**Quantum Processes and Computation**  
**Assignment 2, Friday, 22 Oct, 17:00**

**Deadline:** Week 3 (group 1) or week 4 (groups 2-6)

**Goals:** After completing these exercises you should know how to reason with the transpose, adjoints, and the conjugate, and work with projections, unitaries and isometries. Material covered in book: Chapter 4.

**Note:** Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been slightly modified for the problem sheet. The corresponding exercise number from the book is shown in brackets.

**Exercise 1 (4.10 & 4.16):**
(a) Prove that in relations, the following relations on a set $A$:

\[ \cup :: x \mapsto \{(a, a) \mid a \in A\} \quad \cap :: \forall a \in A : (a, a) \mapsto \star \]

satisfy the yanking equations (eq. 4.11 in the book), and thus that relations has process-state duality.

(b) Show that process-state duality does not hold for functions.

**Exercise 2 (4.12):** Prove that

\[ \begin{array}{c|c|c}
\cup & = & \cap \\
\cap & \setminus & \cup \\
\cap & \cap & \cup \\
\cap & \cup & \cup \\
\end{array} \]

follows from the following 3 equations:

\[ \begin{array}{c|c|c|c|c|c}
\cap & \cap & \cap & \cap & \cap & \cap \\
\cap & \cap & \cap & \cap & \cap & \cap \\
\cap & \cap & \cap & \cap & \cap & \cap \\
\cap & \cap & \cap & \cap & \cap & \cap \\
\end{array} \]

**Hint:** Use the second notation (with the boxes) as it prevents you from accidentally cheating.

**Exercise 3 (4.14 in online version of PQP):** Show that the following are equivalent:

(i) a state and an effect satisfying:

\[ \begin{array}{c|c|c|c|c|c|c}
\cap & \cap & \cap & \cap & \cap & \cap & \cap \\
\cap & \cap & \cap & \cap & \cap & \cap & \cap \\
\cap & \cap & \cap & \cap & \cap & \cap & \cap \\
\cap & \cap & \cap & \cap & \cap & \cap & \cap \\
\end{array} \]

(ii) a state and an effect satisfying:

\[ \begin{array}{c|c|c|c|c|c|c}
\cap & \cap & \cap & \cap & \cap & \cap & \cap \\
\cap & \cap & \cap & \cap & \cap & \cap & \cap \\
\cap & \cap & \cap & \cap & \cap & \cap & \cap \\
\cap & \cap & \cap & \cap & \cap & \cap & \cap \\
\end{array} \]
So in particular, if either eqs. (i) or eqs. (ii) hold, then all equations hold.

Exercise 4 (4.59): An inverse for a process \( f : A \to B \) is a process \( f^{-1} : B \to A \) such that \( f^{-1} \circ f = \text{id}_A \) and \( f \circ f^{-1} = \text{id}_B \). Show that for a process \( f \) the following are equivalent:

- \( f \) is unitary.
- \( f \) is an isometry and has an inverse.
- \( f^\dagger \) is an isometry and has an inverse.

Exercise 5 (4.37): Show that the trace of a process is independent of the particular choice of cup and cap, i.e. that if \( \cup \) and \( \cap \) satisfy the yanking equations, but \( \cup' \) and \( \cap' \) also satisfy it then:

For a process \( f : A \to A \) we define its separable projector by

\[
\begin{align*}
P_f & := f \\
\end{align*}
\]

where

\[
\begin{align*}
\bar{f} & := \overline{f} \\
\end{align*}
\]

Exercise 6 (4.73): Given processes \( f_i : A \to A \) find the process \( g \) such that:

Write \( g \) as a sequential composition of the conjugates, transposes and adjoints of the \( f_i \)'s.

Hint: Doing exercise 4.73 from the book first might reveal whether you understand the concept.

Exercise 7 (4.82): A state \( \psi \) is maximally non-separable if it corresponds to a unitary \( U \) by process-state duality, up to a number:
Show (i) that if one applies a unitary $V$ to one side of a maximally non-separable state:

that one again obtains a maximally non-separable state, and (ii) that this unitary can always be chosen such that the resulting state is the cup (up to a number).