**Quantum Processes and Computation**  
*Assignment 5, Friday, 12 Nov*

**Deadline:** Week 6 (group 1) or week 7 (groups 2-6)

**Goals:** After completing these problems, you will be able to work with (non-deterministic) quantum processes, both in indexed form and with classical wires and spiders.

**Note:** Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been modified for the problem sheet. The corresponding exercise number from the book is shown in brackets. If you are stuck, try looking up the exercise number in the book. Usually the definitions or equations you need are nearby.

**Exercise 1 (7.13):** Let \( \left\{ \frac{1}{\sqrt{2}} \right\} \) form an ONB for any non-trivial system (so that it contains at least two elements). We can use this ONB to form a non-demolition ONB measurement in the following way:

If we sum over all the branches we get a process that is called *decoherence*. Show that decoherence is not equal to the identity by finding an explicit quantum state \( \rho \) such that:

![Diagram of decoherence process]

**Exercise 2:** Suppose we have two different ONB measurements on a qubit. Before we do anything with the quantum system we can flip a coin. If it comes up heads we can perform the first ONB measurement, and if it comes up tails we can do the other one. This is modelled by the quantum process

\[
\left( \frac{1}{2} \psi_1 \right), \left( \frac{1}{2} \psi_2 \right), \left( \frac{1}{2} \phi_1 \right), \left( \frac{1}{2} \phi_2 \right)
\]

where \( (\hat{\psi}_1, \hat{\psi}_2) \) and \( (\hat{\phi}_1, \hat{\phi}_2) \) both form ONB measurements.

(i) Show that the above set of 4 effects indeed forms a quantum process (i.e. that their sum satisfies the causality condition).

(ii) We can generalise the above construction. Suppose we now have \( n \) different ONB measurements which we call \( (\hat{\psi}_1^j, \hat{\psi}_2^j) \) for \( j = 1, \ldots, n \). Find the number \( p \) such that

\[
\left( \frac{1}{2} \psi_1^1, \frac{1}{2} \psi_2^1, \ldots, \frac{1}{2} \psi_1^n, \frac{1}{2} \psi_2^n \right)
\]

is a quantum measurement. Would the number \( p \) change if instead of ONB measurements on a qubit, we would consider ONB measurements on a bigger system?
Exercise 3: Write a non-demolition ONB-measurement in the classical-quantum notation:

\[
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{img1.png}}
\end{array}
\end{array}
\sim \sum_i \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{img2.png}}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{img3.png}}
\end{array}
\end{array}
\]

and show that

\[
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{img4.png}}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{img5.png}}
\end{array}
\end{array}
\]

Exercise 4 (8.32): Prove the generalised copy rule for spiders:

\[
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{img6.png}}
\end{array}
\end{array} = \delta_{i_1 \ldots i_m}^{j_1 \ldots j_n}
\]

where \( \delta_{i_1 \ldots i_m}^{j_1 \ldots j_n} \) is the generalised Kronecker delta that is 1 when all the in- and outputs match and is 0 otherwise.

Exercise 5 (8.37 & 8.38):

(i) Prove using just the spider fusion law that the spider with no legs equals the ‘circle’ (i.e. the dimension):

\[
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{img7.png}}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{img8.png}}
\end{array}
\end{array}
\]

(ii) Not all spiders are causal. Determine which spiders are causal, which can be made causal by rescaling by a number, and which cannot.