

Quantum Processes and Computation Lecture Notes

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MT 2026

1 Lecture 1: Quantum Theory is SCUM

Quantum theory is the theory of the smallest stuff in the universe. At its core are the *Dirac-von Neumann postulates*. We'll give a paraphrased, quantum-computing spin on these, called **SCUM postulates**:

- S. States** of a quantum system correspond to vectors* in a Hilbert space.
- C. Compound systems** are represented by the tensor product.
- U. Unitary maps** evolve states in time.
- M. Measurements** are computed according to the Born rule.

1.1 States

A **Hilbert space** is a vector space over the complex numbers, with an inner product defined between vectors, which we write physics-style as follows:

$$\langle \psi | \phi \rangle \in \mathbb{C}$$

Definition 1.1. A mapping $\langle - | - \rangle$ from a pair of vectors to the complex numbers is called an **inner product** if it's:

1. *Linear* in the second argument: $\langle \psi | \lambda_1 \phi_1 + \lambda_2 \phi_2 \rangle = \lambda_1 \langle \psi | \phi_1 \rangle + \lambda_2 \langle \psi | \phi_2 \rangle$.
2. *Conjugate-symmetric*: $\overline{\langle \phi | \psi \rangle} = \langle \psi | \phi \rangle$.
3. *Positive-definite*: $\langle \psi | \psi \rangle > 0$ when $\psi \neq 0$.

This implies it is *conjugate-linear* in the first argument:

$$\langle \lambda_1 \psi_1 + \lambda_2 \psi_2 | \phi \rangle = \overline{\lambda_1} \langle \psi_1 | \phi \rangle + \overline{\lambda_2} \langle \psi_2 | \phi \rangle$$

Our **main example** is $\mathbb{C}^n$, which is the set of size- n column vectors:

$$\psi = \begin{pmatrix} \psi^0 \\ \psi^1 \\ \vdots \\ \psi^{n-1} \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi^0 \\ \phi^1 \\ \vdots \\ \phi^{n-1} \end{pmatrix}, \quad \dots$$

with inner product:

$$\langle \psi | \phi \rangle := \begin{pmatrix} \overline{\psi^0} & \overline{\psi^1} & \dots & \overline{\psi^{n-1}} \end{pmatrix} \begin{pmatrix} \phi^0 \\ \phi^1 \\ \vdots \\ \phi^{n-1} \end{pmatrix} = \sum_{i=0}^{n-1} \overline{\psi^i} \phi^i$$

We can split the inner product into two parts, multiplying the column vector ϕ with the row vector $\psi^\dagger$, given by the **adjoint** (a.k.a. conjugate-transpose) of ψ :

$$\psi = \begin{pmatrix} \psi^0 \\ \psi^1 \\ \vdots \\ \psi^{n-1} \end{pmatrix} \implies \psi^\dagger = (\overline{\psi^0} \quad \overline{\psi^1} \quad \dots \quad \overline{\psi^{n-1}})$$

This gives us **Dirac bra-ket notation**:

$$\begin{aligned} \psi &\mapsto |\psi\rangle && \text{(ket)} \\ \psi^\dagger &\mapsto \langle\psi| && \text{(bra)} \\ \langle\psi||\phi\rangle &:= \psi^\dagger\phi = \langle\psi|\phi\rangle && \text{(bra-ket)} \end{aligned}$$

We can also write this graphically as:

$$\begin{aligned} |\phi\rangle &\rightsquigarrow \begin{array}{c} \text{---} \phi \text{---} \\ \triangleleft \end{array}^H \\ \langle\psi| &\rightsquigarrow \begin{array}{c} H \\ \text{---} \psi \text{---} \\ \triangleright \end{array} \\ M &\rightsquigarrow \begin{array}{c} H \text{---} \boxed{M} \text{---} K \end{array} \end{aligned}$$

We visualise composition by plugging wires together. For example:

$$\begin{aligned} \langle\psi|\phi\rangle &\rightsquigarrow \begin{array}{c} \triangleleft \phi \text{---} \psi \triangleright \end{array}^H \\ \langle\psi|M|\phi\rangle &\rightsquigarrow \begin{array}{c} \triangleleft \phi \text{---} \boxed{M} \text{---} \psi \triangleright \end{array}^{H \quad K} \\ \langle\psi|CBA|\phi\rangle &\rightsquigarrow \begin{array}{c} \triangleleft \phi \text{---} \boxed{A} \text{---} \boxed{B} \text{---} \boxed{C} \text{---} \psi \triangleright \end{array} \end{aligned}$$

Quantum states are kets, i.e. they are represented as vectors in the Hilbert space. However, there is some redundancy: the *global phase*. But what's that? Well, first what's a phase?

Two ways to write a complex number:

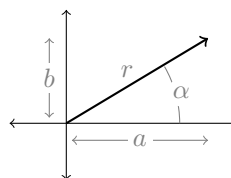
$$\text{Cartesian form : } \lambda = a + ib$$

$$\text{polar form : } \lambda = re^{i\alpha}$$

Polar form is especially useful for quantum theory. It splits the number into a **magnitude** $|\lambda| = r > 0$ and an angle $\alpha \in [0, 2\pi)$ called the **phase**. We have:

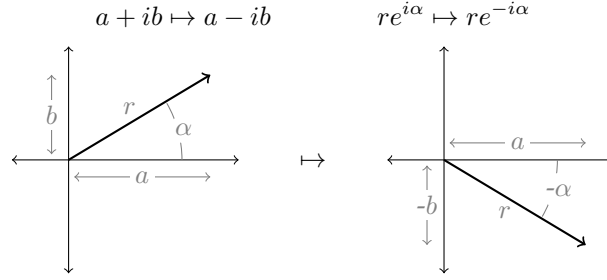
$$e^{i\alpha} = \cos \alpha + i \sin \alpha \implies \begin{cases} a = r \cos(\alpha) \\ b = r \sin(\alpha) \end{cases}$$

which can be visualised in the **complex plane** as follows:



(1)

Complex conjugation $\lambda \mapsto \bar{\lambda}$ flips the sign of b in cartesian form or the sign of α in polar form:



Multiplying phase expressions adds their angles:

$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$$

So:

$$e^{i\alpha} e^{i\alpha} = e^{i\alpha} e^{-i\alpha} = e^{i(\alpha-\alpha)} = e^0 = 1.$$

Two vectors $|\psi\rangle$ and $|\phi\rangle$ are equal *up to a global phase* if $|\phi\rangle = e^{i\alpha}|\psi\rangle$ (notation $|\psi\rangle \sim |\phi\rangle$).

Definition 1.2. A **quantum pure state** is a normalised ($\langle\psi|\psi\rangle = 1$) vector, up to a global phase.

That is, two vectors $|\psi\rangle \sim |\phi\rangle$ represent the *same* quantum state. We will see why when we come to discussing measurements.

In quantum computing, we are most interested in quantum bit (i.e. **qubit**) states. These are states of a 2D space, spanned by:

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

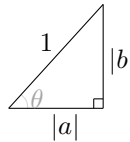
Graphically, we will depict these states as follows:



Let $|\psi\rangle := a|0\rangle + b|1\rangle$. From normalisation:

$$|a|^2 + |b|^2 = 1$$

Sticking those numbers on a triangle:



we see we can always write $|a| := \cos \theta$ and $|b| := \sin \theta$.

We can replace 2 complex numbers with 3 angles:

$$|\psi\rangle = \cos \frac{\theta}{2} e^{i\beta} |0\rangle + \sin \frac{\theta}{2} e^{i\gamma} |1\rangle$$

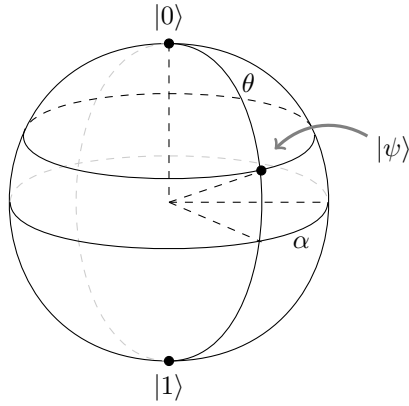
Multiplying by a global phase, we can get rid of β :

$$e^{-i\beta} \left(\cos \frac{\theta}{2} e^{i\beta} |0\rangle + \sin \frac{\theta}{2} e^{i\gamma} |1\rangle \right) = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i(\gamma-\beta)} |1\rangle$$

Letting $\alpha := \gamma - \beta$, we can therefore write a generic qubit state $|\phi\rangle$ conveniently as:

$$|\phi\rangle := \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\alpha} |1\rangle$$

Since the quantum state is now totally described by two angles, we can plot it on a sphere, called the **Bloch sphere**:



This picture is useful for our intuition. For example, the more ‘similar’ two states are, that is, the higher the value of their inner product, the closer they are on the Bloch sphere. In particular, antipodes are always orthogonal states.

$$|\phi_0\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\alpha} |1\rangle$$

$$|\phi_1\rangle = \cos \frac{\theta + \pi}{2} |0\rangle + \sin \frac{\theta + \pi}{2} e^{i\alpha} |1\rangle$$

we have $\langle \phi_0 | \phi_1 \rangle = 0$.

For a qubit, picking an orthonormal basis just comes down to picking a pair of orthogonal normalised states, hence, we can always think of orthonormal bases as different axes cutting through the Bloch sphere.

The standard basis $|0\rangle, |1\rangle$ corresponds to the Z axis of the Bloch sphere. Show that the following states

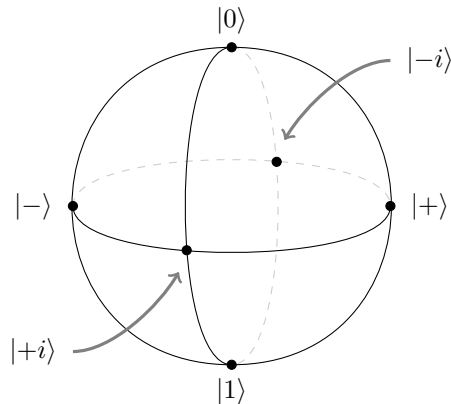
$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|+i\rangle := \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|-i\rangle := \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

correspond to the X and Y axes, respectively. That is, show they are located on the Bloch sphere as follows:



The three pairs of antipodal points each correspond to an orthonormal basis for $\mathbb{C}^2$:

$$\begin{aligned} X\text{-basis} &:= \{|+\rangle, |-\rangle\} \\ Y\text{-basis} &:= \{|+i\rangle, |-i\rangle\} \\ Z\text{-basis} &:= \{|0\rangle, |1\rangle\} \end{aligned}$$

These bases mark out the X, Y, and Z axes of the Bloch sphere. They are eigenvectors of the Pauli X , Y , and Z matrices, respectively.

2 Lecture 2: Compound systems, entanglement, and diagrams

The second SCUM postulate says that systems with Hilbert spaces H and K combine to form a system with Hilbert space $H \otimes K$, their **tensor product**. Two classical bits have four possible values: 00, 01, 10, 11. Correspondingly, two qubits have four basis states:

$$|00\rangle := |0\rangle \otimes |0\rangle, \quad |01\rangle := |0\rangle \otimes |1\rangle, \quad |10\rangle := |1\rangle \otimes |0\rangle, \quad |11\rangle := |1\rangle \otimes |1\rangle.$$

More generally, if $\{|i\rangle\}_{i=0}^{m-1}$ and $\{|j\rangle\}_{j=0}^{n-1}$ are orthonormal bases for H and K , then $\{|i\rangle \otimes |j\rangle\}_{i,j}$ is an orthonormal basis for $H \otimes K$, called the **product basis**. Dimensions multiply:

$$\dim(H \otimes K) = \dim(H) \dim(K), \quad \dim((\mathbb{C}^2)^{\otimes n}) = 2^n.$$

We abbreviate $|i\rangle \otimes |j\rangle$ to $|ij\rangle$ and omit brackets in iterated tensor products.

Product states. The tensor product of two vectors is obtained by pulling out sums and scalars:

$$|\psi\rangle \otimes |\phi\rangle = \left(\sum_i \psi^i |i\rangle \right) \otimes \left(\sum_j \phi^j |j\rangle \right) = \sum_{ij} \psi^i \phi^j |ij\rangle.$$

For two qubits, we just multiply out:

$$\begin{aligned} (\psi^0|0\rangle + \psi^1|1\rangle) \otimes (\phi^0|0\rangle + \phi^1|1\rangle) &= \psi^0\phi^0|00\rangle + \psi^0\phi^1|01\rangle + \psi^1\phi^0|10\rangle + \psi^1\phi^1|11\rangle, \\ \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} \otimes \begin{pmatrix} \phi^0 \\ \phi^1 \end{pmatrix} &= \begin{pmatrix} \psi^0\phi^0 \\ \psi^0\phi^1 \\ \psi^1\phi^0 \\ \psi^1\phi^1 \end{pmatrix}. \end{aligned}$$

The column uses the basis order 00, 01, 10, 11.

Not every vector in the joint space is a product vector. Compare the following unnormalised examples:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \psi^0\phi^0 \\ \psi^0\phi^1 \\ \psi^1\phi^0 \\ \psi^1\phi^1 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The second equation has no solution: the first and last entries force all four coefficients to be nonzero, so the middle entries cannot vanish. After normalisation, a non-product vector gives an **entangled** pure state.

The same construction works for matrices of any size. The **Kronecker product** $A \otimes B$ replaces each entry a_i^j of A by the block $a_i^j B$. For example, for two 2×2 matrices,

$$A = \begin{pmatrix} a_0^0 & a_1^0 \\ a_0^1 & a_1^1 \end{pmatrix}, \quad B = \begin{pmatrix} b_0^0 & b_1^0 \\ b_0^1 & b_1^1 \end{pmatrix},$$

we obtain the 4×4 matrix

$$A \otimes B = \begin{pmatrix} a_0^0 B & a_1^0 B \\ a_0^1 B & a_1^1 B \end{pmatrix} = \begin{pmatrix} a_0^0 b_0^0 & a_0^0 b_1^0 & a_1^0 b_0^0 & a_1^0 b_1^0 \\ a_0^0 b_0^1 & a_0^0 b_1^1 & a_1^0 b_0^1 & a_1^0 b_1^1 \\ a_0^1 b_0^0 & a_0^1 b_1^0 & a_1^1 b_0^0 & a_1^1 b_1^0 \\ a_0^1 b_0^1 & a_0^1 b_1^1 & a_1^1 b_0^1 & a_1^1 b_1^1 \end{pmatrix}.$$

In general, $A \otimes B \neq B \otimes A$: the order of tensor factors matters.

Tensors. Tensors generalise matrices. Rather than just having one input and output system, we can have many, which we can represent by having many lower (input) and upper (output) indices. Such an array is called a **tensor**, depicted as a box with one wire per index:

$$m \left\{ \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{T} \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \right\} n \rightsquigarrow T = \{t_{i_0 \dots i_{m-1}}^{j_0 \dots j_{n-1}} \in \mathbb{C} \mid 0 \leq i_\mu < D_\mu, 0 \leq j_\nu < D'_\nu\}$$

Lower indices label inputs, upper indices label outputs. Kets have no inputs, bras have no outputs, and scalars have neither:

$$|\psi\rangle \rightsquigarrow \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{\psi} \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \quad \langle\psi| \rightsquigarrow \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{\psi} \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \quad \lambda \rightsquigarrow \boxed{\lambda}.$$

Parallel and sequential composition. The tensor product multiplies coefficients, keeping the indices separate:

$$(A \otimes B)_{i,j}^{k,l} = a_i^k b_j^l.$$

Graphically, it is **parallel composition**: stack boxes, with the first factor on top. This works equally well for tensors with several wires:

$$A \otimes B \rightsquigarrow \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{A} \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \quad S \otimes T \rightsquigarrow \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{S} \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{T} \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array}.$$

Sequential composition joins outputs to inputs, read left to right. Each joined pair of indices is **contracted**, meaning it is summed over:

$$BA \rightsquigarrow \text{---} \boxed{A} \text{---} \boxed{B} \text{---} \quad (BA)_i^k = \sum_j a_i^j b_j^k.$$

For tensors we sum over all the connecting wires:

$$TS \rightsquigarrow \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{S} \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \boxed{T} \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \quad (TS)_{i_0 \dots}^{k_0 \dots} = \sum_{j_0 \dots} s_{i_0 \dots}^{j_0 \dots} t_{j_0 \dots}^{k_0 \dots}.$$

A bare wire represents the identity map.

String diagrams and tensor networks. What happens when we compose two matrices A, B in sequence and two more matrices C, D in sequence, then compose the results in parallel?

$$(BA \otimes DC)_{i,j}^{k,l} = (BA)_i^k (DC)_j^l = \sum_x a_i^x b_x^k \cdot \sum_y c_j^y d_y^l = \sum_{xy} a_i^x b_x^k c_j^y d_y^l$$

How about when we do it the other way around?

$$[(B \otimes D)(A \otimes C)]_{i,j}^{k,l} = \sum_{xy} (A \otimes C)_{i,j}^{x,y} (B \otimes D)_{x,y}^{k,l} = \sum_{xy} a_i^x c_j^y b_x^k d_y^l = \sum_{xy} a_i^x b_x^k c_j^y d_y^l$$

We get the same thing! The resulting equation is called the **interchange law**:

$$BA \otimes DC = (B \otimes D)(A \otimes C)$$

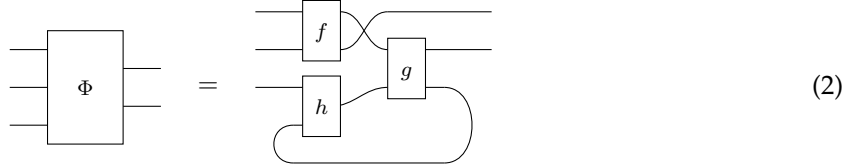
Graphically, both constructions give the same diagram:

$$BA \otimes DC = \begin{array}{|c|} \hline A \\ \hline \end{array} \begin{array}{|c|} \hline B \\ \hline \end{array} \otimes \begin{array}{|c|} \hline C \\ \hline \end{array} \begin{array}{|c|} \hline D \\ \hline \end{array} = \begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array}$$

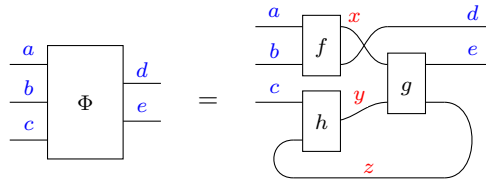
$$(B \otimes D) \circ (A \otimes C) = \left(\begin{array}{|c|} \hline B \\ \hline D \\ \hline \end{array} \right) \circ \left(\begin{array}{|c|} \hline A \\ \hline C \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array}.$$

This is a special case of a **string diagram**, a useful way to describe complex compositions of linear maps. String diagrams are a graphical notation for a general kind of composition of tensors called a **tensor network**.

String diagrams consist of boxes and wires. A box represents a basic tensor being composed, wires that are open at one end represent *free* indices (i.e. inputs and outputs), whereas wires that are connected at both ends correspond to *bound* indices, which get summed over. For example, take this diagram:



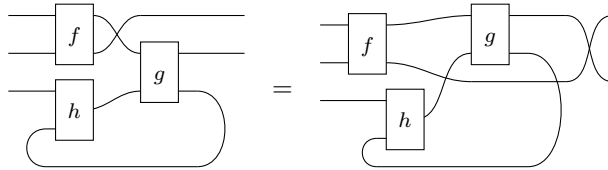
To evaluate it as a tensor network, assign index names to each of the wires:



Then we have:

$$\Phi_{abc}^{de} = \sum_{xyz} f_{ab}^{xd} g_{xy}^{ez} h_{cz}^y \quad (3)$$

Wires can freely cross over each other and even form feedback loops. Also, if we deform the string diagram and draw it differently on the page, it will still describe the same map. For example:



In other words, the only relevant data is which inputs and outputs are connected to which others. This fact can be summarised in the string diagram mantra:

Only connectivity matters!

Sums, cups, and caps. Diagrams represent linear maps, so we can take linear combinations of diagrams with the same input and output spaces. Then, using the tensor network definition above, along with distributing plus over times, it is pretty easy to see that:

Sums distribute over string diagrams.

A sum occurring inside a diagram can be pulled outside the whole diagram. For example, fix an orthonormal basis and consider the (unnormalised) **Bell state**:

$$|\Phi\rangle := \sum_i |ii\rangle = \sum_i \begin{array}{c} \triangleleft_i \\ \triangleleft_i \end{array}.$$

It satisfies the **yanking equation** with its adjoint:

$$(I \otimes \langle\Phi|)(|\Phi\rangle \otimes I) = I \quad \begin{array}{c} \triangleleft_\Phi \\ \triangleleft_\Phi \end{array} = \text{---}$$

Substitute the sums for $|\Phi\rangle$ and $\langle\Phi|$ inside the diagram, then pull both sums outside:

$$\begin{aligned} \begin{array}{c} \triangleleft_\Phi \\ \triangleleft_\Phi \end{array} &= \sum_i \begin{array}{c} \triangleleft_i \\ \triangleleft_i \end{array} \sum_j \begin{array}{c} \triangleleft_j \\ \triangleleft_j \end{array} = \sum_{ij} \begin{array}{c} \triangleleft_i \\ \triangleleft_i \end{array} \begin{array}{c} \triangleleft_j \\ \triangleleft_j \end{array} \\ &= \sum_{ij} \delta_{ij} \begin{array}{c} \triangleleft_j \\ \triangleleft_i \end{array} = \text{---} \end{aligned}$$

The connected bra and ket give $\langle j|i\rangle = \delta_{ij}$. This leaves $\sum_{ij} \delta_{ij} |i\rangle\langle j| = \sum_i |i\rangle\langle i| = I$: the last equality is the **resolution of the identity** in an orthonormal basis.

We can introduce some special notation for $|\Phi\rangle$ and $\langle\Phi|$:

$$\left(\begin{array}{c} \triangleleft_\Phi \\ \triangleleft_\Phi \end{array} \right) := \begin{array}{c} \triangleleft_\Phi \\ \triangleleft_\Phi \end{array} \quad \left(\begin{array}{c} \triangleleft_\Phi \\ \triangleleft_\Phi \end{array} \right) := \begin{array}{c} \triangleleft_\Phi \\ \triangleleft_\Phi \end{array} \quad (4)$$

Now the *yanking equations* above really look like yanking a piece of wire straight:

$$\begin{array}{c} \cup \\ \cup \end{array} = \text{---} \quad \begin{array}{c} \cap \\ \cap \end{array} = \left(\begin{array}{c} \triangleleft_\Phi \\ \triangleleft_\Phi \end{array} \right) \quad (5)$$

In the context of string diagrams we call this special state the **cup** and the effect the **cap**. The cup and cap for qubits have matrices:

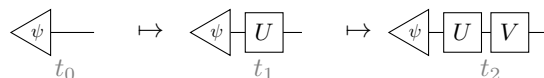
$$\left(\begin{array}{c} \cup \\ \cup \end{array} \right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \left(\begin{array}{c} \cap \\ \cap \end{array} \right) = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \quad (6)$$

We can generalise this to a matrix representation for a cup and cap over any dimension. Can you work out what it is?

3 Lecture 3: UM... Unitaries and Measurements

3.1 Unitaries and time evolution

Time evolution in QT is linear and preserves normalisation:



Theorem 3.1. A linear map $U : \mathcal{H} \rightarrow \mathcal{H}$ preserves normalisation if and only if it is **unitary**, i.e. $U^\dagger U = I = U U^\dagger$.

We'll see why normalisation is important soon, when we get to measurements. But if we take it for granted that states should always be normalised, this is already enough to explain why unitaries are the Right Thing to talk about time evolution. Another way to justify this is connecting to the **Schödinger equation**. Here it is:

$$i \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

H is a self-adjoint ($H = H^\dagger$) linear operator that holds all the physics going on. It's called the **Hamiltonian**. For the most part, we won't worry so much about Hamiltonians (that's a physicist's job), but we will care about time evolution. Solutions to this differential equation are always of the form:

$$|\psi(t)\rangle = e^{-itH} |\psi(0)\rangle$$

The map e^{-itH} is called a **matrix exponential**. We will talk more about these in a few weeks, but for now just note these act like a "normal" exponential when we take the derivative. Hence:

$$i \frac{d}{dt} |\psi(t)\rangle = i \frac{d}{dt} \left(e^{-itH} |\psi(0)\rangle \right) = i(-iH) \left(e^{-itH} |\psi(0)\rangle \right) = H |\psi(t)\rangle$$

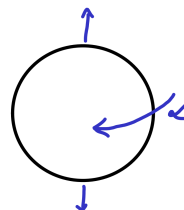
Okay...enough of that. The only important thing for us is this:

Theorem 3.2. A linear map U is unitary iff it can be written as e^{-itH} for some self-adjoint map H .

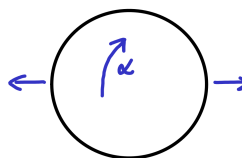
So, U means "evolving according to physics described by H for some fixed time t ".

One-qubit unitaries correspond to rotations of the Bloch sphere:

$$Z[\alpha] := |0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1|$$



$$X[\alpha] := |+\rangle\langle +| + e^{i\alpha}|-\rangle\langle -|$$



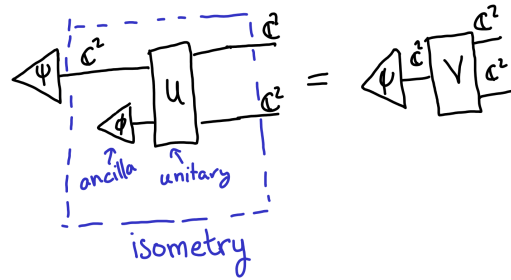
Up to a global phase, we can always write a qubit unitary as a composition of Z and X rotations:

$$\boxed{U} \propto \boxed{Z(\alpha)} \boxed{X(\beta)} \boxed{Z(\gamma)}$$

This is called the **Euler decomposition** of U .

Theorem 3.3. A linear map $V : \mathcal{H} \rightarrow \mathcal{K}$ (where $\mathcal{H}$ and $\mathcal{K}$ could be different Hilbert spaces) preserves normalisation iff it is an isometry: $V^\dagger V = I$.

For example:



3.2 Measurements

A measurement is the only way to get data out of a quantum system. It is non-deterministic and has an **outcome** $j \in \{1, \dots, k\}$. For a state $|\psi\rangle$ and measurement $\mathcal{M}$, QT tells us how to compute:

1. the probability of outcome j
2. the post-measurement state (for a given outcome j)

Definition 3.4. A linear map P is called a **projector** if it is self-adjoint ($P = P^\dagger$) and idempotent ($PP = P$).

A projector splits a space into two orthogonal pieces:

$$\text{im}(P) := \{|\psi\rangle \mid P|\psi\rangle = |\psi\rangle\}$$

$$\text{im}(P)^\perp := \{|\psi\rangle \mid P|\psi\rangle = 0\} = \text{im}(Q) \quad \text{where} \quad Q = I - P$$

We have $\mathcal{H} = \text{im}(P) \oplus \text{im}(Q)$ iff $P + Q = I$. More generally, $\{P_1, \dots, P_k\}$ splits $\mathcal{H}$ into k orthogonal pieces iff $\sum_j P_j = I$.

Definition 3.5. A quantum (von Neumann) measurement is a set of projectors:

$$\mathcal{M} = \{M_1, \dots, M_k\} \quad \text{where} \quad \sum_j M_j = I.$$

The probability of getting the i -th outcome is given by the **Born rule**, named after Max Born:



$$\text{Prob}(i \mid \psi) = \langle \psi \mid M_i \mid \psi \rangle$$

This rule is how QT makes predictions: it assigns probabilities to measurement outcomes. This is the most important thing the theory does, and it is the only part we can test. We can now see why it is important that $|\psi\rangle$ is normalised. This implies that the Born rule probabilities sum to 1:

$$\sum_j \text{Prob}(j | \psi) = \sum_j \langle \psi | M_j | \psi \rangle = \langle \psi | \left(\sum_j M_j \right) | \psi \rangle = \langle \psi | I | \psi \rangle = \langle \psi | \psi \rangle = 1$$

The Born rule also tells us why we don't care about global phases. If $|\psi\rangle \sim |\phi\rangle$, then $|\phi\rangle = e^{i\alpha}|\psi\rangle$ and:

$$\text{Prob}(j | \phi) = \langle \phi | M_j | \phi \rangle = (e^{-i\alpha} \langle \psi |) M_j (e^{i\alpha} |\psi\rangle) = \langle \psi | M_j | \psi \rangle = \text{Prob}(j | \psi)$$

So, $|\psi\rangle$ and $|\phi\rangle$ are *completely indistinguishable*! We may as well call them the same state, so we do.

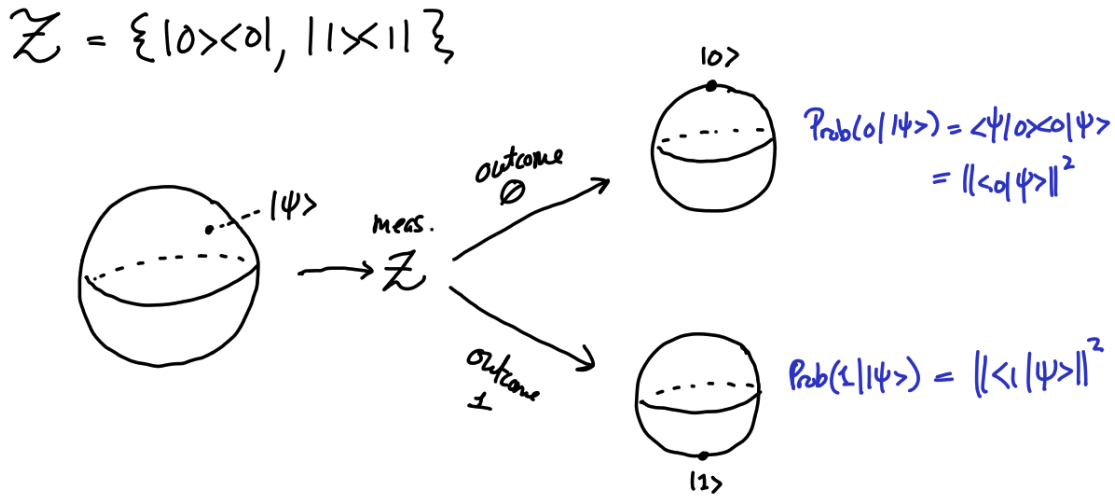
Measuring and getting outcome j *collapses* the quantum state into the j -th subspace given by M_j . We can model that using **Lüder's rule**:

$$|\psi\rangle \xrightarrow{\text{project}} M_i |\psi\rangle \xrightarrow{\text{renormalise}} \frac{1}{\sqrt{\text{Prob}(i | \psi)}} M_i |\psi\rangle$$

The most important example of a measurement is an **ONB measurement**. Any normalised state $|\psi\rangle$ gives a 1D projector $|\psi\rangle\langle\psi|$. If we do this for an ONB, we get a set of projectors forming a quantum measurement, e.g.:

$$\mathcal{Z} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\} \quad \mathcal{X} = \{|+\rangle\langle +|, |-\rangle\langle -|\}$$

Applying Lüder's rule is especially simple for ONB measurements: it just sends the state to the basis element associated with that outcome:



While this is the main kind of measurement we care about, there are other examples where the projectors are not 1D, i.e. they don't *completely* destroy the state, but still project out some info. One example is measuring the first qubit of a 2-qubit quantum state in the computational basis,

$$\mathcal{M} := \left\{ M_0 = \text{---} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \text{---} , M_1 = \text{---} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \text{---} \right\}$$

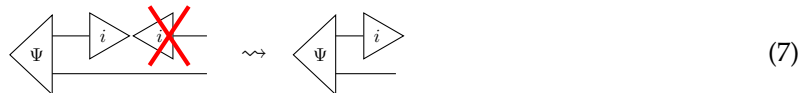
We can check each of the two maps M_0 and M_1 are projectors and also that $M_0 + M_1 = I$.

If after we perform this measurement on a state $|\Psi\rangle$, we get one of two possible results:



In either case, the state of the second qubit could change, depending on which measurement was performed and the measurement outcome. This is sometimes referred to as the **back-action** of a quantum measurement, and we'll talk more about it later.

It is worth noting that after a basis measurement is performed, the measured qubits are no longer entangled with the rest of the state, so we will sometimes simply ignore them:



In fact, sometimes performing a measurement will actually destroy the system (or at least make it unusable to us afterwards). This is called a **demolition measurement**. We'll usually represent a demolition ONB measurement just by plugging in a bra, as in the RHS above.

If we keep the quantum system around after projecting, this is called a **non-demolition measurement**. These are especially important for quantum error correction, which we'll discuss toward the end of the course.