Quantum Processes and Computation

Assignment 4, Friday, 5 Nov

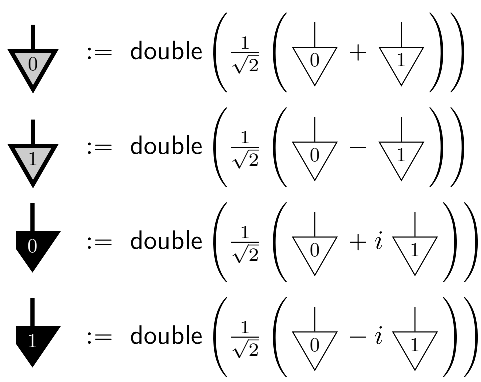
Deadline: Week 5 (group 1) or week 6 (groups 2-6)

Goals: After completing these exercises you should know how to plot states on the Bloch sphere as well as recognise and work with quantum maps. Material covered in book: Chapter 6.

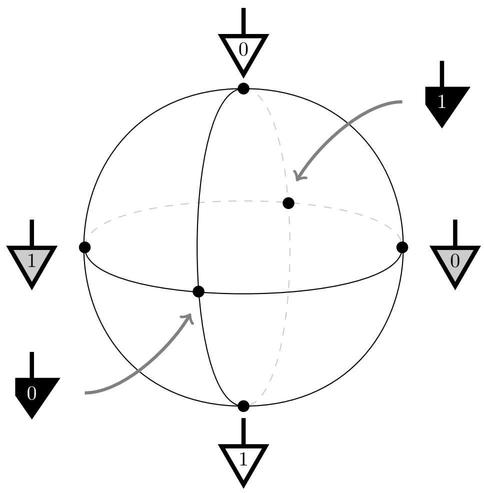
Note: Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been modified for the problem sheet. The corresponding exercise number from the book is shown in brackets. **If you are stuck, try looking up the exercise number in the book. Usually the definitions or equations you need are nearby.**

In section 6.1.2, it was shown that 2D quantum pure states correspond to points on a sphere.

Exercise 1 (6.7) (30 points): Show that the following points:

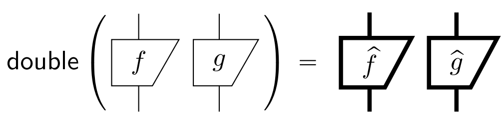


are located on the Bloch sphere as follows:



Exercise 2 (6.10 & 6.22):

(i) Show that doubling preserves parallel composition:

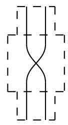


(ii) Show that doubling preserves normalisation: that a state ψ is normalised if and only if its doubled state $\widehat{\psi}$ is normalised.

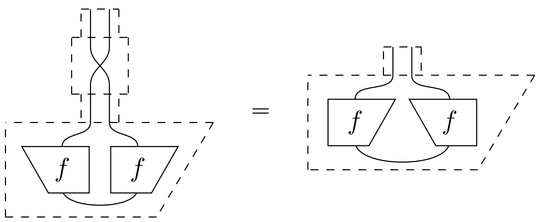
(iii) Show that doubling preserves orthogonality: that states ψ and ϕ are orthogonal if and only if $\widehat{\psi}$ and $\widehat{\phi}$ are orthogonal.

Hint: Use theorem 6.17 for the latter two points.

The transpose of a positive process is again a positive process and by bending some wires we can also take the 'transpose' of a $\otimes$-positive state, i.e. of a quantum state (see **Corollary 6.36**). This transpose acts as a swap of wires on the doubled system:

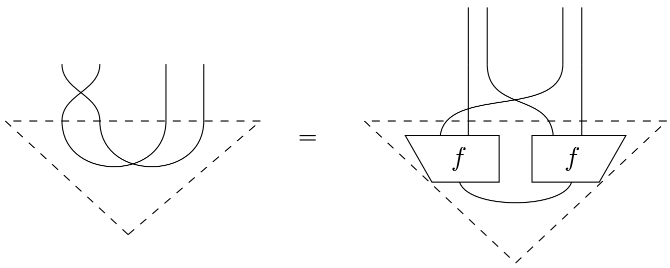


and it indeed sends quantum states to quantum states:



In the next exercise we will show that nevertheless, this swap of wires is *not* a quantum operation.

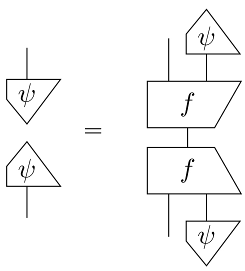
Exercise 3: In this exercise we will show that a swap applied to one pair of the wires of the doubled cup state will result in a state that is no longer $\otimes$-positive, and therefore not a quantum state. We will do this by contradiction. So suppose:



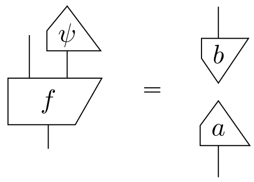
(1)

for some process f.

(i) Let ψ be a normalised state. Show that the equation above implies that



and hence, by Proposition 5.74, that there exist states a and b such that:

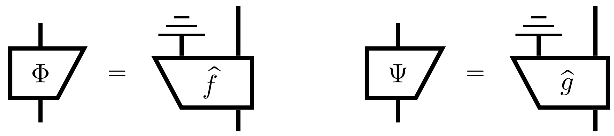


(2)

(ii) Plug ψ into equation 1 and use equation 2 to show that the identity wire disconnects. Conclude that therefore the swap can't be a quantum map.

Note: In proposition 6.48 it is also shown that the swap is not a quantum operation, but it uses a specific counter-example found in **linear maps**. The proof above only uses string diagrams and the property implied by proposition 5.74.

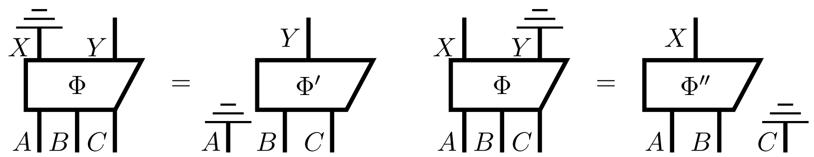
Exercise 4: Let Φ and Ψ be quantum maps with purifications f, respectively g:



Give purifications of $\Phi \circ \Phi$ and $\Psi \circ \Phi$ in terms of f and g.

Exercise 5: Prove that not every quantum map is pure, by showing that if this were the case, the identity wire would separate.

Exercise 6: Construct a causal quantum map $\Phi : A \otimes B \otimes C \to X \otimes Y$ by making a connected circuit diagram involving exactly 3 causal quantum maps Φ_1, Φ_2, Φ_3 such that



Hint: Read Section 6.3.