Ex 2.1

(a) First, let's show \( \mathcal{U} = I \). We'll do this by decomposing the LHS as:

\[
R_1 = \bigcup a \mapsto \{(b, b, a) \mid b \in A^3\}
\]

\[
R_2 = I \cap (a, b, c) \mapsto \{(a \text{ if } b = c) \text{ otherwise}\}
\]

Then:

\[ \mathcal{U} = R_2 \circ R_1 \]

\[ \mathcal{U} \circ a \mapsto \{(b, b, a) \mid b \in A^3\} \xrightarrow{R_2} \{b \mid b = a^2 \circ \xi_0^3\} \]

\[ \mathcal{U} \text{ and } I \text{ both map } a \mapsto \xi_0^3 \text{.} \]

Hence \( \mathcal{U} = I \).

For \( \mathcal{U} = \emptyset \), we have:

\[ \text{LHS} \circ \ast \xrightarrow{U} (a, a) \]

\[ \text{RHS} \circ \ast \xrightarrow{U} (a, a) \xrightarrow{U} (a, a) \text{.} \]

The other two equations in 4.11 are proven by flipping the relations above.
n.b. We can also prove \( \bigcup I = 1 \) by writing the diagram formula for the LHS and following the recipe from PAP (p.65).

**Step 1:** Write LHS as diagram formula:

Let \( R = \bigcup I \), then:

\[
R^A = \bigcup A_2 A_1 \bigcap A_2 A_1
\]

**Step 2:** Replace labels with set elements:

\[
R^A = \bigcup a_2 a_1 \bigcap a_2 a_1, \text{ for } a_1, a_2, a_3 \in A
\]

**Step 3:**

\[
\bigcup : a_1 \rightarrow a_3 \iff \exists a \in A. \left( \bigcup : (a_3, a_2) \rightarrow \star \right) \bigcap : \star \rightarrow (a_2, a_1) \rightarrow \star
\]

\[
\iff \exists a_2. a_1 = a_2 = a_3
\]

\[
\iff a_1 = a_3
\]

Since \( \bigcup : a_1 \rightarrow a \ \forall a \in A \), \( \bigcup = 1 \).
Ex 2.2

\[
\begin{align*}
\text{Diagram 1} &= \text{Diagram 2} \quad \text{or} \quad \text{Diagram 3} = 1
\end{align*}
\]
Ex 2.3

Assume (i) \[ \begin{array}{c}
\Delta \rightarrow a \\
\subseteq \\
\end{array} \] and \[ \begin{array}{c}
\Delta \rightarrow b \\
\subseteq \\
\end{array} \].

Then:

\[ \begin{array}{c}
\Delta \rightarrow a \\
\subseteq \\
\end{array} \quad \text{and} \quad \begin{array}{c}
\Delta \rightarrow b \\
\subseteq \\
\end{array} \]

so we have \[ \Delta \rightarrow c \] \[ \subseteq \Delta \]. From this, we can show:

\[ \begin{array}{c}
\Delta \rightarrow b \\
\subseteq \\
\end{array} \quad \text{and} \quad \begin{array}{c}
\Delta \rightarrow c \\
\subseteq \\
\end{array} \]

So we have shown (ii) \[ \begin{array}{c}
\Delta \rightarrow a \\
\subseteq \\
\end{array} \] \[ \Rightarrow \] \[ \Delta \rightarrow c \]

follows from (i). The proof that (ii) \[ \Rightarrow \] (i) is the same, but with all diagrams flipped vertically (or horizontally!).
Ex 2.4

Thm The following are equivalent:

(i) $f$ is a unitary

(ii) $f$ is an isometry and has an inverse

(iii) $f^*\ d f = f^+\ d f$, so $f$ has an inverse

Pf If $f$ is a unitary, then it is an isometry (by definition) $\Rightarrow f^{-1} = f^*$, so (i) $\Rightarrow$ (ii).

Assume (ii), then:

$$f^* f = \begin{bmatrix} 1 \\ f \\ f^* \end{bmatrix} = \begin{bmatrix} 1 \\ f \\ f^* \end{bmatrix} = \begin{bmatrix} f^*f^{-1} \end{bmatrix} \Rightarrow f = f^*f^{-1}.$$

Since $f^* f = f^* f$, $f$ is a unitary. Hence (i) $\iff$ (ii).

Now, $f$ is a unitary iff $f^*$ is unitary. Hence (i) $\iff$ (iii) by the same proof.
Ex 2.6

\[ P_f = \text{Diagram} \]

\[ \text{Diagram} = \text{Diagram} \]

\[ \text{Diagram} = \text{Diagram} \]

\[ \text{Diagram} = \text{Diagram} \]

\[ \text{Diagram} = \text{Diagram} \]
Ex 2.7

Suppose is maximally non-sep.

Then for some unitary $U$.

For any unitary $V$, is also unitary. Hence is also maximally non-sep.

If we let $V = U^*$, then:

$$\begin{align*}
    \begin{array}{c}
        U \\
        \psi
    \end{array} & \sim \begin{array}{c}
        U \\
        U^*
    \end{array} = 1 .
\end{align*}$$

By bending the input wire up, we get:

$$\begin{align*}
    \begin{array}{c}
        U \\
        \psi
    \end{array} & \sim U .
\end{align*}$$