**Deadline:** Fri Week 6 (Groups 3 and 4), Weds Week 7 (Groups 1 and 2)

**Goals:** After completing these problems, you will be able to work with spiders, unbiased/phase states, and the phase group.

**Note:** Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been modified for the problem sheet. The corresponding exercise number from the book is shown in brackets. If you are stuck, try looking up the exercise number in the book. Usually the definitions or equations you need are nearby.

**Exercise 1 (8.32):** Prove the generalised copy rule for spiders:

\[
\delta_{j_1 \ldots j_n}^{i_1 \ldots i_m}
\]

where \(\delta_{j_1 \ldots j_n}^{i_1 \ldots i_m}\) is the generalised Kronecker delta that is 1 when all the in- and outputs match and is 0 otherwise.

**Exercise 2 (8.39):** The spiders:

\[
\begin{array}{c}
\text{for the 2D basis:} \\
\end{array}
\]

are associated with *bits*. Show that the following family of classical maps:

\[
\text{is also a family of spiders (i.e. that it fits into Definition 8.31) and that it is associated with the ONB of } N\text{-bitstrings:}
\]

\[
\begin{array}{c}
\end{array}
\]
Exercise 3 (8.37 & 8.38):

(i) Prove using just the spider fusion law that the spider with no legs equals the ‘circle’ (i.e. the dimension):

\[ \bigcirc = \bigcirc \]

(ii) Not all spiders are causal classical maps. Determine which spiders are causal, which can be made causal by rescaling by a number, and which cannot.

Exercise 4 (8.15): We saw in the lecture that a classical map has a matrix with only positive numbers. For instance the classical map characterised by

\[
\begin{pmatrix}
\frac{2}{3} & 0 \\
0 & \frac{1}{3}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{3} & 1 \\
0 & \frac{2}{3}
\end{pmatrix}
\]

has the matrix

\[
\begin{pmatrix}
\frac{2}{3} & \frac{1}{3}
\end{pmatrix}
\]

However, if we change the basis the numbers in the matrix no longer have to be positive. Find a basis such that the matrix of \( f \) with respect to this basis contains some negative values.

Exercise 5 (8.26 & 8.55 & 8.69):

(i) Show that for any two linear maps \( f \) and \( g \) of the same type, the diagram:

\[
\begin{pmatrix}
\frac{2}{3} & 0 \\
0 & \frac{1}{3}
\end{pmatrix} + \begin{pmatrix}
\frac{1}{3} & 1 \\
0 & \frac{2}{3}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{2}{3} & \frac{1}{3}
\end{pmatrix}
\]

(ii) For a classical map \( g \), find a pure quantum map \( \hat{f} \) such that

\[
\begin{pmatrix}
\frac{2}{3} & \frac{1}{3}
\end{pmatrix}
\]

Hint: Use the representation

\[
\begin{pmatrix}
\frac{2}{3} & \frac{1}{3}
\end{pmatrix}
\]
(iii) Show that a function map \( f \) (i.e. a deterministic causal classical map) satisfies

\[
\begin{array}{c}
\text{\( \hat{f} \)} \\
\text{\( f \)}
\end{array}
\]

**Exercise 6 (9.2):** Show that a normalised pure state is *unbiased* for an ONB-measurement if and only if for all \( i \) we have:

\[
\frac{1}{\sqrt{D}}
\]

**Exercise 7 (9.21):** In section 9.1, it was shown that any phase state of dimension \( D \) is of the form:

\[
\otimes := \text{double } \left( \sum_{j=0}^{D-1} e^{i\alpha_j} \right)
\]

and furthermore that we can assume, up to a global phase, that \( \alpha_0 = 0 \). Hence, a \( D \)-dimensional phase state is labelled by a vector of \( D - 1 \) phases \( \vec{\alpha} := (\alpha_2, \ldots, \alpha_D) \).

Show that the phase group unit, addition, and inverse are defined in terms of these vectors as follows:

\[
\begin{align*}
\vec{0} & := (0, \ldots, 0) \\
\vec{\alpha} + \vec{\beta} & := (\alpha_1 + \beta_1, \ldots, \alpha_{D-1} + \beta_{D-1}) \\
-\vec{\alpha} & := (-\alpha_1, \ldots, -\alpha_{D-1})
\end{align*}
\]