Quantum Group

INTERACTING FROBENIUS ALGEBRAS AND THE STRUCTURE OF MULTIPARTITE ENTANGLEMENT

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PRG-RR-09-12

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Abstract

Providing a generic description of entanglement in n-qubit systems is a long-standing open problem in quantum information science. A structural scheme for representing arbitrary multipartite entangled states will yield a deeper understanding of how they behave and interact within more general computational models and protocols. Here we provide such a description.

First we show that both the GHZ state and the W state admit a similar algebraic structure, which is only different in one important detail, and that for symmetric tripartite states this algebraic structure exactly characterises the corresponding SLOCC-entanglement classes. Our main theorem states that arbitrary SLOCC entanglement classes for multipartite pure entanglement arise from the interaction of the tripartite GHZ- and W-algebras.

The GHZ- and W-algebra are moreover subject to a purely diagrammatic calculus, and consequently, so are the resulting multipartite entangled states. In this graphical realm, the distinction between the GHZ-structure and W-structure is purely topological, in terms of 'connected vs. disconnected.' The graphical calculus gives rise to a generalised notion of graph state. In this realm, mathematical interaction of structures boils down to 'plugging graphs together'. In addition, the graph presentation teaches us how to prepare these multipartite entangled states.

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1 Introduction

Entanglement is one of the most important concepts in quantum information science. Hence it is somewhat surprising that so little is known about the general structure of multipartite pure entanglement. In [28] Nielsen provided a compelling structural characterisation of bipartite pure entanglement, which unfortunately, does not scale to multiple parties. Nielsen's account of bipartite pure state entanglement did teach us an important lesson: simple numeric measures of entanglement won't tell us the whole story. Indeed, once we consider two qutrits rather than two qubits, it is a *preorder* which classifies the entangled states, and not a total order such as $[0, 1]$. Thus, there are states $|\Psi\rangle$ and $|\Phi\rangle$ such that neither is more entangled than the other, nor are they equally entangled. Here we understand that $|\Psi\rangle$ is more or equally entangled than $|\Phi\rangle$ if, as a computational resource, whatever we can do with $|\Phi\rangle$ we can also do with $|\Psi\rangle$. When we consider four qubit states, this yields a structure significantly more rich than a total ordering. In [17] Dür, Vidal and Cirac showed that even for tripartite states there are two incomparable states, referred to as the GHZ state and the W state, which are both "maximally entangled" in some sense.

A class of multipartite entangled states which have been intensively studied in a structural manner are *graph states*, since they provide a key resource for measurement-based quantum computing (MBQC) [30, 18]. The structural understanding of these graph states, besides the obvious implications for MBQC, has recently also resulted in novel communication protocols [25]. As the name indicates, the structures governing them are (undirected) graphs with one kind of node and one kind of edge. The graph moreover tells us how to prepare these states by interpreting nodes as qubits and by interpreting edges as the application of a control-Z gate.

Here we provide a general structural account of multipartite entanglement which additionally provides a graph-state-like mechanism for describing the preparation and manipulation of a wide variety of entangled states. Our vehicle to do so will be algebraic, namely the theory of *interacting frobenius algebras*. This particular algebraic structure moreover admits a diagrammatic calculus, which translates different behaviours of multipartite entangled states into basic topological properties of graphs. As opposed to ordinary graph states, we will need to consider graphs with two kinds of nodes, embodying respectively the structure of the GHZ state and of the W state. Indeed, key to our approach is the fact that we can reduce the structure of multipartite entanglement to that of interacting GHZ and W states. In that sense, our approach relates the inductive procedure by Lamata, Leon, Salgado and Solano to classify arbitrary multipartite entangled states up to SLOCC-equivalence (SLOCC = stochastic local operations and classical communication) [23, 24]. Our main theorem provides a graph that witnesses each member of the continuous family of SLOCC-classes that they identified for four qubits, as well as any n-partite SLOCC-class they would identify when applying their inductive classifying procedure.

Frobenius algebras are becoming increasingly important in several areas of mathematical physics, and also in many other areas of science. For example, following Atiyah, they provide a very concise presentation of topological quantum field theories [2, 3, 21]. In logic, they provide a bridge between classical logic and linear logic [27]. They also allow diagrammatic axiomatisation of Hilbert space bases and C*-algebras [14, 35]. Other areas of applications include number theory, algebraic geometry, combinatorics, cohomology, quantum groups and coding theory. Historically, they trace back to Ferdinand Georg Frobenius' work on the representation theory of finite groups.

Terms in a frobenius algebra can be represented graphically, with their axioms expressible as a simple set of graph identities [7]. This graphical language traces back to Penrose's work in the early 1970's and was formalised by Joyal and Street [19]. [12] provides a pedestrian tutorial, and [31] provides a survey of the various flavours of this graphical language. The categorical (and hence diagrammatic) definition of frobenius algebras is due to Carboni and Walters [6] and the corresponding "spider" theorem for *special* commutative frobenius algebra (see Theorem 18 below) is due to Lack [22]. There exists a software tool to automate diagrammatic reasoning, nl. `quantomatic` [15], which was crafted by Duncan, Dixon and Kissinger. The results in this paper will now enable one to apply this tool to the study of multipartite entanglement and its applications.

Our principal interest in *commutative* frobenius algebras is that they provide a method for manipulating classical data [13] within the category-theoretic approach to quantum computation which was initiated by Abramsky and Coecke in [1]. Recently, Edwards, Spekkens and one of the authors provided a group-theoretic analysis of GHZ/Mermin-type quantum non-locality [10], based on a category-theoretic axiomatization of the GHZ state, a result upon which we will rely in this paper. Certain kinds of interacting commutative frobenius algebras are actually universal in the definition of two-dimensional quantum (qubit) states [7]. Not only can they generate any quantum state, the algebraic properties witness many of the *behavioural* properties of the states they generate. By considering quantum states as graphs and studying their properties via graph rewriting, we can drastically reduce computational complexity and offer an elegant relationship between the "shape" of a quantum state and its behaviour.

NOTE: *If time permits we could put a paragraph. "In this paper we proceed as follows ..."*

2 Entanglement

For hilbert spaces $\mathcal{H}_i$, $i = 1, ..., n$, let $|\Psi\rangle \in \bigotimes \mathcal{H}_i$ be a state. If there exist states $|\psi_i\rangle \in \mathcal{H}_i$ such that $|\Psi\rangle = \bigotimes |\psi_i\rangle$, $|\Psi\rangle$ is said to be *separable*. If no such states exist, $|\Psi\rangle$ is *entangled*. $|\Psi\rangle$ is a *degenerate* n-partite entangled state if there exist non-trivial $|\Phi_1\rangle$, $|\Phi_2\rangle$ such that $|\Psi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle$. If there are no such states, $|\Psi\rangle$ is a *genuine* n-partite entangled state. In $\mathbb{C}^2 \otimes \mathbb{C}^2$, two examples of genuine bipartite entanglement are the Bell state $|Bell\rangle = |00\rangle + |11\rangle$ and the EPR-state $|EPR\rangle = |10\rangle - |01\rangle$.

2.1 LOCC- vs. SLOCC-equivalence

It is often useful to consider entangled states only up to local operations. If two states can be deterministically inter-converted with only local (one-qubit) physical operations and classical communication, they are said to be *LOCC-equivalent*. If they can be inter-converted, but only with some non-zero probability, they are said to be stochastic LOCC-equivalent, or *SLOCC-equivalent.* These can be formalised by the following pair of results, due respectively to [4] and [17].

Theorem 1. *Two states* $|\Psi\rangle$ *and* $|\Phi\rangle$ *are LOCC-equivalent iff there exist local unitary maps* U_i *such that* $|\Psi\rangle = (U_1 \otimes U_2 \otimes \ldots \otimes U_n)|\Phi\rangle$.

Theorem 2. *Two states* $|\Psi\rangle$ *and* $|\Phi\rangle$ *are SLOCC-equivalent iff there exist local invertible maps* L_i *such that* $|\Psi\rangle = (L_1 \otimes L_2 \otimes \ldots \otimes L_n)|\Phi\rangle$.

These clearly both generate equivalence relations, and from hence forth, we shall use these as the definitions of LOCC and SLOCC. For example, $|Bell\rangle$ and $|EPR\rangle$ are LOCC equivalent, but they are not LOCC-equivalent to $\frac{1}{3}|00\rangle + \frac{2}{3}|11\rangle$, while $|Bell\rangle$, $|EPR\rangle$ and $\frac{1}{3}|00\rangle + \frac{2}{3}|11\rangle$ are all SLOCC equivalent.

2.2 GHZ vs. W states

In [17] it was shown that there are exactly two SLOCC-equivalence classes of genuine tripartite states in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The first is witnessed by a 3-qubit generalisation of the Bell state, called the *Greenberger–Horne–Zeilinger (GHZ) state*:

$$|GHZ\rangle = |000\rangle + |111\rangle \,,$$

and the second is witnessed by the *W state*:

$$|W\rangle = |100\rangle + |010\rangle + |001\rangle \,.$$

To assist with the constructions to follow, we shall prove a pair of technical lemmas.

Lemma 3. *If* $|\Psi\rangle$ *is symmetric and SLOCC-equivalent with* $|GHZ\rangle$*, then it is of the form* $(L \otimes L \otimes L)|GHZ\rangle$ *for some invertible* L.

Proof. Let $|\Psi\rangle := |u_1u_2u_3\rangle + |v_1v_2v_3\rangle$ be symmetric and SLOCC with $|GHZ\rangle$. Fix a non-zero vector $\langle u_1^\perp|$ such that $\langle u_1^\perp|u_1\rangle = 0$. Now,

$$(\langle u_1^\perp| \otimes 1)|\Psi\rangle = \langle u_1^\perp|v_1\rangle|v_2v_3\rangle = \langle u_1^\perp|v_1\rangle|v_3v_2\rangle \,.$$

We are in two dimensions, so if $\langle u_1^\perp|v_1\rangle = 0$, then $|u_1\rangle = k|v_1\rangle$, which would make $|\Psi\rangle$ separable. So $|v_2v_3\rangle = |v_3v_2\rangle$. Now, this is only possible if $\lambda'|v_2\rangle = |v_3\rangle$. Similarly, we have $\lambda|v_1\rangle = |v_2\rangle$. So, $|v_1v_2v_3\rangle = \lambda^2\lambda'|v_1v_1v_1\rangle$. Let $|v\rangle$ be a rescaling such that $|v_1v_2v_3\rangle = |vvv\rangle$. Performing a similar trick, for the $|u_i\rangle$'s, we can find $|u\rangle$ such that:

$$|\Psi\rangle = |uuu\rangle + |vvv\rangle \,.$$

□

Lemma 4. *If* $|\Psi\rangle$ *is symmetric and SLOCC-equivalent with* $|W\rangle$*, then it is of the form* $(L \otimes L \otimes L)|W\rangle$ *for some invertible* L.

Proof. Let $|\Psi\rangle := |u_1v_2v_3\rangle + |v_1u_2v_3\rangle + |v_1v_2u_3\rangle$. We apply the same trick as above:

$$(\langle v_i^\perp| \otimes 1)|\Psi\rangle = \langle v_i^\perp|u_i\rangle|v_jv_k\rangle = \langle v_i^\perp|u_i\rangle|v_kv_j\rangle$$

Again, we know $\langle v_i^\perp|u_i\rangle \neq 0$, so all the $|v_i\rangle$'s are scalar multiples of some vector $|v\rangle$. Take $|v\rangle$ to be normalised and let $\Psi = \lambda_1|u_1vv\rangle + \lambda_2|vu_2v\rangle + \lambda_3|vvu_3\rangle$. Rescale the u_i's to absorb these scalars, and we have:

$$|\Psi\rangle = |u_1'vv\rangle + |vu_2'v\rangle + |vvu_3'\rangle$$

Fix another vector $|v^\perp\rangle$ such that $\{|v\rangle, |v^\perp\rangle\}$ is an ONB. Since $|\Psi\rangle$ is symmetric:

$$|u_i'vv\rangle + |vu_j'v\rangle + |vvu_k'\rangle = |u_j'vv\rangle + |vu_k'v\rangle + |vvu_i'\rangle$$

If we apply $\langle v^\perp vv|$ to both sides:

$$\langle v^\perp|u_i'\rangle = \langle v^\perp|u_j'\rangle \neq 0$$

Let $\langle v^\perp|u_i'\rangle = \lambda$ for $i = 1, 2, 3$. Now, express the identity as a sum of projections:

$$1 = |v\rangle\langle v| + |v^\perp\rangle\langle v^\perp|$$

We now have $|u_i\rangle = 1|u_i\rangle = |v\rangle\langle v|u_i\rangle + |v^\perp\rangle\langle v^\perp|u_i\rangle = \langle v|u_i\rangle|v\rangle + \lambda|v^\perp\rangle$, thus we can write

$$|\Psi\rangle = \sum_i \langle v|u_i\rangle|vvv\rangle + \lambda\left(|v^\perp vv\rangle + |vv^\perp v\rangle + |vvv^\perp\rangle\right)$$

Define a linear map

$$L :: |v\rangle \mapsto |0\rangle, |v^\perp\rangle \mapsto \frac{1}{\lambda}\left(|1\rangle - \frac{\sum_i \langle v|u_i\rangle}{3}|0\rangle\right).$$

Now we have $(L \otimes L \otimes L)\,|\Psi\rangle = |W\rangle$. □

3 Frobenius algebras

We recall the usual notion of unital associative algebra on a vector space A over a field k. Consider a map $(- \cdot -) : A \times A \to A$. We say $(A, \cdot)$ is a *unital associative k-algebra* iff

- $(- \cdot -)$ is bilinear,
- $(|u\rangle \cdot |v\rangle) \cdot |w\rangle = |u\rangle \cdot (|v\rangle \cdot |w\rangle)$ for all $|u\rangle, |v\rangle, |w\rangle \in A$, and
- there exists $|\eta\rangle \in A$ such that $|u\rangle \cdot |\eta\rangle = |\eta\rangle \cdot |u\rangle = |u\rangle$ for all $|u\rangle \in A$.

Since $(- \cdot -)$ is bilinear, there exists a unique $\mu : A \otimes A \to A$ such that

$$\mu(|u\rangle \otimes |v\rangle) = |u\rangle \cdot |v\rangle$$

Taking this to define multiplication, we obtain the following definition.

Definition 5. For some field k, a *unital associative k-algebra* (A, μ, η) is a k-vector space A with maps $\mu : A \otimes A \to A$, $\eta : k \to A$ such that $\mu(1 \otimes \mu) = \mu(\mu \otimes 1)$ and $\mu(1 \otimes \eta) = \mu(\eta \otimes 1) = 1$.

We can also form a counital coassociative k-coalgebra. This is a unital associative k-algebra on the dual space B^*.

$$(B^*, \delta^* : B^* \otimes B^* \to B^*, \epsilon^* : k \to B^*)$$

To clarify, we make a direct definition in terms of B, rather than B^*.

Definition 6. A *counital, coassociative k-coalgebra* (B, δ, ϵ) is a k-vector space A with a map $\delta : B \to B \otimes B$ called the comultiplication and a map $\epsilon : B \to k$ called the counit such that $(1 \otimes \delta)\delta = (\delta \otimes 1)\delta$ and $(\epsilon \otimes 1)\delta = (1 \otimes \epsilon)\delta = 1$.

Let $\sigma_{A,B}$ be the swap map:

$$\sigma_{A,B} : A \otimes B \to B \otimes A :: |u\rangle \otimes |v\rangle \mapsto |v\rangle \otimes |u\rangle$$

Then, a k-algebra (resp. k-coalgebra) is *commutative* (resp. *cocommutative*) iff $\mu = \mu\sigma_{A,A}$ (resp. $\delta = \sigma_{A,A}\delta$).

Definition 7. A *frobenius k-algebra* $(F, \mu, \eta, \delta, \epsilon)$ is a vector space F such that

- (F, μ, η) is a unital associative k-algebra,
- (F, δ, ϵ) is a counital coassociative k-coalgebra, and
- $(\mu \otimes 1)(1 \otimes \delta) = (1 \otimes \mu)(\delta \otimes 1) = \delta\mu$.

Example 8. Let M be the vector space of $n \times n$ matrices. Take μ to be matrix multiplication, which is associative and bilinear. Let η be the $n \times n$ identity matrix, and let $\epsilon : M \to k$ be the trace functional. This data induces a unique map δ such that $(M, \mu, \eta, \delta, \epsilon)$ is a frobenius k-algebra.

Example 9. $(\mathbb{C}^2, \delta^\dagger, \epsilon^\dagger, \delta, \epsilon)$ is a frobenius $\mathbb{C}$-algebra, where $(-)^\dagger$ is the conjugate-transpose, $\{|0\rangle, |1\rangle\}$ is an orthonormal basis, and

- $\delta :: |0\rangle \mapsto |0\rangle \otimes |0\rangle$, $|1\rangle \mapsto |1\rangle \otimes |1\rangle$
- $\epsilon :: |0\rangle \mapsto 1$, $|1\rangle \mapsto 1$

Frobenius algebras can be formed over more structures than just vector spaces, such as relations, projective spaces, and smooth manifolds. We can discuss all of these at once if we develop a (nearly identical) abstract definition of frobenius algebras, using the language of category theory.

3.1 Symmetric monoidal categories

We shall briefly review the concept of symmetric monoidal categories. A gentler, physicist-oriented introduction to this and category theory in general is provided by [12].

A monoidal category is a category $\mathcal{V}$, with a bifunctor $(- \otimes -) : \mathcal{V} \times \mathcal{V} \to V$ that is (weakly) associative and unital. This means the for all objects $A, B, C \in \mathcal{V}$, $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, and there exists an object I such that $A \otimes I \cong A \cong I \otimes A$. These isomorphisms are subject to certain coherence properties (see e.g. Mac Lane [26]).

There are many categories that have such bifunctors, and often many choices of which bifunctor to use as $\otimes$ within a single category. However, an example of particular note is the category of finite-dimensional complex hilbert spaces and linear maps (FHilb). Here, we take $\otimes$ to be the usual tensor product. For any two hilbert spaces A and B, there is an isomorphism

$$\sigma_{A,B} : A \otimes B \to B \otimes A :: \psi \otimes \phi \mapsto \phi \otimes \psi$$

Definition 10. A *symmetric monoidal category* is a monoidal category that contains a natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$ such that σ commutes with associativity and unit isomorphisms, and $\sigma \circ \sigma = 1$.

Another important structure in FHilb we wish to capture is the property of having a dual space. We say A *has a dual* if there exists an object A^* and maps $d_A : I \to A^* \otimes A$ and $e_A : A \otimes A^* \to I$ such that

$$(d_A \otimes 1_A) \circ (1_A \otimes e_A) = 1_A$$

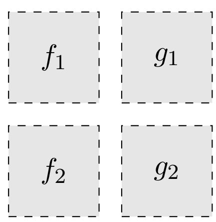
Definition 11. A symmetric monoidal category is *compact closed* if all objects have duals.

3.2 Graphical representation

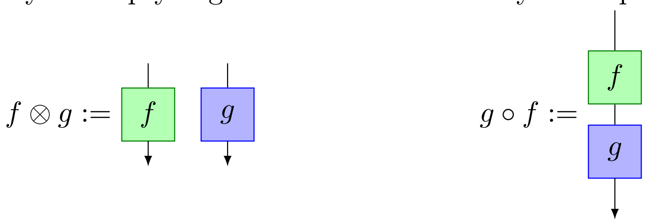
The theory of monoidal categories is in a strong sense a 2-dimensional theory. One interpretation for this dimensionality is that the tensor product provides a spacial dimension, while composition of arrows provides temporal, or causal dimension. The interplay of these two dimensions is represented by the bifunctoriality of the tensor product.

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1) \tag{1}$$

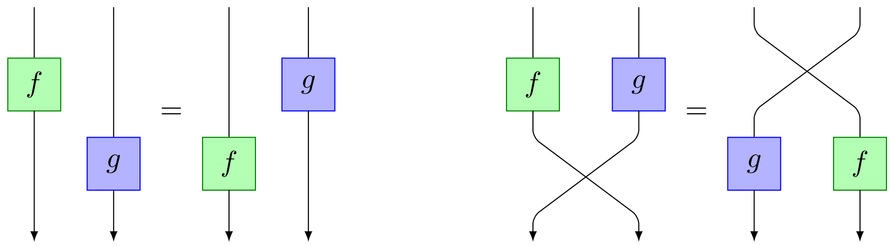
One can interpret this equation by thinking of these four arrows occupying a piece of 2-dimensional space.



From this point of view, the bracketing in eq (1) is a piece of essentially meaningless syntax, which is required to make something that is 2-dimensional by nature expressible as a (1-dimensional) term. To address this issue, we shall introduce a graphical notation for symmetric monoidal categories, similar to that of circuit diagrams. Edges represent objects and nodes represent arrows. Normally, both edges and nodes are labeled, but here we shall consider only graphs where every edge represents the same object, so we shall omit edge labels. Tensoring is done by juxtaposition and composition is performed by *plugging*, or gluing the inputs of one graph to the outputs of another. The identity arrow is represented by an empty edge and the tensor unit by an empty graph.



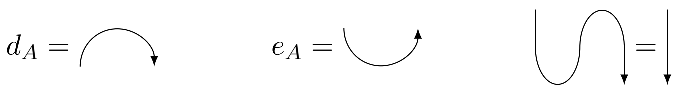
We can express the bifunctoriality of $\otimes$ and the naturality of σ as follows:



Edges, nodes, and edge crossings provide a graphical language for symmetric monoidal categories. This language captures exactly the coherence properties present in a symmetric monoidal category. Selinger states this precisely in [31].

Theorem 12. *(Coherence for symmetric monoidal categories). A well-formed equation between morphisms in the language of symmetric monoidal categories follows from the axioms of symmetric monoidal categories if and only if it holds, up to isomorphism of diagrams, in the graphical language.*

So far, we have introduced a graphical language for describing terms in a symmetric monoidal category. These terms are precisely the directed acyclic graphs generated by the arrows in the category. If the line represents an object A, and A has a dual, we can actually express terms as *arbitrary* graphs. We represent the type A as a line directed down, A^* as a line directed up, and the maps d_A and e_A from the previous section as caps and cups.

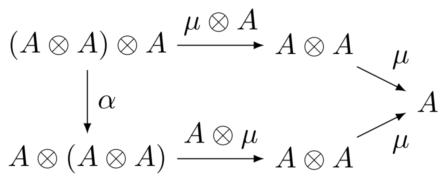


We have another coherence theorem from [20, 31] for this new, bigger diagrammatic language.

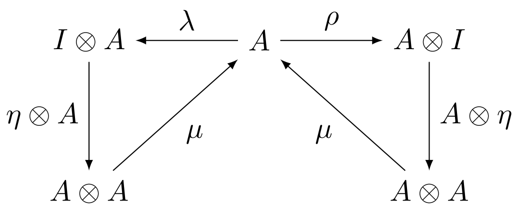
Theorem 13. *(Coherence for compact closed categories). A well-formed equation between morphisms in the language of compact closed categories follows from the axioms of compact closed categories if and only if it holds, up to isomorphism of diagrams, in the graphical language.*

3.3 Internal monoids and comonoids

A monoid *internal* to a monoidal category $(\mathcal{V}, \otimes, I)$, is an object A and a pair of maps $\mu : A \otimes A \to A$ defining multiplication and $\eta : I \to A$ picking out the unit. Multiplication is associative, so this diagram commutes.

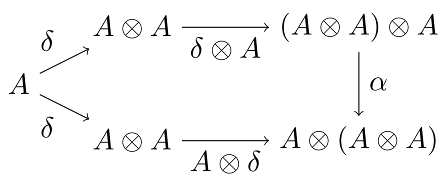


Multiplication is left and right unital, so this diagram also commutes.

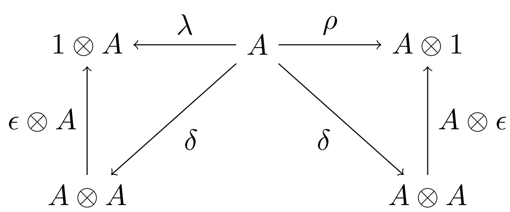


We now have the ability to equip a large variety of objects with a multiplicative structure. In the case where $\mathcal{V} = \text{Set}$, this recovers the usual notion of monoid. Monoids internal to Ab, the category of abelian groups, are rings, and monoids internal to Vect_k are associative k-algebras.

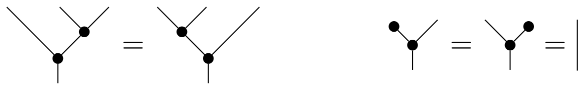
The dual of a monoid is a comonoid. For an object A in $\mathcal{V}$, one can define a *internal comonoid* as a triple $(A, \delta : A \to A \otimes A, \epsilon : A \to I)$ where the following diagrams commute. Coassociativity:



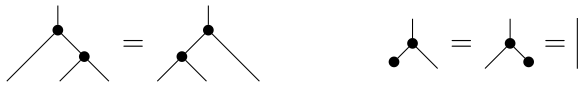
Counit:



Using the graphical language, we can express the axioms of a monoid $(A,$  $)$ as follows.



The axioms of a comonoid are just the previous ones, upside-down.



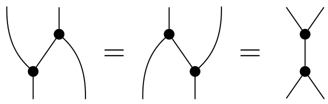
3.4 Internal frobenius algebras

Since the notion of internal monoid and comonoid give us an abstract way to define k-(co)algebras, we also have an abstract way to define frobenius algebras. We do so by essentially re-stating Def 7.

Definition 14. For a monoidal category $\mathcal{V}$ a *frobenius algebra* internal to $\mathcal{V}$ is an object A and four maps $\mu, \eta, \delta, \epsilon$ such that

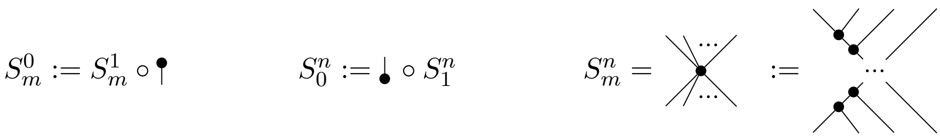
- (A, μ, η) is an internal monoid,
- (A, δ, ϵ) is an internal comonoid, and
- $(\mu \otimes 1) \circ (1 \otimes \delta) = (1 \otimes \mu) \circ (\delta \otimes 1) = \delta \circ \mu$.

Graphically, the third condition depicts as follows.



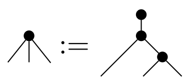
(2)

The frobenius identity guarantees that any *tree* consisting of $\mu, \eta, \delta, \epsilon$ has a unique normal form. By tree, we mean the graph contains no undirected cycles. For this situation, we introduce a special notation for canonical trees, known as *spiders*.



4 GHZ states and W states represent frobenius algebras

Why do we care about frobenius algebras? Let us begin by looking at this graph.



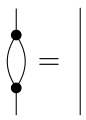
(3)

Let ($\mathbb{C}^2$, , , ,) be a CFA in FHilb. Recall that is a map from $\mathbb{C}^2$ to $\mathbb{C}^2 \otimes \mathbb{C}^2$ and that is a map from the tensor unit $\mathbb{C}$ to $\mathbb{C}^2$. So, (3) is a map $\Psi : \mathbb{C} \to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. We can interpret this map as a ket, simply taking $\Psi(1) = |\Psi\rangle$. The point is, every frobenius algebra has a way to represent states! We chose a tripartite example, because these shall be of particular interest. Up to SLOCC, there are two kinds of tripartite states, and it just so happens that there are two kinds of CFA's that serve to uniquely pick out those states.

4.1 Special and anti-special commutative frobenius algebras

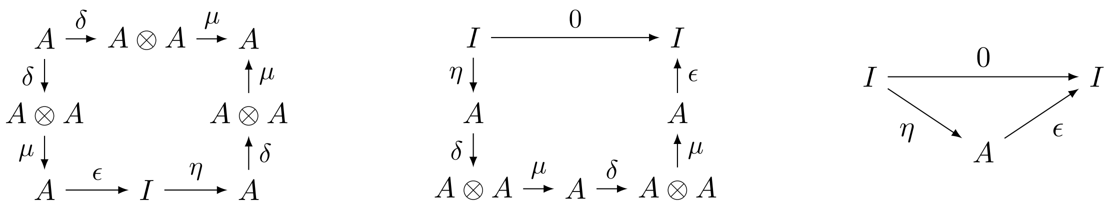
Let $(\mathcal{V}, \otimes, I)$ be a symmetric monoidal category. Also let $\mathcal{V}$ have a zero object that annihilates the tensor (i.e. $A \otimes 0 \cong 0$ for all A). In FHilb, this this object is just the zero space.

Definition 15. A *special frobenius commutative frobenius algebra (SCFA)* is a CFA such that $\mu \circ \delta = 1$. Graphically,

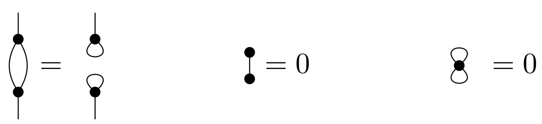


The notion of SCFA is a standard one in the literature e.g. [21]. We introduce here the new notion of ASCFA.

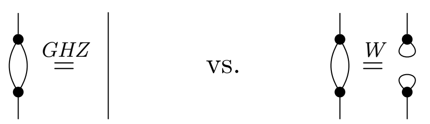
Definition 16. An *anti-special commutative frobenius algebra (ASCFA)* is a commutative frobenius algebra $(A, \delta, \epsilon, \mu, \eta)$ such that the following diagrams commute



Graphically, these conditions are:



Remark 17. The difference between an SCFA and an ASCFA is essentially topological:



in terms of 'connected vs. disconnected'.

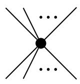
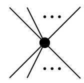
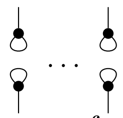
The graphical representations of SCFAs and ASCFa admit very simple normal forms. For SCFAs this was established by Lack in [22], and is spelled out more concretely in [11].

Theorem 18. *For an SCFA on V, any linear function with n inputs and m outputs, obtained from 1_V, μ, δ, $|e\rangle$, $|e'\rangle$ by composition and tensor, and with a connected diagrammatic representation, is equal to a 'spider' S^n_m. On the other hand, for an ASCFA, it is equal to:*

a. a 'spider' S^n_m if there are no loops;

b. $|\iota \ldots \iota\rangle\langle\iota' \ldots \iota'|$ if there is one loop;

c. the 0-map if there is more than one loop.

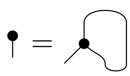
Hence, for $n, m \neq 0$, in the case of an SCFA all connected graphical representations are equal to $S^n_m =$  . Intuitively, this means that we can 'fuse' the dots representing multiplications and comultiplications together along connecting wires. In the case of an ASCFA, all connected graphical representations are either equal to $S^n_m =$  , or $|\iota \ldots \iota\rangle\langle\iota' \ldots \iota'| =$  , or it is the zero map. Hence also here the fusing interpretation still holds, except for where there is a loop, which causes the graph to completely disconnect, and the case of two or more loops stands for zero.

4.2 Special commutative frobenius algebras are GHZ States

Remark 19. Any SCFA in n-dimensional vector space has a co-multiplication that copies n linearly-independent vectors [14], called the *classical points* of the algebra.

$$\delta :: |u_1\rangle \mapsto |u_1 u_1\rangle, \ \ldots, \ |u_n\rangle \mapsto |u_n u_n\rangle$$

Furthermore, a comultiplication on a vector space generates a *unique* special frobenius algebra. This comes from the fact that



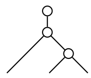
This is just the partial trace of the comultiply, so the unit is uniquely determined. This gives enough data to construct the whole frobenius algebra. We are now ready to state the main result for this section.

Theorem 20. *Each SCFA $\mathcal{G}$ on $\mathbb{C}^2$ canonically induces a symmetric state in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ which is SLOCC-equivalent to $|GHZ\rangle$. Conversely, any symmetric state that is SLOCC-equivalent to $|GHZ\rangle$ arises from a unique SCFA $\mathcal{G}$.*

Proof. (⇒) Let (, , ,) be an SCFA. Define the SCFA $\mathcal{Z}$ as follows:

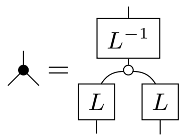
$$:: |0\rangle \mapsto |00\rangle, |1\rangle \mapsto |11\rangle; \qquad := \langle +|; \qquad := (\ )^\dagger; \qquad := (\ )^\dagger$$

It is easy to verify that this is indeed an SCFA. It can also be verified that

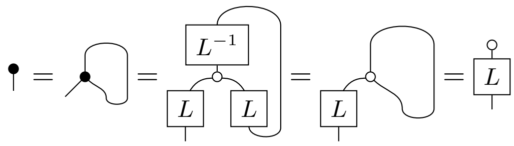


$$= \lambda|GHZ\rangle$$

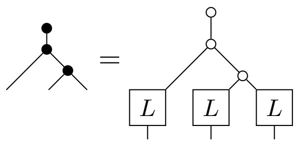
Let $|u\rangle, |v\rangle$ be the pair of linearly independent vectors copied by (cf. Rem 19). Define an invertible map $L :: |0\rangle \mapsto |u\rangle, |1\rangle \mapsto |v\rangle$. We can now define in terms of L and .



This induces as follows.



It then follows that



$$= \lambda(L \otimes L \otimes L)|GHZ\rangle$$

(⇐) In the other direction, we start with a symmetric state $|\Psi\rangle$ that is SLOCC with $|GHZ\rangle$. By Lem 3, we know $|\Psi\rangle$ must be of the following form, for L invertible.

$$|\Psi\rangle := (L \otimes L \otimes L)|GHZ\rangle \tag{4}$$

Let $\epsilon := \langle +|L^{-1}$. Let $cap := (\epsilon \otimes 1 \otimes 1) \circ |\Psi\rangle$. Now, pick a unique *cup* such that *cup* and *cap* form a compact structure. We can then define an ASCFA as follows:

$$\begin{aligned}
\delta &:= (cup \otimes 1 \otimes 1) \circ (1 \otimes |\Psi\rangle) \\
\epsilon &:= \langle +|L^{-1} \\
\mu &:= (1 \otimes cup) \circ (1 \otimes 1 \otimes cup \otimes 1) \circ (|\Psi\rangle \otimes 1 \otimes 1) \\
\eta &:= (1 \otimes \epsilon) \circ cap
\end{aligned}$$

Taking $|\Psi\rangle$ to be of the form of Eqn 4, we obtain the following values:

$$
\begin{aligned}
cap &:= (L \otimes L)(|00\rangle + |11\rangle) \\
cup &:= (\langle 00| + \langle 11|)(L^{-1} \otimes L^{-1}) \\
\delta &:= (L \otimes L)(|00\rangle\langle 0| + |11\rangle\langle 1|)L^{-}1 \\
\epsilon &:= \langle +|L^{-1} \\
\mu &:= L(|0\rangle\langle 00| + |1\rangle\langle 11|)(L^{-1} \otimes L^{-1}) \\
\eta &:= L|+\rangle
\end{aligned}
$$

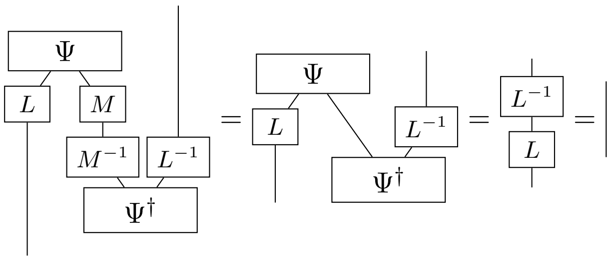
This set of generators does indeed obey the axioms of an ASCFA. □

4.3 Anti-special commutative frobenius algebras are W states

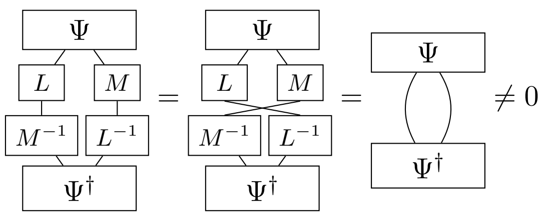
Theorem 21. *Each ASCFA $\mathcal{W}$ on $\mathbb{C}^2$ canonically induces a symmetric state in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ which is SLOCC-equivalent to $|W\rangle$. Conversely, any symmetric state that is SLOCC-equivalent to $|W\rangle$ arises from a unique ASCFA $\mathcal{W}$.*

Proof. (⇒) Let $\mathcal{W} := (\mathbb{C}^2, , , ,)$ be an ASCFA. $\cap := \circ $ and $\cup := \circ $. Since $\cap$ is part of a compact structure, it must be SLOCC with the bell state $|\Psi\rangle = |00\rangle + |11\rangle$. Let it be $(L \otimes M) \circ |\Psi\rangle$. Every cap yields a unique cup, namely $|\Psi\rangle^\dagger \circ (M^{-1} \otimes L^{-1})$.

NOTE: *Here and below, not sure what our convention is; do by $|\Psi\rangle^\dagger$ we mean $\langle\Psi|$?*

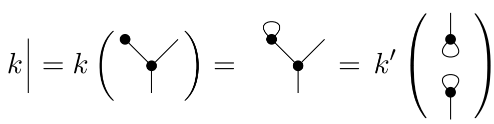


Since an ASCFA is commutative, $\cap = \sigma \circ \cap$. So, $\cup \circ \cap = |\Psi\rangle^\dagger \circ |\Psi\rangle \neq 0$:



Assume $\neq 0$, otherwise Z is trivial. Consider $:=$ $\circ \cap$. Suppose $= 0$. Then, $\circ$ $= \cup \circ \cap = 0$, which is a contradiction. So $\neq 0$.

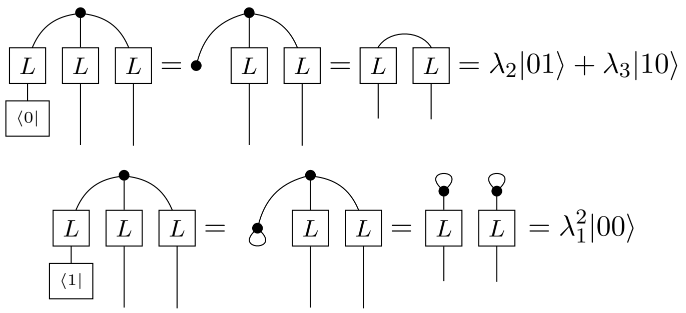
Now, suppose $k\,()$ = . Then k is non-zero and:



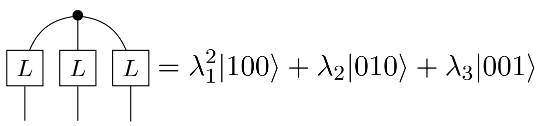
But this is impossible, since the map on the left is rank 2 and the map on the right is rank 1. So and are linearly independent, non-zero vectors. Similarly, and are such. Therefore, we can fix a non-singular map L where $\circ L = \langle 0|$ and $\circ L = \langle 1|$. Since $\{\langle 0|, \langle 1|\}$ is an orthonormal basis, we can determine the value of $L \circ$ by looking at the following projections. Let all λ_i be non-zero scalars.

We can do the same for .

So, $L \circ$ $= \lambda_1|1\rangle$ and $L \circ$ $= \lambda_1|0\rangle$. We use a similar trick to show $(L \otimes L) \circ \cap = \lambda_2|01\rangle + \lambda_3|10\rangle$. Projecting one of the qubits then gives a value for $(L \otimes L \otimes L) \circ$.



Therefore:



We can apply local linear maps of the form $\{|0\rangle \mapsto |0\rangle, |1\rangle \mapsto \frac{1}{\lambda}|1\rangle\}$ to undo the scalars, and we have a W state.

($\Leftarrow$) For the converse, we mirror the construction from the proof of Thm 20. The only difference is the choice of counit. Start with a symmetric state $|\Psi\rangle$ that is SLOCC with $|W\rangle$. By Lem 4, we know $|\Psi\rangle$ must be of the following form, for L invertible.

$$|\Psi\rangle := (L \otimes L \otimes L)|W\rangle \tag{5}$$

Let $\epsilon := \langle 0|L^{-1}$. Let $cap := (\epsilon \otimes 1 \otimes 1) \circ |\Psi\rangle$. Now, pick a unique *cup* such that *cup* and *cap* form a compact structure. We can then define an ASCFA as follows:

$$\begin{aligned}
\delta &:= (cup \otimes 1 \otimes 1) \circ (1 \otimes |\Psi\rangle) \\
\epsilon &:= \langle 0|L^{-1} \\
\mu &:= (1 \otimes cup) \circ (1 \otimes 1 \otimes cup \otimes 1) \circ (|\Psi\rangle \otimes 1 \otimes 1) \\
\eta &:= (1 \otimes \epsilon) \circ cap
\end{aligned}$$

Taking $|\Psi\rangle$ to be of the form of Eqn 5, we obtain the following values:

$$\begin{aligned}
cap &:= (L \otimes L)(|10\rangle + |01\rangle) \\
cup &:= (\langle 10| + \langle 01|)(L^{-1} \otimes L^{-1}) \\
\delta &:= (L \otimes L)(|10\rangle\langle 1| + |01\rangle\langle 1| + |00\rangle\langle 0|)L^-1 \\
\epsilon &:= \langle 0|L^{-1} \\
\mu &:= L(|1\rangle\langle 11| + |0\rangle\langle 01| + |0\rangle\langle 10|)(L^{-1} \otimes L^{-1}) \\
\eta &:= L|1\rangle
\end{aligned}$$

This set of generators does indeed obey the axioms of an SCFA. □

For $|GHZ\rangle$ the corresponding SCFA is:

$$\mu_{\mathcal{G}} = |0\rangle\langle 00| + |1\rangle\langle 11| \qquad |e_{\mathcal{G}}\rangle = |0\rangle + |1\rangle$$
$$\delta_{\mathcal{G}} = |00\rangle\langle 0| + |11\rangle\langle 1| \qquad \langle e'_{\mathcal{G}}| = \langle 0| + \langle 1|$$

and for $|W\rangle$ the corresponding ASCFA is:

$$\mu_{\mathcal{W}} = |0\rangle(\langle 01| + \langle 10|) + |1\rangle\langle 11| \qquad |e_{\mathcal{W}}\rangle = |1\rangle$$
$$\delta_{\mathcal{W}} = |00\rangle\langle 0| + (|01\rangle + |10\rangle)\langle 1| \qquad \langle e'_{\mathcal{W}}| = \langle 0|$$

It follows that

$$\eta_{\mathcal{G}} = |00\rangle + |11\rangle \qquad \eta_{\mathcal{W}} = |01\rangle + |10\rangle \qquad |\iota_{\mathcal{W}}\rangle = |0\rangle \qquad \langle \iota'_{\mathcal{W}}| = \langle 1|\,,$$

so in particular, $\{|\iota_{\mathcal{W}}\rangle, |e_{\mathcal{W}}\rangle\} = \{|0\rangle, |1\rangle\}$ forms a basis.

4.4 Interaction of GHZ and W states

There is a bijective correspondence between the bases of V and SCFAs on V [14]. One extracts the basis corresponding to an SCFA as its *δ-copiable vectors*, that is, the vectors $|i\rangle$ satisfying $\delta|i\rangle = |i\rangle \otimes |i\rangle$. Conversely, given a basis on V, *δ-copiability* of the basis vectors uniquely defines an SCFA. So SCFAs on $\mathbb{C}^2$ have two δ-copyable vectors. For SCFAs, by Thm 18.b, we have $\delta|\iota\rangle = |\iota\rangle \otimes |\iota\rangle$. However, for ASCFAs on $\mathbb{C}^2$ this is the only δ-copyable vector.

NOTE: *Needs to be † to force basis to be normalised.*

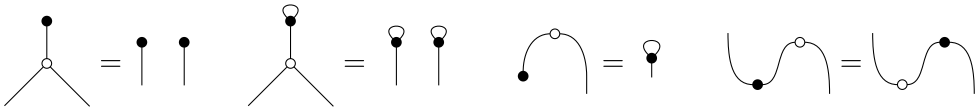
Theorem 22. *There is a bijective correspondence between SCFAs $\mathcal{G}$ and ASFCAs $\mathcal{W}$ on $\mathbb{C}^2$ via the relation:*

$$\delta_{\mathcal{G}}|e_{\mathcal{W}}\rangle = |e_{\mathcal{W}}\rangle \otimes |e_{\mathcal{W}}\rangle \qquad \delta_{\mathcal{G}}|\iota_{\mathcal{W}}\rangle = |\iota_{\mathcal{W}}\rangle \otimes |\iota_{\mathcal{W}}\rangle \tag{6}$$

We call these canonical pairs *MS-pairs.* It then follows that these MS-pairs are in bijective correspondence with normalised bases of $\mathbb{C}^2$. For MS-pairs we moreover have:

$$(\langle e'_{\mathcal{W}}| \otimes 1_V)|\eta_{\mathcal{G}}\rangle = |\iota_{\mathcal{W}}\rangle \tag{7}$$
$$(\langle \epsilon_{\mathcal{W}}| \otimes 1_V)(1_V \otimes |\eta_{\mathcal{G}}\rangle) = (\langle \epsilon_{\mathcal{G}}| \otimes 1_V)(1_V \otimes |\eta_{\mathcal{W}}\rangle) \tag{8}$$

When we combine an SCFA with an ASCFA, then a wealth of multipartite states emerge, as we show below. To distinguish the SCFA from the ASCFA when we combine them, we denote the SCFA by (, , ,) and the ASCFA by (, , ,). Eqs. (6, 7, 8) which connect them now depict as follows:

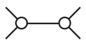
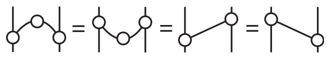
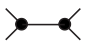
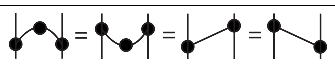
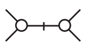
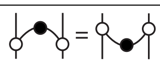
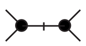
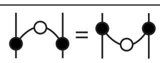
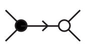
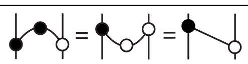
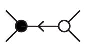
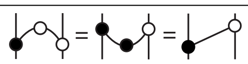


Intuitively, the role played by the 'white dots' is copying the basis vectors , , while the 'black dots' have a control-function, altering the topology in a way which depends on the basis vector (cf. Thm. 18). Combining the two generates a rich class of possible behaviors.

5 Graphs for arbitrary multipartite states

The above diagrams are similar to circuits in that they are obtained by composing components (cf. gates) both in parallel and sequentially, but different in the sense that the number of inputs and outputs of these components do not have to coincide. This flexibility is embodied by the fact that the category FHilb has duals, so we can bend lines, in the sense of section 3.2.

Let (,) be the canonical MS-pair, i.e. the pair generated from the GHZ and W states via the constructions in section 4. By relying on the equalities induced by Theorem 18 and eq. (7), we introduce the following conventions for the edges in the corresponding graphs:

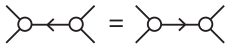
	edge in graph	categorical interpretation	hilbert space interpretation
(a)			$\vert 00\rangle\langle 00\vert + \vert 11\rangle\langle 11\vert$
(b)			$\vert 00\rangle(\langle 01\vert + \langle 10\vert) + (\vert 01\rangle + \vert 10\rangle)\langle 11\vert$
(c)			$\vert 01\rangle\langle 01\vert + \vert 10\rangle\langle 10\vert$
(d)			$1_{\mathbb{C}^2\otimes\mathbb{C}^2} + \vert 11\rangle\langle 11\vert$
(e)			$\vert 00\rangle\langle 00\vert + \vert 11\rangle\langle 01\vert + \vert 10\rangle\langle 10\vert$
(f)			$\vert 01\rangle\langle 01\vert + \vert 00\rangle\langle 10\vert + \vert 11\rangle\langle 11\vert$

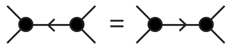
There are a number of differences with 'standard' graph state notation, most importantly that 'loose ends' stand for qubits, not nodes. In other words, a node can either represent a simple scaling factor if it has no incident edges, or it could represent n entangled qubits if it has n external edges.

Another difference is that there now are two kinds of nodes, representing the GHZ-algebra and W-algebra respectively. There are also three kinds of edges, regular ones, edges with a tick, and directed edges. We use this as a notational trick to reflect the identities given in the above table.

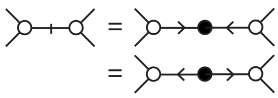
One could eliminate the need for edges (a,b) by assuming that the graphs are always in spider-normal-form (cf. Theorem 18). Note that in this context that the tick on edges (c,d) can be interpreted as 'prohibiting to fuse the dots of the same colour to a single dot'.

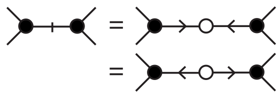
Arrows reflect the direction of composition in the categorical interpretation of the edge. Recall that the graphs in the second column should be read from top to bottom. The arrow in (e) reflects the fact that the white dot is post-composed with the black dot. When the arrow is omitted, this means the direction of composition is irrelevant. I.e.





On the other hand, edges (c,d) with the ticks cannot be directed. They have (equivalent) decompositions into the directed edges:

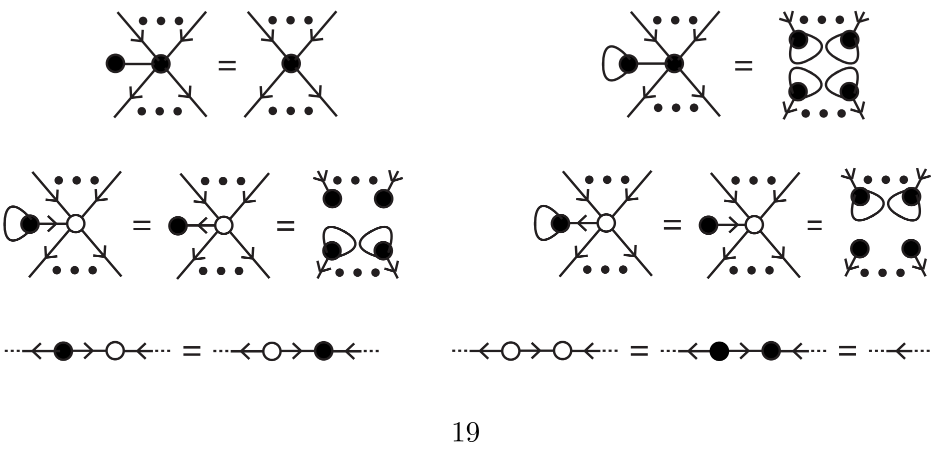




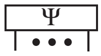
Edges (c,d) are indeed not atomic but compound. However, treating them as basic will simplify graphs representing multipartite states. By convention, we will assign outgoing arrows to the loose ends, representing the 'outputs' of that state.

5.1 Evaluating graphs of multipartite states under post-selection

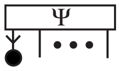
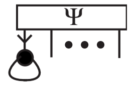
From Theorem 18 and eqs. (6, 7, 8) together we can derive equalities between these graphs. Given a graph, assume that we have assigned directions to all edges, in the sense discussed above. Then the following rules preserve the state which this graph represents:



We will be particularly interested in what a graph representing a multipartite entangled state 'reduces to' when we post-select by means of either  $= \langle e'_{\mathcal{W}}| = \langle 0|$ or  $= \langle \iota'_{\mathcal{W}}| = \langle 1|$ on one of its outputs. Explicitly, given a graph

 $= |\Psi\rangle \in \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n}$

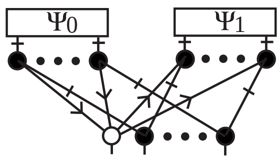
we can compute the graph of two states $|\Psi_0\rangle, |\Psi_1\rangle \in \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-1}$ defined as follows:

$|\Psi_0\rangle =$  $= ((\langle 0| \otimes 1_{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}))|\Psi\rangle \qquad |\Psi_1\rangle =$  $= ((\langle 1| \otimes 1_{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}))|\Psi\rangle$

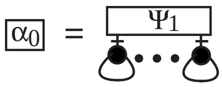
If from the resulting graphs we can deduce what $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are, then we know that the initial graph represented the state $|0\Psi_0\rangle + |1\Psi_1\rangle$.

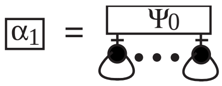
The following theorem provides a converse to this procedure, i.e. it tells us how to produce a graph for the state $\alpha_0|0\Psi_0\rangle + \alpha_1|1\Psi_1\rangle$, which for $\alpha_0, \alpha_1 \neq 0$ is SLOCC-equivalent to $|0\Psi_0\rangle + |1\Psi_1\rangle$, given graphs of the states $|\Psi_0\rangle$ and $|\Psi_1\rangle$.

Theorem 23. *Given the graphs of the states* $|\Psi_0\rangle, |\Psi_1\rangle \in \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-1}$*, then a graph representing the state* $\alpha_0|0\Psi_0\rangle + \alpha_1|1\Psi_1\rangle \in \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n}$ *can be constructed as follows:*

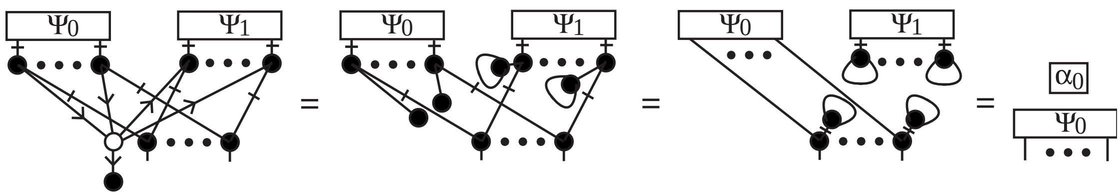


where:





Proof. Postselecting by means of  $= \langle e'_{\mathcal{W}}| = \langle 0|$ yields



i.e. $\alpha_0|0\Psi_0\rangle$, and similarly, post-selecting by means of $= \langle \iota'_{\mathcal{W}}| = \langle 1|$ yields $\alpha_1|1\Psi_1\rangle$. $\square$

5.2 Generating witnesses for SLOCC entanglement classes

For four qubit multipartite states the number of SLOCC-classes is not finite but rather constitutes a continuous family [17]. In [24] the authors presented an inductive procedure to produce witnesses for the SLOCC-classes of arbitrary multipartite states of N qubits. In [23] these were explicitly constructed for the case of four qubits.

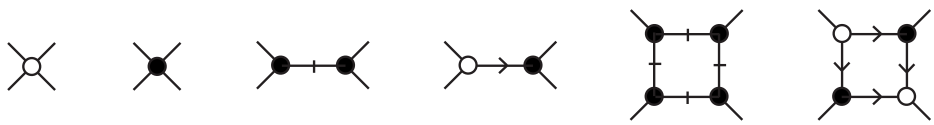
We now sketch this inductive procedure. Assume that we know the SLOCC-classes for $N-1$ qubits. These classes now play the role of the right singular vectors in the singular value decomposition of the N-qubit state, when conceiving it via the Choi-Jamiolkowski isomorphism, as a 1-input $(N-1)$-output linear functional. We use a local linear map to take the left-singular vectors to the computational basis. It is now of the form $|0\Psi_0\rangle + |1\Psi_1\rangle$, where $|\Psi_0\rangle, |\Psi_1\rangle$ are $(N-1)$-partite states that span the right singular subspace. We apply the induction hypothesis to classify the states that span this space. This leads to a classification of the N-qubit SLOCC-classes in terms of the $N-1$-qubit SLOCC-classes.

Theorem 23 above allows us to produce a graphical witness for each SLOCC-class produced in this manner, given we know the graph of the right singular vectors of a state which witness that SLOCC-class.

Corollary 24. *Given the graphs representing the $N-1$ qubit SLOCC-classes the structure of an MS-pair allows to construct the graphs of the SLOCC-classes of N qubits.*

Let us provide some examples here.

Following [24], the continuous family of non-degenerate SLOCC-classes group themselves in eight superclasses, i.e. radically inequivalent ways in which four qubits can be entangled. Each of the following graphs witnesses a different SLOCC-superclass:



The first two are the four qubit GHZ- and W-state respectively for which we have:

- $|GHZ_4\rangle = |0\rangle|000\rangle + |1\rangle|111\rangle$
- $|W_4\rangle = |0\rangle|W\rangle + |1\rangle|000\rangle$

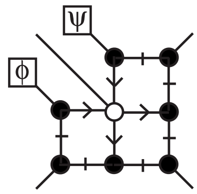
Using the above rules on graphs one verifies that the other four respectively are:

- $|0\rangle \underbrace{(|000\rangle + |101\rangle + |010\rangle)}_{\overset{\text{SLOCC}}{\simeq}|W\rangle} + |1\rangle(|0\rangle \underbrace{(|01\rangle + |10\rangle)}_{\overset{\text{SLOCC}}{\simeq}|Bell\rangle})$
- $|0\rangle|000\rangle + |1\rangle(|1\rangle \underbrace{(|01\rangle + |10\rangle)}_{\overset{\text{SLOCC}}{\simeq}|Bell\rangle})$
- $|0\rangle \underbrace{(|0\rangle(|01\rangle + |10\rangle) + |1\rangle(|00\rangle + |11\rangle))}_{\overset{\text{SLOCC}}{\simeq}|GHZ\rangle} + |1\rangle \underbrace{(|010\rangle + |111\rangle + |101\rangle)}_{\overset{\text{SLOCC}}{\simeq}|W\rangle}$

- $|0\rangle \underbrace{(|000\rangle + |111\rangle)}_{\overset{\text{SLOCC}}{\simeq}|GHZ\rangle} + |1\rangle|010\rangle$

respectively, from which we also easily read the corresponding right singular vectors. Here is an example computation:

Note that the above depicted graphs are in several cases simpler than the ones which would arise via Theorem 23. One of the two remaining superclasses as a whole, which is parameterised by two single qubit states $|\psi\rangle, |\phi\rangle$, depicts as follows:



which corresponds with the parametrized SLOCC-superclass:

- $|0\rangle(\underbrace{(|00\rangle + |1\psi\rangle)}_{\overset{\text{SLOCC}}{\simeq}|Bell\rangle}|\phi\rangle) + |1\rangle|0\rangle|Bell\rangle$

There is in fact a generally applicable procedure to adjoin such variables in order to obtain all SLOCC-classes, once we have found one graph witnessing a SLOCC-class within a superclass, which again directly follows from Theorem 23.

By means of examples like the ones depicted above one can establish the following:

Theorem 25. *Given a MS-pair on $\mathbb{C}^2$ we can construct a witness for each SLOCC-superclass of four qubits, in terms of 1_V, $\mu_\mathcal{G}$, $\delta_\mathcal{G}$, $|e_\mathcal{G}\rangle$, $|e'_\mathcal{G}\rangle$, $\mu_\mathcal{W}$, $\delta_\mathcal{W}$, $|e_\mathcal{W}\rangle$, $|e'_\mathcal{W}\rangle$, composition and tensor. If we freely adjoin the continues variables of these superclasses we obtain the continuous family of all SLOCC-classes.*

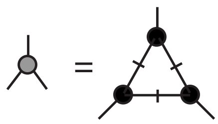
Remark 26 (W states and other entangled states in other categories and toy theories)**.** Let FRel be the category with finite sets as objects, relations between these as morphisms, and the cartesian product as the tensor. Since the concrete form of the ASCFA studied in the paper is *relational* (i.e. it has a matrix form consisting entirely of 0's and 1's), it follows that there exists an ASCFA on the two-element set in FRel! Hence we can model W states in this category. This extends the list of quantum-like features that can be modelled on the two-element set in FRel even further. In [8] it was shown that complementary observables can be modelled on the two-element set and relations on it. Moreover, since the corresponding SCFA lives in FRel, all of the witnesses of four-partite SLOCC-classes we constructed above also live in FRel. On the other hand, these W states on the two-element set in FRel do not seem to canonically lift to the qubit in

Spek [8, 10], the category which models Spekkens' toy qubit theory [32]. This is because their matrix representation involves 3 non-zero entries, and hence violates the knowledge-balance principle. Further analysis of this situation could cast light on the theoretical content of a model that is truly non-local.

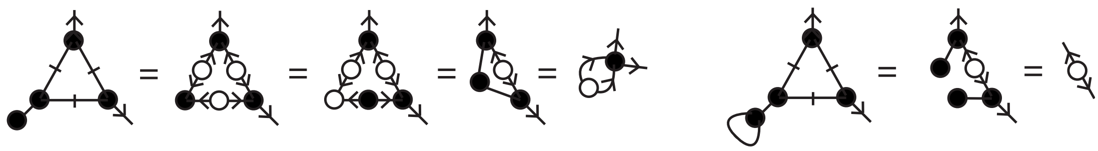
5.3 Generalised graph states

In [7] it was shown how to produce graph states using a pair of *complementary*[1] SCFAs. The authors encode graph states as graphs where adjacent nodes correspond to different SCFAs. It easily follows that in this way one can construct (up to LOCC-equivalence) all graph states which involve no cycles of odd-length. If one adjoins an idempotent to the calculus which transforms one SCFA into the other (e.g. the Hadamard gate in the case of the Z- and the X-observables) then one can construct arbitrary graph states.

Theorem 27. *From an ASCFA (, , ,) in an MS-pair we can construct another SCFA which is complementary to the SCFA (, , ,) in the sense of* [7] *as follows:*



Proof. (Sketch) We will show the result in the specific case where the ASCFA (, , ,) is generated by the W state (in the sense of section 4) and the SCFA (, , ,) is generated by the GHZ state. Then, by rewriting we have:



!TODO: *RESEARCH QUESTION: can we establish this in abstract terms i.e. derive complementarity from rules governing MS-pairs? That would be great; see REMARK below.*

where the first resulting graph is easily seen to be $|01\rangle + |10\rangle$. Hence the state is $|0\rangle(|01\rangle + |10\rangle) + |1\rangle(|00\rangle + |11\rangle)$. By expressing this map in the $\{|+\rangle, |-\rangle\}$ basis, it can be seen that this is the complementary co-multiplication.

$$\delta_X :: |+\rangle \mapsto |++\rangle, |-\rangle \mapsto |--\rangle$$

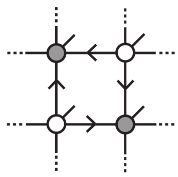
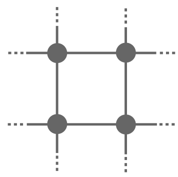
The multiplication is constructed similarly, the co-unit is , and the unit is . □

Remark 28 (Complementarity versus the GHZ/W-dichotomy)**.** The fact that an MS-pair enables us to construct a pair of complementary observables, while the converse doesn't hold, seems to indicate that from a structural perspective, a MS-pair seems to be a more fundamental notion than that of a pair of complementary observables. The fact that we cannot construct a state which is SLOCC-equivalent to the W-state from a pair of complementary observables is a consequence of the results in [9], where it was shown

[1] I.e. those that have mutually unbiased bases as their δ-copiable points.

that one needs to adjoin non-trivial *phases* to a pair of complementary observables in order to construct such a state. In the light of Remark 26 this even more indicates the MS-pairs are structurally more fundamental than pairs of complementary observables.

Corollary 29. *Given an MS-pair on $\mathbb{C}^2$ we can construct all graph states which involve no cycles of odd-length up to LOCC-equivalence.*

MS-pair graph	corresponding graph state
	

While there are graph states which are not of this kind, the simplest being the 'pentagon' graph state, they all are SLOCC-equivalent to one that we can construct. An extension of the graphical language of an MS-pair does allow one to represent any graph state directly. If following [7], we adjoin the idempotent which transforms the two SCFAs into each other, then we can represent all graph states, and as shown by Duncan and Perdrix in [16], the equations governing the corresponding calculus allows one to derive van den Nest's local complementation theorem.

In [7] it was also shown how the diagrammatic language exposes different manners of preparing graph states, be it either by application of entangling gates [30], or by *fusion* [5, 34]. Since the generalised graphs introduced here are also built up from simple components a fusion-like technique applies here too. There will now be two ways of fusing required, *GHZ-fusion*, which is the same as fusion in [5, 34], and also *W-fusion*.

References

[1] S. Abramsky and B. Coecke (2004) *A categorical semantics of quantum protocols.* In: Proceedings of 19th IEEE conference on Logic in Computer Science (LiCS), pages 415–425. IEEE Press. arXiv:quant-ph/0402130.

[2] M. Atiyah (1989) *Topological quantum field theories.* Publications Mathématique de l'Institut des Hautes Études Scientifiques **68** , 175–186.

[3] J. C. Baez and J. Dolan (1995) *Higher-dimensional algebra and topological quantum field theory.* Journal of Mathematical Physics **36**, 60736105. arXiv:q-alg/9503002

[4] C. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin and A. V. Thapliyal (1999) *Exact and asymptotic measures of multipartite pure state entanglement.* Physical Review A. arXiv:quant-ph/9908073v3

[5] D. E. Browne and T. Rudolph (2005) *Resource-efficient linear optical quantum computation.* Physical Review Letters **95**, 010501.

[6] A. Carboni and R. F. C. Walters (1987) *Cartesian bicategories I.* Journal of Pure and Applied Algebra **49**, 11–32.

[7] B. Coecke and R. Duncan (2008) *Interacting quantum observables.* In: Proceedings of the 35th International Colloquium on Automata, Languages and Programming (ICALP), pp. 298–310, Lecture Notes in Computer Science 5126, Springer-Verlag. An extended and improved version *Interacting quantum observables: categorical algebra and diagrammatics* is available as arXiv:0906.4725

[8] B. Coecke and B. Edwards (2008) *Toy quantum categories.* Electronic Notes in Theoretical Computer Science, to appear. `arXiv:0808.1037`

[9] B. Coecke and B. Edwards (2009) *Three qubit entanglement analysed with graphical calculus.* Oxford University Computing Laboratory Research Report PRG-RR-09-03.

[10] B. Coecke, B. Edwards and R. W. Spekkens (2009) *The group theoretic origin of non-locality for qubits.* Oxford University Computing Laboratory Research Report PRG-RR-09-04. http://web.comlab.ox.ac.uk/publications/publication3026-abstract.html

[11] B. Coecke and E. O. Paquette (2008) *POVMs and Naimark's theorem without sums.* Electronic Notes in Theoretical Computer Science **210**, 15–31. arXiv:quant-ph/0608072

[12] B. Coecke and E. O. Paquette (2009) *Categories for the practicing physicist.* In: New Structures for Physics, pp. 167–271, B. Coecke (ed). Springer Lecture Notes in Physics. arXiv:0905.3010v2

[13] B. Coecke, E. O. Paquette and D. Pavlovic (2009) *Classical and quantum structuralism.* To appear in Semantic Techniques for Quantum Computation, I. Mackie and S. Gay (eds), Cambridge University Press.

[14] B. Coecke, D. Pavlovic, and J. Vicary (2008) *A new description of orthogonal bases.* To appear in Mathematical Structures in Computer Science. arXiv:0810.0812

[15] L. Dixon, R. Duncan and A. Kissinger, `quantomatic` software tool, available form http://dream.inf.ed.ac.uk/projects/quantomatic/

[16] R. Duncan and S. Perdrix (2009) *Graph states and the necessity of Euler decomposition.* In: Proceedings of Computability in Europe: Mathematical Theory and Computational Practice (CiE'09), pages 167–177. Lecture Notes in Computer Science 5635, Springer-Verlag. arXiv:0902.0500

[17] W. Dür, G. Vidal and J. I. Cirac (2000) *Three qubits can be entangled in two inequivalent ways.* Physical Review A **62**, 062314. arXiv:quant-ph/0005115

[18] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Van den Nest and H.-J. Briegel (2006) *Entanglement in graph states and its applications.* arXiv:quant-ph/0602096v1

[19] A. Joyal and R. Street (1991) *The Geometry of tensor calculus* I. Advances in Mathematics **88**, 55–112.

[20] G. M. Kelly and M. L. Laplaza (1980) *Coherence for compact closed categories.* Journal of Pure and Applied Algebra **19**, 193–213.

[21] J. Kock (2003) *Frobenius Algebras and 2D Topological Quantum Field Theories.* Cambridge University Press.

[22] S. Lack (2004) *Composing PROPs.* Theory and Applications of Categories **13**, 147–163. http://www.tac.mta.ca/tac/volumes/13/9/13-09abs.html

[23] L. Lamata, J. Leon, D. Salgado and E. Solano (2006) *Inductive classification of multipartite entanglement under SLOCC.* Physical Review A **74**, 052336. arXiv:quant-ph/0603243

[24] L. Lamata, J. Leon, D. Salgado and E. Solano (2007) *Inductive entanglement classification of four qubits under SLOCC.* Physical Review A **75**, 022318. arXiv:quant-ph/0610233

[25] D. Markham and B. C. Sanders (2008) *Graph states for quantum secret sharing.* Physical Review A **78**, 042309. arXiv:0808.1532

[26] S. Mac Lane. (1971) *Categories for the Working Mathematician.* Springer.

[27] P.-A. Melliés (2009) *Categorical semantics of linear logic.* `www.pps.jussieu.fr/~mellies/papers/panorama.pdf`

[28] M. A. Nielsen (1999) *Conditions for a class of entanglement transformations.* Physical Review Letters **83**, 436–439.

[29] R. Penrose (1971) *Applications of negative dimensional tensors.* In: Combinatorial Mathematics and its Applications, pages 221–244. Academic Press.

[30] R. Raussendorf, D. E. Browne and H.-J. Briegel (2003) *Measurement-based quantum computation on cluster states.* Physical Review A **68**, 022312. arXiv:quant-ph/ 0301052.

[31] P. Selinger (2009) *A survey of graphical languages for monoidal categories.* In: New Structures for Physics, pp. 275–337, B. Coecke (ed). Springer Lecture Notes in Physics. http://www.mscs.dal.ca/~selinger/papers.html.

[32] R. Spekkens (2007) *Evidence for the epistemic view of quantum states: A toy theory.* Physical Review A **75**, 032110. arXiv:quant-ph/0401052

[33] R. Street (2007) *Quantum Groups: A Path to Current Algebra*. Cambridge University Press.

[34] F. Verstraete and J. I. Cirac (2004) *Valence-bond states for quantum computation.* Physical Review A **70**, 060302(R). arXiv:quant-ph/0311130

[35] J. Vicary (2008) *Categorical formulation of quantum algebras*. Communications in Mathematical Physics. To appear. `arXiv:0805.0432`