Quantum Processes and Computation

Assignment 2, Wednesday, 28 Oct, 17:00

Deadline: Wednesday, 4 Nov, 17:00

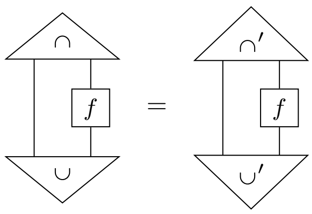
Goals: After completing these exercises you should know how to reason with the transpose, adjoints, and the conjugate, and work with projections, unitaries and isometries. You will also know about orthonormal bases, composition of linear maps, and encoding logic gates as linear maps. Material covered in book: Chapter 4, Sections 5.1, 5.2, 5.3.4 and a bit of 5.3.5.

Note: Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been slightly modified for the problem sheet. The corresponding exercise number from the book is shown in brackets.

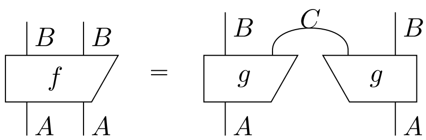
Exercise 1 (4.59): An *inverse* for a process $f : A \to B$ is a process $f^{-1} : B \to A$ such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. Show that for a process f the following are equivalent:

- f is unitary.
- f is an isometry and has an inverse.
- $f^\dagger$ is an isometry and has an inverse.

Exercise 2 (4.37): Show that the trace of a process is independent of the particular choice of cup and cap, i.e. that if $\cup$ and $\cap$ satisfy the yanking equations, but $\cup'$ and $\cap'$ also satisfy it that then:



In the lecture we saw the notion of a positive process. There is also a notion of $\otimes$-positivity. A process f is $\otimes$-positive if there exists a process g such that

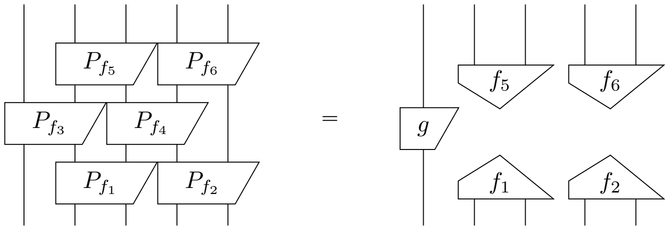


(see section 4.3.6 of the book for more information)

Exercise 3 (4.67): Show that the sequential composition of two $\otimes$-positive processes is again a $\otimes$-positive process.

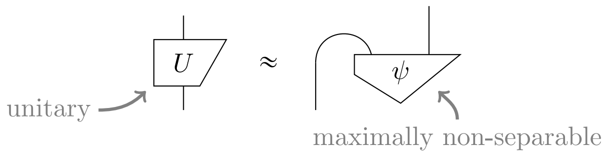
For a process $f : A \to A$ we define its *separable projector* by

Exercise 4 (4.73): Given processes $f_i : A \to A$ find the process g such that:

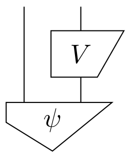


Write g as a sequential composition of the conjugates, transposes and adjoints of the f_i's.
Hint: Doing exercise 4.73 from the book first might reveal whether you understand the concept.

Exercise 5 (4.82): A state ψ is *maximally non-separable* if it corresponds to a unitary U by process-state duality, up to a number:

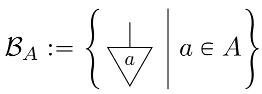


Show (i) that if one applies a unitary V to one side of a maximally non-separable state:



that one again obtains a maximally non-separable state, and (ii) that this unitary can always be chosen such that the resulting state is the cup (up to a number).

Exercise 6 (5.4): We saw in the lecture that for a set A with n elements in **relations** the singletons:

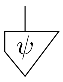
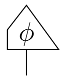


form a basis, that is, that no element can be removed from $\mathcal{B}_A$ without losing the property of being a basis. This basis is also orthonormal. Show that this is the *only* orthonormal basis of A.

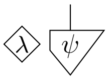
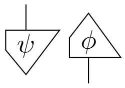
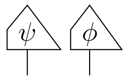
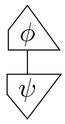
Bonus exercise: The orthonormality condition is actually not necessary for proving the uniqueness of the basis. Show that any basis (not necessarily orthonormal) of A must be the singleton basis.

For the next two exercises, assume we are working in a process theory with bases and sums (e.g. **relations** or **linear maps**).

Exercise 7 (5.54): Let

 $\leftrightarrow \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}$ and  $\leftrightarrow \begin{pmatrix} \phi_0 & \phi_1 \end{pmatrix}$

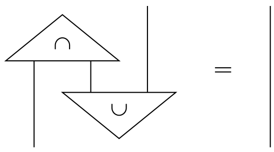
be respectively a 2-dimensional state, and 2-dimensional effect. Let λ be a number. Write the matrices for the processes

(i)  *(ii)*  *(iii)*  *(iv)* 

Exercise 8 (5.58): The matrices for cups and caps in 2 dimensions are:

 $\leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$  $\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$

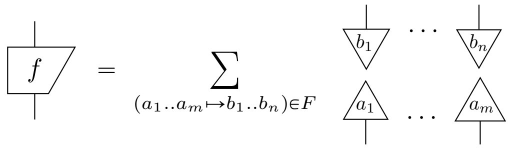
(i) First, verify the yanking equation



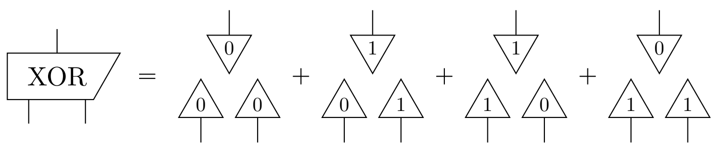
directly using the matrices of the 2-dimensional cup and cup by using the rules for sequential and parallel composition of matrices, i.e. show that $(\cap \otimes \text{id}) \circ (\text{id} \otimes \cup) = \text{id}$ (where id is the 2×2 identity matrix).

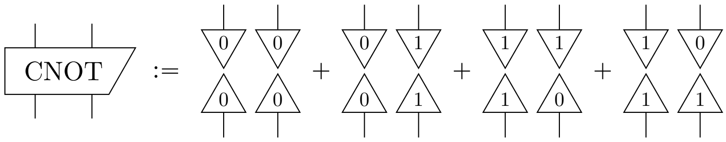
(ii) Second, give the matrices for the cup and cap in 3 dimensions.

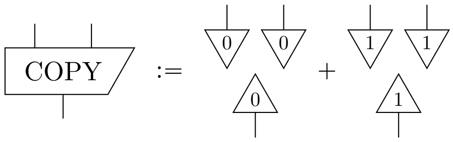
The next excercise is about encoding classical logic gates in the theory of **linear maps**, as explained in Section 5.3.4. Recall that a classical logic gate F can be encoded as a linear map via:



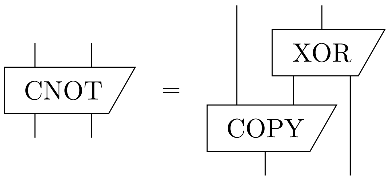
Using this encoding, we defined:





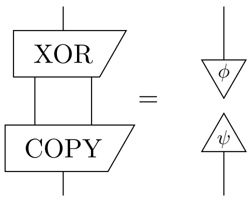


Exercise 9 (5.86): Show that



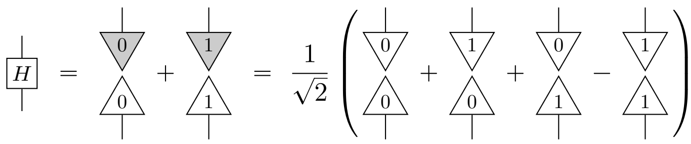
(Hint: try comparing the LHS to the RHS on all basis states, rather than writing out a big sum.)

Next, find ψ and ϕ such that the following equation holds:

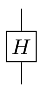


Although it might not look like much now, this equation will turn out to lie at the heart of the notion of *complementarity* which we will cover in great depth in the coming lectures.

Exercise 10 (5.93): In the proof of proposition 5.92, we see the Hadamard process written in matrix form with respect to the Z basis:



From this we can conclude the matrix of H (with respect to the Z-basis) is:

 $\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

What is the matrix of H in the X-basis?