x}$. The free variables of ψ are the variables occurring in $\overline{t}$ and the free variables of φ other than $\overline{x}$.

Let $\mathfrak{A}$ be a structure providing an interpretation of the free variables of φ other than $\overline{x}$. Then for any tuple $\overline{a}$, $\mathfrak{A} \models [\mathbf{lfp}_{R,\overline{x}}\,\varphi](\overline{a})$ if, and only if, $\overline{a} \in \mathbf{lfp}(F_\varphi)$.

As explained above, for any monotone operator F the least fixed point of F always exists. Therefore the semantics of the monotone fixed-point logic is well defined.

The problem with this definition of a fixed-point logic is, that the property of a formula to define a monotone operator on all structures is undecidable. Thus, monotone fixed-point logic has an undecidable syntax.

To avoid this problem, one considers syntactical restrictions of MFP which guarantee monotonicity of the corresponding operators. The most important such restriction is to allow only formulae in the fixed-point rule, which are positive in the relation variable R. It is clear, that if a formula $\varphi(R, \overline{x})$ is positive in R, then the corresponding operator F_φ is monotone. This leads to the definition of the *least fixed-point logic*.

2.2 Definition (Least Fixed-Point Logic). Least fixed-point logic (LFP) *is defined as the extension of first-order logic by the following formula building rule. If $\varphi(R, \overline{x})$ is a formula with free first-order variables $\overline{x} := x_1, \ldots, x_k$ and a free second-order variable R of arity k such that φ is positive in R, then*

$$\psi := [\mathbf{lfp}_{R,\overline{x}}\,\varphi](\overline{t})$$

is also a formula, where $\overline{t}$ is a tuple of terms of the same length as $\overline{x}$. The free variables of ψ are the variables occurring in $\overline{t}$ and the free variables of φ other than $\overline{x}$.

Let $\mathfrak{A}$ be a structure providing an interpretation of the free variables of φ other than $\overline{x}$. Then for any tuple $\overline{a}$, $\mathfrak{A} \models [\mathbf{lfp}_{R,\overline{x}}\,\varphi](\overline{a})$ if, and only if, $\overline{a} \in \mathbf{lfp}(F_\varphi)$.

Clearly, LFP $\subseteq$ MFP. Monotone and least fixed-point logic are examples of fixed-point logics defined by restricting the class of formulae so that the corresponding operators become monotone. To obtain more general fixed-point logics, i.e. logics allowing non-monotone operators also, one has to consider suitable semantics to guarantee the existence of meaningful fixed-points. The simplest such logic is the *inflationary fixed-point logic*.

2.3 Definition (Inflationary Fixed-Point Logic). Inflationary fixed-point logic (IFP) *is defined as the extension of first-order logic by the following formula building rule. If $\varphi(R, \overline{x})$ is a formula with free first-order variables $\overline{x} := x_1, \ldots, x_k$ and a free second-order variable R of arity k, then*

$$\psi := [\mathbf{ifp}_{R,\overline{x}}\,\varphi](\overline{t})$$

is also a formula, where $\overline{t}$ is a tuple of terms of the same length as $\overline{x}$. The free variables of ψ are the variables occurring in $\overline{t}$ and the free variables of φ other than $\overline{x}$.

Let $\mathfrak{A}$ be a structure with universe A providing an interpretation of the free variables of φ other than $\overline{x}$. Consider the following sequence of stages induced by φ on $\mathfrak{A}$.

$$\begin{aligned} R^0 &:= \varnothing \\ R^{\alpha+1} &:= R^\alpha \cup F_\varphi(R^\alpha) \\ R^\lambda &:= \bigcup_{\beta<\lambda} R^\beta \qquad \text{for limit ordinals } \lambda. \end{aligned} \tag{1}$$

Clearly this sequence of sets is increasing and thus leads to a fixed point R^∞. For any tuple $\overline{a} \in A$,

$$\mathfrak{A} \models [\mathbf{ifp}_{R,\overline{x}}\,\varphi](\overline{a}) \text{ if, and only if, } \overline{a} \in R^\infty.$$

As a corollary of the following theorem due to Knaster and Tarski we get that monotone is contained in inflationary fixed-point logic.

2.4 Theorem (Knaster and Tarski). *Every monotone operator $F : \mathcal{P}(M) \to \mathcal{P}(M)$ has a least fixed-point which may be written in the form*

$$\mathbf{lfp}(F) = \bigcap\{P : F(P) \subseteq P\}.$$

Further, this fixed point can also be obtained by the fixed-point of the following sequence of sets

$$\begin{aligned} R^0 &:= \varnothing \\ R^{\alpha+1} &:= F(R^\alpha) \\ R^\lambda &:= \bigcup_{\beta<\lambda} R^\beta \qquad \text{for limit ordinals } \lambda. \end{aligned} \tag{2}$$

As the sequence in the previous theorem is increasing due to the monotonicity of F, the least fixed point reached in this way must also be the inflationary fixed point of F. Therefore it is clear, that regarding expressive power

$$\text{LFP} \quad \subseteq \quad \text{MFP} \quad \subseteq \quad \text{IFP}.$$

3. Comparing the stages of inductive definitions

In this section we introduce the stage comparison method which has been the key to many results about fixed-point logics. (See [9, 5] for instance.) The method will be essential for the explorations in the later sections. Let $\varphi(R, \overline{x})$ be a formula, say in first-order logic. As mentioned above, if φ is positive in R, then its least fixed point is obtained as the fixed point of the sequence of sets as defined by (1). Thus we concentrate on such sequences of sets approximating least or inflationary fixed points.

Let $\mathfrak{A} := (A, \tau)$ be a τ-structure with universe A. Right by definition, the sequence of stages defined in (1) is increasing and thus there is an ordinal $\alpha < |A|^+$ such that $R^\alpha = R^{\alpha+1} = R^\infty$, where $|A|^+$ denotes the least infinite cardinal greater than the cardinality of A. The individual sets occurring in the sequence induced by a formula φ are called the *stages of the induction on* φ and the set R^α is called the α-th stage of the induction. Sometimes we also write φ^α to denote the α-th stage of the induction on φ. As a final bit of notation, we write $R^{<\alpha}$ or $\varphi^{<\alpha}$ for the union of all stages up to α, i.e., $\varphi^{<\alpha} := \bigcup_{\beta<\alpha} \varphi^\beta$, and likewise for $R^{<\alpha}$.

We now define the stage comparison relations for a least or inflationary fixed-point induction.

3.1 Definition. *Let $\varphi(R, \overline{x})$ be a formula and $\overline{a} \in A$. The* rank $|\overline{a}|_\varphi$ of $\overline{a}$ with respect to φ *is defined as the least ordinal α such that $\overline{a} \in \varphi^\alpha$ if such an ordinal α exists and ∞ otherwise.*

The stage comparison relations $\leq_\varphi$ and $\prec_\varphi$ *are defined as*

$$x \leq_\varphi y \iff x, y \in \varphi^\infty \text{ and } |x|_\varphi \leq |y|_\varphi,$$

and

$$x \prec_\varphi y \iff x \in \varphi^\infty \text{ and } |x|_\varphi < |y|_\varphi,$$

where we allow $|y|_\varphi = \infty$.

Clearly, for all $\overline{a}$,

$$\begin{array}{lll} \overline{a} \in \varphi^\infty & \text{if, and only if,} & \overline{a} \leq_\varphi \overline{a} \\ & \text{if, and only if,} & (\mathfrak{A}, \{\overline{u} : \overline{u} \prec_\varphi \overline{a}\}) \models \varphi[\overline{a}]. \end{array}$$

The following theorem, due to Moschovakis shows that if $\varphi(R, \overline{x})$ is in LFP and positive in R, then these relations can be defined in LFP itself and likewise for IFP. See [9] and references therein for the case of LFP and [10] for the IFP-version.

3.2 Theorem (Stage Comparison Theorem).

(i) Let $\varphi(R, \overline{x})$ be a formula in LFP *positive in R. Then $\leq_\varphi$ and $\prec_\varphi$ are both definable in* LFP.

(ii) Let $\varphi(R, \overline{x})$ be a formula in IFP. *Then $\leq_\varphi$ and $\prec_\varphi$ are both definable in* IFP.

We only sketch the proof for the relation $\prec_\varphi$ defining the stages of an inflationary induction on $\varphi(R, \overline{x})$. Consider the formulae ψ defined as

$$\psi(\prec, \overline{x}, \overline{y}) := \varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{x}) \wedge \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}),$$

where $\prec$ is a second-order variable of arity twice the arity of $\overline{x}$ and the formula $\varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{x})$ is obtained from φ by replacing each atom of the form $R\overline{u}$ by the formula $\overline{u} \prec \overline{x}$.

We claim, that the inflationary fixed point of ψ exactly defines $\prec_\varphi$. This is proved by induction on α showing that ψ^α defines the set of pairs $(\overline{x}, \overline{y})$ such that $|\overline{x}|_\varphi < \alpha$ and $|\overline{x}|_\varphi < |\overline{y}|_\varphi$.

A detailed proof of the theorem will be given in the next section, where we show that the relation $\prec_\varphi$ can be defined in LFP, even if the formula φ is not positive in its free second-order variable.

4. Inflationary vs. Least Fixed-Point Inductions

In this section we establish the equivalence of IFP and LFP. Towards this goal, let $\varphi'(R, \overline{x})$ be in LFP, not necessarily positive in R, and consider the formula $\varphi := R\overline{x} \vee \varphi'(\overline{x})$. Clearly, φ and φ' have the same inflationary fixed point. Fix φ for the rest of this section.

Notation. Let $\psi_1(\overline{u})$ and $\psi_2(\overline{u})$ be formulae, which may or may not contain R, where $\overline{u}$ and R have the same arity. We write $\varphi(\overline{x}, R\overline{u}/\psi_1(\overline{u}), \neg R\overline{u}/\psi_2(\overline{u}))$ to denote the formula obtained from φ by replacing each positive occurrence of atoms of the form $R\overline{u}$, where $\overline{u}$ is a tuple of terms, by $\psi_1(\overline{u})$ and each negative occurrence of atoms of the form $R\overline{u}$ by $\neg\psi_2(\overline{u})$. Clearly, the formula is positive both in ψ_1 and ψ_2 and, thus, positive in R if both ψ_1 and ψ_2 are. □

We aim at defining the stage comparison relation $\prec_\varphi$ for φ in LFP. By the stage comparison theorem, $\prec_\varphi$ can be defined as the inflationary fixed-point of the formula

$$\begin{aligned} \varphi'(\prec, \overline{x}, \overline{y}) := {} & \varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}) \wedge \\ & \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}), \end{aligned}$$

where $\prec$ is a second-order variable of appropriate arity.

To turn this into a formula in LFP we have to replace the formula $\neg\overline{u} \prec \overline{x}$ by a definition positive in $\prec$. Essentially, we aim at defining another formula $\vartheta(\ll, \overline{x}, \overline{y})$, negative in $\prec$, with free second-order variables $\ll$ and $\prec$, such that if $\prec$ is interpreted by a given stage $\prec^\alpha$, for some ordinal α, then the least fixed-point $\ll^\infty$ will be $\prec^\alpha$. We could then use $[\mathbf{lfp}\ \vartheta]$ negatively to get the desired positive definition of $\prec$.

Unfortunately, by definition, the relation defined by such a formula must increase with increasing stages $\prec^\alpha$. On the other hand, as ϑ was supposed to be negative - and therefore antitone - in $\prec$, the relation defined by ϑ must decrease with increasing stages $\prec^\alpha$. Thus, in general, we can not hope for such a formula to exist. Instead we will use a formula defining a slightly different relation. But it might be helpful to keep the original idea in mind.

Consider the following formulae:

$$\chi(\overline{x}, \overline{y}) := [\mathbf{lfp}_{\prec, \overline{x}, \overline{y}}\, \chi'(\overline{x}, \overline{y})](\overline{x}, \overline{y}),$$

where

$$\chi'(\overline{x},\overline{y}) := \varphi(\overline{x}, R\overline{u}/u \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \lhd \overline{x}) \wedge \forall\overline{u}(\overline{u} \prec \overline{x} \vee \neg\overline{u} \lhd \overline{x}) \wedge \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \lhd \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x})$$

and

$$\overline{x} \lhd \overline{y} := [\mathbf{lfp}_{\ll,\overline{x},\overline{y}}\,\vartheta(\ll,\overline{x},\overline{y})](\overline{x},\overline{y})$$

where

$$\vartheta(\overline{x},\overline{y}) := \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\}) \wedge \neg\exists\overline{u}(\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})) \wedge \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})).$$

Obviously, the formula χ' is positive in $\prec$ and itself a formula in LFP. Thus the least fixed point of χ' exists. We claim that this fixed point defines exactly the stage comparison relation $\prec_\varphi$ of φ. Before proving this we first have to establish some facts about the sub-formula ϑ. Recall from the beginning of this section that φ is supposed to be of the form $R\overline{x} \vee \varphi'$. This is important for the proofs below as it ensures that whenever a tuple $\overline{x}$ satisfies φ at a stage α, it satisfies φ at all higher stages also.

4.1 Lemma. *Consider the fixed-point induction on ϑ where $\prec$ is interpreted by $\prec^{<\alpha}$, the strict stage comparison relation up to stage α, i.e., $\overline{x} \prec \overline{y}$ if, and only if, $\overline{x} \in \varphi^{<\alpha}$ and $|\overline{x}|_\varphi < |\overline{y}|_\varphi$.*

- *(i) If $\overline{x} \in \varphi^\alpha$ or $\overline{y} \in \varphi^\alpha$, then $(\overline{x},\overline{y}) \in \vartheta^\infty$ if, and only if, $|\overline{x}| < |\overline{y}|$.*
- *(ii) For all $\overline{y}$ such that $|\overline{y}| > \alpha$ there is an $\overline{x}$ such that $|\overline{x}| = \alpha$ and $(\overline{x},\overline{y}) \in \vartheta^\infty$.*
- *(iii) If the fixed-point of $\prec$ has already been reached, i.e. if $\prec^\alpha = \prec^{<\alpha}$, then $\vartheta^\infty = \prec^\alpha$.*

Proof.

1. We first prove by induction on β that for all $\beta < \alpha$, $(\overline{x},\overline{y}) \in \vartheta^\beta$ if, and only if, $\overline{x} \in \varphi^\beta$ and $|\overline{x}| < |\overline{y}|$. This is clear for $\beta = 0$. Now assume that for all $0 \leq \gamma < \beta$ the claim has been proved. We distinguish between the case where $\overline{x} \in \varphi^\beta$ and $\overline{x} \notin \varphi^\beta$.

   - Suppose, $\overline{x} \in \varphi^\beta$. We show that $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \models \vartheta(\overline{x},\overline{y})$, if, and only if, $|\overline{x}| < |\overline{y}|$. We consider the three conjuncts of ϑ separately.

     If $\overline{x} \in \varphi^\beta$, then for all $\overline{u}$, by induction hypothesis, $\overline{u} \ll \overline{x}$ iff $|\overline{u}|_\varphi < |\overline{x}|_\varphi$ and, by assumption, $\neg\overline{u} \prec \overline{x}$ iff $|\overline{u}|_\varphi \geq |\overline{x}|_\varphi$. Thus,

     $$(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \models \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\}).$$

     Now consider $\overline{y}$. If $|\overline{y}|_\varphi > |\overline{x}|_\varphi$, then for all $\overline{u}$, $\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}$ if, and only if, $\overline{u} \ll \overline{x}$ which in turn is true only if $\overline{u} \prec \overline{x}$. Thus $\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ reduces to $\neg\overline{u} \prec \overline{x}$. Therefore there is no $\overline{u}$ satisfying $(\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$ and the second conjunct in ϑ is satisfied. Further, $\overline{y}$ does not satisfy $\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$ as otherwise $|\overline{y}|_\varphi \leq |\overline{x}|_\varphi$. Thus, $(\overline{x},\overline{y}) \in \vartheta^\beta$.

     On the other hand, if $|\overline{y}|_\varphi < |\overline{x}|_\varphi$, then $(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ in the second conjunct reduces to $\overline{u} \prec \overline{y}$ and thus there is a $\overline{u}$ satisfying $\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$, $\overline{u} := \overline{y}$ for instance.

     Finally, suppose $|\overline{x}|_\varphi = |\overline{y}|_\varphi$. By the same argument as above we get that in this case $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \models \varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/(\neg\overline{u} \ll \overline{x} \vee \neg\overline{u} \ll \overline{y}))$ and thus ϑ is not satisfied.

   - Suppose $\overline{x} \notin \varphi^\beta$. We show that $\varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\})$ is not satisfied. By induction hypothesis, $\overline{u} \ll \overline{x}$ defines the set $M := \varphi^{<\beta}$. Further, define $N := \{\overline{u} : \neg\overline{u} \prec \overline{x}\}$. As, by assumption, $\overline{x} \notin \varphi^\beta$, we get $\overline{M} \supseteq N$, where $\overline{M}$ denotes the complement of M.

     Thus $\mathfrak{A} \not\models \varphi(\overline{x}, R\overline{u}/\overline{u} \in M, \neg R\overline{u}/\overline{u} \in \overline{M})$ and, by monotonicity of φ in M and $\overline{M}$, it follows

     $$(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \not\models \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\}.$$

   Taken together we get that $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \models \vartheta(\overline{x},\overline{y})$ for a pair $(\overline{x},\overline{y})$ if, and only if, $\overline{x} \in \varphi^\beta$ and $|\overline{x}|_\varphi < |\overline{y}|_\varphi$.

2. From the first part of the proof we get that $(\overline{x},\overline{y}) \in \vartheta^{<\alpha}$, if, and only if, $\overline{x} \in \varphi^{<\alpha}$ and $|\overline{x}| < |\overline{y}|$. Thus, $\ll^{<\alpha} = \prec^{<\alpha}$. Now consider the next induction step. Again we distinguish between $\overline{x} \in \varphi^\alpha$ and $\overline{x} \notin \varphi^\alpha$.

   - Suppose $\overline{x} \in \varphi^\alpha$. Clearly, $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\alpha}) \models \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\})$.

     If $|\overline{y}|_\varphi \geq |\overline{x}|_\varphi$, then $(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ reduces to $\overline{u} \prec \overline{x}$ and thus $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\alpha}) \models \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$ if, and only if, $|\overline{y}|_\varphi > |\overline{x}|_\varphi$.

     Now suppose $|\overline{y}|_\varphi < |\overline{x}|_\varphi$. Then $|\overline{y}|_\varphi < \alpha$ and there is an $\overline{u}$ satisfying $\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$, again $\overline{y}$ being such a tuple itself. Thus $\vartheta(\overline{x},\overline{y})$ is not satisfied.

   - Now assume $\overline{x} \notin \varphi^\alpha$. Then $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\alpha}) \not\models \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\})$ as $\overline{u} \prec \overline{x}$ defines the set $\overline{u} \in \varphi^{<\alpha}$ and $\neg\overline{u} \prec \overline{x}$ its complement.

Taken together, ϑ^α contains all pairs $(\overline{x}, \overline{y})$ such that $\overline{x} \in \varphi^\alpha$ and $|\overline{x}|_\varphi < |\overline{y}|_\varphi$. This also proves Part (ii) because if there is a tuple $\overline{y}$ of rank greater than α there must also be a tuple $\overline{x}$ of rank exactly α and this pair would be in ϑ^α.

If the fixed-point of $\prec$ has already been reached, i.e. $\prec^\alpha = \prec^{<\alpha}$, then there are no tuples $\overline{x}$ of rank exactly α and thus all the tuples $(\overline{x}, \overline{y}) \in \vartheta^\alpha$ have already been in $\vartheta^{<\alpha}$ and the fixed-point of ϑ has been reached. This proves Part (iii) of the lemma. Thus, from now on, we assume that $\prec^{<\alpha} \subsetneq \prec^\alpha$.

3. We show now that in no stage $\gamma > \alpha$ a pair $(\overline{x}, \overline{y})$ such that $\overline{x}, \overline{y} \in \varphi^\alpha$ and $|\overline{y}|_\varphi \leq |\overline{x}|_\varphi$ can enter the fixed-point. Towards a contradiction let γ be the smallest such stage and let $(\overline{x}, \overline{y})$ be as described. Then the same argument as in the first item of Step 1 yields a contradiction.

4. What is left to be shown is that for no $\overline{x} \notin \varphi^\alpha$ and $\overline{y} \in \varphi^\alpha$ the pair $(\overline{x}, \overline{y})$ enters the fixed-point at some higher stage. Towards a contradiction, let γ be the least such stage, i.e. the least stage such that there is a pair $(\overline{x}, \overline{y}) \in \vartheta^\gamma$ with $\overline{x} \notin \varphi^\alpha$ and $\overline{y} \in \varphi^\alpha$.

   Now, as $\overline{x} \notin \varphi^\alpha$, $\overline{u} \prec \overline{x}$ defines just $\varphi^{<\alpha}$. As γ is supposed to be the least such stage, we get that $\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ defines the set of tuples $\overline{u}$ such that $|\overline{u}|_\varphi \geq |\overline{y}|_\varphi$. Thus, if $|\overline{y}|_\varphi < \alpha$ then there is a tuple $\overline{u}$ satisfying $\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ and thus ϑ is not satisfied by $(\overline{x}, \overline{y})$. On the other hand, if $|\overline{y}|_\varphi = \alpha$, then $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\alpha}) \models \varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$ and again ϑ is not satisfied.

This finishes the proof of the lemma. □

We now prove a technical lemma which will establish the induction step in the proof that the fixed point of χ' defines $\prec_\varphi$.

4.2 Lemma. *Let $\prec^{<\alpha}$ be the strict stage comparison relation up to stage α, i.e., $\overline{x} \prec^{<\alpha} \overline{y}$ if, and only if, $\overline{x} \in \varphi^{<\alpha}$ and $|\overline{x}|_\varphi < |\overline{y}|_\varphi$.*

Then $(\mathfrak{A}, \prec^{<\alpha}) \models \chi'(\overline{x}, \overline{y})$, if, and only if, $\overline{x} \in \varphi^\alpha$ and $|\overline{x}|_\varphi < |\overline{y}|_\varphi$.

Proof. We distinguish between the cases where $\overline{x} \in \varphi^\alpha$ and $\overline{x} \notin \varphi^\alpha$.

- Suppose $\overline{x} \in \varphi^\alpha$. By assumption, $\overline{u} \prec \overline{x}$ defines the set $\{\overline{u} : |\overline{u}|_\varphi < |\overline{x}|_\varphi\}$ and, by Part (i) of Lemma 4.1, $\neg\overline{u} \lhd \overline{x}$ defines its complement. Thus, $(\mathfrak{A}, \prec^{<\alpha}) \models \varphi(\overline{x}, R\overline{u}/u \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \lhd \overline{x})$ and all $\overline{u}$ satisfy $\overline{u} \prec \overline{x} \vee \neg\overline{u} \lhd \overline{x}$.

  Now, $(\mathfrak{A}, \prec^{<\alpha}) \models \varphi(\overline{y}, R\overline{u}/\overline{u} \lhd \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x})$ if, and only if, $|\overline{y}|_\varphi > |\overline{x}|_\varphi$.

  Thus, $(\mathfrak{A}, \prec^{<\alpha}) \models \chi'(\overline{x}, \overline{y})$ if, and only if, $|\overline{y}|_\varphi > |\overline{x}|_\varphi$.

- Suppose $\overline{x} \notin \varphi^\alpha$. Then $\overline{u} \prec \overline{x}$ defines the set $\{\overline{u} : \overline{u} \in \varphi^{<\alpha}\}$. If $\varphi^{<\alpha} = \varphi^\alpha$, i.e. if the fixed-point of φ has been reached, then, by Part (iii) of Lemma 4.1, we get $\lhd = \prec$ and $(\mathfrak{A}, \prec^{<\alpha}) \not\models \varphi(\overline{x}, R\overline{u}/u \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \lhd \overline{x})$. Therefore χ' is not satisfied.

  Otherwise, i.e. if $\varphi^{<\alpha} \subsetneq \varphi^\alpha$, then, by Part (ii) of Lemma 4.1, there is a tuple $\overline{a}$ of rank α with $\overline{a} \lhd \overline{x}$. Thus, the conjunct $\forall\overline{u}(\overline{u} \prec \overline{x} \vee \neg\overline{u}\lhd \prec \overline{x})$ is not satisfied as $\overline{a} \lhd \overline{x}$ but $\overline{a} \not\prec \overline{x}$.

This finishes the proof of the lemma. □

As corollary of this we get that the relation $\prec_\varphi$ is definable in LFP.

4.3 Corollary. *Let $\varphi(R, \overline{x})$ be a formula in LFP. Then the stage comparison relation $\prec_\varphi$ of the inflationary fixed point of φ is definable in LFP.*

Proof. A simple induction on the stages using the previous lemma shows that $\prec_\varphi$ is defined by the formula χ above. □

The equivalence of LFP and IFP follows immediately.

4.4 Theorem. *For every formula in* IFP *there is a formula in* LFP *equivalent to it on all structures.*

Proof. By Corollary 4.3, the relation $\prec_\varphi$ is definable in LFP for every $\varphi(R, \overline{x}) \in$ LFP. Thus, for all $\overline{x}$, $\overline{x} \in \varphi^\infty$ if, and only if, $\mathfrak{A} \models \varphi(\overline{x}, R\overline{u}/\chi(\overline{u}, \overline{x}))$. Thus, the inflationary fixed point of a LFP-formula can be defined in LFP.

For arbitrary formulae $\varphi \in$ IFP, the theorem follows by induction on the number of inflationary fixed points in φ converting them to least fixed points from the inside out. □

The theorem shows that also on infinite structures, least and inflationary fixed-point logic have the same expressive power. But, contrary to the case of finite structures where the translation of IFP-formulae to equivalent LFP-formulae does not alter the fixed-point structure - although the resulting formulae often become more complicated and less understandable - in the general case also their structure in terms of alternations between **lfp**-operators and negation as well as the nesting depth of fixed-point operators becomes more complicated. In the next section we show that this cannot be avoided. Although it might be possible to lower the increase in nesting depth of the resulting LFP-formulae, we will show that in general an increase in the number of alternations is necessary.

5. Normal Forms and Hierarchies

In this section we investigate on the question whether allowing to nest **ifp**-operators inside each other or to use them negatively increases the expressive power of the resulting logic. In particular, we will show that using of fixed-point operators negatively does not increase the expressive power of IFP whereas nesting **ifp**-operators does.

5.1 Theorem. *Each* IFP*-formula is equivalent to an* IFP*-formula where negation occurs only in front of atoms.*

Proof. The theorem is proved by induction on the structure of the IFP-formula. For the case of the **ifp**-operator note that a formula $\neg[\mathbf{ifp}_{R,\overline{x}}\,\varphi]\overline{t}$ is equivalent to the simultaneous fixed point $[\mathbf{ifp}\;Q : S]\overline{t}$ of the system

$$S := \begin{cases} R\overline{x} & \leftarrow \varphi(R,\overline{x}) \\ Q\overline{x} & \leftarrow \forall\overline{y}\,(\varphi(R,\overline{y}) \rightarrow R\overline{y}) \wedge \neg R\overline{x}. \end{cases}$$

On structures with at least two elements this is equivalent to the inflationary fixed point of a single formula whereas on structures with only one element IFP collapses to FO anyway and the theorem is trivial. □

We now turn to the question whether nested **ifp**-operators can be reduced to a single one as it can be done in LFP. For this we first need some technical definitions.

5.2 Definition (Alternation and nesting-depth hierarchy). *Let* $\varphi \in$ LFP *be a formula such that no fixed-point variable is bound twice in it and let* $X_1, \dots, X_k$ *be the fixed-point variables occurring in* φ*. Let for all* i*,* φ_i *be the formula binding* X_i *in* φ*, i.e.* $\varphi_i := [\mathbf{lfp}_{X_i,\overline{x}_i}\ \varphi'_i]$ *for suitable* $\overline{x}_i$ *and* φ'_i*.*

We define a partial order $\sqsubseteq_\varphi$ *on the variables* $X_1, \dots, X_k$ *as*

$X_i \sqsubseteq_\varphi X_j$ *if, and only if,* φ_i *is a sub-formula of* φ_j

(i) The nesting-depth *of* φ *is defined as the maximal cardinality of a subset of* $\{X_1, \dots, X_k\}$ *which can be linearly ordered by* $\sqsubseteq_\varphi$*.*

(ii) The alternation-level *of* φ *is defined as the maximal cardinality of a subset* $\mathcal{M}$ *of* $\{X_1, \dots, X_k\}$ *which can be linearly ordered by* $\sqsubseteq_\varphi$ *such that in addition for all* $X_i, X_j \in \mathcal{M}$*, if* X_i *is a direct predecessor of* X_j *with respect to* $\sqsubseteq_\varphi$*, then* φ_i *occurs negatively in* φ_j*.*

The n*-th level of the alternation hierarchy* $(\mathrm{LFP}^a_n)_{n\in\omega}$ *consists of all formulae of* LFP *with alternation-level* n*. Analogously, the* n*-th level of the nesting-depth hierarchy* $(\mathrm{LFP}^d_n)_{n\in\omega}$ *of* LFP *is defined as the class of formulae in* LFP *of nesting-depth* n*.*

The alternation and nesting-depth hierarchy for IFP *are defined analogously.*

By definition, LFP^a_1 consists of all LFP-formulae where no **lfp**-operator occurs negatively, whereas LFP^a_0 and LFP^d_0 are just the class of first-order formulae.

Theorem 5.1 above shows that the alternation hierarchy of IFP collapses to level one. It follows from results by Moschovakis [9] and Bradfield [2] that in general the alternation hierarchy for LFP is strict on infinite structures, the structure of arithmetic being an example. Note that this definition of the alternation hierarchy is the simplest possible and does not capture the true nature of alternation. Therefore more sophisticated definitions of the alternation hierarchy have been proposed in the literature. (See [6, 11, 2] and references therein.) Here we use the alternation hierarchy only to establish the next theorem showing that in inflationary fixed-point logic, nested fixed points cannot be eliminated in favour of a single one. Therefore we stick to the simplest possible definition of alternation. But the results below can easily be adapted to the other definitions as well. We now prove the main result of this section.

5.3 Theorem. *For every* $n \geq 0$*,*

$$\mathrm{LFP}^a_n \subseteq \mathrm{IFP}^d_n \subseteq \mathrm{LFP}^a_{3^n}$$

Proof. Let $\varphi \in \mathrm{LFP}_n$ be a formula with alternation depth n. It is known that nested **lfp**-operators which all occur positively can be contracted to a single **lfp**-operator increasing the arity. Thus, every formula in LFP_n is equivalent to a formula with n nested fixed-points. By Theorem 5.1 above, this formula again is equivalent to an IFP formula with nesting depth n.

Towards the second containment, note that using the method of Theorem 4.4 to convert an IFP-formula to an equivalent LFP-formula, the translation of each individual **ifp**-operator at most triples the alternation depth. The theorem now follows by induction. □

We immediately get the following corollaries.

5.4 Corollary. *For any structure, the alternation depth hierarchy for LFP collapses if, and only if, the nesting depth hierarchy for IFP collapses.*

An example of a class of structures where the hierarchies are strict is the class of acceptable structures (see [9] for instance.)

5.5 Corollary. *Let* $\mathfrak{A}$ *be acceptable. Then the nesting depth hierarchy for IFP is strict. The same holds, if* $\mathfrak{A}$ *is not acceptable but allows the definition of a coding scheme in IFP.*

Proof. This follows immediately from Corollary 5.5 and the results of Moschovakis [9] and Bradfield [2] on the alternation hierarchy of LFP on acceptable structures. □

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