Quantum Processes and Computation

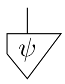
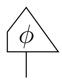
Assignment 3, Friday, 28 Oct, 12:00

Deadline: Fri Week 4 (Groups 3 and 4), Weds Week 5 (Groups 1 and 2)

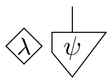
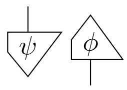
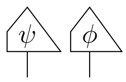
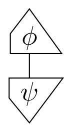
Goals: After completing these exercises you should know about (orthonormal) bases, composition of linear maps, encoding logic gates as linear maps, pure quantum states and maps, and $\otimes$-positivity.
Material covered in book: Sections 5.1, 5.2, 5.3, 6.1, and 6.2.

Note: Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been modified for the problem sheet. The corresponding exercise number from the book is shown in brackets. **If you are stuck, try looking up the exercise number in the book. Usually the definitions or equations you need are nearby.**

Exercise 1 (5.54): Let

 $\leftrightarrow \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}$ and  $\leftrightarrow \begin{pmatrix} \phi_0 & \phi_1 \end{pmatrix}$

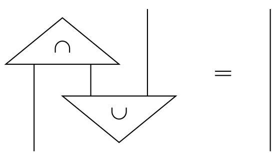
be respectively a 2-dimensional state, and 2-dimensional effect. Let λ be a number. Write the matrices for the processes

(i)  *(ii)*  *(iii)*  *(iv)* 

Exercise 2 (5.58): The matrices for cups and caps in 2 dimensions are:

 $\leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$  $\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$

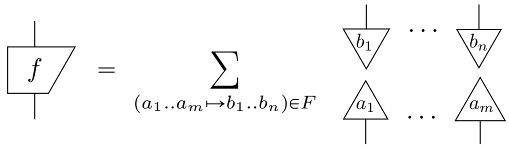
(i) First, verify the yanking equation



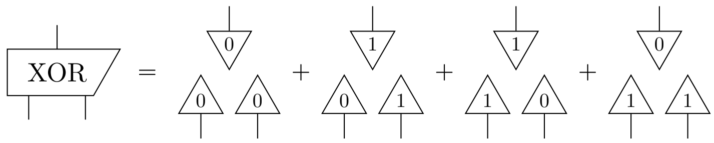
directly using the matrices of the 2-dimensional cup and cup by using the rules for sequential and parallel composition of matrices, i.e. show that $(\cap \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \cup) = \mathrm{id}$ (where id is the 2×2 identity matrix).

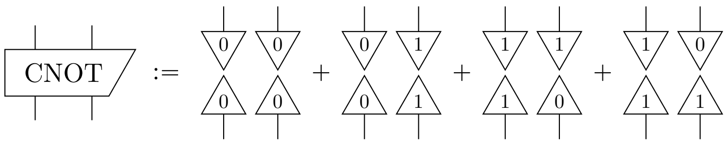
(ii) Second, give the matrices for the cup and cap in 3 dimensions.

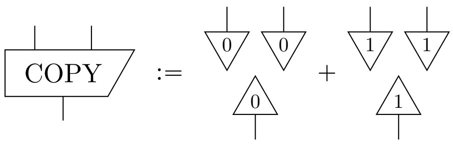
The next excercise is about encoding classical logic gates in the theory of **linear maps**, as explained in Section 5.3.4. Recall that a classical logic gate F can be encoded as a linear map via:



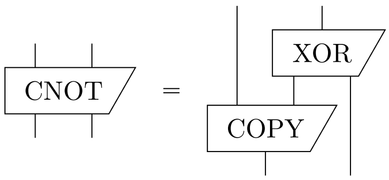
Using this encoding, we defined:





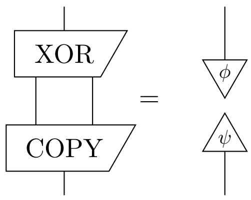


Exercise 3 (5.86): Show that



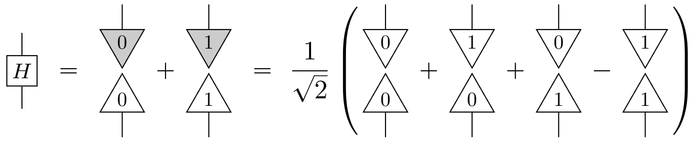
(Hint: try comparing the LHS to the RHS on all basis states, rather than writing out a big sum.)

Next, find ψ and ϕ such that the following equation holds:

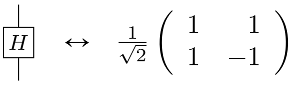


Although it might not look like much now, this equation will turn out to lie at the heart of the notion of *complementarity* which we will cover in great depth in the coming lectures.

Exercise 4 (5.93): In the proof of proposition 5.92, we see the Hadamard process written in matrix form with respect to the Z basis:



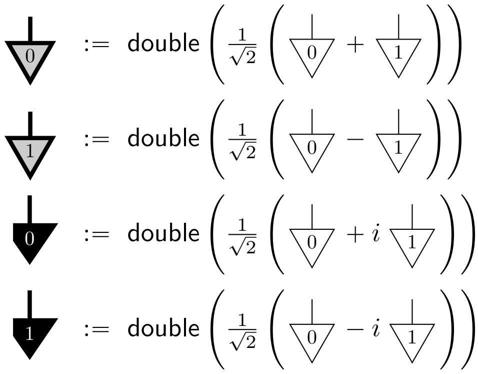
From this we can conclude the matrix of H (with respect to the Z-basis) is:



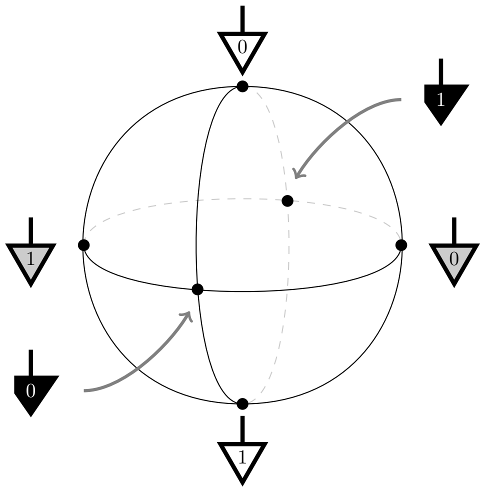
What is the matrix of H in the X-basis?

In section 6.1.2, it was shown that 2D quantum pure states correspond to points on a sphere.

Exercise 5 (6.7): Show that the following points:

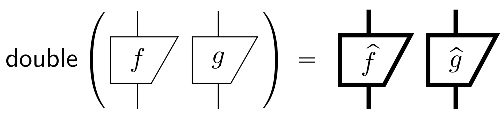


are located on the Bloch sphere as follows:



Exercise 6 (6.10 & 6.22):

(i) Show that doubling preserves parallel composition:

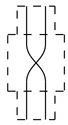


(ii) Show that doubling preserves normalisation: that a state ψ is normalised if and only if its doubled state $\widehat{\psi}$ is normalised.

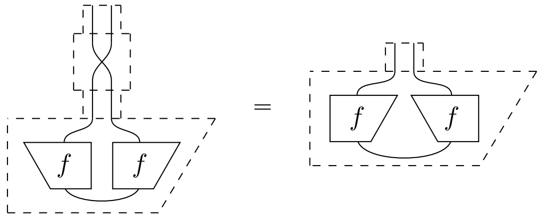
(iii) Show that doubling preserves orthogonality: that states ψ and ϕ are orthogonal if and only if $\widehat{\psi}$ and $\widehat{\phi}$ are orthogonal.

Hint: Use theorem 6.17 for the latter two points.

The transpose of a positive process is again a positive process and by bending some wires we can also take the 'transpose' of a $\otimes$-positive state, i.e. of a quantum state (see **Corollary 6.36**). This transpose acts as a swap of wires on the doubled system:

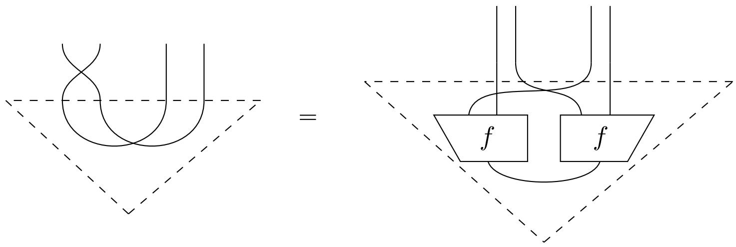


and it indeed sends quantum states to quantum states:



In the next exercise we will show that nevertheless, this swap of wires is *not* a quantum operation.

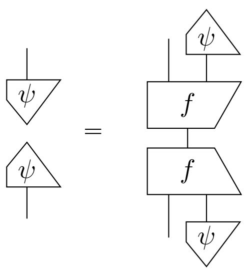
Exercise 7: In this exercise we will show that a swap applied to one pair of the wires of the doubled cup state will result in a state that is no longer ⊗-positive, and therefore not a quantum state. We will do this by contradiction. So suppose:



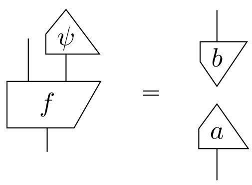
(1)

for some process f.

(i) Let ψ be a normalised state. Show that the equation above implies that



and hence, by Proposition 5.74, that there exist states a and b such that:



(2)

(ii) Plug ψ into equation 1 and use equation 2 to show that the identity wire disconnects. Conclude that therefore the swap can't be a quantum map.

Note: In proposition 6.48 it is also shown that the swap is not a quantum operation, but it uses a specific counter-example found in **linear maps**. The proof above only uses string diagrams and the property implied by proposition 5.74.