Goals: After completing these exercises you should know how to prove equalities between classical-quantum maps using the ZX-calculus, translate to and from quantum circuit notation, and do basic quantum circuit optimisations by hand.

Material covered in book: Chapters 9, 12.1

Note: Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been slightly modified for the problem sheet. The corresponding exercise number from the book is shown in brackets. If you are stuck, try looking up the exercise number in the book. Usually the definitions or equations you need are nearby.

Exercise 1 (9.49, 9.58)

1. Read Section 9.2.7 about teleportation with complementary spiders. To show that it is a valid protocol we must show that the actions of both Aleks and Bob can in fact be performed, i.e. that they are causal cq-maps. We already know that Aleks’s side is causal, so it remains to show that Bob’s side is as well, i.e. that

\[
\hat{U} = \begin{array}{c}
\text{(1)}
\end{array}
\]

is causal, up to a number:

2. Show that furthermore, this map is a controlled isometry, up to a number, i.e.

Exercise 2 (9.47): Read Section 9.2.3 about the controlled-NOT gate (CNOT):

(i) Complete Lemma 9.46 by proving the remaining equalities:
(ii) Use complementarity and strong complementarity to prove that

There are a couple of sets quantum gates that crop up a lot in quantum computation. One of these is the controlled phase gate $\text{CZ}(\alpha)$:

for any phase $\alpha$. This indeed gives a controlled-phase gate:

**Exercise 3 (12.8):** Show using the ZX-calculus that the $\text{CZ}(\alpha)$-gate can be built from CNOTs and $\circ$ phase gates as follows:

The usage of ZX-diagrams to represent quantum circuits is not (yet) standard. A more commonly used notation is known just as *quantum circuit notation*. In this notation, the atomic building blocks are gates instead of spiders. A set of commonly used gates and their notation is:

\[
\begin{align*}
\text{NOT} &= \circ = \pi, \\
\text{CNOT} &= \begin{array}{c}
\end{array}, \\
\text{HAD} &= -1
\end{align*}
\]

(2)

All of these gates are Clifford except for the $T$ gate and its adjoint $T^\dagger$. To make sure you get familiar with this notation, we will use it a few times throughout this exercise sheet.

**Exercise 4:**
(i) In the last exercise sheet you had to find a circuit representation of the controlled-phase gate. The important part of this gate is what we call a phase-gadget:

\[ \alpha = \alpha \]

Show that this equation generalises by proving the following equation in the ZX-calculus:

\[ \alpha = \alpha \]

(ii) Show that a phase gate does not commute past the target of a CNOT:

\[ S \neq S \]

**Hint:** Find some input state under which the two sides aren’t equal.

(iii) Show that a phase gate *does* commute past a phase-gadget:

\[ T = T \]

**Exercise 5:** Consider the following quantum circuit:

\[ T \]

The goal is to find a more optimal implementation of this circuit. Write this circuit as a ZX-diagram, simplify it using diagrammatic reasoning, and then write it again as a quantum circuit in quantum circuit notation using the gate-set of (2). Use whatever rewrite rules you know to minimize the *T-count* of the circuit. This is the amount of T and T\(^\dagger\) gates in the circuit (or in terms of a ZX-diagram, the amount of spiders that have odd multiples of \(\frac{\pi}{4}\) on them). For full marks, find an equivalent circuit that has a T-count of zero.

T-count optimisation is an important problem in large-scale quantum computation as in most fault-tolerant architectures, Clifford gates are cheap to implement, while T gates can require a lot of resources and time to implement (sometimes up to 100 times more than a CNOT).