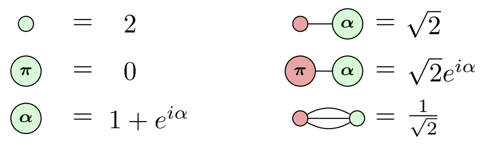
Quantum Software
Assignment 2, Hillary Week 2, 2023

Exercise 1: In the lectures we often ignore scalar factors in ZX-diagrams. We can however represent any scalar we want with a ZX-diagram. For instance, we have:

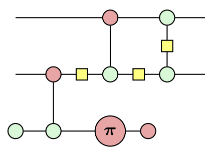


(1)

By combining the diagrams from (1), find a ZX-diagram to represent the following scalar values z:

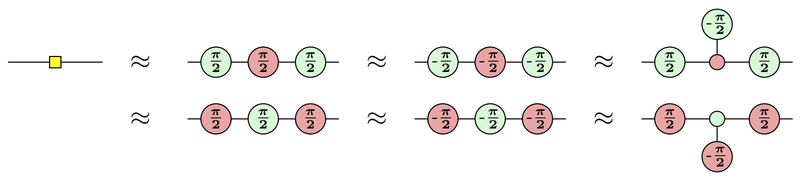
1. $z = -1$.
2. $z = e^{i\theta}$ for any θ.
3. $z = \frac{1}{2}$.
4. $z = \cos\theta$ for any value θ.
5. Find a general description or algorithm to construct the ZX-diagram for any complex number z.

Exercise 2: Using ZX-calculus rewrites, help the poor trapped π phase find it's way to an exit (i.e. an output).



Note that it might be leaving with friends.

Exercise 3: The Euler decomposition from the lecture is just one possible way to write the Hadamard in terms of spiders. There is in fact an entire family of representations that will also be useful to note:



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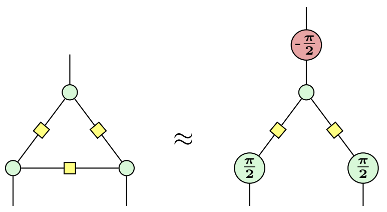
(2)

Prove that all the equations of (2) hold in the ZX-calculus, by using the top-left decomposition and the other rewrite rules of the ZX-calculus we have seen so far.

Exercise 4: Prove the following base-case of the local complementation lemma using the ZX-calculus:



(3)

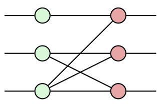
Hint: Push the top Hadamards up and decompose the middle Hadamard using one of Eq. (2) to reveal a place where you can apply strong complementarity. How this result can be used to prove general local complementation is shown in the lecture notes (Lemma 4.2.4).

Exercise 5: Write down a CNOT circuit C with at least 3 qubits and at least 8 CNOT gates. Cut the circuit in two halves containing the first 4 CNOT gates and the last 4 CNOT gates. Simplify to parity normal form in two phases. First, form D_1 by simplifying each half individually to parity normal form. Then form D_2 by simplifying all of D_1 the rest of the way to parity normal form. What can you say about:

1. The number of forward-directed paths from each input to each output in C, D_1, and D_2?
2. The relationship between the bi-adjancency matrices of each half of D_1 and the bi-adjacency matrix of D_2?
3. The relationship between facts 1 and 2?

Note there is some ambiguity of what counts as a "forward-directed path", especially for C. This can be resolved as follows: for each diagram, choose a direction for all of the wires such that each Z spider has at most input and each X spider has at most one output. For CNOT gates, this means the wire connecting the two spiders should go from Z to X, not vice-versa.

Exercise 6: In the lecture we saw how to reduce a parity normal form diagram to a CNOT circuit if its biadjacency matrix is invertible. Apply the Gaussian elimination procedure to the following diagram which does *not* have an invertible matrix to see what it reduces to:



Use this to argue that the diagram is not unitary.

Exercise 7: In *Picturing Quantum Software*, an algorithm is given for reducing a phase-free ZX-diagram to generalised parity form. Create a variation of this algorithm that allows Z and X phases that are integer multiples of π.