Quantum Processes and Computation

Assignment 1, Wednesday 14 Oct 2020

Deadline: Wednesday 21 Oct 2020, 17:00

Goals: After completing these exercises successfully you should be able to perform simple diagrammatic computations and reason with cups, caps, and process-state duality in string diagrams.

Note: Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been slightly modified for the problem sheet. The corresponding exercise number from the book is shown in brackets.

Exercise 1 (3.4): We saw in the lecture that **functions** and **relations** are examples of process theories. Give two other examples of a process theory. For each one answer the following questions:

1. What are the system-types?
2. What are the processes?
3. What does it mean to compose them sequentially or in parallel?
4. When should two processes be considered equal?

Hint: Note that a single process is not a process theory. In particular, almost any process theory will have an infinite amount of system types (e.g. A, $A \otimes A$, $A \otimes A \otimes A, \ldots$). Also: Be creative! You don't have to restrict yourself to mathematics.

Exercise 2 (3.10): Please read Section 3.1.3 about diagrams as diagram formulas. Draw the diagrams corresponding to the following diagram formulas:

1. $f_{B_1 C_2}^{C_4} g_{C_4}^{D_3}$
2. $f_{A_1}^{A_1}$
3. $g_{B_1}^{A_1} f_{A_1}^{B_1}$
4. $1_{A_1}^{A_6} 1_{A_2}^{A_5} 1_{A_3}^{A_4}$.

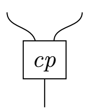
Use the convention that inputs and outputs are numbered from left-to-right.

Exercise 3 (3.12): Give the diagrammatic equations of a process $*$ taking two inputs and one output that express the algebraic properties of being

1. associative: $x * (y * z) = (x * y) * z$
2. commutative: $x * y = y * x$
3. having a unit: there exists a process e (with no inputs) such that $x * e = e * x = x$

Note: x, y and z should not appear in your final diagrams. They are however useful in trying to figure out what the diagrammatic equation should be.

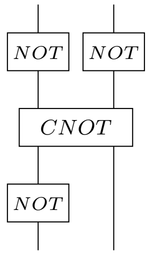
Exercise 4 (3.15): Using the copy operation:

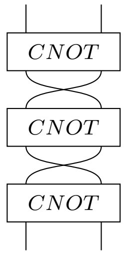


$:: \quad n \mapsto (n, n)$

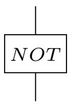
write down the diagram representing distributivity: $(x + y) * z = (x * z) + (y * z))$? Here, $+$ and $*$ are processes that take two inputs and and one output.

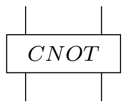
Exercise 5 (3.30): First compute the values of the following functions, then give the commonly used name of these functions:

$(a):$ 

$(b):$ 

where:

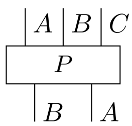


$:: \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$ and 

$:: \begin{cases} (0,0) \mapsto (0,0) \\ (0,1) \mapsto (0,1) \\ (1,0) \mapsto (1,1) \\ (1,1) \mapsto (1,0) \end{cases}$

Exercise 6 (3.31): Suppose A, B, C, and D are sets and P is a relation given by:

$$\begin{aligned} A &= \{a_1, a_2, a_3\} \\ B &= \mathbb{B} \\ C &= \{\mathbf{red}, \mathbf{green}\} \\ D &= \mathbb{N} \end{aligned}$$



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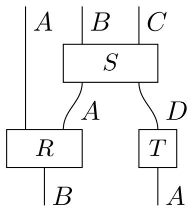


Compute P first for R, S, T given by:

$$R :: \begin{cases} 1 \mapsto (a_1, a_1) \\ 1 \mapsto (a_1, a_2) \end{cases} \qquad S :: \begin{cases} (a_1, 5) \mapsto (0, \mathbf{red}) \\ (a_1, 5) \mapsto (1, \mathbf{red}) \\ (a_2, 6) \mapsto (1, \mathbf{green}) \end{cases} \qquad T :: \begin{cases} a_1 \mapsto 200 \\ a_3 \mapsto 5 \end{cases}$$

and then for R, S, T given by:

$$R :: \begin{cases} 0 \mapsto A \times \{a_2, a_3\} \\ 1 \mapsto A \times \{a_2, a_3\} \end{cases} \qquad S :: \begin{cases} (a_1, 0) \mapsto \mathbb{B} \times \{\mathbf{red}, \mathbf{green}\} \\ (a_1, 1) \mapsto \mathbb{B} \times \{\mathbf{red}, \mathbf{green}\} \\ (a_1, 2) \mapsto \mathbb{B} \times \{\mathbf{red}, \mathbf{green}\} \\ \quad \vdots \end{cases} \qquad T :: \begin{cases} a_1 \mapsto \mathbb{N} \\ a_2 \mapsto \mathbb{N} \\ a_3 \mapsto \mathbb{N} \end{cases}$$

Hint: This exercise is in fact well-defined, and does not contain typos. Please read Section 3.3.3 if you are confused.

Exercise 7 (3.38 & 3.40): Suppose that there is a zero process $0 : A \to B$ for all possible types A and B (see Section 3.4.2).

(a) Show that for each type the zero process is unique. I.e. show that if $0' : A \to B$ is also a process satisfying the requirements of a zero process that then $0' = 0$.

(b) We call two processes f and g with the same inputs and outputs *equal up to a number* (written $f \approx g$) if there exist non-zero numbers λ, μ such that $\lambda f = \mu g$. Suppose a process theory has *no zero divisors*. That is, it satisfies the following property: $\lambda f = 0$ if and only if $\lambda = 0$ or $f = 0$. Show that $f \approx 0$ if and only if $f = 0$.

Exercise 8 (4.10 & 4.16):

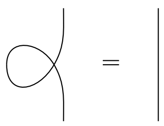
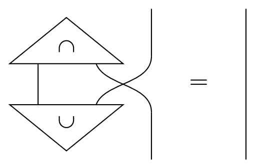
(a) Prove that in **relations**, the following relations on a set A:

 $:: * \mapsto \{(a,a) \mid a \in A\}$  $:: \forall a \in A : (a,a) \mapsto *$

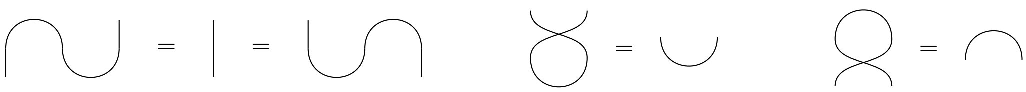
satisfy the *yanking equations* (eq. 4.11 in the book), and thus that **relations** has process-state duality.

(b) Show that process-state duality does not hold for **functions**.

Exercise 9 (4.12): Prove that

 or written differently: 

follows from the following 3 equations:

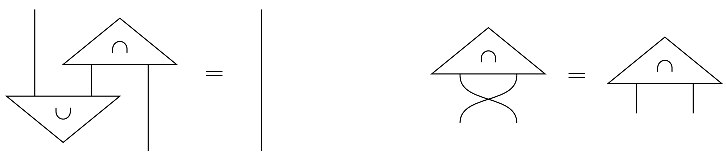


Hint: Use the second notation (with the boxes) as it prevents you from accidentally cheating.

Exercise 10 (4.14 in online version of PQP): Show that the following are equivalent:

(i) a state and an effect satisfying:

(ii) a state and an effect satisfying:



So in particular, if either eqs. (i) or eqs. (ii) hold, then all equations hold.