Quantum Processes and Computation

Assignment 4, Wednesday, November 18, 2020

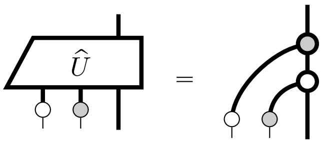
Deadline: Wednesday, November 25, 17:00

Goals: After completing these exercises you should know how to prove equalities between classical-quantum maps using the ZX-calculus, translate to and from quantum circuit notation, and do basic quantum circuit optimisations by hand.
Material covered in book: Chapters 9, 12.1

Note: Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been slightly modified for the problem sheet. The corresponding exercise number from the book is shown in brackets. **If you are stuck, try looking up the exercise number in the book. Usually the definitions or equations you need are nearby.**

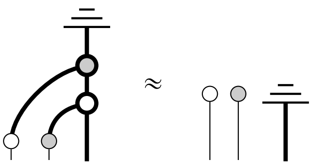
Exercise 1 (9.49, 9.58)

1. Read Section 9.2.7 about teleportation with complementary spiders. To show that it is a valid protocol we must show that the actions of both Aleks and Bob can in fact be performed, i.e. that they are causal cq-maps. We already know that Aleks's side is causal, so it remains to show that Bob's side is as well, i.e. that

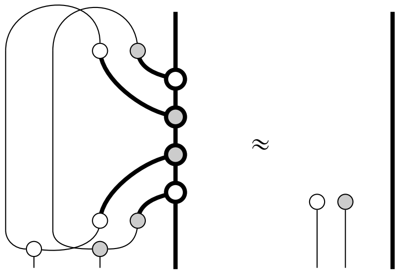


(1)

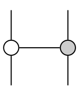
is causal, up to a number:



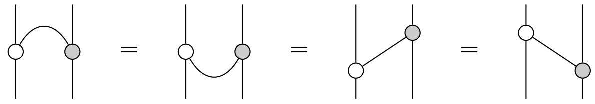
2. Show that furthermore, this map is a controlled isometry, up to a number, i.e.



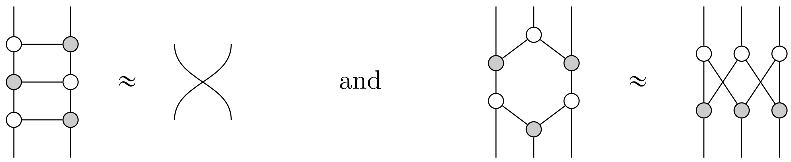
Exercise 2 (9.47): Read Section 9.2.3 about the controlled-NOT gate (CNOT):



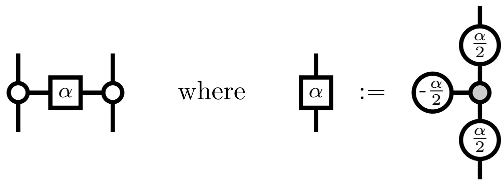
(i) Complete Lemma 9.46 by proving the remaining equalities:



(ii) Use complementarity and strong complementarity to prove that

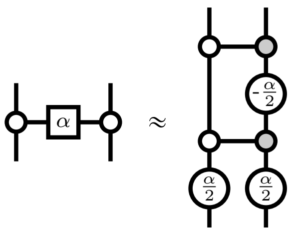


There are a couple of sets quantum gates that crop up a lot in quantum computation. One of these is the *controlled phase* gate CZ(α):

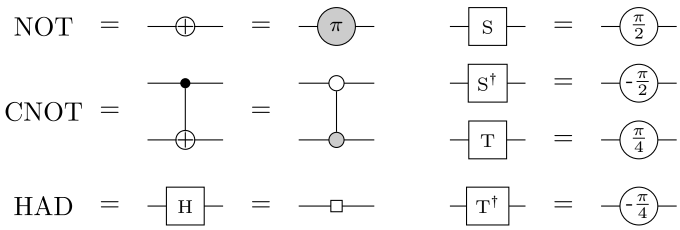


for any phase α. This indeed gives a controlled-phase gate:

Exercise 3 (12.8): Show using the ZX-calculus that the CZ(α)-gate can be built from CNOTs and ○ phase gates as follows:



The usage of ZX-diagrams to represent quantum circuits is not (yet) standard. A more commonly used notation is known just as *quantum circuit notation*. In this notation, the atomic building blocks are gates instead of spiders. A set of commonly used gates and their notation is:

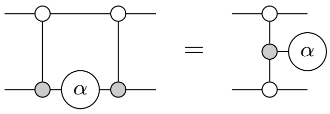


(2)

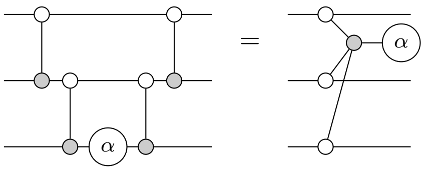
All of these gates are Clifford except for the T gate and its adjoint $T^\dagger$. To make sure you get familiar with this notation, we will use it a few times throughout this exercise sheet.

Exercise 4:

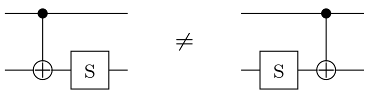
(i) In the last exercise sheet you had to find a circuit representation of the controlled-phase gate. The important part of this gate is what we call a *phase-gadget*:



Show that this equation generalises by proving the following equation in the ZX-calculus:

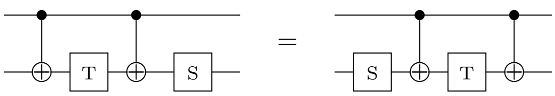


(ii) Show that a phase gate does not commute past the target of a CNOT:

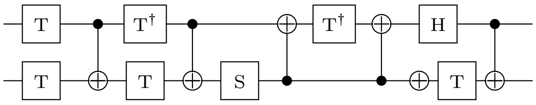


Hint: Find some input state under which the two sides aren't equal.

(iii) Show that a phase gate *does* commute past a phase-gadget:



Exercise 5: Consider the following quantum circuit:



The goal is to find a more optimal implementation of this circuit. Write this circuit as a ZX-diagram, simplify it using diagrammatic reasoning, and then write it again as a quantum circuit in quantum circuit notation using the gate-set of (2). Use whatever rewrite rules you know to minimize the *T-count* of the circuit. This is the amount of T and T† gates in the circuit (or in terms of a ZX-diagram, the amount of spiders that have odd multiples of $\frac{\pi}{4}$ on them). For full marks, find an equivalent circuit that has a T-count of zero.

T-count optimisation is an important problem in large-scale quantum computation as in most fault-tolerant architectures, Clifford gates are cheap to implement, while T gates can require a lot of resources and time to implement (sometimes up to 100 times more than a CNOT).