Quantum Processes & Computation

Model solutions, Sheet 6

Oxford MT 2022
Ex 6.1

1. For the $1^{st}$ input wire is classical in $\{\hat{\Psi}_i\}$ basis, $2^{nd}$ is classical in $\{\hat{\Phi}_i\}$ basis, so causality means:

For bastard spiders with 1 quantum leg: $\hat{\Phi} = \hat{\Psi} = \hat{\Upsilon} = \hat{\Gamma}$ (*)

So:

\[ \begin{array}{c}
\text{Complementarity (\#)} \\
\end{array} \]
6.2

(i) Need to show:

![Diagram A = B = C = D]

Starting with A, and using spider fusion, \( U = \omega = \omega \), and \( \Gamma = \Theta = \Theta \), we have:

![Diagram showing the fusion process]

(ii)

![Diagram showing \( \alpha_m \), \( \approx \), \( \cong \), and \( \approx \)]

\( \approx \) or:

![Diagram showing additional equivalence relationships]

SC: “strong completion,” C: “completion” (implied by SC)
6.3

Lemma 1

Proof

Lemma 1
(ii) Need to show:

\[ \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \neq \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \]

But: \( \frac{\pi}{2} \neq \frac{\pi}{2} \). In fact, they are orthogonal:

\[ \left( \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \right)^{\dagger} \cdot \left( \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \right) = \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} = 1 \implies \left( \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \right)^{\dagger} \cdot \left( \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \right) = 0. \]

(iii) 

\[ \begin{array}{c}
\text{LT} \\
\text{S}
\end{array} = \begin{array}{c}
\frac{\pi}{4} \\
\frac{\pi}{2}
\end{array} \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \begin{array}{c}
\frac{\pi}{4} \\
\frac{\pi}{2}
\end{array} \]

\[ \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} = \begin{array}{c}
\frac{\pi}{4} \\
\frac{\pi}{2}
\end{array} \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \begin{array}{c}
\frac{\pi}{4} \\
\frac{\pi}{2}
\end{array} \]
Lemma 2 ("gadget" fusion)

\[ \approx \]

Proof

There is not a unique solution. Here's a possibility:

\[ \text{T-count } 6 \leq 7 \]

\[ \approx \]

\[ \approx \]

\[ \approx \]

\[ \approx \]

\[ \approx \]

\[ \approx \]

\[ \approx \]
Note: in this question, we write non-deterministic processes using indices, rather than cq-maps/classical wires.

E.g. a Z qubit measurement is written as the non-deterministic process $(\overset{\rightarrow}{\bigotimes})^k \approx (\overset{\rightarrow}{\bigotimes^k})$.

(a)

- If measurement gives outcome $k=0$:

  \[
  * = \begin{array}{c}
  \begin{array}{c}
  \text{measure } Z \\
  \text{CNOT} \\
  \text{prepare } Z_{\pi/4}
  \end{array}
  \end{array}
  \]

  \[
  = \begin{array}{c}
  \begin{array}{c}
  \pi/4 \\
  \pi/4
  \end{array}
  \end{array}
  = T
  \]

- If measurement gives outcome $k=1$:

  \[
  * = \begin{array}{c}
  \begin{array}{c}
  \text{measure } Z \\
  \text{CNOT} \\
  \text{prepare } Z_{\pi/4}
  \end{array}
  \end{array}
  \]

  \[
  \begin{array}{c}
  \begin{array}{c}
  \pi/4 \\
  -\pi/4
  \end{array}
  \end{array}
  \]

  \[
  = \begin{array}{c}
  \begin{array}{c}
  \pi/4 \\
  -\pi/4
  \end{array}
  \end{array}
  = T^t
  \]
(b) Classically-Controlled $S$

\[
\begin{align*}
\text{measure } Z && \quad \text{prepare } Z_\frac{\pi}{4} && \quad \text{CNOT} \\
\end{align*}
\]

\[
\begin{align*}
\text{for } k \in \mathbb{Z} \\
\end{align*}
\]

\[
\begin{align*}
\left(-1\right)^{\frac{k\pi}{4}} &= \frac{\pi}{4} - k\frac{\pi}{2} \\
\end{align*}
\]

\[
\begin{align*}
* &= \frac{\pi}{4} - k\frac{\pi}{2} = \frac{\pi}{4} - k\frac{\pi}{2} + k\frac{\pi}{2} = \frac{\pi}{4} = T.
\end{align*}
\]

(c) We already know we can counts 1 $\frac{\pi}{4}$ into 1 $\frac{\pi}{4}$ using free operations. We can also make $S^+$ using free ops:

\[
\begin{align*}
\alpha_2 &= \frac{\pi}{4} \\
\end{align*}
\]

So, with $4 \times \frac{\pi}{4}$ and free ops:

implemented as in part (b) using $1 \times \frac{\pi}{4}$

\[
\begin{align*}
3 \times \frac{\pi}{4} \{ \text{free} \} = \alpha_{\text{rm}} = \\
\end{align*}
\]
(d)

**LEM 3**

\[-j \pi_2, -l \frac{\pi}{2}, (-1)^{\frac{k}{4}} \frac{\pi}{2}, (-1)^{\frac{h}{2}} \frac{\pi}{4}, (-1)^{\frac{g}{2}} \frac{\pi}{4} = \]

**Proof**

\[-(-1)^{\frac{x}{2}} \pi = \frac{-x}{2} + \frac{x \pi}{2} \]

**LEM 2**

\[\frac{\pi}{4} - k \frac{\pi}{2} + k \frac{\pi}{2} \]

\[\approx \]

\[\approx \]

\[\approx \]

\[\approx \]
6.5 (cont'd)

Now, starting with $P_{\frac{\pi}{4}}$, we can get:

\[
\begin{align*}
&\sim \\
&\sim \\
&= (-1)^{\frac{a+b+c}{4}} \frac{\pi}{4}
\end{align*}
\]
Hence, we can apply Lemma 3, for \( j = a, k = a + b + c, l = c \):
6.5 (cont'd)

\( 1 \times Z_{\pi/4} \times \frac{\pi}{2} \approx \frac{k\pi}{4} \)

\( 1 \times P_{\pi/4} \approx \frac{\pi}{4} \)

\[ \sim \]

\[ \frac{\pi}{2} k\pi \]